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Truncated Cauchy Power Kumaraswamy Generalized Family of Distributions:Theory and Applications

Ibrahim Elbatal [1], Laba Handique [2,*], Subrata Chakraborty [3]

[1] *Department of Mathematics and Statistics - College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU)- Saudi Arabia*
[2] *Department of Statistics, Darrang College, Tezpur, Assam, India*
[3] *Department of Statistics,Dibrugarh University, Assam, India*

Abstract A new family called the Truncated Cauchy Power Kumaraswamy -G family of distributions is proposed. Some special models of this family are introduced. Statistical properties of the family such as expansion of density function, moments, incomplete moments, mean deviation, bonferroni and Lorenz curves are proposed. We discuss the method of maximum likelihood to estimate the model parameters and study its performance by simulation. Real data sets are modeled to illustrate the importance and flexibility of the proposed model in comparison to some known ones yielded favourable results.

Keywords Truncated Cauchy Power family, Kumaraswamy family, Entropy, Moments, Maximum likelihood estimation.

AMS 2010 subject classifications 60E05, 62F10, 62G05.

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1. Introduction

Several statisticians have been interested in defining new generators or generalized classes of univariate continuous distributions by introducing additional shape parameter(s) to a baseline model. These extended distributions give greater flexibility for applications in several fields such as medical sciences, environmental, engineering, biological studies, life testing problems, demography, actuarial and economics. Many generalized of families have been developed and applied to describe various phenomena in real data. Some examples of these families are beta-generated [11], generalized Kumaraswamy [10], Marshall Olkin alpha power-G [26], Generalized Marshall-Olkin-Kumaraswamy-G family [7], Beta generated kumaraswamy-Marshall Olikinn G family of distributions [14], Beta generalized Marshall Olikin G family of distributions [15],Beta generated Kumaraswamy - G family of distributions [16], Kumaraswamy Marshal-Olkin family of distributions [3], Kumaraswamy Poisson-G family [8], transmuted odd Fréchet- G family by [5], Exponentiated Generalized Marshall-Olkin family [18], Poisson Transmuted-G [19] among others .

Cordeiro et al.[10] introduced a new family of generalized distributions called Kumaraswamy generalized $(Kw - G)$ distributions. From an arbitrary parent cdf $G(x)$, the cumulative distribution function (cdf) of the $Kw - G$ distribution is defined by

$$H_{kw}(x; \lambda, \theta, \xi) = 1 - (1 - G(x; \xi)^{\lambda})^{\theta}, x > 0, \tag{1}$$

where $\lambda > 0$ and $\theta > 0$ are two extra parameters induced for controlling the skewness and tail density. The $Kw - G$ distribution with compact distribution function (1) is equally well suited for the censored data. The corresponding

*Correspondence to: Laba Handique (Email: handiquelaba@gmail.com). Department of Statistics, Darrang College, Tezpur, Assam, India.

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probability density function (pdf) is given by

$$h_{kw}(x;\lambda,\theta,\xi)=\lambda\theta g(x;\xi)G(x;\xi)^{\lambda-1}(1-G(x;\xi)^{\lambda})^{\theta-1}, \tag{2}$$

where $G(x;\xi)$ is the baseline cumulative distribution function depending on a vector of parameter ξ. One major benefit of the $Kw-G$ family of generalized distributions is its ability to fit skewed data that cannot be properly fitted by existing distributions.

The Cauchy distribution plays an important role and has applications in different fields such as econometrics, engineering, spectroscopy, biological analysis, reliability , queuing theory and stochastic modeling of decreasing hazard rate life devices. There are many generalization and extension forms of Cauchy distribution in the statistical literature , for examples , Rider [29] presented generalized Cauchy distribution, A truncated Cauchy distribution by Nadarajah et al. [25], The existence of the moments of the Cauchy distribution by Ohakwe et al. [28], Jacob et al. [20] studied On half-Cauchy distribution, Kumaraswamy- half- Cauchy distribution by Hamedani et al. [13], Alshawarbeh et al. [4] presented properties of Beta-Cauchy distribution, The power-Cauchy negative- binomial distribution by Zubair et al. [30], Cordeiro et al. [10] introduced the beta Half-Cauchy distribution, The odd power Cauchy family by Alizadeh et al. [2], among others.

Recently, Aldahlan et al. [1] proposed the Truncated Cauchy Power-G ($TCP-G$) family. They defined on the basis on the truncated Cauchy distribution with the interval (0, 1) and the $exp-G$ family. The cdf of the $TCP-G$ family is given by

$$F(x;\alpha)=\frac{4}{\pi}\arctan(H(x)^{\alpha}),\quad x\in\mathbb{R} \tag{3}$$

where $\alpha>0$.The corresponding the probability density function (pdf) and hazard rate functions (hrf) are

$$f\,(x;\alpha)=\frac{4\alpha h(x)(H(x))^{\alpha-1}}{\pi\left[1+(H(x))^{2\alpha}\right]}, \tag{4}$$

and

$$\tau(x;\alpha)=\frac{f\,(x;\alpha)}{\overline{F}(x;\alpha)}=\frac{4\alpha h(x)(H(x))^{\alpha-1}}{\pi\left[1+(H(x))^{2\alpha}\right]\left[1-\frac{4}{\pi}\arctan(H(x)^{\alpha})\right]}. \tag{5}$$

respectively.

The main aim of this paper is to present a new wider and flexible family of distributions based on $Kw-G$ family and TCP_G family . We construct a new family called Truncated Cauchy Power kumaraswamy- G ($TCPK_w-G$) family of distributions by inserting (1) into (3) , the cdf and pdf of the $TCPK_w-G$ are given, respectively, by

$$F(x;\lambda,\theta,\alpha,\xi)=\frac{4}{\pi}\arctan\left[1-(1-G(x;\xi)^{\lambda})^{\theta}\right]^{\alpha},x>0, \tag{6}$$

and

$$\begin{aligned} f\,(x;\lambda,\theta,\alpha,\xi) &= 4\alpha\lambda\theta g(x;\xi)G(x;\xi)^{\lambda-1}(1-G(x;\xi)^{\lambda})^{\theta-1}\\ &\quad\times\frac{(1-(1-G(x;\xi)^{\lambda})^{\theta})^{\alpha-1}}{\pi\left[1+(1-(1-G(x;\xi)^{\lambda})^{\theta})^{2\alpha}\right]}. \end{aligned} \tag{7}$$

Henceforth, a random variable X has pdf (7) will be defined as $X\backsim TCPK_w(\lambda,\theta,\alpha,\xi)$. The survival function for the $TCPK_w-G$ family is given by

$$\overline{F}(x;\lambda,\theta,\alpha,\xi)=1-\frac{4}{\pi}\arctan\left[1-(1-G(x;\xi)^{\lambda})^{\theta}\right]^{\alpha},$$

This paper is organized as follows. A useful expansion for the pdf and cdf of $TCPK_w$ family and special models are presented in Section 2. Some of mathematical properties including quantile function , moments, incomplete moments , mean deviations , Lorenz and Bonferroni curves , residual life and reversed residual life functions are provided in Section 3. The entropy is discussed in Section 4. Parameter estimation by the maximum likelihood method is investigated in Section 5. In the section 6 a simulation study to asses estimation methods is presented. Applications to real data set to illustrate the flexibility of the proposed family are provided in Section 7.

