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A New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling

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Abstract This paper introduces a new flexible four-parameter lifetime model. Various of its structural properties are derived. The new density is expressed as a linear mixture of well-known exponentiated Weibull density. The maximum likelihood method is used to estimate the model parameters. Graphical simulation results to assess the performance of the maximum likelihood estimation are performed. We proved empirically the importance and flexibility of the new model in modeling four various types of data.

Keywords Marshall-Olkin Family; Lehmann Weibull Distribution; Order Statistics, Maximum Likelihood Estimation; Simulation; Generating Function; Quantile function; Moments.

AMS 2010 subject classifications 62N01; 62N02; 62E10

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1. Introduction and motivation

Based on Weibull[52] and Lehmann[40], consider a baseline reliability function (RF) of the Lehmann Weibull (LW) distribution

$$\overline{G}_{\beta,a_1,a_2}(y)=[1-G_{a_1,a_2}(y)]^{\beta}=\exp\left[-\beta\left(a_2 y\right)^{a_1}\right]|_{(y\geq 0 \text{ and } \beta,a_1,a_2>0)}, \tag{1}$$

with probability density function (PDF)

$$g_{\beta,a_1,a_2}(y)=\beta a_2^{a_1} a_1 y^{a_1-1}\exp\left[-\beta\left(a_2 y\right)^{a_1}\right]|_{(y\geq 0 \text{ and } \beta,a_1,a_2>0)}, \tag{2}$$

For $\beta=a_1=1$ we have the Exponential (Exp) model. For $\beta=a_1=1$ we have the one-parameter W model. For $\beta=1,a_1=2$ we have the Rayleigh (R) model. For $a_1=1$ we have the Lehmann Exp (LExp) model. For $a_1=1$ we have the two-parameter Lehmann W model. For $a_1=2$ we have the Lehmann R (LR) model. The RF of the Marshall-Olkin-G (MO-G) family of distributions is defined by

$$\overline{F}_{\alpha,\psi}(y)=1-\frac{1-G_{\underline{\psi}}(y)}{1-\overline{\alpha}G_{\underline{\psi}}(y)}|_{(y\in\Re,\ \alpha>0)}, \tag{3}$$

where α is a positive shape parameters and $\overline{\alpha}=1-\alpha$. For $\alpha\in(0,1)$, MOL-G family reduces to the complementary geometric L-G (CGcL-G) family. For $\alpha=1$, MOL-G family reduces to the standard G family. The corresponding PDF of (3) is given by

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$$f_{\alpha,\psi}\left(y\right)=\alpha g_{\underline{\psi}}\left(y\right)\left[1-\overline{\alpha}G_{\underline{\psi}}\left(y\right)\right]^{-2}|_{(y\in\Re,\ \alpha>0)}. \tag{4}$$

In this paper, we propose and study a new generated Weibull model called the Marshall-Olkin generalized-Weibull (MOL-W) distribution and give a comprehensive description of its mathematical properties. In fact, the MOL-W model is motivated by its importance flexibility in application. By means of two applications, it is noted that the MOL-W model provides better fits than other models each having the same number of parameters. By inserting (1) in (3), we obtain the cumulative distribution function (CDF) of the MOL-G class

$$F_{\alpha,\beta,a_1,a_2}\left(y\right)=\frac{\overbrace{1-\exp\left[-\beta\left(a_2y\right)^{a_1}\right]}^{A_{\beta,a_1,a_2}(y)}}{\underbrace{1-\overline{\alpha}\exp\left[-\beta\left(a_2y\right)^{a_1}\right]}_{B_{\alpha,\beta,a_1,a_2}(y)}}|_{(y\geq0\text{ and }a_1,a_2,\alpha,\beta>0)}, \tag{5}$$

The corresponding PDF of (5) is given by

$$f_{\alpha,\beta,a_1,a_2}\left(y\right)=\frac{\alpha\beta a_1a_2y^{a_1-1}\exp\left[-\beta\left(a_2y\right)^{a_1}\right]}{\left[1-\overline{\alpha}\exp\left[-\beta\left(a_2y\right)^{a_1}\right]\right]^2}|_{(y\geq0\text{ and }a_1,a_2,\alpha,\beta>0)}. \tag{6}$$

Some other extensions of the W distribution can also be used in this comparison, but are not limited to Mudholkar[46], Mudholkar et al.[47], Alizadeh et al.[3], Alizadeh et al.[5], Alizadeh et al.[4], Yousof et al.[56], Yousof et al.[57], Cordeiro et al.[14], Yousof et al.[54], Yousof et al.[59], Yousof et al.[60], Brito et al.[11], Aryal et al.[7], Aryal and Yousof[8], Korkmaz et al.[37], Korkmaz et al.[33], Korkmaz et al.[36], Korkmaz et al.[34], Korkmaz et al.[35], Yousof et al.[58], Hamedani et al.[23], Hamedani et al.[21], Hamedani et al.[22], Mansour et al.[42], Mansour et al.[43] and Mansour et al.[44]. For $a_2=1$, the MOL-W reduces to three-parameter MOL-W model. Table 1 give some sub-models from the MOL-W model. Equation (5) and (6) can be also derived based on Yousof et al.[55]. Figure 1 gives some plots of the MOL-W PDF and HRF. From Figure 1 (left panel) we conclude that the PDF MOL-W distribution exhibits various important shapes with different Kurtosis. From Figure 1 (right panel) we conclude that the HRF MOL-W distribution exhibits constant hazard rate ($\alpha=1,\beta$=1$,a_1=1,a_2=1$)**,** upside down-constant ($\alpha=0.5,\beta$=0.5$,a_1=1.01,a_2=1$)**,** decreasing hazard rate ($\alpha=0.5,\beta$=5$,a_1=1,a_2=0.2$)**,** increasing-constant hazard rate ($\alpha=0.5,\beta$=0.15$,a_1=1.25,a_2=1$)**,** increasing hazard rate ($\alpha=2,\beta$=1$,a_1=1.5,a_2=1$)**, J-**hazard rate ($\alpha=0.5,\beta$=1$,a_1=20,a_2=1$) and decreasing hazard rate ($\alpha=0.2,\beta$=1$,a_1=0.1,a_2=1$)**.** Table 1 and Figure 1 refer to the wide flexibility of the new mode. We are motivated to introduce the MOL-W model since its HRF can have many useful shapes. We are motivated to introduce the MOL-W model since its HRF can have many useful shapes as illustrated in Figure 1(right panel). The MOL-W model is a potential model for modeling the "symmetric bimodal" real data, the "asymmetric bimodal heavy tailed right skewed" real data, "asymmetric bimodal right skewed" real data and "asymmetric bimodal heavy tailed left skewed" real data as illustrated in Section 6.

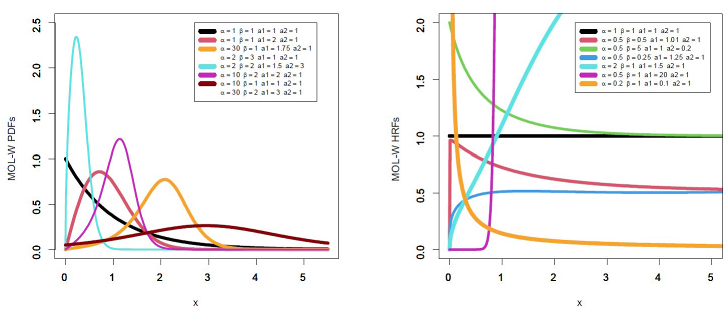

Figure 1. Plots of the MOL-W PDF and HRF.

Table 1: Some sub-models from the MOL-W model.

α	β	a_1	a_2	Reduced model	CDF
			1	three-parameter MOL-W	$\frac{1-\exp[-\beta(y)^{a_1}]}{1-\overline{\alpha}\exp[-\beta(y)^{a_1}]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGcL-W	$\frac{1-\exp[-\beta(y)^{a_1}]}{1-\overline{\alpha}\exp[-\beta(y)^{a_1}]}$
		1		MOL-Exp	$\frac{1-\exp[-\beta(a_2y)]}{1-\overline{\alpha}\exp[-\beta(a_2y)]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGcL-Exp	$\frac{1-\exp[-\beta(a_2y)]}{1-\overline{\alpha}\exp[-\beta(a_2y)]}$
			2	MOL-R	$\frac{1-\exp\left[-\beta(a_2y)^2\right]}{1-\overline{\alpha}\exp\left[-\beta(a_2y)^2\right]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGcL-R	$\frac{1-\exp\left[-\beta(a_2y)^2\right]}{1-\overline{\alpha}\exp\left[-\beta(a_2y)^2\right]}$
1			1	L-W	$1-\exp\left[-\beta\left(y\right)^{a_1}\right]$
1		1		L-Exp	$1-\exp\left[-\beta\left(a_2y\right)\right]$
1			2	L-R	$1-\exp\left[-\beta\left(a_2y\right)^2\right]$
	1		1	MO-W	$\frac{1-\exp[-(y)^{a_1}]}{1-\overline{\alpha}\exp[-(y)^{a_1}]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGc-W	$\frac{1-\exp[-(y)^{a_1}]}{1-\overline{\alpha}\exp[-(y)^{a_1}]}$
	1	1		MO-Exp	$\frac{1-\exp[-(a_2y)]}{1-\overline{\alpha}\exp[-(a_2y)]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGc-Exp	$\frac{1-\exp[-(a_2y)]}{1-\overline{\alpha}\exp[-(a_2y)]}$
	1		2	MO-R	$\frac{1-\exp\left[-(a_2y)^2\right]}{1-\overline{\alpha}\exp\left[-(a_2y)^2\right]}$
$\alpha^*\|_{[\alpha^*\in(0,1)]}$				CGc-R	$\frac{1-\exp\left[-(a_2y)^2\right]}{1-\overline{\alpha}\exp\left[-(a_2y)^2\right]}$
1	1			W	$1-\exp\left[-\left(a_2y\right)^2\right]$
1	1		1	W	$1-\exp\left[-\left(y\right)^{a_1}\right]$
1	1	1		Exp	$1-\exp\left[-\left(a_2y\right)\right]$
1	1		2	R	$1-\exp\left[-\left(a_2y\right)^2\right]$

