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Necessary and Sufficient Optimality Conditions for Semi-infinite Programming with Multiple Fuzzy-valued Objective Functions

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Abstract This paper deals with semi-infinite programming with multiple fuzzy-valued objective functions. Firstly, some types of efficient solutions are proposed and illustrated in some examples. Then, necessary and sufficient Karush-Kuhn-Tucker optimality conditions for semi-infinite programming with multiple fuzzy-valued objective functions are established.

Keywords Multiobjective Semi-infinite Programming, Fuzzy-valued Objective Functions, Efficient Solutions, Karush-Kuhn-Tucker Optimality Conditions.

AMS 2010 subject classifications 90C46, 90C34, 90C70.

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1. Introduction

Some optimization problems in practice handle with the infinite number of constraints and they are called the semi-infinite programming problems. These classes could be served in formulating many problems in moment robust optimizations and their applications in [15], FIR filter design in [20], robot trajectory planning [29], air pollution control in [30]. Hence, semi-infinite programming problems have been investigated recently by many researchers, see e.g. the papers [4, 5, 10, 12, 13, 16, 21, 24, 25, 26, 27, 28] and references therein. Sometimes semi-infinite programming with infinite-dimensional decision spaces are labeled as problems of infinite programming, see e.g. [17]. In some optimization problems in the real world, the coefficients of objective functions and constraint functions are not known precisely. These imprecision are used to be treated by quantitatively by employing the concepts of randomness and fuzziness. The randomness is formulated by probability theory and employed to describe the chance events. The fuzziness amounts to a type of imprecision which is associated with fuzzy sets, in which there is no clear transition from membership to nonmembership [33]. To manipulate fuzzy concepts arising in many decision processes, fuzzy optimization problems have been studied numerously in the recent time. The papers [31, 32] investigated Karush-Kuhn-Tucker (KKT) sufficent optimality conditions for smooth optimization problems with one and multiple fuzzy-valued objective functions. In [18, 19], optimality conditions for fuzzy optimization problems were established by utilizing generalized Hukuhara derivatives. The interval and fuzzy directional gH-derivatives and differentiability were proposed in [23] and applied in considering KKT optimality conditions for both interval-valued and fuzzy-valued constrained optimization problems. The KKT optimality conditions in an optimization problem with interval-valued objective function on Hadamard manifolds were studied in [3]. However, to the best of our knowledge, there is no paper dealing with semi-infinite programming with fuzzy-valued objective function. Furthermore, in the case that the number of constraints a finite set, the necessary

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optimality conditions for the constrained optimization problems with fuzzy-valued objective function were not investigated in [31, 32].

The above observations motivate us to establish KKT optimality conditions for some types of efficient solutions of semi-infinite programming with multiple fuzzy-valued objective functions in this paper. The structure of the paper is as follows. We first retraces basic concepts, some preliminaries and presents some binary relations in section 2. Then, both KKT necessary and sufficient optimality conditions for some types of efficient solutions of semi-infinite programming with multiple fuzzy objective functions are established. Some examples are provided to illustrate the results of the paper.

2. Preliminaries

In the sequel, let $\mathbb{R}^n$ be a finite-dimensional Euclidean space. The notation $\langle \cdot, \cdot \rangle$ is used to denote the inner product. For a given subset $X \subseteq \mathbb{R}^n$, $\mathrm{int}X$, $\mathrm{cl}X$, $\mathrm{aff}X$, and $\mathrm{co}X$ stand for its interior, closure, affine hull, convex hull of X, respectively (resp). The cone and the convex cone (containing the origin) generated by X are expressed resp by $\mathrm{cone}X$, $\mathrm{pos}X$. We write $\langle X^*, x\rangle \leq 0$ when $\langle x^*, x\rangle \leq 0$ for all $x^* \in X^*$, where X^* is a subset of the dual space of $\mathbb{R}^n$. The negative polar cone and strictly negative polar cone of X are defined resp by

$$X^- := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle \leq 0 \ \forall x \in X\},$$

$$X^s := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0 \ \forall x \in X\}.$$

It should be noted that if $0 \in X$, then $X^s = \emptyset$. Moreover, we can check that if $X^s \neq \emptyset$ then $\mathrm{cl}X^s = X^-$. Indeed, let $x^* \in \mathrm{cl}X^s$. Then, there exists a sequence $x^*_\ell \to x^*$ satisfying $\langle x^*_\ell, x\rangle < 0$ for all $x \in X$. Letting ℓ to infinity, one has $\langle x^*, x\rangle \leq 0$ for all $x \in X$, leading that $x^* \in X^-$. Conversely, let $x^* \in X^-$. Then, $\langle x^*, x\rangle \leq 0$ for all $x \in X$. We deduce from $X^s \neq \emptyset$ that there is $\bar{x}^* \in X^s$ such that $\langle \bar{x}^*, x\rangle < 0$ for all $x \in X$. Setting $x^*_\ell = x^* + \frac{1}{\ell}\bar{x}^*$, we get $x^*_\ell \to x^*$ and $x^*_\ell \in X^s$ since

$$\langle x^*_\ell, x\rangle = \langle x^*, x\rangle + \frac{1}{\ell}\langle \bar{x}^*, x\rangle < 0 \ \forall x \in X,$$

i.e., $x^* \in \mathrm{cl}X^s$.

The contingent cone [1] of X at the point $\bar{x} \in \mathrm{cl}X$ is

$$\mathcal{T}(X, \bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_\ell \downarrow 0, \exists x_\ell \to x, \bar{x} + \tau_\ell x_\ell \in X \forall \ell\}.$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, where $\mathbf{a} = (a_1, ..., a_m)$ and $\mathbf{b} = (b_1, ..., b_m)$. Recall the following notations.

(i) $\mathbf{a} \leqq \mathbf{b} \Leftrightarrow a_i \leq b_i \ \forall i \in I$;
(ii) $\mathbf{a} \leq \mathbf{b} \Leftrightarrow a_i \leq b_i \ \forall i \in I$ and $a_{i_0} < b_{i_0}$ for at least one $i_0 \in I$;
(iii) $\mathbf{a} < \mathbf{b} \Leftrightarrow a_i < b_i \ \forall i \in I$.

It is easy to see that $\mathbf{a} \leqq \mathbf{b} \Rightarrow \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a} < \mathbf{b}$. Moreover, if $m = 1$ then $\mathbf{a} \leqq \mathbf{b} \Leftrightarrow a_1 \leq b_1$ and $\mathbf{a} \leq \mathbf{b} \Leftrightarrow \mathbf{a} < \mathbf{b} \Leftrightarrow a_1 < b_1$.

Let $\mathbf{K}_C$ designate the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathbf{K}_C = \{[x^L, x^R] \mid x^L, x^R \in \mathbb{R} \text{ and } x^L \leq x^R\}.$$

The width of $X \in \mathbf{K}_C$ is defined by $\mu(X) = x^R - x^L$. The definition brings us

$$X + Y := \{x + y \mid x \in X, y \in Y\} = [x^L + y^L, x^R + y^R],$$

$$\lambda X := \lambda[x^L, x^R] = \begin{cases} [\lambda x^L, \lambda x^R], & \text{if } \lambda \geq 0, \\ [\lambda x^R, \lambda x^L], & \text{if } \lambda < 0. \end{cases}$$

Hence, $-X = (-1)X = [-x^R, -x^L]$ and $X - Y = X + (-1)Y = [x^L - y^R, x^R - y^L]$.

A fuzzy set $\widetilde{X}$ on $\mathbb{R}$ is defined by a function $\mu_{\widetilde{X}} : \mathbb{R} \to [0,1]$, which is called a membership function. The α-level set of $\widetilde{X}$, indicated by $\widetilde{X}_\alpha$, is defined as $\widetilde{X}_\alpha := \{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) \geq \alpha\}$, $\forall \alpha \in (0,1]$. The support of $\widetilde{X}$ is the set $\mathrm{supp}(\widetilde{X}) := \{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) > 0\}$. The zero-level set of $\widetilde{X}$ is defined as the closure of the support of $\widetilde{X}$, i.e., $\widetilde{X}_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) > 0\}$.

Definition 1
A fuzzy number is a fuzzy set $\widetilde{X}$ with membership function $\mu_{\widetilde{X}}$ satisfying the following conditions:

(i) $\mu_{\widetilde{X}}$ is normal, that is, there exists $\bar{x} \in \mathbb{R}$ such that $\mu_{\widetilde{X}}(\bar{x}) = 1$;

(ii) $\mu_{\widetilde{X}}$ is quasiconcave, i.e.,
$$\mu_{\widetilde{X}}(\lambda x + (1-\lambda)x') \geq \max\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{X}}(x')\}$$
for all $\lambda \in [0,1]$, for all $x, x' \in \widetilde{X}$;

(iii) $\mu_{\widetilde{X}}$ is upper semicontinuous, i.e., $\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0,1]$;

(iv) $\widetilde{X}_0$ is a compact subset of $\mathbb{R}$.

The set of all fuzzy numbers on $\mathbb{R}$ is signified by $\mathbf{F}(\mathbb{R})$.