2. Linear Representation of the $TCPK_{\mathrm{w}} - G$ Density Function

In this section we will used the useful power series to derive the expansion of the $TCPK_w - G$ density. If $\mid z \mid < 1$ and $b > 0$ is a real non-integer ,then the following power series are hold.

$$(1+z)^{-b} = \sum_{k=0}^{\infty} \binom{-b}{k} z^k, \tag{8}$$

and

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j, \tag{9}$$

applying (8) to the last term in (7) , we get

$$f_{\ TCPK_w}(x;\lambda,\theta,\alpha,\xi) = \frac{4\alpha\lambda\theta}{\pi} \sum_{i=0}^{\infty} (-1)^i g(x;\xi) G(x;\xi)^{\lambda-1} (1-G(x;\xi)^{\lambda})^{\theta-1} \\ (1-(1-G(x;\xi)^{\lambda})^{\theta})^{\alpha(2i+1)-1} \tag{10}$$

applying (9) in the last term of (10) becomes

$$f_{\ TCPK_w}(x;\lambda,\theta,\alpha,\xi) = \frac{4\alpha\lambda\theta}{\pi} \sum_{i,j=0}^{\infty} (-1)^{i+j} \tbinom{\alpha(2i+1)-1}{j} g(x;\xi) G(x;\xi)^{\lambda-1} \\ (1-G(x;\xi)^{\lambda})^{\theta(j+1)-1} \tag{11}$$

again using (9) into (11) the $TCPK_w - G$ density can be expressed as infinite linear combination of exp - G density functions

$$f_{\ TCPK_w}(x;\theta,\lambda,\alpha,\xi) = \sum_{k=0}^{\infty} \vartheta_k \ \pi_{(\lambda(k+1))}(x), \tag{12}$$

where $\pi_{(\lambda(k+1))}(x) = (\lambda(k+1)) g(x) G^{\lambda(k+1)-1}(x)$ is the exp - G pdf with power parameter $\lambda(k+1)$ and

$$\vartheta_k = \frac{4\alpha\lambda\theta(-1)^k}{\pi} \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\alpha(2i+1)-1}{j} \binom{\theta(j+1)-1}{k}.$$

Thus the $TCPK_w - G$ density can be written as an infinite mixture of the exponentiated - G densities with parameter $(\lambda(k+1))$, Thus, several mathematical and statistical properties of the $TCPK_w$ distribution can be determined obviously from those of exp - G distribution. Similarly, the cdf of the $TCPK_w$ family can also be expressed as a mixture of exp - G cdfs where

$$F_{\ TCPK_w}(x;\lambda,\theta,\alpha,\xi) = \sum_{k=0}^{\infty} \vartheta_k \ \Pi_{(\lambda(k+1))}(x).$$

where $\Pi_{(\lambda(k+1))}(x)$ is the exp- G cdf with power parameter $(\lambda(k+1))$.

2.1. Special models

In this section we introduced three special models of the $TCPK_w$ family of distributions, to ensure tractability we choose the cdf $G(x)$ and pdf $g(x)$ with nice compact form to provide three sub models of this family by taking the baseline distributions: Lomax, Exponential and Rayleigh distributions. The cdf and pdf of thses baseline models are listed in the following table

$Model$	$Cdf: G(x;\xi)$	$pdf: g(x;\xi)$
$Lomax$	$1-(1+\frac{x}{\beta})^{-\mu}$	$\frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-\mu-1}$
$Exponential$	$1-e^{-ax}$	ae^{-ax}
$Rayleigh$	$1-e^{-\frac{\rho}{2}x^2}$	$\rho x e^{-\frac{\rho}{2}x^2}$

2.1.1. Truncated Cauchy Power Kumaraswamy Lomax (TCPK$_w$L) distribution: The cdf and pdf of TCPK$_w$L distribution are

$$F(x;\lambda,\theta,\alpha,\beta,\mu)=\frac{4}{\pi}\arctan\left[1-\left[1-(1-(1+\frac{x}{\beta})^{-\mu})^{\lambda}\right]^{\theta}\right]^{\alpha},x>0,$$

and

$$f\,(x;\lambda,\theta,\alpha,\beta,\mu) = 4\alpha\lambda\theta\frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-\mu-1}(1-(1+\frac{x}{\beta})^{-\mu})^{\lambda-1}$$

$$\times\frac{\left[1-(1-(1+\frac{x}{\beta})^{-\mu})^{\lambda}\right]^{\theta-1}\left[1-\left[1-(1-(1+\frac{x}{\beta})^{-\mu})^{\lambda}\right]^{\theta}\right]^{\alpha-1}}{\pi\left[1+\left[1-\left[1-(1-(1+\frac{x}{\beta})^{-\mu})^{\lambda}\right]^{\theta}\right]^{2\alpha}\right]}.$$

2.1.2. Truncated Cauchy Power Kumaraswamy Exponential (TCPK$_w$E) distribution: The cdf and pdf of the TCPK$_w$E model (for $x>0$) are

$$F(x;\lambda,\theta,\alpha,a)=\frac{4}{\pi}\arctan\left[1-(1-(1-e^{-ax})^{\lambda})^{\theta}\right]^{\alpha},x>0,$$

and

$$f\,(x;\lambda,\theta,\alpha,a) = 4\alpha\lambda\theta a e^{-ax}(1-e^{-ax})^{\lambda-1}(1-(1-e^{-ax})^{\lambda})^{\theta-1}$$

$$\times\frac{(1-(1-(1-e^{-ax})^{\lambda})^{\theta})^{\alpha-1}}{\pi\left[1+(1-(1-(1-e^{-ax})^{\lambda})^{\theta})^{2\alpha}\right]}.$$

2.1.3. Truncated Cauchy Power Kumaraswamy Rayleigh (TCPK$_w$R) distribution: The cdf and pdf of the TCPK$_w$R model (for $x>0$) are

$$F(x;\lambda,\theta,\alpha,\rho)=\frac{4}{\pi}\arctan\left[1-(1-G(1-e^{-\frac{\rho}{2}x^2})^{\lambda})^{\theta}\right]^{\alpha},x>0,$$

and

$$f\,(x;\lambda,\theta,\alpha,\rho) = 4\alpha\lambda\theta\rho x e^{-\frac{\rho}{2}x^2}(1-e^{-\frac{\rho}{2}x^2})^{\lambda-1}(1-(1-e^{-\frac{\rho}{2}x^2})^{\lambda})^{\theta-1}$$

$$\times\frac{(1-(1-(1-e^{-\frac{\rho}{2}x^2})^{\lambda})^{\theta})^{\alpha-1}}{\pi\left[1+(1-(1-(1-e^{-\frac{\rho}{2}x^2})^{\lambda})^{\theta})^{2\alpha}\right]}.$$

We have provided two plots in Figures (1) and (2) for the pdf and two plots in Figures (3) and (4) for the hzarad rate function (hrf) of TCPK$_w$E$(\lambda,\theta,\alpha,a)$ for different choices of parameters. From the plots its apparent that this particular distribution can be positively skewed to symmetric and the hrf is increasing with different shapes.

