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Prime Optimization

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Abstract This is a pioneering work, introducing a novel class of optimization of objective functions over subsets of prime only integer points. We show a rich variety of Prime Optimization and mixed problems.

Keywords Complexity, Constraints, Optimization, Primes, Target function

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1. Introduction

It is well-known that an optimization problem can be represented in the following way: given a function f : **G** $\rightarrow$ **R** from some set **G** to the real numbers; sought: an element $x_0 \in$ **G** such that $f(x_0) \leq f(x)$ for all $x \in$ **G**,("minimization"), or such that $f(x_0) \geq f(x)$ for all $x \in$ **G** ("maximization"). Typically, **G** is some subset of the Euclidean space $\mathbf{R}^n$, specified by a set of constraints and the function f is called an objective function or target function. Its well-known in Optimization Theory the case when **G** is some subset of integer points: Integer Optimization (see, e.g., [5, 7, 10]). A general model of mixed-integer optimization could be written as: max/min $f(x)$ subject to $g_1(x) \leq 0$, ... , $g_m(x) \leq 0$, $x \in \mathbf{R}^k \times \mathbf{Z}^s$, where $f, g_1, ..., g_m : \mathbf{R}^n \rightarrow \mathbf{R}$ are arbitrary nonlinear functions. We are going to be focused on Primes: an infinite countable subset $\mathbf{P} \subset \mathbf{Z}$ of Integers and introduce a novel class of optimization: optimization of real-valued functions over subsets of prime only points and mixed problems. Recall that a natural number is called a prime number (or a prime) if it is greater than 1 and cannot be written as the product of two smaller natural numbers The first 25 prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. In the 1970s, when it was publicly announced that prime numbers could be used as a basis for the creation of public key cryptography algorithms, these applications have led to significant study of algorithms for computing with prime numbers, and in particular of primality testing, methods for determining whether a given number is prime [1, 4]. Prime numbers are also used in computing for checksums, hash tables, and pseudorandom generators. Prime numbers are of central importance to Number Theory but also have many applications to the other areas within mathematics including abstract algebra and elementary geometry (see, e.g., [6, 9, 14]). The purpose of this paper is to introduce and describe wide variety of optimization problems of real-valued functions over subsets of prime only points: Prime Optimization and mixed problems. Prime Optimization certainly would serve needs of at least Number Theory and its applications.

2. Some simple Prime Optimization Problems

We start out with the following simple cases to demonstrate Prime Optimization over Euclidean space subsets.

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2.1. *One-dimensional Prime Optimization*

epop21a = $\{\max\ e^x$ subject to $6 \leq x \leq 18,\ x \in \mathbf{P}\}$.
argmax(epop21a) = 17.

epop21b = $\{\min\ e^x$ subject to $6 \leq x \leq 18,\ x \in \mathbf{P}\}$.
argmin(epop21b) = 7.

2.2. *Two-dimensional Prime Optimization*

epop22 = $\{\max\ (x + y)$ subject to $x + 7y \leq 91, 3x + y \leq 33,\ x, y \in \mathbf{P}\}$.

Its clear that feasible region of epop22 is bounded by the line, passing through the points (0, 13), (7, 12) and the line, passing through the points (7, 12), (11, 0), as well as by the lines $x = 0$ and $y = 0$, so argmax(epop22) = (7, 11).

2.3. *Three-dimensional Prime Optimization(Contribution to the Number Theory)*

epop23 = $\{\min\ |x^n + y^m - z^k|$ subject to $x, y, z \in \mathbf{P},\ n, m, k \in \mathbf{N}\}$.

Due to Fermat Last Theorem and $\mathbf{P} \subset \mathbf{Z}$,

epop23 $>$ 0 for n = m = k $>$ 2 .

3. Euclidean space Prime Optimization

Now, let us introduce in details classes of Prime Optimization Problems in a Euclidean space.

3.1. *Single variable. Polynomial Prime Optimization*

epop31 = $\{\max\quad c_n x^n + ... + c_1 x$ subject to
$a_{1n}x^n + ... + a_{11}x \leq b_1, \quad \cdots, \quad a_{mn}x^n + ... + a_{m1}x \leq b_m,$
$x \in \mathbf{P},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.

