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Exponentiated Extended Chen Distribution: Regression Model and Estimations

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Abstract In this paper, we introduce a new four-parameter generalized version of the Chen model called the exponentiated extended Chen distribution. Some results about the reliability characteristics of hazard rate function as well as some mathematical properties are provided. The maximum likelihood estimators and five approaches based on the concept of minimum spacing distance estimators are given for estimation of the model parameters and their performances in estimating of parameters are compared by means of Monte Carlo simulations. Also, a multiple regression model with the censored data based on proposed distribution is introduced.

Keywords Chen distribution; Maximum likelihood estimator; Multiple regression; Characterizations

AMS 2010 subject classifications 62E15, 60E05, 62F10

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1. Introduction

[13] proposed an extension of the generalized exponential (GE) distribution, which is very flexible, positively skewed, and has increasing, decreasing, unimodal and bathtub shaped hazard rate functions (hrf). It includes GE, exponential, generalized Pareto [9], and Pareto distributions. Recently, a generalization of Chen distribution [3], is introduced in [2] with the following cumulative distribution function (cdf):

$$F_{ECh}(x) = \left(1 - e^{\lambda(1-e^{x^\beta})}\right)^\alpha, \quad x \geq 0, \quad \alpha > 0, \ \lambda > 0, \ \beta > 0. \tag{1}$$

It is called the exponentiated Chen (ECh) family. For $\alpha = 1$, the ECh distribution is reduced to the Chen distribution. Various properties of the ECh distribution and estimation methods for the parameters of this distributions are studied in [6]. [11] generalized a class of extended-Weibull distributions. According to Table 1 of their paper, by changing the generator function $\Phi(x; \boldsymbol{\eta})$, a lot of extension of distributions in the class of extended-Weibull distributions can be obtained. Based on [11], we introduce the exponentiated extended Chen (EE-Ch) distribution with support $S_x = (0, \infty)$ if $\gamma < 0$ and $S_x = (0, \psi)$ if $\gamma > 0$ where

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$\psi = \left[\log(\frac{1}{\gamma\lambda}+1)\right]^{\frac{1}{\beta}}$ via the following cdf:

$$F(x;\alpha,\beta,\lambda,\gamma) = \begin{cases} \left\{1-[1-\gamma\lambda(e^{x^{\beta}}-1)]^{\frac{1}{\gamma}}\right\}^{\alpha} & \text{if} \quad \gamma \neq 0 \\ \left(1-e^{-\lambda(e^{x^{\beta}}-1)}\right)^{\alpha} & \text{if} \quad \gamma = 0. \end{cases} \tag{2}$$

We denote this distribution as EE-Ch$(\alpha,\beta,\lambda,\gamma)$. This distribution includes some well-known distributions such as Chen and extended Chen distributions. This new model inherits the necessity of existence of parameters α, β and λ from ECh model (1). However, one can pay attention to the existence of parameter γ in model (2). Through Lemma 1 to Lemma 3, it is apparent that the the parameter γ can control the flexibiliity of the new proposed distribution for both pdf and hrf functions of the model. Also, as it is seen in our analyzing the real data set in Example (5), our proposed model with reasonable value of the parameter γ provides a better fit than the other rival models in fitting to the left skewed data set in this real example. So, it is not correct to ignor the effects made by this parameter in fitting the model EE-Ch to some real data sets. One can see that this distribution has a lot of flexibility to fit to real data sets rather than the other competing models. In the real data section, we show that the EE-Ch has a better fit than the other considered models for fitting to strengths of glass fibers and modeling a medical data known as myelogenous leukemia data. So, this model can be fitted to various fields of science such as engineering and medicine. As we mentioned earlier in the paper, for a good reference to this work, one can see the paper of [11]. Also, some researches related to this work were considered in the litreture such as [13] and [10]. For estimating the parameters of model, we use the method which is known as minimum spacing distance (MSDE) as well as the traditional maximum likelihood estimation (MLE). MSDE was introduced by [17] and followed in some researches such as [20] and [18] and so on.
The paper is organized as follows: In Section 2, we provide some properties of the pdf, hazard rate function (hrf), rth non-central moment and moment generating function (mgf) of the EE-Ch distribution. Furthermore, in this section, we derive the quantile measure and provide some asymptotes for cdf, pdf and hrf functions of the proposed distribution. Section 3 presents various characterizations of the proposed distribution. In Section 4, we discuss maximum likelihood estimation (MLE) of the EE-Ch parameters from one observed sample. Application of the this new model using a real data set is considered in Section 5. In Section 6, we introduce a multiple regression model where the error term of the model follows the shifted-log-EE-Ch distribution (SLEE-Ch) and we illustrate how this new model can be applied to myelogenous leukemia data. Finally, we conclude the paper in Section 7.