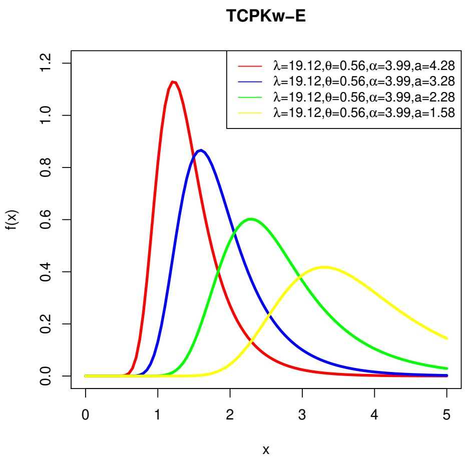

Figure 1. pdf of TCPKw-E

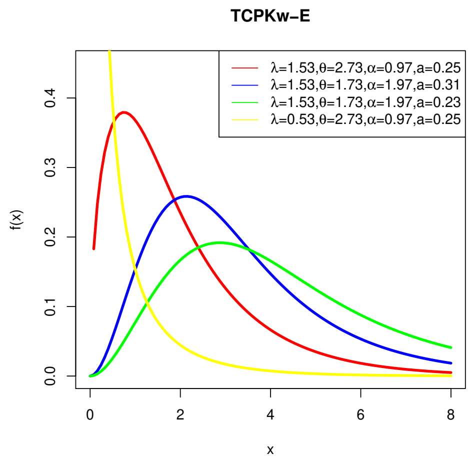

Figure 2. pdf of TCPKw-E

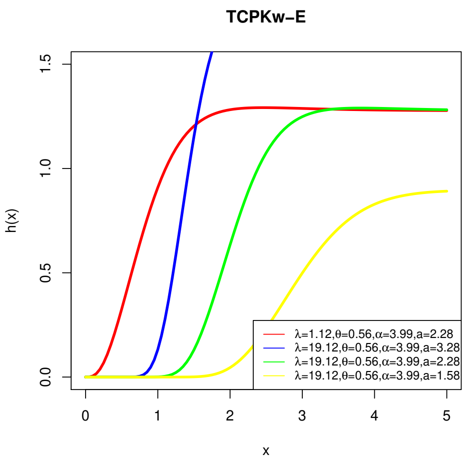

Figure 3. hrf of TCPKw-E

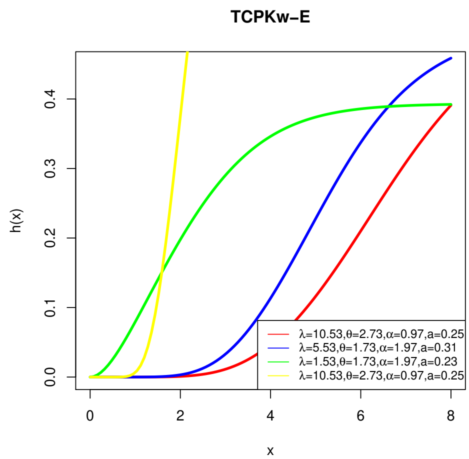

Figure 4. hrf of TCPKw-E

3. Statistical Properties

In this section we studied the statistical properties of the $TCPK_w$ distribution, specifically quantile function , moments, incomplete moments , mean deviation , Lorenz and Bonferroni curves , residual life and reversed residual life functions and order statistics .

3.1. Quantile Function

Quantile functions are used in theoretical aspects, statistical applications and Monte Carlo methods. Monte-Carlo simulations employ quantile functions to produce simulated random variables for classical and new continuous distributions. The $TCPK_w$ quantile function , say $x = Q(u)$ can be obtained by inverting (6) as follows

$$F^{-1}(u) = Q_G(u) = G^{-1}\left\{1 - \left[1 - \left(\tan(\frac{u\,\pi}{4})\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{\theta}}\right\}^{\frac{1}{\lambda}}. \tag{13}$$

where $Q_{G(u)}$ denotes the quantile function corresponding to $G(x)$. Let us recall that $Q(u)$ is characterized by the non-linear equation $F(Q(u)) = Q(F(u)) = u$, $u \in (0,1)$. One of the earliest skewness measures to be suggested is the Bowley skewness [22] defined by

$$SK = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

On the other hand , the Moors kurtosis [24] based on quantiles is given by

$$KU = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

where $Q(\cdot)$ represents the quantile function. The measures SK and KU are less sensitive to outliers and they exist even for distributions without moments. For symmetric unimodal distributions, positive kurtosis indicates heavy tails and peakedness relative to the normal distribution, whereas negative kurtosis indicates light tails and flatness. For the normal distribution, $SK = KU = 0$. We have plotted skweness and kurtosis of TCPKwE distribution for varying λ and θ keeping other parameters fixed in figures (5) and (6). Note that the parameter a has no effect of the both skewness as well as kurtosis.

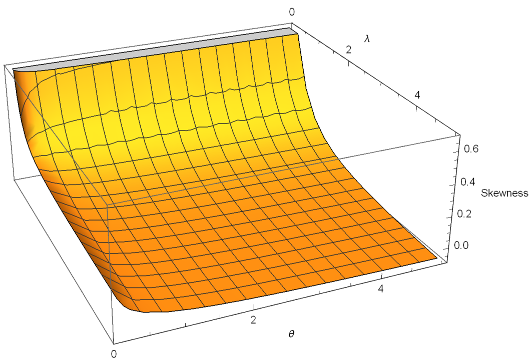

Figure 5. Skewness of TCPKw-E

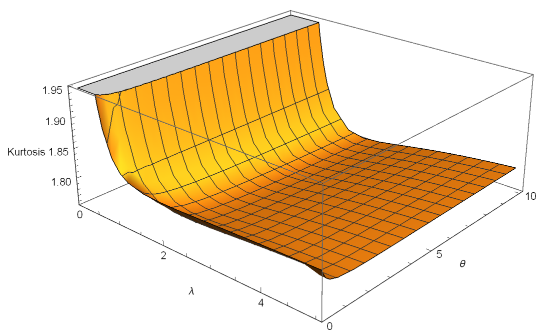

Figure 6. Kurtosis of TCPKw-E

3.2. *Moments and moment generating functions*

In this sub- section, the ordinary moment and moment generating functions of $TCPK_w$ family are derived. The different orders for the moments is very useful to determine the expected life time of a device , skewness and kurtosis in a given set of observations arising in reliability applications.

3.2.1. Moments: Let $Y_{(\lambda(k+1))}$ be a random variable having the $exp - G$ pdf $\pi_{(\lambda(k+1))}$ with power parameter $(\lambda(k+1))$. The r^{th} moment of $TCPK_w$ family of distributions can be obtained from (12)

$$\mu_r^{/} = E(X^r) = \sum_{k=0}^{\infty} \vartheta_k \, E(Y^r_{(\lambda(k+1))}) \tag{14}$$

where $Y_{(\lambda(k+1)}$ denotes the the exponentiated - G distribution with power parameter $\lambda(k+1)$.Another formula for the r^{th} moment follows from (12) as

$$\mu_r^{/} = E(X^r) = \sum_{k=0}^{\infty} \vartheta_k E(Y^r_{(\lambda(k+1))})$$

where

$$\begin{aligned} E(Y_\eta^r) &= \eta \int_{-\infty}^{\infty} x^r g(x) G(x)^{\eta-1}, \eta > 0 \\ &= \eta \int_0^1 u^{\eta-1} Q_G(u)^r du, \end{aligned}$$

where $Q_G(u) = G^{-1}(u)$ can be calculated numerically.
Now we introduce two formulae for the moment generating function . The first formula can be calculated from equation (12) as follows

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \vartheta_k \; M_{(\lambda(k+1))}(t), \tag{15}$$

where $M_{(\lambda(k+1))}(t)$ is the moment generating function of $Y_{(\lambda(k+1))}$. Consequently, $M_X(t)$ is obtained from mgf of from the exp- G generating function. The second formula for the $M_X(t)$ can be written as

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \vartheta_k \; \eta\,(t, \lambda(k+1))$$

where $\eta\,(t, w) = \int_0^1 e^{tQ_G(u)} u^w du$ which can be calculated numerically from the baseline quantile function , i.e., $Q_G(u) = G^{-1}(u)$.