By introducing the slack variables $w_1 \geq 0, ..., w_m \geq 0$, the above inequalities can be converted into the following equations:

$a_{1n}x^n + ... + a_{11}x + w_1 = b_1, \quad \cdots, \quad a_{mn}x^n + ... + a_{m1}x + w_m = b_m.$

More sophisticated problems would contain rational functions.

3.2. *Single variable. Nonlinear Prime Optimization*

epop32 = $\{\min\ e^x - \log(x)$ subject to $x^2 \leq a,\ x \in \mathbf{P},\ a \in \mathbf{R}\}$.

3.3. *Several variables. Linear Prime Optimization*

epop33a = $\{\max\quad c_1x_1 + ... + c_nx_n$ subject to
$a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \quad \cdots, \quad a_{m1}x_1 + ... + a_{mn}x_n \leq b_m,$
$x_j \in \mathbf{P},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.

By introducing the slack variables $w_1 \geq 0, ..., w_m \geq 0$, the above inequalities can be converted into the following equations:

$a_{11}x_1 + ... + a_{1n}x_n + w_1 = b_1, \quad \cdots, \quad a_{m1}x_1 + ... + a_{mn}x_n + w_m = b_m.$

epop33b = $\{\max\quad c_1x_1 + ... + c_nx_n$ subject to

$a_{11}x_1 + ... + a_{1n}x_n = b_1, \quad \cdots, \quad a_{m1}x_1 + ... + a_{mn}x_n = b_m,$
$x_j \in \mathbf{P},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, (Ax = b), 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$. (See [9] as well.)

3.4. Several variables. Quadratic Prime Optimization

epop34 = $\{\max \quad x_1^2 + ... + x_n^2 - x_1x_2$ subject to
$a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \quad \cdots, \quad a_{m1}x_1 + ... + a_{mn}x_n \leq b_m,$
$x_j \in \mathbf{P},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$

3.5. Several variables. Nonlinear Prime Optimization

epop35 = $\{\min\ e^x - \log(y)$ subject to $x^2 + y^2 \leq a^2,\ x, y \in \mathbf{P},\ a \in \mathbf{R}\}.$

4. Complex Prime Optimization

Let us introduce Complex Prime Optimization. Complex Optimization is considered in [17], and, in particular, in [17]is introduced a novel concept of Integer Complex Optimization on the base of Gaussian Integers. Recall that the Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{a + b\mathbf{i}\ |a, b \in \mathbf{Z}\}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane $\mathbf{C}$, the Gaussian integers constitute the two-dimensional integer lattice (see [11, 12, 13, 15, 16]). Let us introduce the following subset of Gaussian Integers:
$\mathbf{Z_P}[\mathbf{i}] := \{a + b\mathbf{i} \mid a, b \in \mathbf{P}\}$,
$\mathbf{Z_P}[\mathbf{i}] \subset \mathbf{Z}[\mathbf{i}]$.
Considering optimization problems using subsets of $\mathbf{Z_P}[\mathbf{i}]$, we obtain a novel class of optimization: Complex Prime Optimization.

4.1. Prime Optimization and Riemann hypothesis

The Riemann hypothesis, considered one of the greatest unsolved problems in mathematics is a conjecture that the Riemann zeta function(see, e.g., [3, 6]):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s \in \mathbf{C},$$

has its zeros only at the negative even integers(trivial zeros) and complex numbers with real part $\frac{1}{2}$(nontrivial zeros). In the theory of the Riemann zeta function, the set: $\{s \in \mathbf{C} : \mathrm{Re}(s) = \frac{1}{2}\}$ is called the critical line. Let us introduce the following two Prime Optimization problems:
poz1 = $min|\zeta(s)|$ subject to $\frac{1}{2} < \mathrm{Re}(s) < M, 1 \leq |\mathrm{Im}(s)| < M$ (to exclude trivial zeros),
poz2 = $min|\zeta(s)|$ subject to $-M < \mathrm{Re}(s) < \frac{1}{2}, 1 \leq |\mathrm{Im}(s)| < M, \quad M > 0, M \in \mathbf{R}, s \in \mathbf{Z_P}[\mathbf{i}]$.
If there exists $M > 0$, such that poz1(argmin(poz1)) = 0 or poz2(argmin(poz2)) = 0 it would mean that Riemann conjecture is wrong. That is why together with the facts that according to the Euler product formula:

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}},$$

and the infinite product in the right hand side extends over all prime numbers p, $\mathbf{P} \subset \mathbf{Z}$ and PRIMES is in P(see, e.g., [1]), it would encourage and stimulate researchers to explore and develop Prime Optimization.