2. Some properties of EE-Ch distribution

This section provides some properties of the pdf and hrf and presents the quantile measure of the proposed model. Further, we obtain some general features of EE-Ch distribution such as raw moments and moment generating function(mgf). Also the asymptotic properties of the pdf, cdf and hrf of the new distribution at 0 and ∞ are considered. The pdf of EE-Ch distribution is

$$f(x;\alpha,\beta,\lambda,\gamma) = \begin{cases} \alpha\beta\lambda x^{\beta-1}e^{x^{\beta}}(1-\gamma w)^{\frac{1}{\gamma}-1}\left[1-(1-\gamma w)^{\frac{1}{\gamma}}\right]^{\alpha-1} & \text{if} \quad \gamma \neq 0 \\ \alpha\beta\lambda x^{\beta-1}e^{x^{\beta}}e^{-w}\left(1-e^{-w}\right)^{\alpha-1} & \text{if} \quad \gamma = 0, \end{cases} \tag{3}$$

where $w = \lambda\left(e^{x^{\beta}}-1\right)$.

Lemma 1
Let $f(x)$ be the pdf of the EE-Ch$(\alpha,\beta,\lambda,\gamma)$. Then we have

$$\lim_{x\to 0^+} f(x) = \begin{cases} 0 & \text{if } (\alpha=1,\beta>1) \text{ or } (\alpha>1,\beta\geq 1) \\ \infty & \text{if } (\alpha=1,\beta<1) \text{ or } (\alpha<1,\beta\leq 1) \\ \lambda & \text{if } \alpha=1,\beta=1 \end{cases}$$

$$\lim_{x\to c^-} f(x) = \begin{cases} 0 & \text{if } \gamma<1 \\ \infty & \text{if } \gamma>1, \end{cases}$$

where $c=\psi$ for $\gamma>0$, and $c=\infty$ for $\gamma\leq 0$.

The hazard rate function of the EE-Ch$(\alpha,\gamma,\lambda,\beta)$ distribution has the following form:

$$h(x) = \begin{cases} \dfrac{\alpha\beta\lambda x^{\beta-1}e^{x^\beta}(1-\gamma w)^{\frac{1}{\gamma}-1}\left[1-(1-\gamma w)^{\frac{1}{\gamma}}\right]^{\alpha-1}}{1-\left[1-(1-\gamma w)^{\frac{1}{\gamma}}\right]^{\alpha}} & \text{if } \gamma\neq 0 \\ \dfrac{\alpha\beta\lambda x^{\beta-1}e^{x^\beta}e^{-w}\left(1-e^{-w}\right)^{\alpha-1}}{1-(1-e^{-w})^{\alpha}} & \text{if } \gamma=0. \end{cases} \tag{4}$$

In Figure 1, we plot the pdf and hrf of EE-Ch distribution for some different values of model parameters.

Lemma 2
For $\gamma<0$, the hrf of EE-Ch distribution cannot be bathtub shaped or increasing when $\alpha\geq 1$ and $\beta\geq 1$.

Proof
The limiting behaviour of hrf is

$$\lim_{x\to 0^+} h(x) = \begin{cases} \lambda & \text{if } \alpha=1,\beta=1 \\ 0 & \text{if } \alpha\geq 1,\beta\geq 1. \end{cases}$$

Also, $\lim_{x\to\infty} h(x)=0$. Therefore, the proof is completed. □

Lemma 3
For $\gamma>0$, the hrf of EE-Ch distribution cannot be upside-down bathtub shaped or decreasing when $\alpha\geq 1$ and $\beta\geq 1$.

Proof
The limiting behavior of hrf at ψ is $\lim_{x\to\psi^-} h(x)=\infty$. Therefore, the proof is completed. □

The pth percentile of EE-Ch$(\alpha,\beta,\lambda,\gamma)$ is

$$x_p = \begin{cases} \left[\log\left(\dfrac{1-\left(1-\sqrt[\alpha]{p}\right)^{\gamma}}{\gamma\lambda}+1\right)\right]^{\frac{1}{\beta}} & \text{if } \gamma\neq 0 \\ \left[\log\left(-\dfrac{\log\left(1-\sqrt[\alpha]{p}\right)}{\lambda}+1\right)\right]^{\frac{1}{\beta}} & \text{if } \gamma=0. \end{cases} \tag{5}$$

Using integral transformation theorem one can generate data from an EE-Ch$(\alpha,\beta,\lambda,\gamma)$ distribution by applying the above percentile function.

Lemma 4
For $\gamma>0$, the rth non-central moment, $\mu^{(r)}$, of an EE-Ch$(\alpha,\beta,\lambda,\gamma)$ distribution is

$$\mu^{(r)} = \psi^r - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{n_2}(-1)^{n_1+n_2+n_3}\binom{\alpha}{n_1}\binom{\frac{n_1}{\gamma}}{n_2}\binom{n_2}{n_3}(\gamma\lambda)^{n_2} I_{\{(n_2-n_3),\beta/r,\psi^r\}}, \tag{6}$$

where $I_{\{n,t,y\}}=\int_0^y e^{nx^t}dx=\sum_{j=0}^{\infty}\frac{n^j\,y^{t\,j+1}}{j!\,(t\,j+1)}$.