The condition (ii) leads that $\widetilde{X}_\alpha$ is a convex set for each $\alpha \in [0,1]$. Combining this with conditions (iii) and (iv) tells us that $\widetilde{X}_\alpha$ is a compact and convex subset of $\mathbb{R}$ for each $\alpha \in [0,1]$. In other words, $\widetilde{X}_\alpha = [\widetilde{x}^L_\alpha, \widetilde{x}^R_\alpha] \in \mathbf{K}_C$.

Definition 2
A fuzzy number $\widetilde{X}$ is said to be a canonical number in the case when the functions $\eta_1(\alpha) = \widetilde{x}^L_\alpha$ and $\eta_2(\alpha) = \widetilde{x}^R_\alpha$ are continuous on $[0,1]$, where $[\widetilde{x}^L_\alpha, \widetilde{x}^R_\alpha] = \widetilde{X}_\alpha$. The set of all canonical fuzzy numbers on $\mathbb{R}$ is denoted by $\mathbf{F}_C(\mathbb{R})$.

Remark 1
Recollect the following fuzzy numbers.

(i) A fuzzy number $\widetilde{X}$ is a crisp number with value a if its membership function is
$$\mu_{\widetilde{X}}(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise.} \end{cases}$$
Then the crisp number with value a is denoted by $\widetilde{\mathbf{I}}_{\{a\}}$.

(ii) A fuzzy number $\widetilde{X}$ is said to be triangular fuzzy number, indicated by $\widetilde{X} = (a^L, a, a^R)$ with $a^L \leq a \leq a^R$, if the membership function is defined as
$$\mu_{\widetilde{X}}(x) = \begin{cases} \frac{x-a^L}{a-a^L}, & \text{if } a^L < x \leq a, \\ \frac{a^R-x}{a^R-a}, & \text{if } a \leq x < a^R, \\ 0, & \text{otherwise.} \end{cases}$$
The α-level set of $\widetilde{X}$ is $\widetilde{X}_\alpha = [a^L + \alpha(a - a^L), a^R + \alpha(a - a^R)]$. Notice that if $a^L = a = a^R$, then the triangular fuzzy number $\widetilde{X}$ is a crisp number.

(iii) A fuzzy number $\widetilde{X}$ is said to be trapezoidal fuzzy number, denoted by $\widetilde{X} = (a^L, \underline{a}, \overline{a}, a^R)$ with $a^L \leq \underline{a} \leq \overline{a} \leq a^R$, if the membership function is defined as
$$\mu_{\widetilde{X}}(x) = \begin{cases} \frac{x-a^L}{\underline{a}-a^L}, & \text{if } a^L < x \leq \underline{a}, \\ 1, & \text{if } \underline{a} \leq x \leq \overline{a}, \\ \frac{a^R-x}{a^R-\overline{a}}, & \text{if } \overline{a} \leq x < a^R, \\ 0, & \text{otherwise.} \end{cases}$$
The α-level set of $\widetilde{X}$ is $\widetilde{X}_\alpha = [a^L + \alpha(\underline{a} - a^L), a^R + \alpha(\overline{a} - a^R)]$. Note that if $\underline{a} = \overline{a} = a$, then the trapezoidal fuzzy number $\widetilde{X}$ is a triangular fuzzy number.

For $\widetilde{X}, \widetilde{Y} \in \mathbf{F}_C(\mathbb{R})$, the notion "$\widetilde{\cdot}$" stands for the binary operation "$\widetilde{+}$" or "$\widetilde{\times}$" between $\widetilde{X}$ and $\widetilde{Y}$, where the membership function [33] for $\widetilde{X}\widetilde{\cdot}\widetilde{Y}$ is defined by

$$\mu_{\widetilde{X}\widetilde{\cdot}\widetilde{Y}}(z) = \sup_{z = x\widetilde{\cdot}y} \min\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(y)\}.$$

Proposition 1
Let $\widetilde{X}, \widetilde{Y}$ be in $\mathbf{F}_C(\mathbb{R})$. Then,

(i) $\widetilde{X}\widetilde{+}\widetilde{Y} \in \mathbf{F}_C(\mathbb{R})$ and $(\widetilde{X}\widetilde{+}\widetilde{Y})_\alpha = [\widetilde{x}^L_\alpha + \widetilde{y}^L_\alpha, \widetilde{x}^R_\alpha + \widetilde{y}^R_\alpha]$.
(ii) $\widetilde{X}\widetilde{\times}\widetilde{Y} \in \mathbf{F}_C(\mathbb{R})$ and
$(\widetilde{X}\widetilde{\times}\widetilde{Y})_\alpha = [\min\{\widetilde{x}^L_\alpha\widetilde{y}^L_\alpha, \widetilde{x}^L_\alpha\widetilde{y}^R_\alpha, \widetilde{x}^R_\alpha\widetilde{y}^L_\alpha, \widetilde{x}^R_\alpha\widetilde{y}^R_\alpha\}, \max\{\widetilde{x}^L_\alpha\widetilde{y}^L_\alpha, \widetilde{x}^L_\alpha\widetilde{y}^R_\alpha, \widetilde{x}^R_\alpha\widetilde{y}^L_\alpha, \widetilde{x}^R_\alpha\widetilde{y}^R_\alpha\}]$.

The Hausdorff metric $D_H(\mathcal{X}, \mathcal{Y})$ of two sets $\mathcal{X}, \mathcal{Y}$ in $\mathbb{R}^n$ is defined by

$$D_H(\mathcal{X}, \mathcal{Y}) = \max\{\sup_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}} \|x - y\|, \sup_{y\in\mathcal{Y}} \inf_{x\in\mathcal{X}} \|x - y\|\}.$$

If $\widetilde{X}, \widetilde{Y}$ are in $\mathbf{F}(\mathbb{R})$, then, $D_H(\widetilde{X}_\alpha, \widetilde{Y}_\alpha) = \max\{|\widetilde{x}^L_\alpha - \widetilde{y}^L_\alpha|, |\widetilde{x}^R_\alpha - \widetilde{y}^R_\alpha|\}$. The metric $D_{\mathbf{F}}$ on $\mathbf{F}(\mathbb{R})$ is defined by

$$D_{\mathbf{F}}(\widetilde{X}, \widetilde{Y}) = \sup_{\alpha\in[0,1]} D_H(\widetilde{X}_\alpha, \widetilde{Y}_\alpha), \forall \widetilde{X}, \widetilde{Y} \in \mathbf{F}(\mathbb{R}).$$

Let $\widetilde{\psi} : \mathbb{R}^n \to \mathbf{F}_C(\mathbb{R})$ be a fuzzy-valued function defined on $\mathbb{R}^n$. Then, for each $x \in \mathbb{R}^n$, $(\widetilde{\psi}(x))_\alpha = [(\widetilde{\psi}(x))^L_\alpha, (\widetilde{\psi}(x))^R_\alpha]$ for each $\alpha \in [0, 1]$ and we can determine two real-valued functions $\widetilde{\psi}^L_\alpha(x) = (\widetilde{\psi}(x))^L_\alpha$, $\widetilde{\psi}^R_\alpha(x) = (\widetilde{\psi}(x))^R_\alpha$. Let $\bar{x} \in \mathbb{R}^n$ and $\widetilde{X} \in \mathbf{F}_C(\mathbb{R})$. We write $\lim_{x\to\bar{x}} \widetilde{\psi}(x) = \widetilde{X}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that, for $0 < \|x - \bar{x}\| < \delta \Rightarrow D_{\mathbf{F}}(\widetilde{\psi}(x), \widetilde{X}) < \epsilon$. The right-hand limit $\lim_{x\to\bar{x}^+} \widetilde{\psi}(x)$ of the fuzzy-valued function $\widetilde{\psi} : \mathbb{R} \to \mathbf{F}_C(\mathbb{R})$ can be defined similarly. The fuzzy-valued function $\widetilde{\psi}$ is said to be continuous at $\bar{x}$ if $\lim_{x\to\bar{x}} \widetilde{\psi}(x) = \widetilde{\psi}(\bar{x})$.

Proposition 2
[31] Let $\widetilde{\psi} : \mathbb{R}^n \to \mathbf{F}_C(\mathbb{R})$ be a function with fuzzy values. If $\widetilde{\psi}$ is continuous at $\bar{x}$, then $\widetilde{\psi}^L_\alpha$, $\widetilde{\psi}^R_\alpha$ are continuous at $\bar{x}$ for all $\alpha \in [0, 1]$.

Definition 3
[31] Let X be an open subset of $\mathbb{R}^n$. A fuzzy-valued function $\widetilde{\psi} : X \to \mathbf{F}_C(\mathbb{R})$ is called level-wise continuously differentiable at $\bar{x} \in X$ if the real-valued functions $\widetilde{\psi}^L_\alpha$ and $\widetilde{\psi}^R_\alpha$ are continuously differentiable at $\bar{x}$ for all $\alpha \in [0, 1]$.

Definition 4
[22]. Let $X \subset \mathbb{R}^n$ be a convex set, $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in X$.

(i) ψ is convex at $\bar{x}$ if $\psi(\lambda\bar{x} + (1 - \lambda)x) \leq \lambda\psi(\bar{x}) + (1 - \lambda)\psi(x)\ \forall x \in X, \forall\lambda \in [0, 1]$.
(ii) ψ is strictly convex at $\bar{x}$ if $\psi(\lambda\bar{x} + (1 - \lambda)x) < \lambda\psi(\bar{x}) + (1 - \lambda)\psi(x)\ \forall x \in X \setminus \{\bar{x}\}, \forall\lambda \in (0, 1)$.
(ii) ψ is convex/strictly convex on X if ψ is convex/strictly convex on each point of X.