3.3. *Incomplete Momemts*

The first incomplete moment is used to find the mean deviations, Bonferroni and Lorenz curves. These are useful in many areas of applied studied including econometrics, engineering, medicine etc.. The s^{th} incomplete moments of X defined by $\chi_s(t)$ for any real $s > 0$ can be expressed from (12) as

$$\chi_s(t) = \int_{-\infty}^{t} x^s f\,(x) dx = \sum_{k=0}^{\infty} \vartheta_k \; \int_{-\infty}^{t} x^s \; \chi_{s,(\lambda(k+1)}(t) dx \tag{16}$$

where

$$\chi_{s,b}(t) = \int_0^{G(t)} u^{b-1} Q_G(u)^s du$$

and $\chi_{s,b}(t)$ can be evaluated numerically.

The mean deviations give important information about characteristic of population and also have been applied of income fields and property in economics. If X has the $TCPK_w$ family of distribution. The mean deviations about the mean $\mu = E(X)$ and the mean deviations about the median M are defined by

$$\delta_\mu(x) = E \mid X - \mu_1' \mid = 2\mu_1' F(\mu_1') - 2\chi(\mu_1') \tag{17}$$

and

$$\delta_M(x) = E \mid X - M \mid = \mu_1' - 2\chi_1(M) \tag{18}$$

respectively, where $\mu_1^{/} = E(X)$, $M = \text{median}(X) = Q(\frac{1}{2})$, and $\chi_1(t)$ is the first complete moment given by (16) with $s = 1$.

Lorenz and Bonferroni curves are used to income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. For a positive random variable X, The Lorenz and Bonferroni curves , for a given probability p , are given by $L(p) = \frac{\chi_1(q)}{\mu_1'}$ and $B(p) = \frac{\chi_1(q)}{p\mu_1'}$ respectively, where $\mu_1' = E(X)$, and $q = Q(p)$ is the quantile function of X at p.

3.4. Residual life and reversed residual life functions.

Suppose that a component survives up to time $t \geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X - t|X > t$. The r^{th} moment of the residual life is given by (See Navarro et al., 1998)

$$\begin{aligned} \mu_r(t) &= E((X-t)^r \mid X > t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} (x-t)^r f\,(x)dx, r \geq 1 \\ &= \frac{1}{\overline{F}(t)} \sum_{k=0}^{\infty} \vartheta_k^* \int_t^{\infty} x^r \pi_{(\lambda(k+1))}(x)dx \end{aligned} \tag{19}$$

where $\vartheta_k^* = \vartheta_k \sum_{m=0}^{r} \binom{r}{m} (-t)^{r-m}$. The mean residual life (or the life expectancy at age t) represents the expected remaining lifetime of a component or device that has survived up to age t. The MRL of $TCPK_w$ family of distributions can be obtained by setting $r = 1$ in the Equation (19), defined as

$$\mu(t) = E(X_t) = E(X \mid X > t).$$

In reliability theory , the additional life time given that the component has already failed by time t, is called reversed residual life function (RRL) of the component . represent the reversed residual lifetime . The conditional random variable $X_{(t)} = t - X \mid X < t$ shows that the time elapsed since the the failure of X given that it failed at or before t. The r^{th} moment of the reversed residual life (or inactivity time) can be obtained by the well known formula

$$\begin{aligned} m_r(t) &= E((t-X)^r \mid X \leq t) = \frac{1}{F(t)} \int_0^{t} (t-x)^r f\,(x)dx, r \geq 1 \\ &= \frac{1}{F(t)} \sum_{k=0}^{\infty} \vartheta_k^* \int_0^{t} x^r \pi_{(\lambda(k+1))}(x)dx \end{aligned} \tag{20}$$

an alternative ageing measure widely used in application is the mean past lifetime (MPL) function The mean inactivity time (MIT) of the $TCPK_w$ family of distributions can be determined by setting $n = 1$ in (20), where

$$m(t) = E(X_{(t)}) = E(t - X \mid X < t).$$

4. Entropy

The entropy is significant to measure the amount of uncertainty associated with a random variable X. It is very useful in communication, physics and probability, the Rényi entropy is defined by $(\rho > 0, \rho \neq 1)$

$$I_R(\rho) = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} f^{\,\rho}(x)dx \right]. \tag{21}$$

Using (7), applying the same procedure of the useful expansion (12) and after some simplifications, we get

$$f^{\,\rho}(x) = \sum_{k=0}^{\infty} \Lambda_k \; g(x)^{\rho} \; G(x)^{\lambda(k+\rho)-\rho}$$

where

$$\Lambda_k = \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{-\rho}{i} \binom{\alpha(2i+\rho)-\rho}{j} \binom{\theta(\rho+j)-\rho}{k}.$$

Thus Rényi entropy of $TCPK_w$ family is defined as

$$I_R(\rho) = \frac{\rho}{1-\rho} \log\left(\frac{4\alpha\lambda\theta}{\pi}\right) + \frac{1}{1-\rho} \log\left\{ \sum_{k=0}^{\infty} \Lambda_k \int_{-\infty}^{\infty} g(x)^{\rho} \; G(x)^{\lambda(k+\rho)-\rho} dx \right\}. \tag{22}$$

5. Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. Here the MLEs of the parameters for complete samples is considered. Let $x_1, ..., x_n$ be a random sample of size n from the $TCPK_w$ distribution given by (7). Let $\Psi = (\lambda, \theta, \alpha, \xi)^T$ be $q \times 1$ vector of parameters. The log-likelihood function is given by

$$
\begin{aligned}
L_n &= n\log\left(\frac{4\lambda}{\pi}\right) + n\log(\theta) + n\log(\alpha) + \sum_{i=1}^{n}\log g(x_i;\xi) + (\lambda-1)\sum_{i=1}^{n}\log G(x_i;\xi) \\
&\quad +(\theta-1)\sum_{i=1}^{n}\log(1-t_i) + (\alpha-1)\sum_{i=1}^{n}\log(1-(1-t_i)^{\theta}) \\
&\quad -\sum_{i=1}^{n}\log\left[1+(1-(1-t_i)^{\theta})^{2\alpha}\right]. \qquad (23)
\end{aligned}
$$

The components of score function $U(\Psi) = (U_\lambda, U_\theta, U_\alpha, U_\xi)$ are

$$
\begin{aligned}
U_\lambda &= \frac{\partial L_n}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log G(x_i;\xi) - (\theta-1)\sum_{i=1}^{n}\frac{\log(G(x_i;\xi))t_i}{1-t_i} \\
&\quad +(\alpha-1)\sum_{i=1}^{n}\frac{\theta\log(G(x_i;\xi))t_i(1-t_i)^{\theta-1}}{1-(1-t_i)^{\theta}} \\
&\quad -\sum_{i=1}^{n}\frac{2\alpha\theta\log(G(x_i;\xi))t_i(1-t_i)^{\theta-1}(1-(1-t_i)^{\theta})^{2\alpha-1}}{1+(1-(1-t_i)^{\theta})^{2\alpha}}, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
U_\theta &= \frac{\partial L_n}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-t_i) \\
&\quad +(\alpha-1)\sum_{i=1}^{n}\frac{\log(1-t_i)(1-t_i)^{\theta}}{1-(1-t_i)^{\theta}} \\
&\quad -\sum_{i=1}^{n}\frac{2\alpha\log(1-t_i)(1-t_i)^{\theta}(1-(1-t_i)^{\theta})^{2\alpha-1}}{[1+(1-(1-t_i)^{\theta})^{2\alpha}]}, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
U_\alpha &= \frac{\partial L_n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1-(1-t_i)^{\theta}) \\
&\quad -\sum_{i=1}^{n}\frac{2\log(1-(1-t_i)^{\theta})(1-(1-t_i)^{\theta})^{2\alpha}}{1+(1-(1-t_i)^{\theta})^{2\alpha}}, \qquad (26)
\end{aligned}
$$