4.2. Linear Complex Prime Optimization

cpop42a = $\{\max \quad |c_1z_1 + ... + c_nz_n|$ subject to

$|a_{11}z_1 + ... + a_{1n}z_n| \leq b_1, \quad \cdots, \quad |a_{m1}z_1 + ... + a_{mn}z_n| \leq b_m,$
$z_j \in \mathbf{Z_P}[\mathbf{i}],\ a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{C}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$

cpop42b = $\{\max \quad |c_1z_1 + ... + c_nz_n|$ subject to
$a_{11}z_1 + ... + a_{1n}z_n = b_1, \quad \cdots, \quad a_{m1}z_1 + ... + a_{mn}z_n = b_m,$
$z_j \in \mathbf{Z_P}[\mathbf{i}],\ a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{C}, (Az = b), 1 \leq i \leq m,\ 1 \leq j \leq n,$
$n \in \mathbf{N}, m \in \mathbf{N}\}.$

4.3. Nonlinear Complex Prime Optimization

cpop43a = $\{\max \quad |z_1^4 + ... + z_n^4|$ subject to
$b_1 \leq |a_{11}z_1 + ... + a_{1n}z_n| \leq c_1, \quad \cdots, \quad b_m \leq |a_{m1}z_1 + ... + a_{mn}z_n| \leq c_m,$
$z_j \in \mathbf{Z_P}[\mathbf{i}],\ a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$

cpop43b = $\{\min\ e^z - \log(c)$ subject to $|z^2 + c^2| \leq a,\ z, c \in \mathbf{Z_P}[\mathbf{i}],\ a \in \mathbf{R}\}.$

Similarly for Eisenshtein Integers: complex numbers of the form: z = a + bω, where a and b are integers and ω^2 + ω + 1 = 0, we can introduce a subset: $\mathbf{E_P}$ **[i]** $:= \{a + b\omega | a, b \in \mathbf{P}\ \}$, and consider the corresponding optimization.

5. Quaternionic Prime Optimization

Let us introduce Quaternionic Prime Optimization. Recall that quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R},\ b \in \mathbf{R},\ c \in \mathbf{R},\ d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers [2, 8]. The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\|pq\| = \|p\|\|q\|, p \in \mathbf{H}, q \in \mathbf{H}, \|q\|^2 = a^2 + b^2 + c^2 + d^2$. This norm makes it possible to define the distance $d(p, q) = \|p - q\|$ which makes **H** into a metric space. Quaternionic Optimization is considered in [17] and in particular, in[17] is introduced a novel concept of Integer Quaternionic Optimization on the base of Lipschits quaternions $\mathbf{L} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z}\}$. Let us introduce the following subset of Lipschits quaternions:

$\mathbf{L_P} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}\ | a, b, c, d \in \mathbf{P}\}, \mathbf{L_P} \subset \mathbf{L}.$

Considering optimization problems using subsets of $\mathbf{L_P}$, we obtain a novel class of optimization: Quaternionic Prime Optimization.

5.1. Linear Quaternionic Prime Optimization

qpop51a = $\{\max \quad ||c_1q_1 + ... + c_nq_n||$ subject to
$||a_{11}q_1 + ... + a_{1n}q_n|| \leq b_1, \quad \cdots, \quad ||a_{m1}q_1 + ... + a_{mn}q_n|| \leq b_m,$
$q_j \in \mathbf{L_P},\ a_{ij} \in \mathbf{H},\ b_i \in \mathbf{R},\ c_j \in \mathbf{H}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$

By introducing the slack variables $w_1 \geq 0, ..., w_m \geq 0$ the above inequalities can be converted into the following equations:

$$||a_{11}q_1 + ... + a_{1n}q_n|| + w_1 = b_1, \quad \cdots, \quad ||a_{m1}q_1 + ... + a_{mn}q_n|| + w_m = b_m.$$

qpop51b = $\{\max \quad ||c_1q_1 + ... + c_nq_n||$ subject to
$a_{11}q_1 + ... + a_{1n}q_n = b_1, \quad \cdots, \quad a_{m1}q_1 + ... + a_{mn}q_n = b_m,$
$q_j \in \mathbf{L_P},\ a_{ij} \in \mathbf{H},\ b_i \in \mathbf{H},\ c_j \in \mathbf{H}, (Aq = b),$
$1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$