Remark 2
[22]. Let $X \subset \mathbb{R}^n$ be an open convex set, $\psi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in X$.

(i) If ψ is convex at $\bar{x}$ then $\psi(x) - \psi(\bar{x}) \geq \langle\nabla\psi(\bar{x}), x - \bar{x}\rangle\ \forall x \in X$.
(ii) If ψ is strictly convex at $\bar{x}$ then $\psi(x) - \psi(\bar{x}) > \langle\nabla\psi(\bar{x}), x - \bar{x}\rangle\ \forall x \in X \setminus \{\bar{x}\}$.

Definition 5
Let $X = [x^L, x^R], Y = [y^L, y^R]$ be two sets in $\mathbf{K}_C$.

(i) $X \leq_{LR} Y$ if $x^L \leq y^L$ and $x^R \leq y^R$.
(ii) $X <_{LR} Y$ if $X \leq_{LR} Y$ and $X \neq Y$.

(iii) $X <^s_{LR} Y$ if $x^L < y^L$ and $x^R < y^R$.

Definition 6
Let $\widetilde{X}, \widetilde{Y} \in \mathbf{F}_C(\mathbb{R})$ and $\widetilde{X}_\alpha = [\widetilde{X}^L_\alpha, \widetilde{X}^R_\alpha]$ and $\widetilde{Y}_\alpha = [\widetilde{Y}^L_\alpha, \widetilde{Y}^R_\alpha]$ for all $\alpha \in [0,1]$.

(i) [31] $\widetilde{X} \leqq \widetilde{Y}$ if $\widetilde{X}_\alpha \leq_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$.
(ii) [31] $\widetilde{X} \leq \widetilde{Y}$ if $\widetilde{X}_\alpha \leq_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$ and $\exists \bar{\alpha} \in [0,1]$ such that $\widetilde{X}_{\bar{\alpha}} <_{LR} \widetilde{Y}_{\bar{\alpha}}$.
(iii) [18] $\widetilde{X} \leqq^s \widetilde{Y}$ if $\widetilde{X}_\alpha \leq_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$ and $\exists \bar{\alpha} \in [0,1]$ such that $\widetilde{X}_{\bar{\alpha}} <^s_{LR} \widetilde{Y}_{\bar{\alpha}}$.
(iv) [32] $\widetilde{X} < \widetilde{Y}$ if $\widetilde{X}_\alpha <_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$.
(v) $\widetilde{X} \leq^s \widetilde{Y}$ if $\widetilde{X}_\alpha <_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$ and $\exists \bar{\alpha} \in [0,1]$ such that $\widetilde{X}_{\bar{\alpha}} <^s_{LR} \widetilde{Y}_{\bar{\alpha}}$.
(vi) $\widetilde{X} <^s \widetilde{Y}$ if $\widetilde{X}_\alpha <^s_{LR} \widetilde{Y}_\alpha$ $\forall \alpha \in [0,1]$.

Remark 3
Let $\widetilde{X}, \widetilde{Y} \in \mathbf{F}_C(\mathbb{R})$. It is easy to check that

(i) $\widetilde{X} <^s \widetilde{Y} \Rightarrow \widetilde{X} \leq^s \widetilde{Y} \Rightarrow \widetilde{X} \leqq^s \widetilde{Y} \Rightarrow \widetilde{X} \leq \widetilde{Y} \Rightarrow \widetilde{X} \leqq \widetilde{Y}$.
(ii) $\widetilde{X} <^s \widetilde{Y} \Rightarrow \widetilde{X} \leq^s \widetilde{Y} \Rightarrow \widetilde{X} < \widetilde{Y} \Rightarrow \widetilde{X} \leq \widetilde{Y} \Rightarrow \widetilde{X} \leqq \widetilde{Y}$.

Definition 7
Let $X \subset \mathbb{R}^n$ be a convex set, $\widetilde{\psi} : X \to \mathbf{F}_C(\mathbb{R})$ and $\bar{x} \in X$.

(i) [31] We say that $\widetilde{\psi}$ is convex at $\bar{x}$ if

$$\widetilde{\psi}(\lambda\bar{x} + (1-\lambda)x) \leqq (\widetilde{\mathbf{I}}_{\{\lambda\}} \widetilde{\times} \widetilde{\psi}(\bar{x})) \widetilde{+} (\widetilde{\mathbf{I}}_{\{1-\lambda\}} \widetilde{\times} \widetilde{\psi}(x)),$$

for each $\lambda \in [0,1]$ and each $x \in X$. We also say that $\widetilde{\psi}$ is convex on X if $\widetilde{\psi}$ is convex at each point of X.

(ii) $\widetilde{\psi}$ is said to be strongly convex at $\bar{x}$ if

$$\widetilde{\psi}(\lambda\bar{x} + (1-\lambda)x) <^s (\widetilde{\mathbf{I}}_{\{\lambda\}} \widetilde{\times} \widetilde{\psi}(\bar{x})) \widetilde{+} (\widetilde{\mathbf{I}}_{\{1-\lambda\}} \widetilde{\times} \widetilde{\psi}(x)),$$

for each $\lambda \in (0,1)$ and each $x \in X \setminus \{\bar{x}\}$. We also say that $\widetilde{\psi}$ is strongly convex on X if $\widetilde{\psi}$ is strongly convex at each point of X.

In Definition 7, only two binary relations were utilized. Using the others binary relations in Definition 6, the others definition of convexity of fuzzy functions could be defined similarly.

Remark 4
Let $X \subset \mathbb{R}^n$ be a convex set, $\widetilde{\psi} : X \to \mathbf{F}_C(\mathbb{R})$ and $\bar{x} \in X$.

(i) [31] $\widetilde{\psi}$ is convex at $\bar{x}$ if and only if $(\widetilde{\psi})^L_\alpha$ and $(\widetilde{\psi})^R_\alpha$ are convex at $\bar{x}$ for all $\alpha \in [0,1]$.
(ii) $\widetilde{\psi}$ is strongly convex at $\bar{x}$ if and only if $(\widetilde{\psi})^L_\alpha$ and $(\widetilde{\psi})^R_\alpha$ are strictly convex at $\bar{x}$ for all $\alpha \in [0,1]$.

Lemma 1
[22] Let $\{K_t, t \in T\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \text{pos}\left(\bigcup_{t\in T} K_t\right)$. Then, for any $k \in K \setminus \{0\}$, there exist $\ell \leq n$, $\{t_j\}_{j=1,...,\ell} \subset T$, $\lambda = (\lambda_{t_1}, ..., \lambda_{t_\ell}) \in \mathbb{R}^\ell_+$ and $k_{t_j} \in K_{t_j} (j = 1, ..., \ell)$ such that $k = \sum_{j=1}^{\ell} \lambda_{t_j} k_{t_j}$.

Lemma 2
[6] Suppose that U, V are arbitrary (not necessary finite) index sets, $a_u = a(u) = (a_1(u), ..., a_n(u))$ maps S onto $\mathbb{R}^n$, and so does a_v. Assume further that $\text{co}\{a_u, u \in U\} + \text{pos}\{a_v, v \in V\}$ is a closed set. Then, the following two statements are equivalent:

$$I: \quad \begin{cases} \langle a_u, x\rangle < 0, u \in U, U \neq \emptyset \\ \langle a_v, x\rangle \leq 0, v \in V \end{cases} \text{has no solution } x \in \mathbb{R}^n;$$

$$II: \quad 0 \in \text{co}\{a_u, u \in U\} + \text{pos}\{a_v, v \in V\}.$$

Lemma 3
[8] If X is a nonempty compact subset of $\mathbb{R}^n$, then,

(i) $\mathrm{co}X$ is a compact set;
(ii) If $0 \notin \mathrm{co}X$, then $\mathrm{pos}X$ is a closed cone.

3. KKT optimality conditions

In this section, we consider the semi-infinite programming with multiple fuzzy-valued objective functions as follows:

(P) $\widetilde{\min}(\widetilde{f}_1(x), ..., \widetilde{f}_m(x))$
$\quad s.t.\ g_t(x) \leq 0,\ \ t \in T,$

where $\widetilde{f}_i : \mathbb{R}^n \to \mathbf{F}_C(\mathbb{R})$ $(i \in I := \{1, ..., m\})$ are level-wise continuously differentiable fuzzy-valued functions and $g_t : \mathbb{R}^n \to \mathbb{R}$ $(t \in T)$ are continuously differentiable functions. The index set T is an arbitrary nonempty set, possibly infinite. The feasible solution set of (P) is indicated by

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0,\ t \in T\}.$$

Designate $\mathbb{R}_+^{|T|}$ the set of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \mathcal{S}$, we denote by $T(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The collection of active constraint multipliers at $\bar{x} \in \mathcal{S}$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $K \subset T(\bar{x})$ such that $\lambda_t > 0$ for all $t \in K$ and $\lambda_t = 0$ for all $t \in T \setminus K$.

Definition 8
Let $\bar{x} \in \mathcal{S}$ and $\mathbf{U}(\bar{x})$ be the set of neighborhoods of $\bar{x}$.