$$U_\xi = \frac{\partial L_n}{\partial \xi_k} = \sum_{i=1}^{n} \frac{g'(x_i;\xi)}{g(x_i;\xi)} + (\lambda-1)\sum_{i=1}^{n} \frac{G'(x_i;\xi)}{G(x_i;\xi)} + (\theta-1)\sum_{i=1}^{n} \frac{\lambda t_i G'(x_i;\xi)}{1-t_i} + (\alpha-1)\sum_{i=1}^{n} \frac{\theta\lambda t_i G'(x_i;\xi)(1-t_i)^{\theta-1}}{1-(1-t_i)^{\theta}} - \sum_{i=1}^{n} \frac{2\alpha\theta\lambda t_i G'(x_i;\xi)(1-t_i)^{\theta-1}(1-(1-t_i)^{\theta})^{2\alpha-1}}{1+(1-(1-t_i)^{\theta})^{2\alpha}}. \quad (27)$$

where $t_i = G(x_i;\xi)^\lambda$, $g'(x_i;\xi) = \frac{\partial g(x_i;\xi)}{\partial \xi_k}$ and $G'(x_i;\xi) = \frac{\partial G(x_i;\xi)}{\partial \xi_k}$. The maximum likelihood estimation (MLE) of parameters is obtained by setting $\frac{\partial L_n}{\partial \lambda} = \frac{\partial L_n}{\partial \theta} = \frac{\partial L_n}{\partial \alpha} = \frac{\partial L_n}{\partial \xi_k} = 0$ and one can solve these equations simultaneously to get the $MLE(\widehat{\Psi})$. But here solution of the ML equations is not trivial and hence is done using numerically optimization methods available in R software.

6. Simulation

To see how the MLEs perform, a simulation study is undertaken using the statistical software R. Sampling is done from the TCPKw-E distribution by the inversion method for two sets of parameters (i) $\lambda = 1.5$, $\theta = 1.5$, $\alpha = 2.8$, $a = 0.9$ and (ii) $\lambda = 1.2$, $\theta = 1.3$, $\alpha = 3.1$, $a = 0.5$ with sample sizes from small $n = 50$ to large $n = 500$. We have replicated 1000 times and evaluated the average bias and mean square error (MSE) as

$$Bias(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\theta}_j - \theta\right) \text{ and } MSE(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\theta}_j - \theta\right)^2.$$

The results graphically shown in the figures (7) to (10), for (i) and in the figures (11) to (14) for (ii) reveal that the biases and MSE decreases with increase in the sample size, thus establishing the asymptotic unbiasedness and consistency.