2. Properties

2.1. Moments

First we have

$$
\begin{aligned}
A_{\beta,a_1,a_2}\left(y\right) = & \quad 1+\sum_{k_1=0}^{\infty}\left(-1\right)^{1+k_1}\binom{\beta}{k_1}\left\{\exp\left[-\left(a_2y\right)^{a_1}\right]\right\}^{k_1} \\
= & \quad \sum_{k_1=0}^{\infty}\zeta_{k_1}\left\{\exp\left[-\left(a_2y\right)^{a_1}\right]\right\}^{k_1},
\end{aligned}
\tag{7}
$$

where $\zeta_0=2$, $\zeta_{k_1}=\left(-1\right)^{1+k_1}\binom{\beta}{k_1}|_{(k_1\geq 1)}$ and

$$
\begin{aligned}
B_{\alpha,\beta,a_1,a_2}(y) = &\quad 1-\overline{\alpha}-\sum_{k_1=0}^{\infty}(-1)^{k_1}\binom{\beta}{k_1}\{\exp[-(a_2y)^{a_1}]\}^{k_1} \\
= &\quad \sum_{k_1=0}^{\infty}\eta_{k_1}\{\exp[-(a_2y)^{a_1}]\}^{k_1},
\end{aligned} \tag{8}
$$

where $\eta_0=\alpha$ and $\eta_{k_1}=\overline{\alpha}(-1)^{1+k_1}\binom{\beta}{k_1}$, using (7) and (8) the CDF of the MOL-G family in (5) can be expressed as

$$
F_{\alpha,\beta,a_1,a_2}(y)=\sum_{k_1=0}^{\infty}\frac{\zeta_{k_1}\{\exp[-(a_2y)^{a_1}]\}^{k_1}}{\eta_{k_1}\{\exp[-(a_2y)^{a_1}]\}^{k_1}}=\sum_{k_1=0}^{\infty}\tau_{k_1}\{\exp[-(a_2y)^{a_1}]\}^{k_1},
$$

where $\tau_0=\frac{\zeta_o}{\eta_0}$ and for $k_1\geq 1$ we have

$$
\tau_{k_1}=\frac{1}{\eta_0}\left(\zeta_{k_1}-\frac{1}{\eta_0}\sum_{r=1}^{k_1}\eta_r\tau_{k_1-r}\right),
$$

the PDF of the MOL-W model can also be expressed as a mixture of exponentiated W (Exp-W) PDF. By differentiating $F_{\alpha,\beta,a_1,a_2}(y)$, we obtain the same mixture representation

$$
f_{\alpha,\beta,a_1,a_2}(y)=\sum_{k_1=0}^{\infty}\tau_{(1+k_1)}\pi_{1+k_1}(y), \tag{9}
$$

where $\pi_\omega(y)$ is the Exp-W PDF with power parameter (ω). Equation (9) reveals that the MOL-W density function is a linear combination of Exp-W densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the Exp-W distribution. The r$^{[th]}$ ordinary moment of Y is given by

$$
\mu_r'=\mathbf{E}(Y^r)=\int_{-\infty}^{\infty}y^r f(y)\,dy,
$$

then we obtain

$$
\mu_r'=\frac{1}{a_2^r}\mathbf{\Gamma}\left(\frac{r}{a_1}+1\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,r)}|_{(r>-a_1)}, \tag{10}
$$

where $\varrho_{k_1,h}^{(1+k_1,r)}=\tau_{(1+k_1)}\ \varrho_h^{(1+k_1,r)}$ and $\varrho_w^{(\mathbf{C},\tau)}=\mathbf{C}\frac{(-1)^w}{(w+1)^{-\left(\frac{\tau}{a_1}+1\right)}}\binom{\mathbf{C}-1}{w}$ setting $r=1,2,3,4$ in (11) we get

$$
\begin{aligned}
\mathbf{E}(Y) &= \mu_1'=\frac{1}{a_2}\mathbf{\Gamma}\left(\frac{1}{a_1}+1\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,1)}|_{(1>-a_1)},\\
\mathbf{E}(Y^2) &= \mu_2'=\frac{1}{a_2^2}\mathbf{\Gamma}\left(\frac{1}{a_1}+1\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,2)}|_{(2>-a_1)},\\
\mathbf{E}(Y^3) &= \mu_3'=\frac{1}{a_2^3}\mathbf{\Gamma}\left(\frac{3}{a_1}+1\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,3)}|_{(3>-a_1)},
\end{aligned}
$$

and

$$
\mathbf{E}(Y^4)=\mu_4'=\frac{1}{a_2^4}\mathbf{\Gamma}\left(\frac{4}{a_1}+1\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,4)}|_{(4>-a_1)}.
$$

The last expressions can be computed numerically . The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The moment generating function (MGF) $M_Y(\tau)=\mathbf{E}\left(e^{\tau Y}\right)$ of Y. Clearly, the first one can be derived from equation (9) as

$$
M_Y(\tau)=\mathbf{\Gamma}\left(\frac{r}{a_1}+1\right)\sum_{k_1,h,r=0}^{\infty}\frac{1}{a_2}\varrho_{k_1,h,r}^{(1+k_1,r)}|_{(r>-a_1)},
$$

where $r!\varrho_{k_1,h,r}^{(1+k_1,r)} = \tau^r \varrho_{k_1,h}^{(1+k_1,r)}$. The s$^{[th]}$ incomplete moment, say $I_s(\tau)$, of Y can be expressed from (9) as

$$I_s(\tau) = \int_{-\infty}^{\tau} y^s f(y)\,dy = \frac{1}{a_2^s}\gamma\left(\frac{r}{a_1}+1,(a_2\tau)^{a_1}\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,r)}|_{(s>-a_1)}, \tag{11}$$

setting $s = 1,2,3,4$ in (11) we get

$$I_1(\tau) = \int_{-\infty}^{\tau} y f(y)\,dy = \frac{1}{a_2}\gamma\left(\frac{1}{a_1}+1,(a_2\tau)^{a_1}\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,1)}|_{(1>-a_1)},$$

$$I_2(\tau) = \int_{-\infty}^{\tau} y^2 f(y)\,dy = \frac{1}{a_2^2}\gamma\left(\frac{2}{a_1}+1,\left(\frac{1}{\tau}\right)^{a_1}\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,2)}|_{(2>-a_1)},$$

$$I_3(\tau) = \int_{-\infty}^{\tau} y^3 f(y)\,dy = \frac{1}{a_2^3}\gamma\left(\frac{3}{a_1}+1,(a_2\tau)^{a_1}\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,3)}|_{(3>-a_1)},$$

and

$$I_4(\tau) = \int_{-\infty}^{\tau} y^4 f(y)\,dy = \frac{1}{a_2^4}\gamma\left(\frac{4}{a_1}+1,(a_2\tau)^{a_1}\right)\sum_{k_1,h=0}^{\infty}\varrho_{k_1,h}^{(1+k_1,4)}|_{(4>-a_1)}.$$

2.2. Entropies

The Rényi entropy of a random variable Y represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$E_\vartheta(Y) = \frac{1}{1-\vartheta}\log\int_{-\infty}^{\infty}[f(x)]^\vartheta\,dy|_{(\vartheta>0 \text{ and } \vartheta\neq 1)}.$$

Using the PDF (6), we can write

$$[f(x)]^\vartheta = \sum_{s,k=0}^{\infty}\phi_{s,k}\left[a_2^{a_1}a_1 y^{a_1-1}\exp\left[-(a_2y)^{a_1}\right]\right]^\vartheta\left\{1-\exp\left[-(a_2y)^{a_1}\right]\right\}^k$$

where

$$\phi_{s,k} = \frac{(\alpha\beta)^\vartheta(-1)^{s+k}\alpha^s\Gamma([\vartheta+s]\beta-\vartheta+1)}{s!k!\Gamma([\vartheta+s]\beta-\vartheta-k+1)}(-2\vartheta)_s$$

Then

$$E_\vartheta(Y) = \frac{1}{1-\vartheta}\log\left[\sum_{s,k=0}^{\infty}\phi_{s,k}\,\mathbf{I}_{(0,\infty)}^{(\vartheta)}\right]|_{(\vartheta>0 \text{ and } \vartheta\neq 1)}.$$

where

$$\mathbf{I}_{(0,\infty)}^{(\vartheta)} = \int_{-\infty}^{\infty}\left(\left[a_2^{a_1}a_1y^{a_1-1}\exp\left[-(a_2y)^{a_1}\right]\right]^\vartheta\left\{1-\exp\left[-(a_2y)^{a_1}\right]\right\}^k\right)dy$$

The ζ-entropy, say $H_\zeta(Y)$, can be obtained as

$$H_\zeta(Y) = \frac{1}{\zeta-1}\log\left\{1-\left[\sum_{s,k=0}^{\infty}\Psi_{s,k}^{*}\mathbf{I}_{(0,\infty)}^{(\zeta)}\right]\right\}|_{(\zeta>0 \text{ and } \zeta\neq 1)},$$

where

$$\Psi_{s,k}^{*} = \frac{(\alpha\beta)^\zeta(-2\zeta)_s(-1)^{s+k}\overline{\alpha}^s\Gamma([\zeta+s]\beta-\zeta+1)}{s!k!\Gamma([\zeta+s]\beta-\zeta-k+1)}.$$

$$\mathbf{I}_{(0,\infty)}^{(\zeta)} = \int_{-\infty}^{\infty}\left[a_2^{a_1}a_1y^{a_1-1}\exp\left[-(a_2y)^{a_1}\right]\right]^\zeta\left\{1-\exp\left[-(a_2y)^{a_1}\right]\right\}^k dy$$