(i) [32] $\bar{x}$ is a locally strongly efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}SE(P, 1)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U \setminus \{\bar{x}\}$ satisfying $\widetilde{f}_i(x) \leqq \widetilde{f}_i(\bar{x})\ \forall i \in I$.
(ii) [32, 19] $\bar{x}$ is a locally (Pareto) type-1 efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}E(P, 1)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ fulfilling

$$\begin{cases} \widetilde{f}_i(x) \leqq \widetilde{f}_i(\bar{x}) & \forall i \in I, \\ \widetilde{f}_{i_0}(x) \leq \widetilde{f}_{i_0}(\bar{x}) & \text{for at least one } i_0 \in I. \end{cases}$$

(iii) [32, 19] $\bar{x}$ is a locally weakly type-1 efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}WE(P, 1)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ fulfilling $\widetilde{f}_i(x) \leq \widetilde{f}_i(\bar{x})\ \forall i \in I$.
(iv) [19] $\bar{x}$ is a locally (Pareto) type-2 efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}E(P, 2)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ fulfilling

$$\begin{cases} \widetilde{f}_i(x) \leqq \widetilde{f}_i(\bar{x}) & \forall i \in I, \\ \widetilde{f}_{i_0}(x) \leqq^s \widetilde{f}_{i_0}(\bar{x}) & \text{for at least one } i_0 \in I. \end{cases}$$

(v) [19] $\bar{x}$ is a locally weakly type-2 efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}WE(P, 2)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ fulfilling $\widetilde{f}_i(x) \leqq^s \widetilde{f}_i(\bar{x})\ \forall i \in I$.
(vi) $\bar{x}$ is a locally (Pareto) type-3 efficient solution of (P), denoted by $\bar{x} \in \mathrm{loc}E(P, 2)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ fulfilling

$$\begin{cases} \widetilde{f}_i(x) \leqq \widetilde{f}_i(\bar{x}) & \forall i \in I, \\ \widetilde{f}_{i_0}(x) <^s \widetilde{f}_{i_0}(\bar{x}) & \text{for at least one } i_0 \in I. \end{cases}$$

(vii) $\bar{x}$ is a locally weakly type-3 efficient solution of (P), denoted by $\bar{x} \in \text{loc}WE(P,3)$, if there exists $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ satisfying $\widetilde{f}_i(x) <^s \widetilde{f}_i(\bar{x})\ \forall i \in I$.

If $U = \mathbb{R}^n$, the word "locally" is omitted. In this case, the strongly efficient solution sets is denoted by $SE(P,1)$ and so are the other efficient solution sets.

Remark 5
If $\widetilde{f}_i : \mathbb{R}^n \to \mathbf{F}_C(\mathbb{R})$ satisfying $\widetilde{f}_i = \chi_{\{f\}}$ $(i \in I := \{1,...,m\})$, i.e., $\widetilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is a crisp function, then

(i) fuzzy type-1 efficient $\equiv$ fuzzy type-2 efficient $\equiv$ fuzzy type-3 efficient $\equiv$ crisp efficient solution, see e.g. [14];
(ii) fuzzy weakly type-1 efficient $\equiv$ fuzzy weakly type-2 efficient $\equiv$ fuzzy weakly type-3 efficient $\equiv$ crisp weakly efficient solution, see e.g. [14];
(iii) fuzzy strongly efficient $\equiv$ crisp strictly efficient solution, see Definition 3.2 in [9].

Remark 6
The following relations are immediate:

(i) [32] $SE(P,1) \subseteq E(P,1) \subseteq WE(P,1)$;
(ii) [19] $E(P,1) \subseteq E(P,2) \subseteq WE(P,1)$ and $E(P,1) \subseteq WE(P,1) \subseteq WE(P,2)$;
(iii) $WE(P,1) \subseteq WE(P,2) \subseteq WE(P,3)$;
(iv) $E(P,1) \subseteq E(P,2) \subseteq E(P,3) \subseteq WE(P,3)$;
(v) If $m = 1$, then $E(P,1) \equiv WE(P,1)$, $E(P,2) \equiv WE(P,2)$ and $E(P,3) \equiv WE(P,3)$.

The concepts of efficient solutions in Definition 8 are distinct as in the following examples.

Example 1
Consider the following problem (P)

$$\widetilde{\min}\ (\widetilde{f}_1(x), \widetilde{f}_2(x)) = ((1,1,1)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x_1^2\}}\widetilde{+}(0,0,1)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x_2^2\}}, (1,1,1)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x_1^2\}}\widetilde{+}(1,2,4,6)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x_2\}})$$

s.t. $g_t(x) = -x_1 + t \leq 0, \quad t \in [-1,0]$.
Then, $\mathcal{S} = \mathbb{R}_+ \times \mathbb{R}$ and, for $x \in \mathcal{S}$,

$$(\widetilde{f}_1)_\alpha(x) = [x_1^2, x_1^2 + (1-\alpha)x_2^2],$$

$$(\widetilde{f}_2)_\alpha(x) = \begin{cases} [x_1^2 + (1-\alpha)x_2, x_1^2 + (6-2\alpha)x_2], & \text{if } x_2 \geq 0; \\ [x_1^2 + (6-2\alpha)x_2, x_1^2 + (1-\alpha)x_2], & \text{if } x_2 < 0. \end{cases}$$

Picking $\bar{x} = (0,1) \in \mathcal{S}$, one has

$$(\widetilde{f}_1)_\alpha(x) = [x_1^2, x_1^2 + (1-\alpha)x_2^2] \not<^s_{LR} [0, 1-\alpha] = (\widetilde{f}_1)_\alpha(\bar{x})\ \forall \alpha \in [0,1], \tag{1}$$

leading that there is no $x \in \mathcal{S}$ fulfilling $\widetilde{f}_1(x) \leqq \widetilde{f}_1(\bar{x})$, and hence, $\bar{x} \in WE(P,2) \subset WE(P,3)$. However, for $\hat{x} = (0,-1) \in \mathcal{S}$, we have

$$\begin{cases} (\widetilde{f}_1)_\alpha(\hat{x}) = [0, -(1-\alpha)] <_{LR} [0, 1-\alpha] = (\widetilde{f}_1)_\alpha(\bar{x})\ \forall \alpha \in [0,1], \\ (\widetilde{f}_2)_\alpha(\hat{x}) = [-(6-\alpha), -(1+\alpha)] <_{LR} [1+\alpha, 6-2\alpha] = (\widetilde{f}_2)_\alpha(\bar{x})\ \forall \alpha \in [0,1], \end{cases}$$

leading that there is $\hat{x} \in \mathcal{S}$ such that $\widetilde{f}_i(\hat{x}) \leq \widetilde{f}_i(\bar{x}), \forall i \in I$. Thus, $\bar{x} \notin WE(P,1)$ and hence, $WE(P,1) \subsetneqq WE(P,2)$.

Further, we also get

$$\begin{cases} (\widetilde{f}_1)_\alpha(\hat{x}) = [0, -(1-\alpha)] \leq_{LR} [0, 1-\alpha] = (\widetilde{f}_1)_\alpha(\bar{x})\ \forall \alpha \in [0,1], \\ (\widetilde{f}_2)_\alpha(\hat{x}) = [-(6-\alpha), -(1+\alpha)] \leq_{LR} [1+\alpha, 6-2\alpha] = (\widetilde{f}_2)_\alpha(\bar{x})\ \forall \alpha \in [0,1], \end{cases}$$

and, for $\bar{\alpha} = 1$,

$$(\widetilde{f}_2)_{\bar{\alpha}}(\hat{x}) = [-5, -2] <^s_{LR} [2, 4] = (\widetilde{f}_2)_{\bar{\alpha}}(\bar{x})$$

entailing that there is $\hat{x} \in \mathcal{S}$ such that $\widetilde{f}_i(\hat{x}) \leqq \widetilde{f}_i(\bar{x}), \forall i \in I$ and $\widetilde{f}_2(\hat{x}) \leqq^s \widetilde{f}_2(\bar{x})$. Thus, $\bar{x} \notin E(P, 2)$ and hence, $E(P, 2) \subsetneqq WE(P, 2)$.

On the other hand, we also arrive at

$$\begin{cases} (\widetilde{f}_2)_\alpha(\hat{x}) = [-(6-\alpha), -(1+\alpha)] <_{LR} [1+\alpha, 6-2\alpha] = (\widetilde{f}_2)_\alpha(\bar{x}) \ \forall \alpha \in [0, 1], \\ (\widetilde{f}_2)_{\bar{\alpha}}(\hat{x}) = [-5, -2] <^s_{LR} [2, 4] = (\widetilde{f}_2)_{\bar{\alpha}}(\bar{x}), \bar{\alpha} = 1, \end{cases}$$

which in turn implies the existence of $\hat{x} \in \mathcal{S}$ such that $\widetilde{f}_i(\hat{x}) \leqq \widetilde{f}_i(\bar{x}), \forall i \in I$ and $\widetilde{f}_2(\hat{x}) <^s \widetilde{f}_2(\bar{x})$. So, $\bar{x} \notin E(P, 3)$, and hence, $E(P, 3) \subsetneqq WE(P, 3)$.