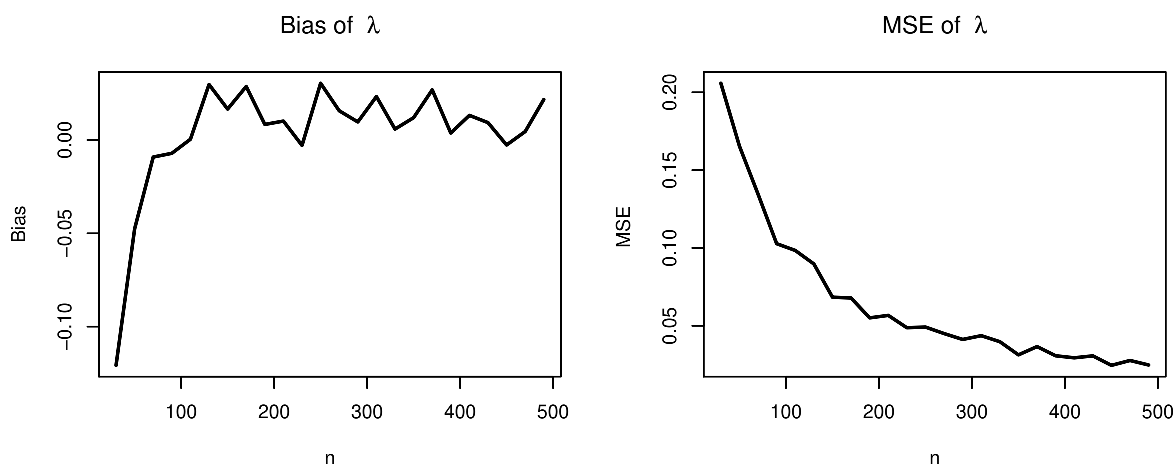

Figure 7. Bias (left) and MSE (right) of the parameter λ

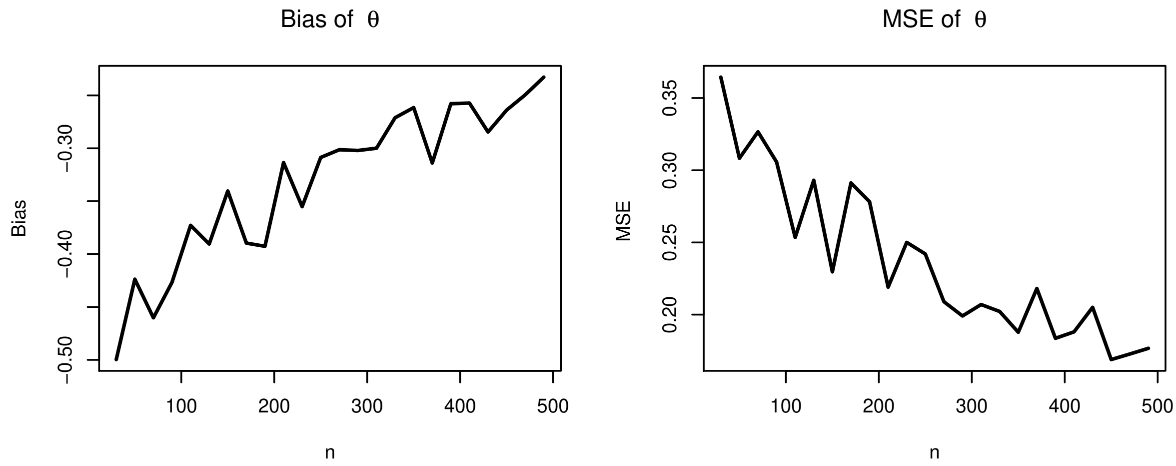

Figure 8. Bias (left) and MSE (right) of the parameter θ

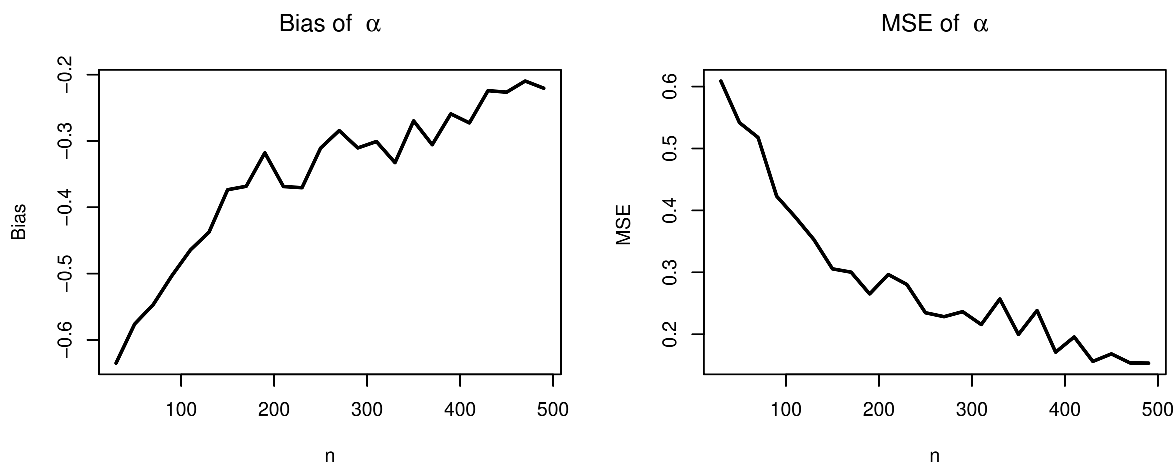

Figure 9. Bias (left) and MSE (right) of the parameter α

7. Applications

Here in order to show that the distributions from the proposed family can provide better model than the corresponding distributions exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E) [23], generalized Marshall-Olkin exponential (GMO-E) [21], Kumaraswamy exponential (Kw-E) [10], Beta exponential (BE) [11], Kumaraswamy Marshall-Olkin exponential (KwMO-E) [3] and Marshall-Olkin Kumaraswamy exponential (MOKw-E) [14] distribution by considering two failure time data sets from literature. Goodness of fit statistics namely the Kolmogorov-Smirnov (K-S) statistics, Anderson-Darling (A) and Cramer von-mises (W) are used along with model selection criteria AIC, BIC, CAIC and HQIC for the purpose of comparing the fitted models. Large sample standard errors of the mles for each considered distribution are also provided. Fitted density and the fitted cdf are presented in Figures (17), (18) and (19), (20) to see the closeness of the proposed distributions to these data.

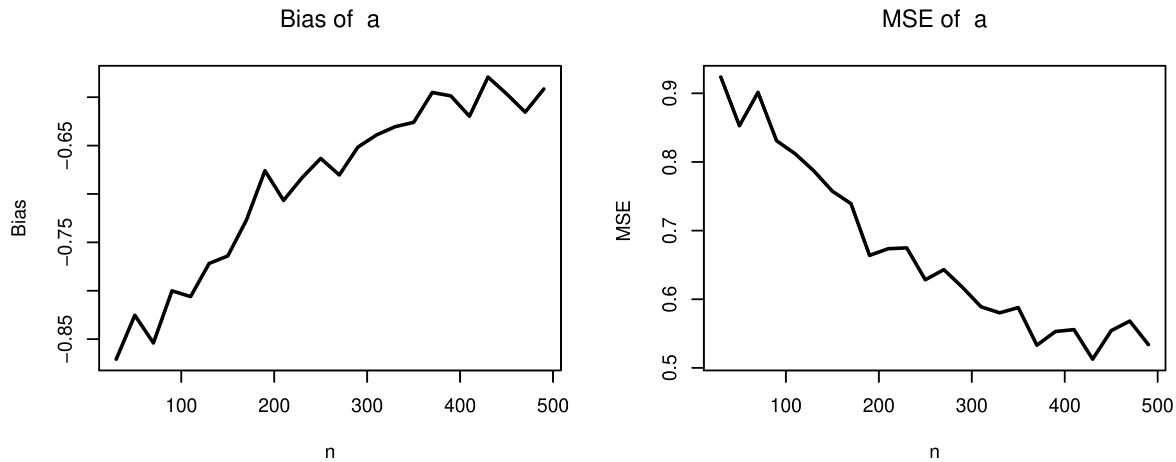

Figure 10. Bias (left) and MSE (right) of the parameter a

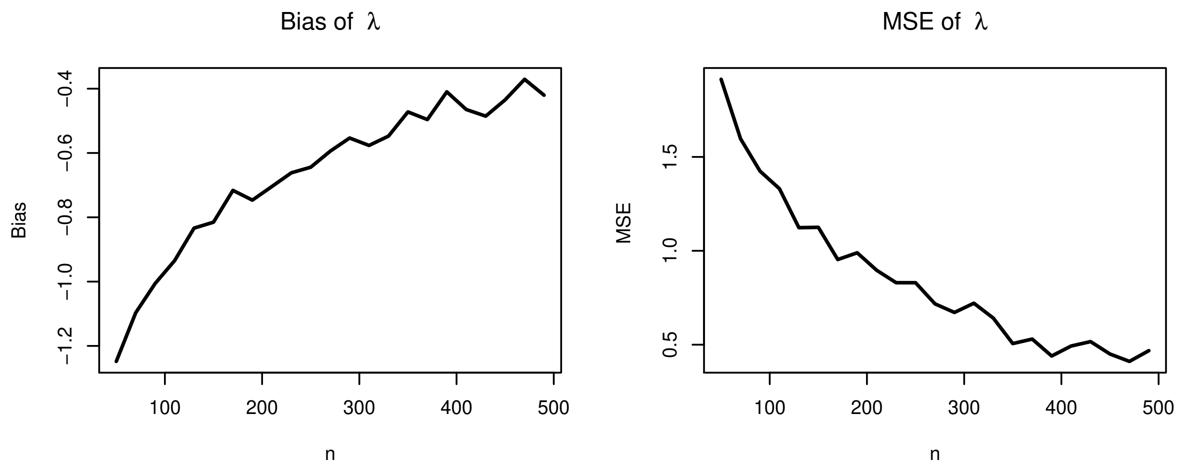

Figure 11. Bias (left) and MSE (right) of the parameter λ

The first data set is about relief times (in minutes) of patients receiving an analgesic [12], the second set presents survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli [6]. Both the data sets are positively skewed as expected from the nature of life time data and second set having higher kurtosis (see Table 1).

Data Sets	n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Max.
I	20	1.10	1.90	1.70	0.70	1.59	2.34	1.47	2.05	4.10
II	72	0.10	1.85	1.56	1.20	1.78	4.15	1.08	2.30	7.00

Table 1. Descriptive Statistics

More over the TTT plots for the data sets Figure (15) and (16) indicate that the both data sets have increasing hazard rate.