2.3. Order statistics

Suppose $Y_1, \ldots, Y_m$ is any random sample from any MOL-W distribution. Let $Y_{\hbar:m}$ denote the i$^{[th]}$ order statistic. The PDF of $Y_{\hbar:m}$ can be expressed as

$$f_{\hbar:m}(y) = \frac{f(y)}{\mathbf{B}(\hbar, m-\hbar+1)} \sum_{s=0}^{m-\hbar} (-1)^s \binom{m-\hbar}{s} F(y)^{s+\hbar-1}. \tag{13}$$

Then

$$f_{\hbar:m}(y) = \sum_{r,k_1=0}^{\infty} \vartheta_{r,k_1}\, \pi_{r+1+k_1}(y), \tag{14}$$

where

$$\vartheta_{r,k_1} = \frac{m!\,(r+1)\,(\hbar-1)!\,\tau_{r+1}}{(r+1+k_1)} \sum_{s=0}^{m-\hbar} \frac{(-1)^s\,\xi_{s+\hbar-1,k_1}}{(m-\hbar-s)!\,s!},$$

$\tau_{(1+k_1)}$ is given in Section 3 and the quantities $\xi_{s+\hbar-1,k_1}$ can be determined with $\xi_{s+\hbar-1,0} = w_0^{s+\hbar-1}$ and recursively for $k_1 \geq 1$

$$\xi_{s+\hbar-1,k_1} = (k_1\,\tau_0)^{-1} \sum_{m=1}^{k_1} \tau_m [m\,(s+\hbar) - k_1]\ \xi_{s+\hbar-1,k_1-m}.$$

Based on (14) we have

$$\mathbf{E}\left(Y_{\hbar:m}^{\zeta}\right) = \frac{1}{a_2^{\zeta}} \mathbf{\Gamma}\left(\frac{\zeta}{a_1}+1\right) \sum_{r,k_1,h=0}^{\infty} \varrho_{r,k_1,h}^{(r+1+k_1,\zeta)}|_{(\zeta>-a_1)},$$

where $\varrho_{r,k_1,h}^{(r+1+k_1,\zeta)} = \vartheta_{r,k_1}\,\varrho_h^{(r+1+k_1,\zeta)}$.

2.4. Residual life and reversed residual life functions

The m$^{[th]}$ moment of the residual life, say

$$\nu_m(\tau) = \mathbf{E}\left[(Y-\tau)^m\ |_{(Y>\tau \text{ and } m=1,2,\ldots)}\right],$$

the m$^{[th]}$ moment of the residual life of Y is given by

$$\nu_m(\tau) = \frac{\int_\tau^\infty (Y-\tau)^m dF(y)}{1-F(\tau)},$$

therefore

$$\nu_m(\tau) = \frac{\gamma\left(\frac{m}{a_1}+1, (a_2\tau)^{a_1}\right)}{a_2^m\,[1-F(\tau)]} \sum_{k_1,h=0}^{\infty} \sum_{r=0}^{m} \varrho_{k_1,h,r}^{(1+k_1,m)^{(\nu_m)}}\ |_{(m>-a_1)},$$

where

$$\varrho_{k_1,h,r}^{(1+k_1,m)^{(\nu_m)}}\,(1-\tau)^{-m} = \varrho_{k_1,h}^{(1+k_1,m)}.$$

The mean residual life (MRL) at age τ can be defined as

$$\nu_1(\tau) = \mathbf{E}\left[(Y-\tau)\ |_{(Y>\tau \text{ and } m=1)}\right],$$

which represents the expected additional life length for a unit which is alive at age τ. The MRL of Y can be obtained by setting $m=1$ in the last equation. The m$^{[th]}$ moment of the reversed residual life, say V

$$V_m(\tau) = \mathbf{E}\left[(\tau-Y)^m\ |_{(Y\leq\tau, \tau>0 \text{ and } m=1,2,\ldots)}\right],$$

we obtain

$$V_m(\tau) = \frac{\int_0^\tau (\tau-y)^m dF(y)}{F(\tau)}.$$

Then, the m$^{[th]}$ moment of the reversed residual life of Y becomes

$$V_m(\tau)=\frac{\gamma\left(\frac{m}{a_1}+1,(a_2\tau)^{a_1}\right)}{a_2^m F(\tau)}\sum_{k_1,h=0}^{\infty}\sum_{r=0}^{m}\varrho_{k_1,h,r}^{(1+k_1,m)^{(V_m)}}|_{(m>-a_1)},$$

where

$$\varrho_{k_1,h,r}^{(1+k_1,m)^{(V_m)}}=(-1)^r\binom{m}{r}\tau^{m-r}\varrho_{k_1,h}^{(1+k_1,m)}.$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by

$$V_1(\tau)=\mathbf{E}[(\tau-Y)\,|_{(Y\leq\tau\text{ and }m=1)}],$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,\tau)$.The MIT of the MOL-W distribution of distributions can be obtained easily by setting $m=1$ in the above equation.

3. Copula

In this Section, we derive some new bivariate MOL-W (B-MOL-W) type distributions using Farlie Gumbel Morgenstern (FGM) Copula (see Farlie[17], Gumbel[45], Gumbel[19], Morgenstern[20], Johnson[28] and Johnson[29]), modified FGM Copula, Clayton Copula and Renyi's entropy (Pougaza[48]). The Multivariate MOL-W (M-MOL-W) type is also presented. However, future works may be allocated to the study of these new models. First, we consider the joint CDF of the FGM family, where

$$\mathbf{H}_\zeta(t,\nu)=t\nu\left(1+\zeta t^{\cdot}\nu^{\cdot}\right)|_{t^{\cdot}=1-t,\nu^{\cdot}=1-\nu},$$

and the marginal function $t=F_1$, $\nu=F_2$, $\zeta\in(-1,1)$ is a dependence parameter and for every $t,\nu\in(0,1)$, $\mathbf{H}(t,0)=\mathbf{H}(0,\nu)=0$ which is "grounded minimum" and $\mathbf{H}(t,1)=t$ and $\mathbf{H}(1,\nu)=\nu$ which is "grounded maximum", $\mathbf{H}(t_1,\nu_1)+\mathbf{H}(t_2,\nu_2)-\mathbf{H}(t_1,\nu_2)-\mathbf{H}(t_2,\nu_1)\geq 0$.

3.1. Via FGM family

A Copula is continuous in t and ν; actually, it satisfies the stronger Lipschitz condition, where

$$|\mathbf{H}(t_2,\nu_2)-\mathbf{H}(t_1,\nu_1)|\leq|t_2-t_1|+|\nu_2-\nu_1|.$$

For $0\leq t_1\leq t_2\leq 1$ and $0\leq\nu_1\leq\nu_2\leq 1$, we have

$$\Pr(t_1\leq t\leq t_2,\nu_1\leq\nu\leq\nu_2)=\mathbf{H}(t_1,\nu_1)+\mathbf{H}(t_2,\nu_2)-\mathbf{H}(t_1,\nu_2)-\mathbf{H}(t_2,\nu_1)\geq 0.$$

Then, setting

$$t^{\cdot}=1-\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}|_{[t^{\cdot}=(1-t)\in(0,1)]}$$

and

$$\nu^{\cdot}=1-\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}|_{[\nu^{\cdot}=(1-\nu)\in(0,1)]},$$

we can esaily get the get the joint CDF of the MOL-W using the FGM family

$$\mathbf{H}_\zeta(t,\nu)=\frac{A_{\beta_1,a_1,a_2}(t)\,A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)\,B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\left(1+\zeta\left\{\begin{array}{c}\left[1-\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\right]\\\times\left[1-\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\right]\end{array}\right\}\right).$$

The joint PDF can then be derived from $c_\zeta(t,\nu)=1+\zeta t^{\cdot}\nu^{\cdot}|_{(t^{\cdot}=1-2t\text{ and }\nu^{\cdot}=1-2\nu)}$ or from $c_\zeta(t,\nu)=f(x_1,x_2)=\mathbf{H}(F_1,F_2)\,f_1f_2$.

3.2. Via modified FGM family

The modified FGM copula is defined as $\mathbf{H}_{\zeta}(t,\nu)=t\nu\left[1+\zeta W\left(t\right)M\left(\nu\right)\right]|_{\zeta\in(-1,1)}$ or $\mathbf{H}_{\zeta}(t,\nu)=t\nu+\zeta\dot{W}_t\dot{M}_\nu|_{\zeta\in(-1,1)}$, where $\dot{W}_t=tW\left(t\right)$, and $\dot{M}_\nu=\nu M\left(\nu\right)$ and $W\left(t\right)$ and $M\left(\nu\right)$ are two continuous functions on $(0,1)$ with $W\left(0\right)=W\left(1\right)=M\left(0\right)=M\left(1\right)=0$. Let

$$\Upsilon_1\left(\dot{W}_t\right)=\inf\left\{\dot{W}_t:\frac{\partial}{\partial t}\dot{W}_t\right\}|_{\varpi_{1,t}}<0,\Upsilon_2\left(\dot{W}_t\right)=\sup\left\{\dot{W}_t:\frac{\partial}{\partial t}\dot{W}_t\right\}|_{\varpi_{1,t}}<0,$$

$$\varphi_1\left(\dot{M}_\nu\right)=\inf\left\{\dot{M}_\nu:\frac{\partial}{\partial\nu}\dot{M}_\nu\right\}|_{\varpi_{2,\nu}}>0,\varphi_2\left(\dot{M}_\nu\right)=\sup\left\{\dot{M}_\nu:\frac{\partial}{\partial\nu}\dot{M}_\nu\right\}|_{\varpi_{2,\nu}}>0.$$