Example 2

Consider the following problem (P)

$\widetilde{\min}\ \widetilde{f}(x) = (-1, -1, 0)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x\}}$

s.t. $\ g_t(x) = -x + t \leq 0, \ \ t \in [-1, 0]$.

Then, $\mathcal{S} = \mathbb{R}_+$ and $\widetilde{f}_\alpha(x) = [-1, -\alpha]x$. Let $\bar{x} = 0 \in \mathcal{S}$. Since there exist $x = 1 \in \mathcal{S}$ such that

$$\begin{cases} \widetilde{f}_\alpha(x) = [-1, -\alpha] \leq_{LR} [0, 0] = \widetilde{f}_\alpha(\bar{x}) \ \forall \alpha \in [0, 1], \\ \widetilde{f}_{\bar{\alpha}}(x) <^s_{LR} \widetilde{f}_{\bar{\alpha}}(\bar{x}), \ \bar{\alpha} = \frac{1}{2} \in [0, 1], \end{cases}$$

one derives the existence of $x = 1 \in \mathcal{S}$ such that $\widetilde{f}(x) \leqq^s (\widetilde{f}_i)(\bar{x})$. Thus, $\bar{x} \notin WE(P, 2)$. Nevertheless, since

$$\widetilde{f}_{\bar{\alpha}}(x) = [-1, 0] \not<^s_{LR} [0, 0] = \widetilde{f}_{\bar{\alpha}}(\bar{x}), \ \bar{\alpha} = 0 \in [0, 1] \ \forall x \in \mathcal{S},$$

one has, for all $x \in \mathcal{S}$, $\widetilde{f}_\alpha(x) \not<^s_{LR} \widetilde{f}_\alpha(\bar{x}) \ \forall \alpha \in [0, 1]$. This implies that there is no $x \in \mathcal{S}$ satisfying $\widetilde{f}(x) <^s \widetilde{f}(\bar{x})$, i.e., $\bar{x} \in WE(P, 3)$. Hence, $WE(P, 2) \subsetneqq WE(P, 3)$. Furthermore, by invoking Remark 6 (v), one yields $E(P, 2) \subsetneqq E(P, 3)$.

Example 3

Consider the following problem (P)

$\widetilde{\min}\ \widetilde{f}(x) = (-1, -1, 0)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x\}}$

s.t. $\ g_t(x) = -x + t \leq 0, \ \ t \in [-1, 0]$.

Then, $\mathcal{S} = \mathbb{R}_+$ and $\widetilde{f}_\alpha(x) = [-1, -\alpha]x$. Let us choose $\bar{x} = 0 \in \mathcal{S}$. Since there exist $x = 1 \in \mathcal{S}$ such that

$$\begin{cases} \widetilde{f}_\alpha(x) = [-1, -\alpha] \leq_{LR} [0, 0] = \widetilde{f}_\alpha(\bar{x}) \ \forall \alpha \in [0, 1], \\ \widetilde{f}_{\bar{\alpha}}(x) <^s_{LR} \widetilde{f}_{\bar{\alpha}}(\bar{x}), \ \bar{\alpha} = \frac{1}{2} \in [0, 1], \end{cases}$$

one infers the existence of $x = 1 \in \mathcal{S}$ such that $\widetilde{f}(x) \leqq^s (\widetilde{f}_i)(\bar{x})$. Thus, $\bar{x} \notin WE(P, 2)$. Nonetheless, since

$$\widetilde{f}_{\bar{\alpha}}(x) = [-1, 0] \not<^s_{LR} [0, 0] = \widetilde{f}_{\bar{\alpha}}(\bar{x}), \ \bar{\alpha} = 0 \in [0, 1], \forall x \in \mathcal{S},$$

one gets, for all $x \in \mathcal{S}$, $\widetilde{f}_\alpha(x) \not<^s_{LR} \widetilde{f}_\alpha(\bar{x}) \ \forall \alpha \in [0, 1]$. This implies that there is no $x \in \mathcal{S}$ satisfying $\widetilde{f}(x) <^s \widetilde{f}(\bar{x})$, i.e., $\bar{x} \in WE(P, 3)$. Hence, $WE(P, 2) \subsetneqq WE(P, 3)$. In addition, by virtue of Remark 6 (v), one yields $E(P, 2) \subsetneqq E(P, 3)$.

Example 4

Consider the following problem (P)

$\widetilde{\min}\widetilde{f}(x) = (-2, -1, 0, 0)\widetilde{\times}\widetilde{\mathbf{I}}_{\{x\}}$,

s.t. $\ g_t(x) = -x + t \leq 0, \ \ t \in [-1, 0]$.

Then, $\mathcal{S} = \mathbb{R}_+$ and $\widetilde{f}_\alpha(x) = [-2 + \alpha, 0]x, \forall \alpha \in [0, 1]$. Let $\bar{x} = 0 \in \mathcal{S}$. Since there exists $x = 1 \in \mathcal{S}$ such that

$$\begin{cases} \widetilde{f}_\alpha(x) = [-2 + \alpha, 0] \leq_{LR} [0, 0] = \widetilde{f}_\alpha(\bar{x}) \ \forall \alpha \in [0, 1], \\ \widetilde{f}_{\bar{\alpha}}(x) = [-\frac{3}{2}, 0] <_{LR} [0, 0] = \widetilde{f}_{\bar{\alpha}}(\bar{x}), \ \bar{\alpha} = \frac{1}{2} \in [0, 1], \end{cases}$$

one gets that $\widetilde{f}(x) \leq \widetilde{f}(\bar{x})$, which in turn shows that $\bar{x} \notin WE(P,1)$. But, there is no $x \in \mathcal{S}$ satisfying $\widetilde{f}(x) \leqq^s \widetilde{f}(\bar{x})$, since

$$\widetilde{f}_\alpha(x) = [(-2+\alpha)x, 0] \not<^s_{LR} [0,0] = \widetilde{f}_\alpha(\bar{x})\ \forall \alpha \in [0,1].$$

Thus, $\bar{x} \in WE(P,2)$. Therefore, $WE(P,1) \subsetneqq WE(P,2)$. Additionally, employing Remark 6 (v) gives us $E(P,1) \subsetneqq E(P,2)$.

Example 5
Consider the following problem (P)

$$\widetilde{\min}\ \widetilde{f}(x) = (0,0,1,1)$$
$$\text{s.t.}\quad g_t(x) = -x + t \leq 0,\quad t \in [-1,0].$$

Then, $\mathcal{S} = \mathbb{R}_+$ and $\widetilde{f}_\alpha(x) = [0,1], \forall \alpha \in [0,1]$. Taking $\bar{x} = 0 \in \mathcal{S}$ and $x = 1 \in \mathcal{S} \setminus \{\bar{x}\}$, one has

$$\widetilde{f}_\alpha(x) = [0,1] \leq_{LR} [0,1] = \widetilde{f}_\alpha(\bar{x})\ \forall \alpha \in [0,1], \tag{2}$$

leading that $\widetilde{f}(x) \leqq \widetilde{f}(\bar{x})$ and hence, $\bar{x} \notin SE(P,1)$. However, for any $x \in \mathcal{S}$,

$$\widetilde{f}_\alpha(x) \not<_{LR} \widetilde{f}_\alpha(\bar{x})\ \forall \alpha \in [0,1],$$

deducing that there is no $x \in \mathcal{S}$ such that $\widetilde{f}_{\bar{\alpha}}(x) <_{LR} \widetilde{f}_{\bar{\alpha}}(\bar{x})$ for some $\bar{\alpha} \in [0,1]$, i.e., $\widetilde{f}(x) \leq \widetilde{f}(\bar{x})$. Thus, $\bar{x} \in E(P,1)$, and hence, $SE(P,1) \subsetneqq E(P,1)$.

Now, we recall the following constraint qualification in [6], which are similar to Abadie constraint qualification in the literature. For others constraint qualifications and their relations, see e.g. [7] and references therein.

Definition 9
The (ACQ) holds at $\bar{x} \in \mathcal{S}$ if $\left(\bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x})\right)^- \subseteq \mathcal{T}(\mathcal{S},\bar{x})$ and $\Delta := \text{pos} \bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x})$ is a closed set.

In the following, we will establish the Karush-Kuhh-Tucker necessary optimality condition for a locally weakly type-3 efficient solution of (P). In view of Remark 6, this necessary optimality is also the necessary optimality for others efficient solutions of (P). The KKT necessary condition could be employed to reject a feasible point as an efficient solution. It is also utilized as a condition in strong duality relations between the primal problem and the dual problem in optimization.