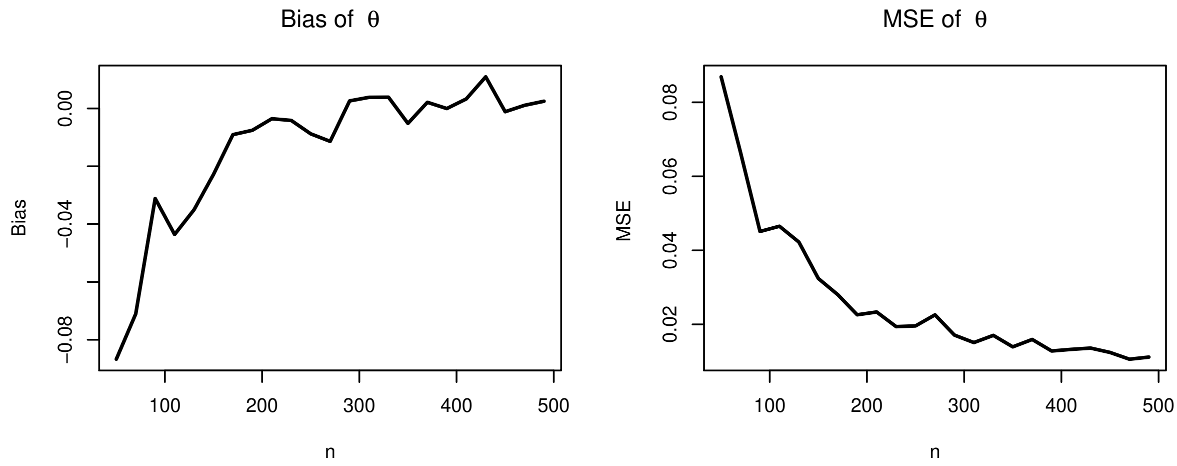

Figure 12. Bias (left) and MSE (right) of the parameter θ

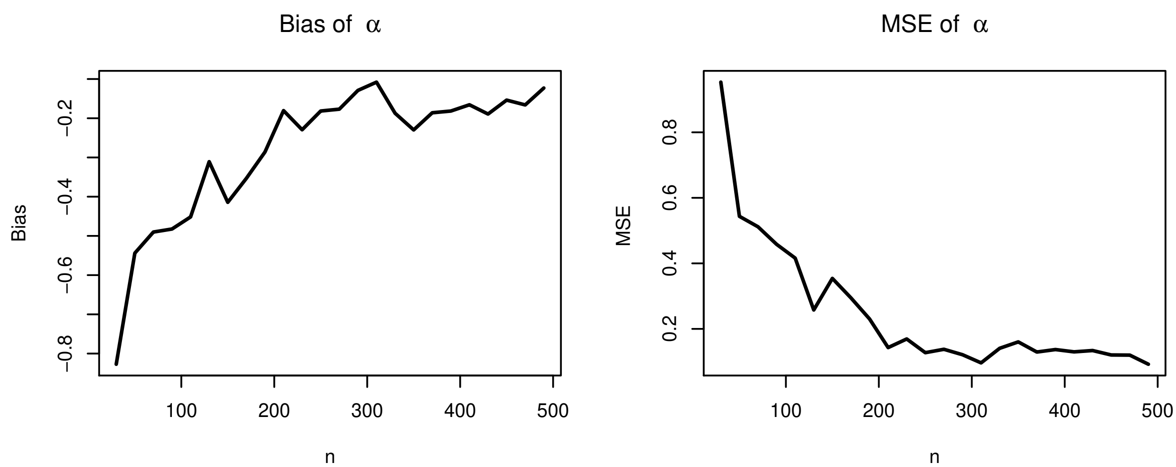

Figure 13. Bias (left) and MSE (right) of the parameter α

$Models$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{a}$
$Exp\,(a)$	–	–	–	$0.53\,(0.12)$
$M-E\,(a)$	–	–	–	$0.96\,(0.15)$
$MO-E\,(\alpha,a)$	–	–	$54.47\,(35.58)$	$2.31\,(0.37)$
$GMO-E\,(\theta,\alpha,a)$	–	$0.52\,(0.25)$	$89.46\,(66.27)$	$3.16\,(0.77)$
$Kw-E\,(\lambda,\theta,a)$	$83.75\,(42.36)$	$0.56\,(0.32)$	–	$3.33\,(1.18)$
$B-E\,(\lambda,\theta,a)$	$81.63\,(120.41)$	$0.54\,(0.32)$	–	$3.51\,(1.41)$
$KwMO-E\,(\lambda,\theta,\alpha,a)$	$34.82\,(22.31)$	$0.29\,(0.23)$	$28.86\,(9.14)$	$4.89\,(3.17)$
$MOKw-E\,(\lambda,\theta,\alpha,a)$	$33.23\,(57.83)$	$0.57\,(0.72)$	$0.13\,(0.33)$	$1.66\,(1.81)$
$TCPKw-E\,(\lambda,\theta,\alpha,a)$	$19.12\,(4.75)$	$0.56\,(0.59)$	$3.99\,(1.75)$	$3.28\,(3.47)$

Table 2. MLEs, standard errors (in parentheses) for first data set

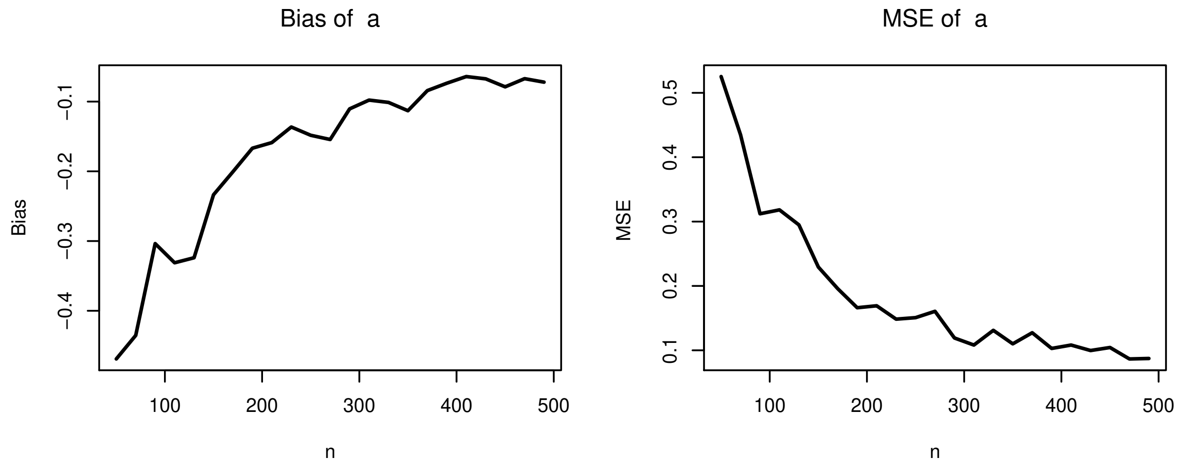

Figure 14. Bias (left) and MSE (right) of the parameter a

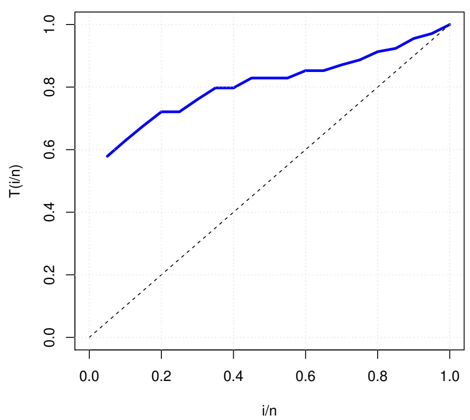

Figure 15. TTT Plot first data set

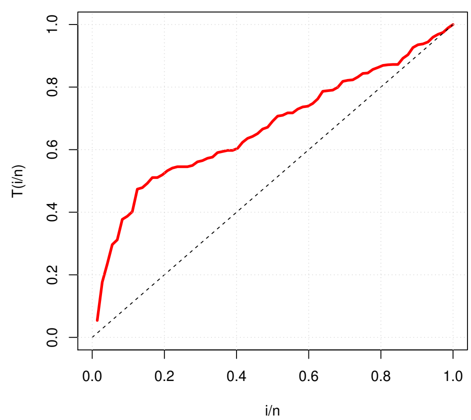

Figure 16. TTT Plot 2nd data set

$Models$	AIC	BIC	CAIC	HQIC	A	W	KS $(p - value)$
$Exp\,(a)$	67.67	68.67	67.89	67.87	4.60	0.96	$0.44\,(0.004)$
$M - E\,(a)$	54.32	55.31	54.54	54.50	2.76	0.53	$0.32\,(0.07)$
$MO - E\,(\alpha, a)$	43.51	45.51	44.22	43.90	0.81	0.14	$0.18\,(0.55)$
$GMO - E\,(\theta, \alpha, a)$	42.75	45.74	44.25	43.34	0.51	0.08	$0.15\,(0.78)$
$Kw - E\,(\lambda, \theta, a)$	41.78	44.75	43.28	42.32	0.45	0.07	$0.14\,(0.86)$
$B - E\,(\lambda, \theta, a)$	43.48	46.45	44.98	44.02	0.70	0.12	$0.16\,(0.80)$
$KwMO - E\,(\lambda, \theta, \alpha, a)$	42.88	46.84	45.55	43.60	1.08	0.19	$0.15\,(0.86)$
$MOKw - E\,(\lambda, \theta, \alpha, a)$	41.58	45.54	44.25	42.30	0.60	0.11	$0.14\,(0.87)$
$TCPKw - E\,(\lambda, \theta, \alpha, a)$	**39.53**	**43.52**	**42.20**	**40.31**	**0.23**	**0.04**	$\mathbf{0.11}\,(\mathbf{0.94})$