Then,

$$1\leq\min\left\{\Upsilon_1\left(\dot{W}_t\right)\Upsilon_2\left(\dot{W}_t\right),\varphi_1\left(\dot{M}_\nu\right)\varphi_2\left(\dot{M}_\nu\right)\right\}<\infty,$$

where

$$t\frac{\partial}{\partial t}W\left(t\right)=\frac{\partial}{\partial t}\dot{W}_t-W\left(t\right),$$

$$\varpi_{1,t}=\left\{t:t\in(0,1)\,|_{\frac{\partial}{\partial t}\dot{W}_t\ \text{ exists}}\right\}$$

and

$$\varpi_{2,\nu}=\left\{\nu:\nu\in(0,1)\,|_{\frac{\partial}{\partial\nu}\dot{M}_\nu\ \text{ exists}}\right\}.$$

3.2.1. Type-I Consider the following functional form for both $W\left(t\right)$ and $M\left(\nu\right)$. Then, the B-MOL-W-FGM (Type-I) can be derived from

$$\mathbf{H}_{\zeta}(t,\nu)=\frac{A_{\beta_1,a_1,a_2}\left(t\right)A_{\beta_2,a_1,a_2}\left(\nu\right)}{B_{\alpha_1,\beta_1,a_1,a_2}\left(t\right)B_{\alpha_2,\beta_2,a_1,a_2}\left(\nu\right)}+\zeta\left\{\begin{array}{c}\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\left[1-\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\right]\\\times\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\left[1-\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\right]\end{array}\right\}|_{\zeta\in(-1,1)}.$$

3.2.2. Type-II Let $W\left(t\right)$ and $M\left(\nu\right)$ be two functional form satisfying all the conditions stated earlier where $W\left(t\right)^{\cdot}|_{(\zeta_1>0)}=t^{\zeta_1}\left(1-t\right)^{1-\zeta_1}$ and $M\left(\nu\right)^{\cdot}|_{(\zeta_2>0)}=\nu^{\zeta_2}\left(1-\nu\right)^{1-\zeta_2}$. Then, the corresponding B-MOL-W-FGM (Type-**II**) can be derived from $\mathbf{H}_{\zeta,\zeta_1,\zeta_2}(t,\nu)=t\nu\left[1+\zeta W\left(t\right)^{\cdot}\ M\left(\nu\right)^{\cdot}\right]$. Thus

$$\begin{aligned}\mathbf{H}_{\zeta,\zeta_1,\zeta_2}(t,\nu)\ &=\ \frac{A_{\beta_1,a_1,a_2}\left(t\right)A_{\beta_2,a_1,a_2}\left(\nu\right)}{B_{\alpha_1,\beta_1,a_1,a_2}\left(t\right)B_{\alpha_2,\beta_2,a_1,a_2}\left(\nu\right)}\\&\times\left[1+\zeta\left(\begin{array}{c}\left\{\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\right\}^{\zeta_1}\left\{\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\right\}^{\zeta_2}\\\times\left(1-\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\right)^{1-\zeta_1}\left(1-\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\right)^{1-\zeta_2}\end{array}\right)\right].\end{aligned}$$

3.2.3. Type-III Let $\ddot{W}\left(t\right)=t\left[\log\left(1+t^{\cdot}\right)\right]$ and $\ddot{M}\left(\nu\right)=\nu\left[\log\left(1+\nu^{\cdot}\right)\right]$ for all $W\left(t\right)$ and $M\left(\nu\right)$ which satisfies all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the B-MOL-W-FGM (Type-**III**) from

$$\mathbf{H}_{\zeta}(t,\nu)=t\nu\left(1+\zeta\ddot{W}\left(t\right)\ \ddot{M}\left(\nu\right)\right).$$

Then

$$\mathbf{H}_{\zeta}(t,\nu)=\frac{A_{\beta_1,a_1,a_2}\left(t\right)A_{\beta_2,a_1,a_2}\left(\nu\right)}{B_{\alpha_1,\beta_1,a_1,a_2}\left(t\right)B_{\alpha_2,\beta_2,a_1,a_2}\left(\nu\right)}\left[1+\zeta\left(\begin{array}{c}\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\\\times\left[\log\left(2-\frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}\right)\right]\\\times\left[\log\left(2-\frac{A_{\beta_2,a_1,a_2}(\nu)}{B_{\alpha_2,\beta_2,a_1,a_2}(\nu)}\right)\right]\end{array}\right)\right].$$

3.3. B-MOL-W and M-MOL-W type via Clayton Copula

The Clayton Copula can be considered as

$$\mathbf{H}(\nu_1,\nu_2) = \left[(1/\nu_1)^{\zeta} + (1/\nu_2)^{\zeta} - 1\right]^{-\zeta^{-1}} |_{\zeta\in(0,\infty)}.$$

Setting $\nu_1 = \frac{A_{\beta_1,a_1,a_2}(t)}{B_{\alpha_1,\beta_1,a_1,a_2}(t)}$ and $\nu_2 = \frac{A_{\beta_2,a_1,a_2}(x)}{B_{\alpha_2,\beta_2,a_1,a_2}(x)}$, the B-MOL-W type can be derived from $\mathbf{H}(\nu_1,\nu_2) = \mathbf{H}(F_{\underline{\mathbf{\Phi}}_1}(\nu_1), F_{\underline{\mathbf{\Phi}}_1}(\nu_2))$. Then

$$\mathbf{H}(\nu_1,\nu_2) = \left\{\left(\frac{B_{\alpha_1,\beta_1,a_1,a_2}(t)}{A_{\beta_1,a_1,a_2}(t)}\right)^{\zeta} + \left(\frac{B_{\alpha_2,\beta_2,a_1,a_2}(x)}{A_{\beta_2,a_1,a_2}(x)}\right)^{\zeta} - 1\right\}^{-\zeta^{-1}} |_{\zeta\in(0,\infty)}.$$

Similarly, the M-MOL-W can be derived from

$$\mathbf{H}(\nu_i) = \left(\sum_{i=1}^{d}\left(\frac{B_{\alpha_i,\beta_i,a_1,a_2}(t_i)}{A_{\beta_i,a_1,a_2}(t_i)}\right)^{\zeta} + 1 - d\right)^{-\zeta^{-1}}.$$

3.4. B-MOL-W type via Renyi's entropy

Using the theorem of Pougaza[48] where $\mathbf{H}(t,\nu) = x_2 t + x_1\nu - x_1 x_2$, the associated B-MOL-W can be derived from

$$\mathbf{H}(t,\nu) = x_2\frac{A_{\beta_1,a_1,a_2}(x_1)}{B_{\alpha_1,\beta_1,a_1,a_2}(x_1)} + x_1\frac{A_{\beta_2,a_1,a_2}(x_2)}{B_{\alpha_2,\beta_2,a_1,a_2}(x_2)} - x_1 x_2.$$

4. Estimation

Let $Y_1,\ldots,Y_m$ be a random sample from the MOL-W distribution with parameters α,β and a_1. Let $\underline{\mathbf{\Psi}} = (\alpha,\beta,a_1)^{\intercal}$ be the 3×1 parameter vector. For determining the MLE of $\underline{\mathbf{\Psi}}$, we have the log-likelihood function

$$\begin{aligned}\ell &= \ell(\underline{\mathbf{\Psi}}) = m\log\alpha + m\log\beta + m\log a_1 + m a_1\log a_2 \\ &\quad + (a_1-1)\sum_{\hbar=1}^{m}\log(y_\hbar) - \beta\sum_{\hbar=1}^{m} y_\hbar^{a_1} - 2\sum_{\hbar=1}^{m}\log s_\hbar,\end{aligned}$$

where $s_\hbar = 1 - \overline{\alpha}\exp\left[-\beta(a_2 y)^{a_1}\right]$. The components of the score vector are

$$U_\alpha = \frac{m}{\alpha} - 2\sum_{\hbar=1}^{m}\frac{z_\hbar}{s_\hbar}, U_\beta = \frac{m}{\beta} - \sum_{\hbar=1}^{m} y_\hbar^{a_1} - 2\bar{\alpha}\sum_{\hbar=1}^{m}\frac{p_\hbar}{s_\hbar},$$

$$U_{a_1} = \frac{m}{a_1} + \sum_{\hbar=1}^{m}\log(y_\hbar) - \beta\sum_{\hbar=1}^{m} y_\hbar^{a_1}\log(y_\hbar) - 2\sum_{\hbar=1}^{m}\frac{d_\hbar}{s_\hbar}.$$

and

$$U_{a_2} = \frac{m a_1}{a_2} - 2\sum_{\hbar=1}^{m}\frac{q_\hbar}{s_\hbar},$$

where $z_\hbar = \frac{\partial}{\partial\alpha}s_\hbar$, $p_\hbar = \frac{\partial}{\partial\beta}s_\hbar$, $d_\hbar = \frac{\partial}{\partial a_1}s_\hbar$ and $q_\hbar = \frac{\partial}{\partial a_2}s_\hbar$. Setting the nonlinear system of equations $U_\alpha = U_\beta =$ and $U_{a_1} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\underline{\mathbf{\Psi}}} = (\widehat{\alpha},\widehat{\beta},\widehat{a_1},\widehat{a_2})^{\intercal}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ.