Proposition 3
Assume that $\bar{x} \in locWE(P,3)$ and (ACQ) holds at $\bar{x}$. Then, there exist nonnegative real-valued functions $\xi_i^L, \xi_i^R (i \in I)$ defined in $[0,1]$ with $\sum_{i\in I} \xi_i^L(\alpha) + \xi_i^R(\alpha) = 1$ for all $\alpha \in [0,1]$ and nonnegative functions $\lambda_t (t \in T)$ defined in $[0,1]$ with $\lambda := (\lambda_t)_{t\in T} \in \Lambda(\bar{x})$ such that

$$\sum_{i\in I} \left(\xi_i^L(\alpha)\nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) + \xi_i^R(\alpha)\nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) + \sum_{t\in T} \lambda_t(\alpha)\nabla g_t(\bar{x}) = 0, \forall \alpha \in [0,1].$$

Proof
We derive from $\bar{x} \in locWE(P,3)$ the existence $U \in \mathbf{U}(\bar{x})$ such that there is no $x \in \mathcal{S} \cap U$ satisfying $\widetilde{f}_i(x) <^s \widetilde{f}_i(\bar{x})\ \forall i \in I$, or equivalently,

$$(\widetilde{f}_i)_\alpha(x) <^s_{LR} (\widetilde{f}_i)_\alpha(\bar{x})\ \forall i \in I, \forall \alpha \in [0,1]. \tag{3}$$

We first justify that

$$\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) \cup \nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right)^s \cap \mathcal{T}(\mathcal{S},\bar{x}) = \emptyset,\ \forall \alpha \in [0,1]. \tag{4}$$

There are only two possible cases here:

Case 1. $\nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x) = 0$ (or $\nabla(\widetilde{f}_i)^R_{\bar\alpha}(\bar x) = 0$) for some $i_0 \in I$ and $\bar\alpha \in [0,1]$. Then, one has $\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_{\alpha}(\bar x) \cup \nabla(\widetilde{f}_i)^R_{\alpha}(\bar x)\right)^s = \emptyset$, leading that (4) holds.

Case 2. $\nabla(\widetilde{f}_i)^L_{\alpha}(\bar x) \neq 0$ and $\nabla(\widetilde{f}_i)^R_{\alpha}(\bar x) \neq 0$ for all $i \in I$, for all $\alpha \in [0,1]$. Suppose to the contrary that (4) is false. Then, there exists $\bar\alpha \in [0,1]$ such that

$$\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x) \cup \nabla(\widetilde{f}_i)^R_{\bar\alpha}(\bar x)\right)^s \cap \mathcal{T}(\mathcal{S}, \bar x) \neq \emptyset$$

Therefore, we ensure the existence of

$$d \in \left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x) \cup \nabla(\widetilde{f}_i)^R_{\bar\alpha}(\bar x)\right)^s \cap \mathcal{T}(\mathcal{S}, \bar x)$$

for some $\bar\alpha \in [0,1]$. This implies that $\langle \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x), d\rangle < 0$ and $\langle \nabla(\widetilde{f}_i)^R_{\bar\alpha}(\bar x), d\rangle < 0$ for all $i \in I$. As $d \in \mathcal{T}(\mathcal{S}, \bar x)$, there exist $\tau_\ell \downarrow 0$ and $d_\ell \to d$ such that $\bar x + \tau_\ell d_\ell \in \mathcal{S}$ for all ℓ. We derive from the fact $(\widetilde{f}_i)^L_{\bar\alpha}(i \in I)$ are continuously differentiable at $\bar x$ that

$$(\widetilde{f}_i)^L_{\bar\alpha}(\bar x + \tau_\ell d_\ell) = (\widetilde{f}_i)^L_{\bar\alpha}(\bar x) + \tau_\ell \langle \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x), d_\ell\rangle + o(\tau_\ell \|d_\ell\|) \ \forall i \in I.$$

Consequently, for all $i \in I$,

$$\frac{(\widetilde{f}_i)^L_{\bar\alpha}(\bar x + \tau_\ell d_\ell) - (\widetilde{f}_i)^L_{\bar\alpha}(\bar x)}{\tau_\ell} = \langle \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x), d_\ell\rangle + \frac{o(\tau_\ell \|d_\ell\|)}{\tau_\ell} \to \langle \nabla(\widetilde{f}_i)^L_{\bar\alpha}(\bar x), d\rangle < 0,$$

when $\ell \to \infty$. Thus, for each $i \in I$, there exists $\ell_i^L > 0$ such that

$$\frac{(\widetilde{f}_i)^L_{\bar\alpha}(\bar x + \tau_\ell d_\ell) - (\widetilde{f}_i)^L_{\bar\alpha}(\bar x)}{\tau_\ell} < 0 \ \forall \ell > \ell_i^L.$$

Setting $\bar\ell^L = \max_{i\in I} \ell_i^L$, we have

$$(\widetilde{f}_i)^L_{\bar\alpha}(\bar x + \tau_\ell d_\ell) < (\widetilde{f}_i)^L_{\bar\alpha}(\bar x) \ \forall \ell > \bar\ell^L, \forall i \in I.$$

Similarly, there exists $\bar l^R > 0$ such that

$$(\widetilde{f}_i)^R_{\bar\alpha}(\bar x + \tau_\ell d_\ell) < (\widetilde{f}_i)^R_{\bar\alpha}(\bar x) \ \forall \ell > \bar\ell^R, \forall i \in I.$$

Designating $\bar\ell := \max\{\bar\ell^L, \bar\ell^R\}$, we assure the existence of $\ell > \bar\ell$ large enough such that $\bar x + \tau_\ell d_\ell \in \mathcal{S} \cap U$ and

$$\begin{cases} (\widetilde{f}_i)^L_{\bar\alpha}(\bar x + \tau_\ell d_\ell) < (\widetilde{f}_i)^L_{\bar\alpha}(\bar x) \ \forall i \in I, \\ (\widetilde{f}_i)^R_{\bar\alpha}(\bar x + \tau_\ell d_\ell) < (\widetilde{f}_i)^R_{\bar\alpha}(\bar x) \ \forall i \in I, \end{cases}$$

i.e., $(\widetilde{f}_i)_{\bar\alpha}(\bar x + \tau_\ell d_\ell) <^s_{LR} (\widetilde{f}_i)_{\bar\alpha}(\bar x), \forall i \in I$, which contradicts (3). Therefore, (4) holds for *Case 2*, and hence, (4) holds for the both possible cases.

We deduce from (4)and (ACQ) that, $\forall \alpha \in [0,1]$,

$$\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_{\alpha}(\bar x) \cup \nabla(\widetilde{f}_i)^R_{\alpha}(\bar x)\right)^s \cap \left(\bigcup_{t\in T(\bar x)} \nabla g_t(\bar x)\right)^- \subset \left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_{\alpha}(\bar x) \cup \nabla(\widetilde{f}_i)^R_{\alpha}(\bar x)\right)^s \cap \mathcal{T}(\mathcal{S}, \bar x) = \emptyset.$$

This leads that there is no $d \in \mathbb{R}^n$ such that, for all $\alpha \in [0,1]$,

$$\begin{cases} \langle \nabla(\widetilde{f}_i)^L_\alpha(\bar{x}), d\rangle < 0 \ \forall i \in I, \\ \langle \nabla(\widetilde{f}_i)^R_\alpha(\bar{x}), d\rangle < 0 \ \forall i \in I, \\ \langle \nabla g_t(\bar{x}), d\rangle \leq 0, \forall t \in T(\bar{x}). \end{cases}$$

Moreover, it follows from Lemma 3 that $\mathrm{co}\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) \cup \nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right)$ is a compact set, and thus, $\mathrm{co}\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) \cup \nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) + \Delta$ is closed. Thanks to Lemma 2, one has

$$0 \in \mathrm{co}\left(\bigcup_{i\in I} \nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) \cup \nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) + \mathrm{pos} \bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x}),$$

for all $\alpha \in [0,1]$. From Lemma 1, there exist $\xi^L_i(\alpha), \xi^R_i(\alpha) \in \mathbb{R}_+$ with $\sum_{i\in I}\left(\xi^L_i(\alpha) + \xi^R_i(\alpha)\right) = 1$ and $\lambda(\alpha) \in \Lambda(\bar{x})$ such that

$$\sum_{i\in I}\left(\xi^L_i(\alpha)\nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) + \xi^R_i(\alpha)\nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) + \sum_{t\in T}\lambda_t(\alpha)\nabla g_t(\bar{x}) = 0,$$

for all $\alpha \in [0,1]$. Denoting $\xi^L_i, \xi^R_i : [0,1] \to \mathbb{R}_+ (i \in I)$ with $\sum_{i\in I}\left(\xi^L_i(\alpha) + \xi^R_i(\alpha)\right) = 1$ for all $\alpha \in [0,1]$ and $\lambda := (\lambda_t)_{t\in T}$ with $\lambda_t : [0,1] \to \mathbb{R}_+ (t \in T)$, the conclusion is obtained. □

In the next part, the KKT sufficient optimality condition for the weakly type-3 efficient solution and the strongly efficient solution of (P) are established under some convexity assumptions. It is well known that the KKT sufficient optimality condition gives the test for a feasible point to be an optimal solution of optimization problems, which is a necessary condition in building algorithms to solve optimization problems.

Proposition 4
Let $\bar{x} \in \mathcal{S}$. Assume that, for each $i \in I$, there exist nonnegative real-valued functions ξ^L_i, ξ^R_i defined in $[0,1]$ with $\sum_{i\in I}\left(\xi^L_i(\alpha) + \xi^R_i(\alpha)\right) = 1$ for all $\alpha \in [0,1]$ and nonnegative functions $\lambda_t (t \in T)$ defined in $[0,1]$ with $\lambda := (\lambda_t)_{t\in T} \in \Lambda(\bar{x})$ such that

$$\sum_{i\in I}\left(\xi^L_i(\alpha)\nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) + \xi^R_i(\alpha)\nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) + \sum_{t\in T}\lambda_t(\alpha)\nabla g_t(\bar{x}) = 0 \tag{5}$$

for all $\alpha \in [0,1]$. Then,

(i) If $\widetilde{f}_i (i \in I)$ are convex at $\bar{x}$ and $g_t (t \in T)$ are convex at $\bar{x}$, then $\bar{x} \in WE(P,3)$;
(ii) If $\widetilde{f}_i (i \in I)$ are strongly convex at $\bar{x}$ and $g_t (t \in T)$ are convex at $\bar{x}$, then $\bar{x} \in SE(P,1)$.