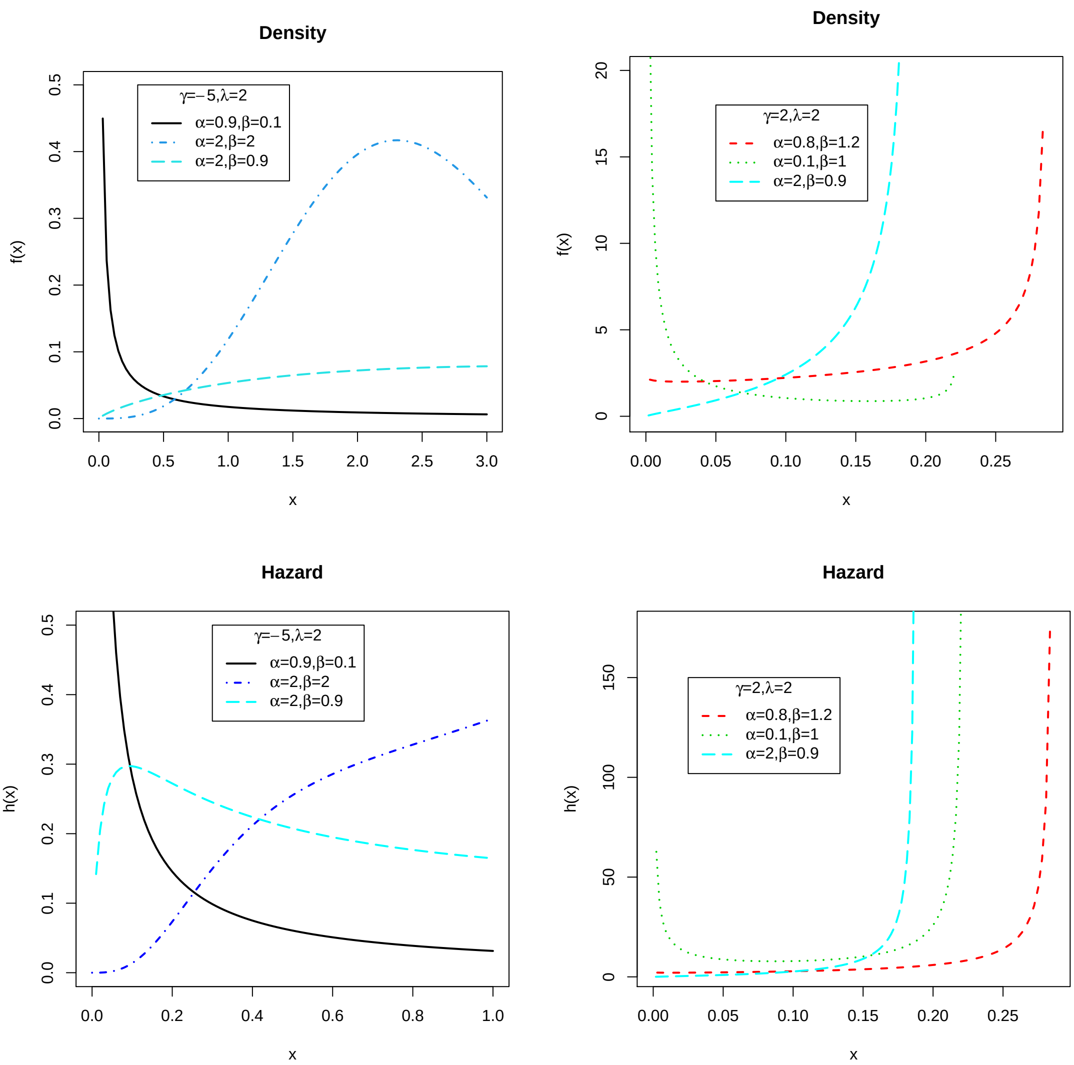

Figure 1. pdf and hrf of EE-Ch distribution for some different values of model parameters.