5. Graphical assessment

Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the MLEs. The assessment was based on the following algorithm:

1. Use

$$Y_U = \frac{1}{a_2}\left\{-\frac{1}{\beta}\ln\left[\frac{1-U}{1-(1-\alpha)U}\right]\right\}^{\frac{1}{a_1}}$$

we generate 1000 samples of size m from the MOL-W distribution;
2. Compute the MLEs for the 1000 samples, say

$$\left[\widehat{\alpha_\hbar}, \widehat{\beta_\hbar}, \widehat{(a_1)}_\hbar, \widehat{(a_1)}_\hbar\right]|_{(\hbar=1,2,\ldots,1000)},$$

3. Compute the SEs of the MLEs for the 1000 samples, say

$$\left[S_{\widehat{\alpha_\hbar}}, S_{\widehat{\beta_\hbar}}, S_{\widehat{(a_1)}_\hbar}, S_{\widehat{(a_2)}_\hbar}\right]|_{(\hbar=1,2,\ldots,1000)}.$$

4. Compute the biases and mean squared errors given for $\underline{\mathbf{\Psi}} = \alpha, \beta, a_1, a_2$. We repeated these steps for $m = 50, 100, \ldots, 500$ with $\alpha = \beta = a_1 = a_2 = 1$, so computing biases$\left(\mathrm{B}_{\underline{\mathbf{\Psi}}}(m)\right)$, mean squared errors (MSEs) $(\mathrm{MSE}_h(m))$ for α, β, a_1, a_2 and $m = 50, 100, \ldots, 500$ where

$$\mathrm{B}_{\underline{\mathbf{\Psi}}}(m)|_{(\underline{\mathbf{\Psi}}=\alpha,\beta,a_1,a_2)} = \frac{1}{1000}\sum_{\hbar=1}^{1000}\left(\widehat{\underline{\mathbf{\Psi}}}_\hbar - \underline{\mathbf{\Psi}}\right),$$

and

$$\mathrm{MSE}_{\underline{\mathbf{\Psi}}}(m)|_{(\underline{\mathbf{\Psi}}=\alpha,\beta,a_1,a_2)} = \frac{1}{1000}\sum_{\hbar=1}^{1000}\left(\widehat{\underline{\mathbf{\Psi}}}_\hbar - \underline{\mathbf{\Psi}}\right)^2$$

Figure 2 (left panel) shows how the four biases vary with respect to m. Figure 2 (right panel) shows how the four MSEs vary with respect to m. The broken lines in Figure 2 corresponds to the biases being 0. From Figure 2, the biases for each parameter are generally negative and decrease to zero as $m \to \infty$, the MSEs for each parameter decrease to zero as $m \to \infty$.

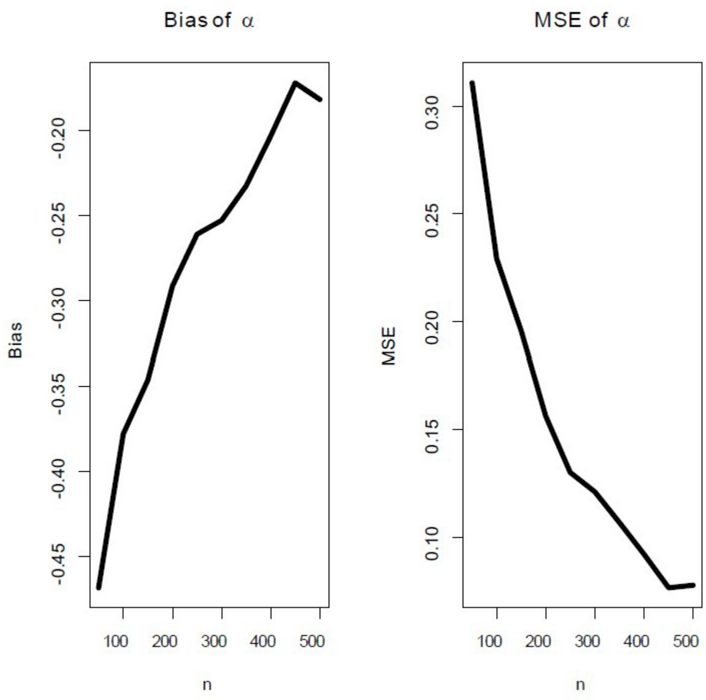

Figure 2. biasesand mean squared errors for the parameter α.

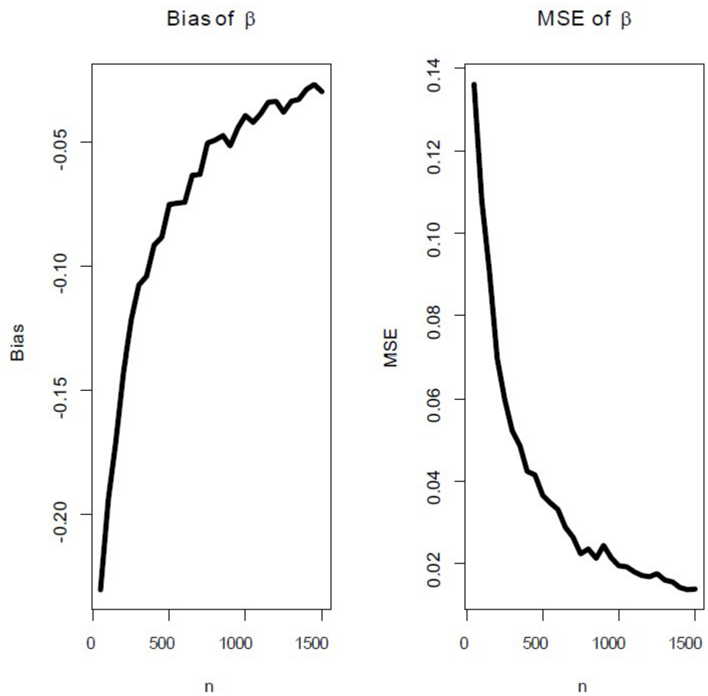

Figure 3. biases and mean squared errors for the parameter β.

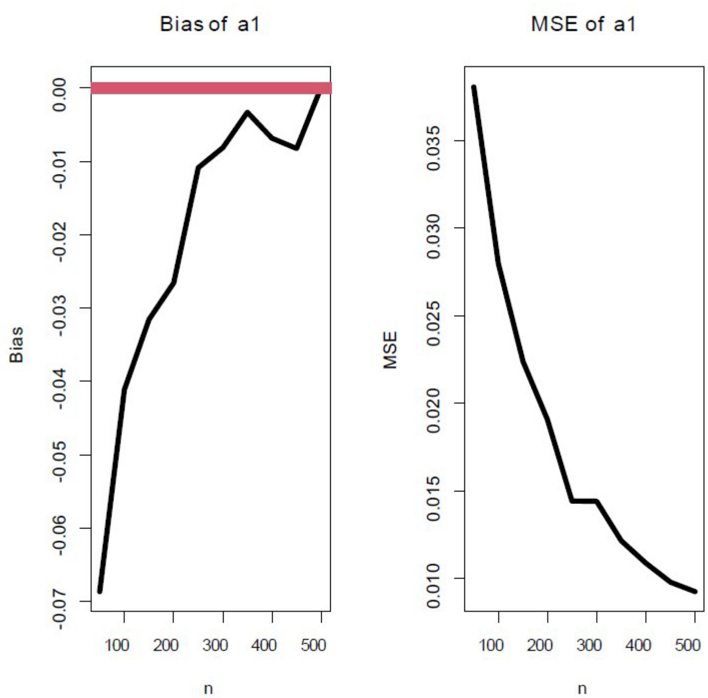

Figure 4. biases and mean squared errors for the parameter a_1.

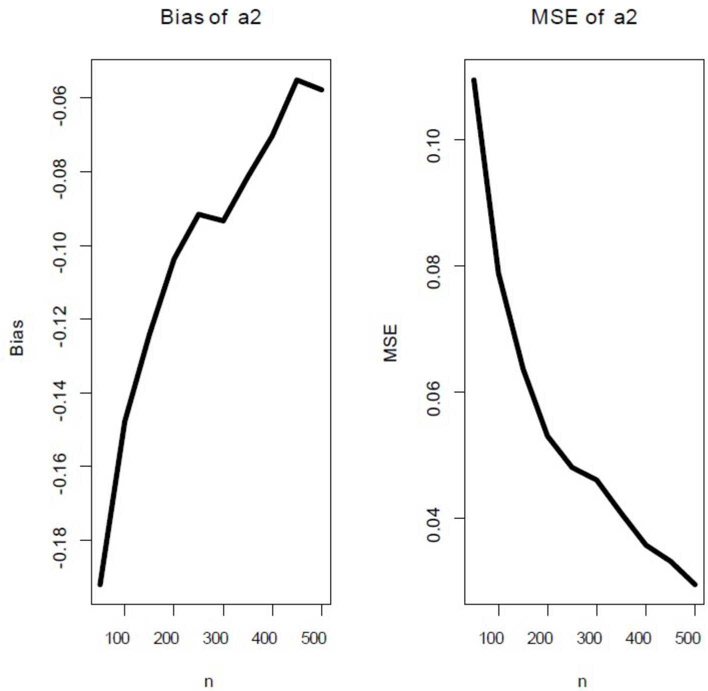

Figure 5. biases and mean squared errors for the parameter a_2

6. Applications

In this section, we provide four applications of the OLEW distribution to show empirically its potentiality. In order to compare the fits of the MOL-W distribution with other competing distributions, we consider the Cramér-von Mises $\left(CVM_{(\text{statistic})}\right)$ and the Anderson-Darling $\left(AD_{(\text{statistic})}\right)$. These two statistics are widely used to determine how closely a specific CDF fits the empirical distribution of a given data set. These statistics are given by

$$\text{CVM}_{(\text{statistic})} = \left[(1/12m) + \sum\nolimits_{s=1}^{m} \left[z_{\hbar} - (2s-1)/2m\right]^2\right](1+1/2m),$$

and

$$\text{AD}_{(\text{statistic})} = \left(1+\frac{9}{4m^2}+\frac{3}{4m}\right)\left\{m+\frac{1}{m}\sum\nolimits_{s=1}^{m}(2s-1)\log\left[z_{\hbar}\left(1-z_{m-s+1}\right)\right]\right\},$$

respectively, where $z_{\hbar} = F(y_s)$ and the y_s's values are the ordered observations. The smaller these statistics are, the better the fit. The required computations are carried out using the R software. The MLEs and the corresponding standard errors (in parentheses) of the model parameters are given in Tables 2, 4, 6 and 8. The numerical values of the statistics $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$ are listed in Tables 3, 5, 7 and 9. The total time in test (TTT) plot (a), nonparametric Kernel density estimation (KDE) plot (b), box plot (c), quantile-quantile (QQ) plot (d), estimated PDF (EPDF) plot (e), estimated CDF (ECDF) plot (f), probability-probability (P-P) plot (g), estimated HRF (EHRF) plot (h) for data sets **I**, **II, III** and **IV** of the proposed model are displayed in Figures 6, 7, 8 and 9. Based on Tables 3, 5, 7 and 9 and Figures 6, 7, 8 and 9, the MOL-W model is a potential model for modeling the ”symmetric bimodal” real data, the ”asymmetric bimodal heavy tailed right skewed” real data, ”asymmetric bimodal right skewed” real data and ”asymmetric bimodal heavy tailed left skewed” real data as illustrated in Section 6.