Proof
Since, $\bar{x} \in \mathcal{S}$ fulfilling (5), there exists a finite subset J_α of $T(\bar{x})$ such that

$$\sum_{t\in J_\alpha}\lambda_t(\alpha)\nabla g_t(\bar{x}) = -\sum_{i\in I}\left(\xi^L_i(\alpha)\nabla(\widetilde{f}_i)^L_\alpha(\bar{x}) + \xi^R_i(\alpha)\nabla(\widetilde{f}_i)^R_\alpha(\bar{x})\right) \tag{6}$$

for each $\alpha \in [0,1]$.
(i) Reasoning ad absurdum, assume that $\bar{x}$ is not a weakly type-3 efficient solution of (P). Then, there exists $\hat{x} \in \mathcal{S}$ satisfying $\widetilde{f}_i(\hat{x}) <^s \widetilde{f}_i(\bar{x}), \forall i \in I$, or equivalently,

$$(\widetilde{f}_i)^L_\alpha(\hat{x}) < (\widetilde{f}_i)^L_\alpha(\bar{x}) \text{ and } (\widetilde{f}_i)^R_\alpha(\hat{x}) < (\widetilde{f}_i)^R_\alpha(\bar{x}),$$

for all $i \in I$ and for all $\alpha \in [0,1]$. The above inequalities together with the fact that, for each $i \in I$, ξ_i^L, ξ_i^R are nonnegative real-valued functions in $[0,1]$ satisfying $\sum\limits_{i\in I} \left(\xi_i^L(\alpha) + \xi_i^R(\alpha)\right) = 1$ for all $\alpha \in [0,1]$ implies that

$$\sum_{i\in I} \left(\xi_i^L(\alpha)((\widetilde{f}_i)_\alpha^L(\hat{x}) - (\widetilde{f}_i)_\alpha^L(\bar{x})) + \xi_i^R(\alpha)((\widetilde{f}_i)_\alpha^R(\hat{x}) - (\widetilde{f}_i)_\alpha^R(\bar{x}))\right) < 0, \tag{7}$$

for all $\alpha \in [0,1]$. It follows from the convexity of $\widetilde{f}_i (i \in I)$ at $\bar{x}$, Remark 2 and Remark 4, one has

$$(\widetilde{f}_i)_\alpha^L(\hat{x}) - (\widetilde{f}_i)_\alpha^L(\bar{x}) \geq \langle \nabla(\widetilde{f}_i)_\alpha^L(\bar{x}), \hat{x} - \bar{x}\rangle, \text{ and } (\widetilde{f}_i)_\alpha^R(\hat{x}) - (\widetilde{f}_i)_\alpha^R(\bar{x}) \geq \langle \nabla(\widetilde{f}_i)_\alpha^R(\bar{x}), \hat{x} - \bar{x}\rangle,$$

for all $i \in I$ and $\alpha \in [0,1]$. Hence, combining the above inequalities, (6) and (7) leads us that

$$\sum_{t\in J_\alpha} \lambda_t(\alpha)\langle \nabla g_t(\bar{x}), \hat{x} - \bar{x}\rangle$$

$$= -\sum_{i\in I} \left(\xi_i^L(\alpha)\langle \nabla(\widetilde{f}_i)_\alpha^L(\bar{x}), \hat{x} - \bar{x}\rangle + \xi_i^R(\alpha)\langle \nabla(\widetilde{f}_i)_\alpha^R(\bar{x}), \hat{x} - \bar{x}\rangle\right) > 0, \alpha \in [0,1]. \tag{8}$$

On the other hand, since $\hat{x} \in \mathcal{S}$ and $g_t(\bar{x}) = 0$ for all $t \in J_\alpha$ for each $\alpha \in [0,1]$, we get $g_t(\hat{x}) \leq g_t(\bar{x}), \forall t \in J_\alpha$ for each $\alpha \in [0,1]$. Therefore, by the convexity of $g_t (t \in T)$ at $\bar{x}$, one concludes that for each $\alpha \in [0,1]$,

$$\sum_{t\in J_\alpha} \lambda_t(\alpha)\langle \nabla g_t(\bar{x}), \hat{x} - \bar{x}\rangle \leq \sum_{t\in J_\alpha} \lambda_t(\alpha)(g_t(\hat{x}) - g_t(\bar{x})) \leq 0,$$

which contradicts (8).

(ii) Arguing by contradiction, suppose that $\bar{x}$ is not a strongly efficient solution of (P). Then, there exists a $\hat{x} \in \mathcal{S} \setminus \{\bar{x}\}$ satisfying $(\widetilde{f}_i)(\hat{x}) \leqq (\widetilde{f}_i)(\bar{x}), \forall i \in I$, or equivalently,

$$(\widetilde{f}_i)_\alpha^L(\hat{x}) \leq (\widetilde{f}_i)_\alpha^L(\bar{x}) \text{ and } (\widetilde{f}_i)_\alpha^R(\hat{x}) \leq (\widetilde{f}_i)_\alpha^R(\bar{x})$$

for all $i \in I$ and $\alpha \in [0,1]$. The above inequalities along with the fact that, for each $i \in I$, ξ_i^L, ξ_i^R are nonnegative real-valued functions in $[0,1]$ with $\sum\limits_{i\in I} \left(\xi_i^L(\alpha) + \xi_i^R(\alpha)\right) = 1$ for all $\alpha \in [0,1]$ deduces that

$$\sum_{i\in I} \left(\xi_i^L(\alpha)((\widetilde{f}_i)_\alpha^L(\hat{x}) - (\widetilde{f}_i)_\alpha^L(\bar{x})) + \xi_i^R(\alpha)((\widetilde{f}_i)_\alpha^R(\hat{x}) - (\widetilde{f}_i)_\alpha^R(\bar{x}))\right) \leq 0 \tag{9}$$

for all $\alpha \in [0,1]$. It follows from the strong convexity of $\widetilde{f}_i (i \in I)$ at $\bar{x}$, $\hat{x} \neq \bar{x}$, Remark 2 and Remark 4 that for all $i \in I$, it holds:

$$(\widetilde{f}_i)_\alpha^L(\hat{x}) - (\widetilde{f}_i)_\alpha^L(\bar{x}) > \langle \nabla(\widetilde{f}_i)_\alpha^L(\bar{x}), \hat{x} - \bar{x}\rangle, \alpha \in [0,1]$$
$$(\widetilde{f}_i)_\alpha^R(\hat{x}) - (\widetilde{f}_i)_\alpha^R(\bar{x}) > \langle \nabla(\widetilde{f}_i)_\alpha^R(\bar{x}), \hat{x} - \bar{x}\rangle, \alpha \in [0,1].$$

Hence, the above inequality, (6) and (8) tell us that, for each $\alpha \in [0,1]$,

$$\sum_{t\in J_\alpha} \lambda_t(\alpha)\langle \nabla g_t(\bar{x}), \hat{x} - \bar{x}\rangle$$

$$= -\sum_{i\in I} \left(\xi_i^L(\alpha)\langle \nabla(\widetilde{f}_i)_\alpha^L(\bar{x}), \hat{x} - \bar{x}\rangle + \xi_i^R(\alpha)\langle \nabla(\widetilde{f}_i)_\alpha^R(\bar{x}), \hat{x} - \bar{x}\rangle\right) > 0. \tag{10}$$

On the other hand, since $\hat{x} \in \mathcal{S}$ and $g_t(\bar{x}) = 0$ for all $t \in J_\alpha$ for each $\alpha \in [0,1]$, one yields $g_t(\hat{x}) \leq g_t(\bar{x}), \forall t \in J_\alpha$ for each $\alpha \in [0,1]$. By invoking the convexity of $g_t (t \in T)$ at $\bar{x}$, one has

$$\sum_{t\in J_\alpha} \lambda_t(\alpha)\langle \nabla g_t(\bar{x}), \hat{x} - \bar{x}\rangle \leq \sum_{t\in J_\alpha} \lambda_t(\alpha)(g_t(\hat{x}) - g_t(\bar{x})) \leq 0,$$

contradicting with (10). □

Example 6
Let $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{R}^n \to \mathbf{F}_C(\mathbb{R})$ be defined respectively by

$$\widetilde{f}_1(x) = (\widetilde{\mathbf{I}}_{\{x_1\}} \widetilde{+} (-4,-3,-2)) \widetilde{\times} (\widetilde{\mathbf{I}}_{\{x_1\}} \widetilde{+} (-4,-3,-2)) \\ \widetilde{+} (\widetilde{\mathbf{I}}_{\{x_2\}} \widetilde{+} (-5,-4,-2)) \widetilde{\times} (\widetilde{\mathbf{I}}_{\{x_2\}} \widetilde{+} (-5,-4,-2)),$$

and

$$\widetilde{f}_2(x) = \left(\widetilde{\mathbf{I}}_{\{x_1\}} \widetilde{+} \left(-\frac{7}{2},-3,-2\right)\right) \widetilde{\times} \left(\widetilde{\mathbf{I}}_{\{x_1\}} \widetilde{+} \left(-\frac{7}{2},-3,-2\right)\right) \\ \widetilde{+} (\widetilde{\mathbf{I}}_{\{x_2\}} \widetilde{+} (-5,-4,-2)) \widetilde{\times} (\widetilde{\mathbf{I}}_{\{x_2\}} \widetilde{+} (-5,-4,-2)).$$