5.2. Polynomial Quaternionic Prime Optimization

qpop52 = $\{\max \quad ||c_nq^n + ... + c_1q\ ||$ subject to

$$||a_{1n}q^n + ... + a_{11}q\ || \le b_1, \quad \cdots, \quad ||a_{mn}q^n + ... + a_{m1}q\ || \le b_m,$$
$$q \in \mathbf{L_P},\ a_{ij} \in \mathbf{H},\ b_i \in \mathbf{R},\ c_j \in \mathbf{H}, 1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}\}.$$

By introducing the slack variables $w_1 \ge 0, ..., w_m \ge 0$ the above inequalities can be converted into the following equations:

$$||a_{1n}q^n + ... + a_{11}q|| + w_1 = b_1, \quad \cdots, \quad ||a_{mn}q^n + ... + a_{m1}q|| + w_m = b_m.$$

5.3. *Nonlinear Quaternionic Prime Optimization*

qpop53a = $\{\max \quad ||q_1^4 + ... + q_n^4||$ subject to
$$b_1 \le ||a_{11}q_1 + ... + a_{1n}q_n|| \le c_1, \quad \cdots, \quad b_m \le ||a_{m1}q_1 + ... + a_{mn}q_n|| \le c_m,$$
$$q_j \in \mathbf{L_P},\ a_{ij} \in \mathbf{H}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, b_i \ge 0, 1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}\}.$$

qpop53b = $\{\min \ ||e^p - \log(q)\ ||$ subject to $||p|| \le a, ||q|| \le b,$
$p, q \in \mathbf{L_P},\ a, b \in \mathbf{R}\}.$

5.4. *Mixed Prime-Integer-Real-Complex-Quaternionic Optimization*

qpop54 = $\{\min \ xz^2||p^2 - pq + r^2|||\mathbf{i}z_1^4 - z_2^2 z_3| - x^2 + y^3 t^2$ subject to
$$xy \ge N,\ a_1 \le ||p|| \le b_1, a_2 \le ||q|| \le b_2, a_3 \le ||r|| \le b_3,\ a_4 \le |z_1| \le b_4,$$
$$a_5 \le |z_2| \le b_5, a_6 \le |z_3| \le b_6, a_7 \le x \le b_7,\ a_8 \le y \le b_8, a_9 \le z \le b_9,\ a_{10} \le t \le b_{10},$$
$$p \in \mathbf{H}, q \in \mathbf{L}, r \in \mathbf{L_P}, z_1 \in \mathbf{C}, z_2 \in \mathbf{Z}[\mathbf{i}], z_3 \in \mathbf{Z_P}[\mathbf{i}],\ x \in \mathbf{Z},\ y \in \mathbf{Z},$$
$$z \in \mathbf{P},\ t \in \mathbf{R},\ a_i, b_i \in \mathbf{R},\ N \in \mathbf{N},\ a_i \ge 0, 1 \le i \le 10\}.$$

The corresponding Optimization Problems can be introduced for two other infinite countable subsets of Integers as well: Odd numbers and Even numbers.

6. Open Problems

Despite wide proliferation of Integer Optimization, it would be preferable to develop specific methods and algorithms for the Prime Optimization problems to serve the needs of the Number Theory and other fields and applications. The corresponding complexity evaluations for the Prime Optimization problems would be developed as well: for example in binary encoded length of the coefficients(see, e.g., [5, 10]), and in particular, finding conditions for the polynomial-time optimization. Recall that PRIMES is in P(see, e.g., [1, 5]).

Prime Optimization ideas may be further extended for octonions and other hypercomplex systems, forming Hypercomplex Prime Optimization, as well as useful for similar approaches in other subfields of the Optimization Theory.

7. Conclusion

We described a rich variety of problems of optimization of target functions over subsets of prime only integer points and the corresponding open problems: complexity, extension for octonions and other hypercomplex systems, forming Hypercomplex Prime Optimization. It would inspire and motivate researchers to develop the corresponding new methods and algorithms.

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