6.1. Modeling failure times

The data consist of 84 observations. The data are given in Appendix (a). Here, we shall compare the fits of the MOL-W distribution with those of other competitive models, namely: the Odd Lindley Exponentiated W (OLE-W), Burr X Exp W (BrXE-W) (Khalil[30]), Poisson Topp Leone-W (PTL-W) (Merovci[41]), MO extended-W (MOE-W) (Ghitany[18]), Gamma-W (Ga-W) (Provost[49]), Kumaraswamy-W (Kw-W) (Cordeiro[13]), Beta-W (B-W) (Lee et al.[39]), Transmuted modified-W (TM-W) (Khan and King[32]), Modified beta-W (MB-W) (Khan[31]) Mcdonald-W (Mc-W) (Cordeiro[12]), transmuted exponentiated generalized W (TExG-W) (Yousof[53]) distributions, whose PDFs (for $y > 0$). Based on the figures in Table 3 we conclude that the new lifetime model provides adequate fits as compared to other W models with small values for $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$. The MOL-W is the best model with $\text{CVM}_{(\text{statistic})}$=0.0682 and $\text{AD}_{(\text{statistic})}$=0.5469.

Table 2: MLEs (standard errors in parentheses) for data set **I**.

Distribution	Estimates				
OLE-W(a, α, β)	0.15935	0.7322	0.765		
	(0.3712)	(1.778)	(0.041)		
BrXE-W(a_1, α, β)	0.63684	4.2622	0.5364		
	(0.356)	(1.757)	(0.0997)		
PTL-W(λ, α, a_2)	-5.78175	4.22865	0.65801		
	(1.395)	(1.167)	(0.039)		
MOE-W(γ, β, α)	488.899	0.2832	1261.97		
	(189.358)	(0.013)	(351.07)		
Ga-W(α, β, γ)	2.376973	0.848094	3.534401		
	(0.378)	(0.00053)	(0.665)		
MOL-W(α, β, a_1, a_2)	**6.5216**	**9.1693**	**1.4987**	**0.1444**	
	(9.8181)	**(7.813)**	**(0.6014)**	**(0.0816)**	
Kw-W(α, β, a, a_2)	14.4331	0.2041	34.6599	81.8459	
	(27.095)	(0.042)	(17.527)	(52.014)	
B-W(α, β, a, a_2)	1.36	0.2981	34.1802	11.4956	
	(1.002)	(0.06)	(14.838)	(6.73)	
TM-W($\alpha, \beta, \gamma, \lambda$)	0.2722	1	4.6×10^{-6}	0.4685	
	(0:014)	(5.2×10^{-5})	(1.9×10^{-4})	(0.165)	
MB-W(α, β, a, a_2, c)	10.1502	0.1632	57.4167	19.3859	2.0043
	(18.697)	(0.019)	(14.063)	(10.019)	(0.662)
Mc-W(α, β, a, a_2, c)	1.9401	0.306	17.686	33.6388	16.7211
	(1.011)	(0.045)	(6.222)	(19.994)	(9.722)
TExG-W($\alpha, \beta, \lambda, a, a_2$)	4.2567	0.1532	0.0978	5.2313	1173.33
	(33.401)	(0.017)	(0.609)	(9.792)	(6.999)

Table 3: $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$ for data set I.

Distribution	$\text{CVM}_{(\text{statistic})}$	$\text{AD}_{(\text{statistic})}$
MOL-W(α, β, a_1, a_2)	**0.0682**	**0.5469**
OLE-W(a, α, β)	0.0723	0.6086
BrXE-W(a_1, α, β)	0.0744	0.6420
PTL-W(λ, α, a_2)	0.1397	1.1939
MOE-W(γ, β, α)	0.3995	4.4477
Ga-W(α, β, γ)	0.2553	1.9489
Kw-W(α, β, a_1, a_2)	0.1852	1.5059
B-W(α, β, a_1, a_2)	0.4652	3.2197
TM-W($\alpha, \beta, \gamma, \lambda$)	0.8065	11.2047
MB-W($\alpha, \beta, a_1, a_2, c$)	0.4717	3.2656
Mc-W($\alpha, \beta, a_1, a_2, c$)	0.1986	1.5906
TExG-W($\alpha, \beta, \lambda, a_1, a_2$)	1.0079	6.2332

6.2. Modeling cancer data

This data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang[38]. This data are given in Appendix (b). We compare the fits of the MOL-W distribution with other competitive models, namely: The TMW, MBW, transmuted additive W distribution (TA-W) (Elbatal and Aryal[15]), and the W ([52]) distributions with corresponding densities (for $y > 0$). Based on the figures in Table 5 we conclude

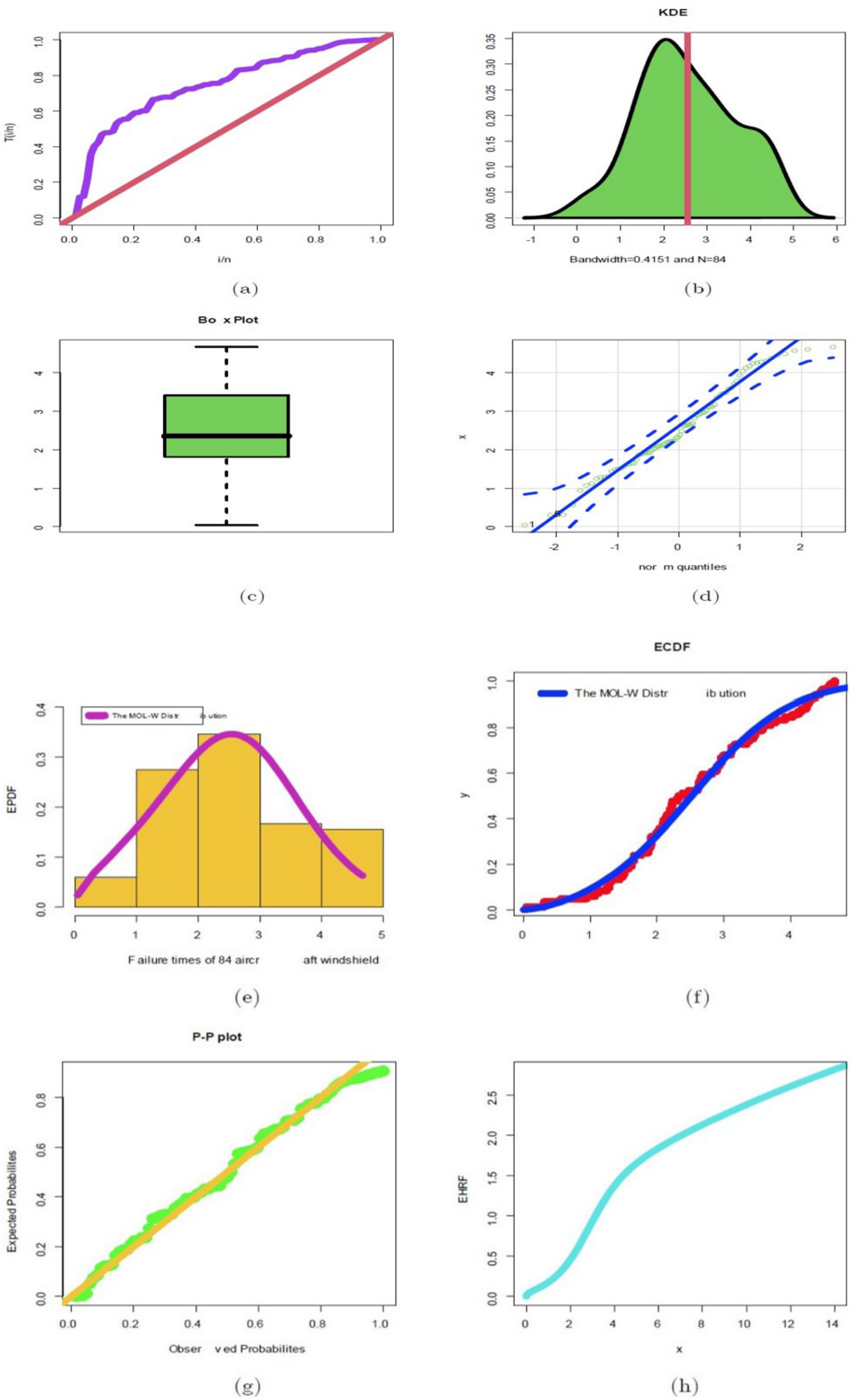

Figure 6. TTT plot (a), KDE plot (b), box plot (c), QQ plot (d), EPDF plot (e), ECDF plot (f), P-P plot (g), EHRF plot (h) for data set **I**.

that the proposed MOL-W lifetime model is much better than the W, TM-W, MB-W, TA-W, ETG-R models with small values for $\text{CVM}_{(\text{statistic})}$ =0.0836 and $\text{AD}_{(\text{statistic})}$ =0.5182 in modeling cancer patients data.