Table 3. Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for first data set

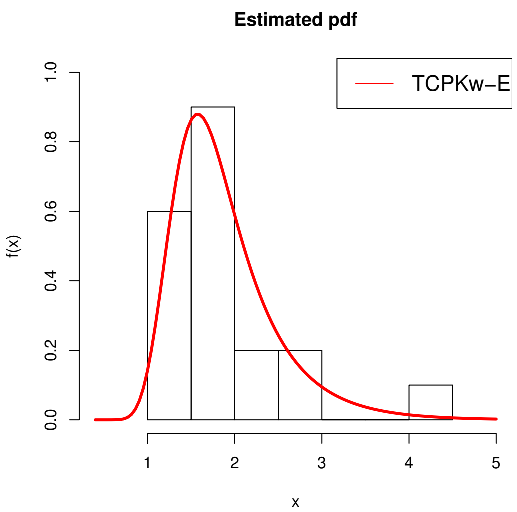

Figure 17. Observed histogram and estimated pdf for first data set

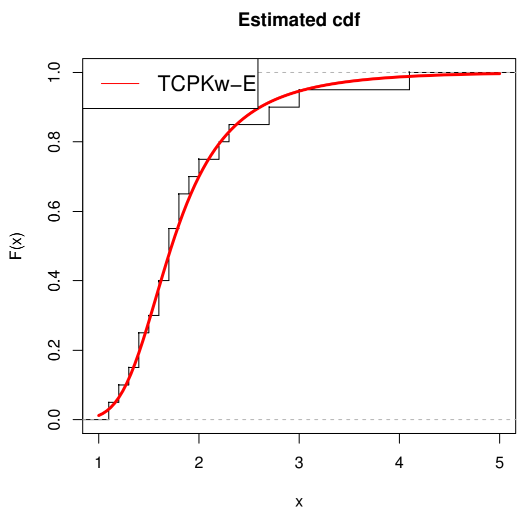

Figure 18. Observed ogive and estimated cdf for first data set.

$Models$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{a}$
$Exp\,(a)$	–	–	–	0.54 (0.06)
$M-E\,(a)$	–	–	–	0.92 (0.07)
$MO-E\,(\alpha,a)$	–	–	8.77 (3.55)	1.37 (0.195)
$GMO-E\,(\theta,\alpha,a)$	–	0.17 (0.07)	47.63 (44.90)	4.46 (1.32)
$Kw-E\,(\lambda,\theta,a)$	3.30(1.10)	1.10(0.765)	–	1.03 (0.61)
$B-E\,(\lambda,\theta,a)$	0.81 (0.69)	3.46 (1.00)	–	1.33 (0.85)
$KwMO-E\,(\lambda,\theta,\alpha,a)$	3.47(0.86)	3.30 (0.77)	0.37 (0.13)	0.29 (1.11)
$MOKw-E\,(\lambda,\theta,\alpha,a)$	2.71 (1.31)	1.98 (0.78)	0.01 (0.002)	0.09 (0.05)
$TCPKw-E\,(\lambda,\theta,\alpha,a)$	1.53 (4.40)	1.73 (6.48)	1.97 (5.71)	0.56 (2.02)

Table 4. MLEs, standard errors (in parentheses) for 2nd data set

$Models$	AIC	BIC	CAIC	HQIC	A	W	KS $(p-value)$
$Exp\,(a)$	234.63	236.91	234.68	235.54	6.53	1.25	0.27 (0.06)
$M-E\,(a)$	210.40	212.68	210.45	211.30	1.52	0.25	0.14 (0.13)
$MO-E\,(\alpha,a)$	210.36	214.92	210.53	212.16	1.18	0.17	0.10 (0.43)
$GMO-E\,(\theta,\alpha,a)$	210.54	217.38	210.89	213.24	1.02	0.16	0.09 (0.51)
$Kw-E\,(\lambda,\theta,a)$	209.42	216.24	209.77	212.12	0.74	0.11	0.08 (0.50)
$B-E\,(\lambda,\theta,a)$	207.38	214.22	207.73	210.08	0.98	0.15	0.11 (0.34)
$KwMO-E\,(\lambda,\theta,\alpha,a)$	207.82	216.94	208.42	211.42	0.61	0.11	0.08 (0.73)
$MOKw-E\,(\lambda,\theta,\alpha,a)$	209.44	218.56	210.04	213.04	0.79	0.12	0.10 (0.44)
$TCPKw-E\,(\lambda,\theta,\alpha,a)$	**206.27**	**215.38**	**206.87**	**209.90**	**0.48**	**0.07**	**0.08 (0.78)**

Table 5. Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for 2nd data set

In terms of the goodness of fit tests as well as all the criteria of model selection it has been observed that the $TCPKw-E$ beats all other models compared for both the data sets.more over the plots in Figures (17), (18), (19) and (20) also revels the closeness between the observed data and the best fitted model.

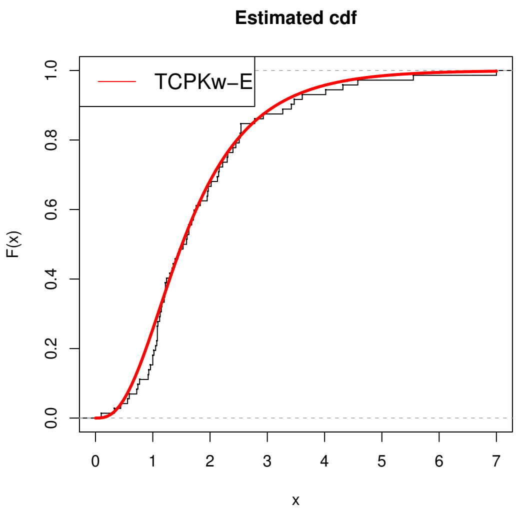

Figure 19. Plots of the observed histogram and estimated pdf for 2nd data set

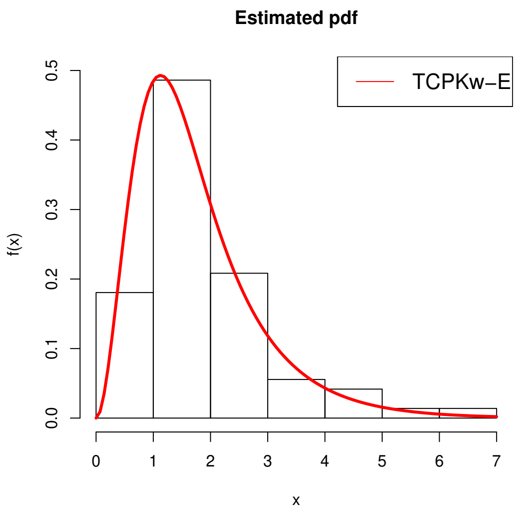

Figure 20. Plots of the observed ogive and estimated cdf for 2nd data set.

8. Conclusion

Truncated Cauchy Power Kumaraswamy-G family of distributions is introduced. various relevant structural properties of the family are derived. Maximum likelihood estimation is developed and assessed through Monte Carlo simulation. Two real data sets are modeled using a member of the family in comparison to some known distributions clearly established the superiority of the proposed family.

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