Consider the following problem (P)
$\widetilde{\min}(\widetilde{f}_1(x), \widetilde{f}_2(x))$
s.t. $g_t(x) = 6 - tx_1 + (t-1)x_2 \leq 0, \quad t \in T = [0,1].$
Then, $\mathcal{S} = \{x \in \mathbb{R}^2 \mid 6 - x_2 \leq 0, 6 - x_1 \leq 0\} = [6,+\infty) \times [6,+\infty)$ and

$$(\widetilde{f}_1)_\alpha(x) = [(x_1 - 4 + \alpha)^2 + (x_2 - 5 + \alpha)^2, (x_1 - 2 - \alpha)^2 + (x_2 - 2 - 2\alpha)^2],$$

$$(\widetilde{f}_2)_\alpha(x) = \left[\left(x_1 - \frac{7}{2} + \frac{1}{2}\alpha\right)^2 + (x_2 - 5 + \alpha)^2, (x_1 - 2 - \alpha)^2 + (x_2 - 2 - 2\alpha)^2\right].$$

Hence,

$$\nabla(\widetilde{f}_1)^L_\alpha(x) = (2(x_1 - 4 + \alpha), 2(x_2 - 5 + \alpha)),$$
$$\nabla(\widetilde{f}_1)^R_\alpha(x) = (2(x_1 - 2 - \alpha), 2(x_2 - 2 - 2\alpha)),$$
$$\nabla(\widetilde{f}_2)^L_\alpha(x) = \left(2\left(x_1 - \frac{7}{2} + \frac{1}{2}\alpha\right), 2(x_2 - 5 + \alpha)\right),$$
$$\nabla(\widetilde{f}_2)^R_\alpha(x) = (2(x_1 - 2 - \alpha), 2(x_2 - 2 - 2\alpha)).$$

Let $\bar{x} = (6,6) \in \mathcal{S}$. Since, $\forall x \in \mathcal{S}$,

$$(\widetilde{f}_1)_\alpha(x) = [(x_1 - 4 + \alpha)^2 + (x_2 - 5 + \alpha)^2, (x_1 - 2 - \alpha)^2 + (x_2 - 2 - 2\alpha)^2]$$
$$\not<^s_{LR} (\widetilde{f}_1)_\alpha(\bar{x}) = [(2 + \alpha)^2 + (1 + \alpha)^2, (4 - \alpha)^2 + (4 - 2\alpha)^2],$$
$$(\widetilde{f}_2)_\alpha(x) = \left[\left(x_1 - \frac{7}{2} + \frac{1}{2}\alpha\right)^2 + (x_2 - 5 + \alpha)^2, (x_1 - 2 - \alpha)^2 + (x_2 - 2 - 2\alpha)^2\right]$$
$$\not<^s_{LR} (\widetilde{f}_2)_\alpha(\bar{x}) = \left[\left(\frac{5}{2} + \frac{1}{2}\alpha\right)^2 + (1 + \alpha)^2, (4 - \alpha)^2 + (4 - 2\alpha)^2\right],$$

for all $\alpha \in [0,1]$, one has, for all $x \in \mathcal{S}$,

$$\begin{cases} \widetilde{f}_1(x) \not<^s \widetilde{f}_1(\bar{x}), \\ \widetilde{f}_2(x) \not<^s \widetilde{f}_2(\bar{x}), \end{cases}$$

i.e., $\bar{x} \in WE(P,3)$. By some calculations, we have

$$\nabla(\widetilde{f}_1)^L_\alpha(\bar{x}) = (2(2 + \alpha), 2(1 + \alpha)), \nabla(\widetilde{f}_1)^R_\alpha(\bar{x}) = (2(4 - \alpha), 2(4 - 2\alpha)),$$
$$\nabla(\widetilde{f}_2)^L_\alpha(\bar{x}) = \left(2\left(\frac{5}{2} + \frac{1}{2}\alpha\right), 2(1 + \alpha)\right), \nabla(\widetilde{f}_2)^R_\alpha(\bar{x}) = (2(4 - \alpha), 2(4 - 2\alpha)),$$
$$\mathcal{T}(\mathcal{S}, \bar{x}) = \mathbb{R}^2_+, T(\bar{x}) = T, \nabla g_t(\bar{x}) = (-t, t - 1),$$

$$\bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x}) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = -1, x_1 \le 0, x_2 \le 0\},$$

$$\left(\bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x})\right)^- = \mathbb{R}^2_+ \subseteq \mathcal{T}(\mathcal{S}, \bar{x}), \text{pos} \bigcup_{t\in T(\bar{x})} \nabla g_t(\bar{x}) = -\mathbb{R}^2_+$$

is a closed set. Hence, (ACQ) holds at $\bar{x}$ and all assumptions in Proposition 3 are satisfied. Now, let $\xi_i^L, \xi_i^R : [0,1] \to \mathbb{R}_+ (i = 1, 2)$ and $\lambda_t : [0,1] \to \mathbb{R}_+$ be defined by

$$\xi_1^L(\alpha) = \frac{4}{5}, \xi_1^R(\alpha) = 0 \ \forall \alpha \in [0,1],$$

$$\xi_2^L(\alpha) = 0, \xi_2^R(\alpha) = \frac{1}{5} \ \forall \alpha \in [0,1],$$

$$\lambda_t(\alpha) = \begin{cases} 2(4+\alpha), & \text{if } t = \frac{3}{5}, \\ 0, & \text{otherwise,} \end{cases} \quad \forall \alpha \in [0,1].$$

Then, the functions $\xi_i^L, \xi_i^R (i = 1, 2)$ are nonnegative real-valued functions defined on $[0,1]$ with $\sum_{i=1}^{2} \left(\xi_i^L(\alpha) + \xi_i^R(\alpha)\right) = 1$ for all $\alpha \in [0,1]$ and $\lambda = (\lambda_t)_{t\in T} \in \Lambda(\bar{x})$. Moreover, for all $\alpha \in [0,1]$, $\sum_{i=1}^{2} \left(\xi_i^L(\alpha)\nabla(\widetilde{f}_i)_\alpha^L(\bar{x}) + \xi_i^R(\alpha)\nabla(\widetilde{f}_i)_\alpha^R(\bar{x})\right) + \sum_{t\in T} \lambda_t(\alpha)\nabla g_t(\bar{x})$

$$\begin{aligned}
&= \frac{4}{5}(2(2+\alpha), 2(1+\alpha)) + 0 + 0 + \frac{1}{5}(2(4-\alpha), 2(4-2\alpha)) + 2(4+\alpha).\left(-\frac{3}{5}, -\frac{2}{5}\right) \\
&= \left(\frac{8(2+\alpha)+2(4-\alpha)}{5}, \frac{8(1+\alpha)+2(4-2\alpha)}{5}\right) + 2(4+\alpha).\left(-\frac{3}{5}, -\frac{2}{5}\right) \\
&= \left(\frac{24+6\alpha}{5}, \frac{16+4\alpha}{5}\right) + 2(4+\alpha).\left(-\frac{3}{5}, -\frac{2}{5}\right) = (0,0),
\end{aligned}$$

i.e., the conclusion of Proposition 3 is fulfilled.

On the other hand, we can verify that $(\widetilde{f}_i)_\alpha^L, (\widetilde{f}_i)_\alpha^R (i = 1, 2)$ are convex at $\bar{x} = (6, 6)$ for all $\alpha \in [0,1]$ and $g_t (t \in T)$ are convex at $\bar{x}$. Hence, all assumptions in Proposition 4 (i) hold. Then, it follows that $\bar{x} \in WE(P, 3)$. Furthermore, $(\widetilde{f}_i)_\alpha^L, (\widetilde{f}_i)_\alpha^R (i = 1, 2)$ are also strictly convex at $\bar{x} = 0$. Thus, we deduce from Proposition 4 (ii) that $\bar{x} \in SE(P, 1)$.

4. Conclusions

In this paper, the Karush-Kuhn-Tucker necessary and sufficient optimality conditions for semi-infinite programming with multiple fuzzy-valued objective functions are investigated. The outcome of the paper extends the results in [31, 32] from optimization problems with multiple fuzzy-valued objective functions to semi-infinite programming problems with multiple fuzzy-valued objective functions. In the case that T is a finite set, the main results in the paper are also new since the necessary optimality conditions were not investigated in [31, 32]. Moreover, our approach in this paper is different from that of [18, 19]. Considering the optimality conditions for fuzzy semi-infinite programming problems with nonsmooth functions by using gH-derivatives [18, 19, 23] or generalized subdifferentials could be an interesting subject for the future research.

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