Table 4: MLEs (standard errors in parentheses) for data set **II**.

Distribution	Estimates				
W(α, β)	9.5593	1.0477			
	(0.853)	(0.068)			
MOL-W(α, β, a_1, a_2)	**0.4998**	**0.2172**	**1.187**	**0.2578**	
	(0.1774)	**(0.053)**	**(0.0035)**	**(0.004)**	
TM-W($\alpha, \beta, \gamma, \lambda$)	0.1208	0.8955	0.0002	0.2513	
	(0.024)	(0.626)	(0.011)	(0.407)	
MB-W($\alpha, \beta, a_1, a_2, c$)	0.1502	0.1632	57.4167	19.386	2.0043
	(22.437)	(0.044)	(37.317)	(13.49)	(0.789)
TA-W($\alpha, \beta, \gamma, a, \lambda$)	0.1139	0.9722	3.0936×10^{-5}	1.0065	-0.163
	(0.032)	(0.125)	(6.106×10^{-3})	(0.035)	(0.28)

Table 5: $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$ for data set **II**.

Distribution	$\text{CVM}_{(\text{statistic})}$	$\text{AD}_{(\text{statistic})}$
MOL-W(α, β, a_1, a_2)	**0.0836**	**0.5182**
W(α, β)	0.1055	0.6628
TM-W($\alpha, \beta, \gamma, \lambda$)	0.1251	0.7603
MB-W($\alpha, \beta, a_1, a_2, c$)	0.1068	0.7207
TA-W($\alpha, \beta, \gamma, a, \lambda$)	0.1129	0.7033

6.3. Modeling survival times

The second real data set corresponds to the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal [9].This data are given in Appendix (c).We shall compare the fits of the MOL-W distribution with those of other competitive models, namely: Odd Lindley exponentiated W (OLEW), the Odd W-W (OW-W) (Bourguignon et al.[10]), the gamma exponentiated-exponential (GaE-E) (Ristic and Balakrishnan [50]) distributions, whose PDFs (for $y > 0$).Based on the figures in Table 7 we conclude that the proposed MOL-W model is much better than all other models with $CVM_{(statistic)} = 0.19885$and $AD_{(statistic)} = 1.15606$.

Table 6: MLEs (standard errors in parentheses) for data set **III**.

Distribution	Estimates			
OLE-W(a, α, β)	0.0018	0.0716	0.2813	
	(0.0004)	(0.025)	(0.009)	
OW-W(β, γ, λ)	11.1576	0.0881	0.4574	
	(4.5449)	(0.036)	(0.08)	
GaE-E(λ, α, a_1)	2.1138	2.6006	0.0083	
	(1.3288)	(0.5597)	(0.005)	
MOL-W(α, a, a_1, a_2)	**2.64**	**0.297**	**1.426**	**0.0172**
	(0.00)	**(0.361)**	**(0.00)**	**(0.017)**

Table 7: $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$ for data set **III**.

Distribution	$\text{CVM}_{(\text{statistic})}$	$\text{AD}_{(\text{statistic})}$
MOL-W(α, β, a_1, a_2)	**0.1988**	**1.1561**
OLE-W(a, α, β)	0.2517	1.4750
OW-W(β, γ, λ)	0.4494	2.4764
GaE-E(λ, α, a)	0.3150	1.7208

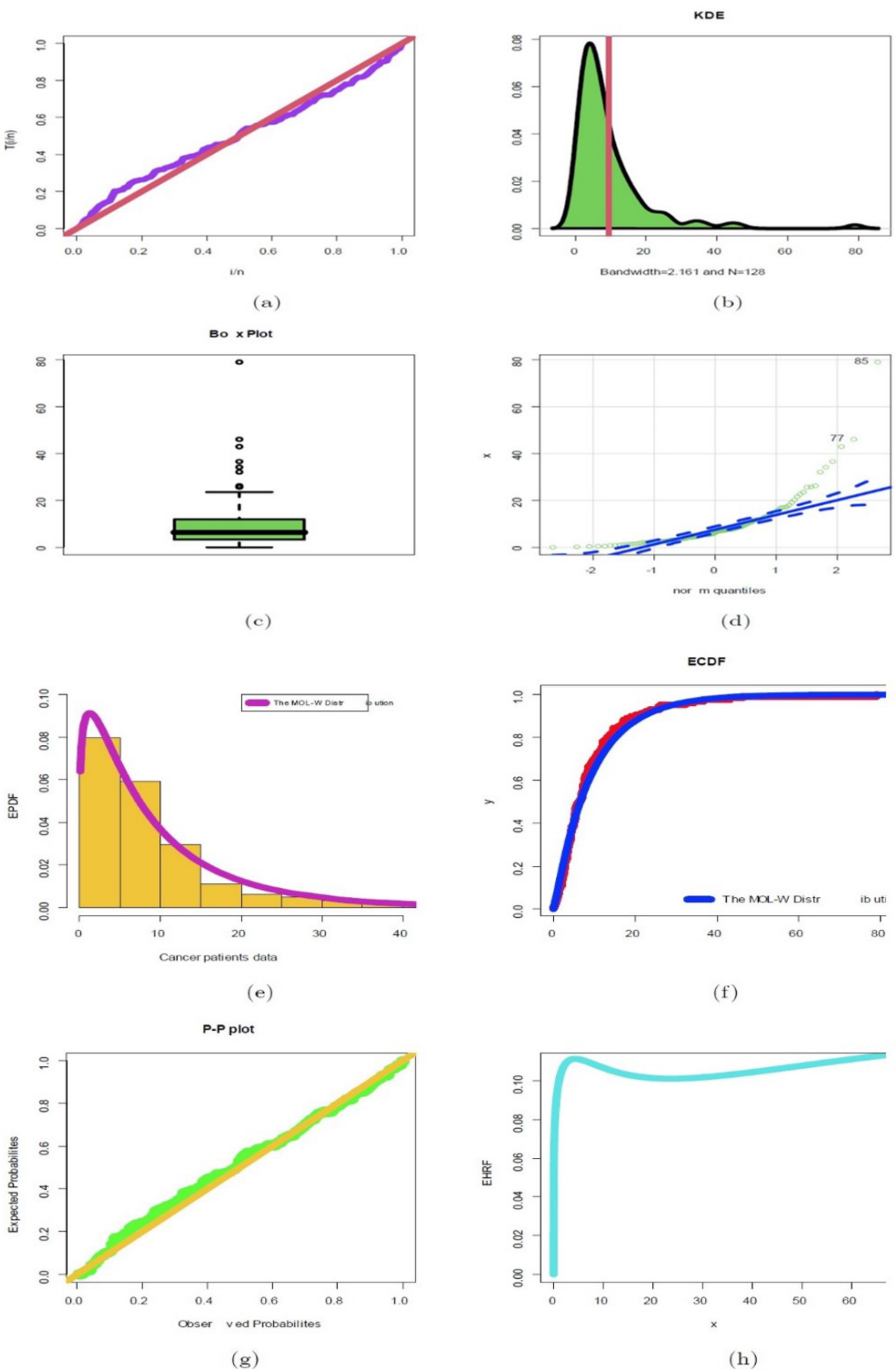

Figure 7. TTT plot (a), KDE plot (b), box plot (c), QQ plot (d), EPDF plot (e), ECDF plot (f), P-P plot (g), EHRF plot (h) for data set **II**.

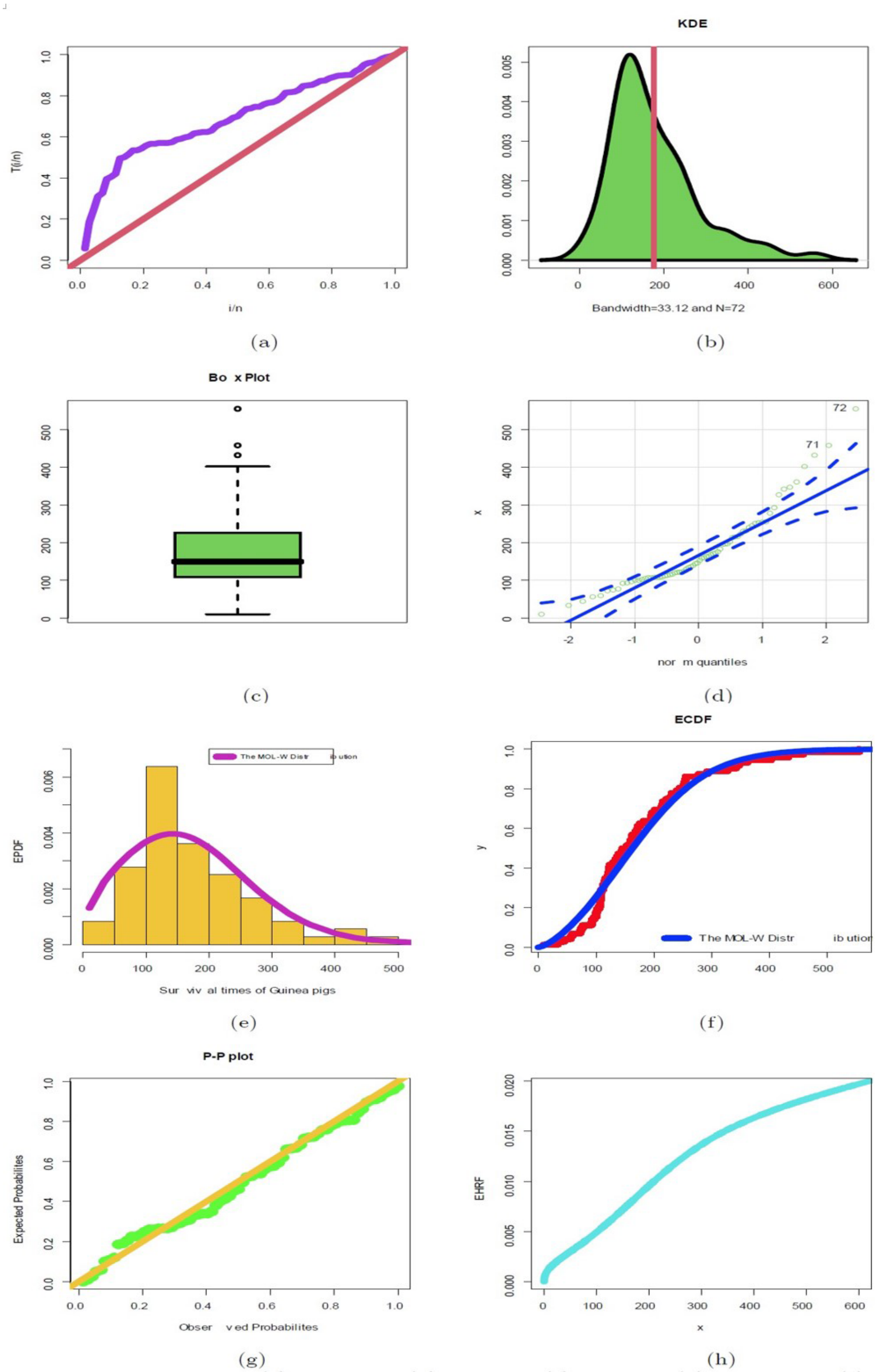

Figure 8. TTT plot (a), KDE plot (b), box plot (c), QQ plot (d), EPDF plot (e), ECDF plot (f), P-P plot (g), EHRF plot (h) for data set **III**.

6.4. *Application 4: Glass fibers data*

This data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. This data are given in Appendix(d).These data have also been analyzed by Smith and Naylor[51].For this data set,we shall compare the fits of the new distribution with some competitive models like OLEW,E-W,T-W.Based on the Table 9 we conclude that the proposed MOL-W model is the best model with $CVM_{statistic} = 0.10565$ and $AD_{statistic} = 0.59106$. Many other useful version can be used in more comparisons see Al-Babtain et al.[1],Al-Babtain et al.[2],Ibrahim et al.[24],Ibrahim and Yousof[25], Ibrahim and Yousof[26]Ibrahim et al.[27], Alshkaki[6]and Esmaeili et al.[16].

Table 8: MLEs (standard errors in parentheses) for data set **IV**.

Distribution	Estimates			
OLE-W(a, α, β)	0.50878	2.534	1.7122	
	(0.397)	(1.8298)	(0.0959)	
E-W(a, α, β)	0.671	7.285	1.718	
	(0.249)	(1.707)	(0.086)	
T-W(a, α, β)	-0.5010	5.1498	0.6458	
	(0.2741)	(0.6657)	(0.0235)	
OLL-W(a_1, α, β)	0.9439	6.0256	0.6159	
	(0.2689)	(1.3478)	(0.0164)	
MOL-W(α, β, a_1, a_2)	**16.6312**	**30.6553**	**3.2027**	**0.3065**
	(20.7)	**(0.000)**	**(0.945)**	**(0.000)**

Table 9: $\text{CVM}_{(\text{statistic})}$ and $\text{AD}_{(\text{statistic})}$ for data set **IV**.

Distribution	$\text{CVM}_{(\text{statistic})}$	$\text{AD}_{(\text{statistic})}$
MOL-W(α, β, a_1)	**0.1057**	**0.5911**
OLEW(a, α, β)	0.2711	1.4965
E-W(a, α, β)	0.636	3.484
T-W(a, α, β)	1.0358	0.1691
OLL-W(a_1, α, β)	1.2364	0.2194

7. Concluding remarks

This paper introduces a new four-parameter lifetime model called the Marshall-Olkin Lehmann Weibull (MOL-W) model. Various of its structural properties are derived. The new PDF is expressed as a linear mixture of well-known exponentiated Weibull PDF. The PDF of the MOL-W distribution exhibits various important shapes with different kurtosis. The HRF of the MOL-W distribution exhibits "constant hazard rate ($\alpha = 1, \beta{=}1, a_1 = 1, a_2 = 1$)**", "**upside down-constant ($\alpha = 0.5, \beta{=}0.5, a_1 = 1.01, a_2 = 1$)"**, "**decreasing hazard rate ($\alpha = 0.5, \beta{=}5, a_1 = 1, a_2 = 0.2$)**",** **"**increasing-constant hazard rate ($\alpha = 0.5, \beta{=}0.15, a_1 = 1.25, a_2 = 1$)**",** "increasing hazard rate ($\alpha = 2, \beta{=}1, a_1 = 1.5, a_2 = 1$)**", "J-**hazard rate ($\alpha = 0.5, \beta{=}1, a_1 = 20, a_2 = 1$)**"** and "decreasing hazard rate ($\alpha = 0.2, \beta{=}1, a_1 = 0.1, a_2 = 1$)**".** We proved the wide flexibility of the new model numerically and graphically. Simple type Copula-based construction is presented to derive many bivariate and multivariate type models. The maximum likelihood method is used to estimate the model parameters. Graphical simulation results to assess the performance of the maximum likelihood estimation are performed. We proved empirically the importance and flexibility of the new Lehmann Weibull model in modeling various types of data. The new distribution has a high ability to model different types of real data sets such as the "symmetric bimodal" real data, the "asymmetric bimodal heavy tailed right skewed" real data, "asymmetric bimodal right skewed" real data and "asymmetric bimodal heavy tailed left skewed" real data.

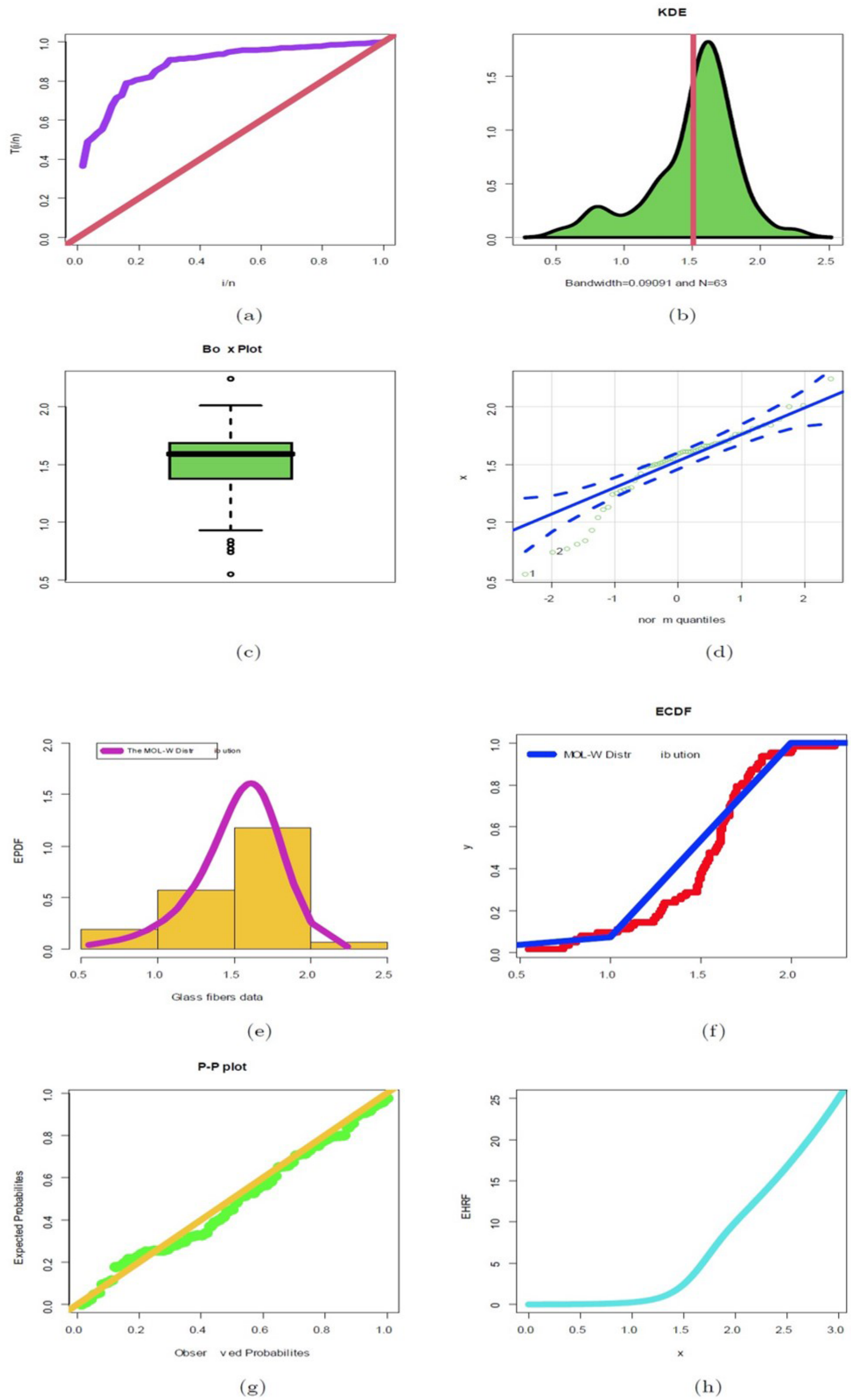

Figure 9. TTT plot (a), KDE plot (b), box plot (c), QQ plot (d), EPDF plot (e), ECDF plot (f), P-P plot (g), EHRF plot (h) for data set **IV**.

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Appendix

(a):

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 2.324, 3.376, 4.663.

(b):

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

(c):

10, 33, 44, 56, 59, 72,74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

(d):

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

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