Proof

Since $\gamma > 0$, we can easily show that $\left|\left(1-\gamma\lambda\left(e^{x^{\beta}}-1\right)\right)^{1/\beta}\right| < 1$ for all $x \in (0, \psi)$. Therefore, by using the binomial series expansion we can compute $\mu^{(r)}$ as follows:

$$\begin{aligned} \mu^{(r)} &= \int_0^{\psi^r} \left(1-\left\{1-\left[1-\gamma\lambda\left(e^{x^{\beta/r}}-1\right)\right]^{\frac{1}{\gamma}}\right\}^{\alpha}\right)dx \\ &= \psi^r - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{n_2}(-1)^{n_1+n_2+n_3}\binom{\alpha}{n_1}\binom{\frac{n_1}{\gamma}}{n_2}\binom{n_2}{n_3}(\gamma\lambda)^{n_2}\int_0^{\psi^r} e^{(n_2-n_3)x^{\beta/r}}dx, \end{aligned}$$

and the proof is completed. □

Lemma 5
For $\gamma > 0$, the moment generating function $M(s)$ of EE-Ch distribution is

$$M(s) = e^{s\psi} - \sum_{n=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{n_2}\frac{s^n}{n!}(-1)^{n_1+n_2+n_3}\binom{\alpha}{n_1}\binom{\frac{n_1}{\gamma}}{n_2}\binom{n_2}{n_3}(\gamma\lambda)^{n_2} I_{\{(n_2-n_3),\beta/n,\psi^n\}}. \quad (7)$$

Proof
Based on the Maclaurin series expansion of the moment generating function $M(s) = \sum_{n=0}^{\infty}\frac{s^n}{n!}\mu^{(n)}$, and using Lemma 4, the proof is completed. □

Proposition 1
The asymptotic of cdf, pdf and hrf of EE-Ch $(\alpha, \gamma, \lambda, \beta)$ distribution as $x \to 0$ are given by

$$F(x) \sim \lambda^{\alpha}\ x^{\alpha\beta}, \quad f(x) \sim \alpha\beta\lambda^{\alpha}\ x^{\alpha\beta-1}, \quad h(x) \sim \alpha\beta\lambda^{\alpha}\ x^{\alpha\beta-1}.$$

Proposition 2
The asymptotic of cdf, pdf and hrf of EE-Ch $(\alpha, \gamma, \lambda, \beta)$ distribution as $x \to \infty$ are given by

$$1 - F(x) \sim \alpha\lambda \mathrm{e}^{\ x^{\beta}}, \quad f(x) \sim \alpha\beta\lambda^{\alpha}x^{\beta\ -1}\mathrm{e}^{\ x^{\beta}}, \quad h(x) \sim \beta\ x^{\beta\ -1}.$$

3. Characterizations results

In this section, we characterize the EE-Ch distribution in the following directions: (i) based on the ratio of two truncated moments and (ii) in terms of the hazard function. We present our characterizations (i) and (ii) in two main subsections, $\gamma = 0$ and $\gamma \neq 0$.

3.1. Characterization results, $\gamma = 0$

3.1.1. Characterizations based on two truncated moments This sub-subsection is devoted to the presentation of certain characterizations of EE-Ch distribution, for $\gamma = 0$, based on a simple relationship between two truncated moments. Our first characterization employs a theorem due to [8], see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 3
Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[1 - e^{-\lambda\left(e^{x^{\beta}}-1\right)}\right]^{1-\alpha}$ and $q_2(x) = q_1(x)\, e^{-\lambda\left(e^{x^{\beta}}-1\right)}$ for $x > 0$. The random variable X has pdf (2.1), for $\gamma = 0$, if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2}e^{-\lambda\left(e^{x^{\beta}}-1\right)}, \quad x > 0.$$

Proof
Suppose the random variable X has pdf (2.1), for $\gamma = 0$, then

$$(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] = \alpha e^{-\lambda\left(e^{x^{\beta}}-1\right)}, \quad x > 0,$$

and

$$(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] = \frac{\alpha}{2}e^{-2\lambda\left(e^{x^{\beta}}-1\right)}, \quad x > 0.$$

Further,

$$\xi(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\lambda\left(e^{x^\beta}-1\right)} < 0, \quad for \quad x > 0.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \lambda\beta x^{\beta-1} e^{x^\beta}, \quad x > 0,$$

and consequently

$$s(x) = \lambda(e^{x^\beta} - 1), \quad x > 0.$$

Now, in view of Theorem 1, X has pdf (2.1), for $\gamma = 0$. □

Corollary 1
Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The random variable X has pdf (2.1), for $\gamma = 0$, if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \lambda\beta x^{\beta-1} e^{x^\beta}, \quad x > 0.$$

Corollary 2
The general solution of the differential equation in Corollary 1 is

$$\xi(x) = e^{\lambda\left(e^{x^\beta}-1\right)} \left[-\int \lambda\beta x^{\beta-1} e^{x^\beta} e^{-\lambda\left(e^{x^\beta}-1\right)} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D \right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, ξ) which satisfy conditions of Theorem 1.

3.1.2. Characterization in terms of hazard function The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this sub-subsection, we present a non-trivial characterization of EE-Ch distribution, for $\gamma = 0, \alpha = 1$, in terms of the hazard function.

Proposition 4
Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (2.1), for $\gamma = 0, \alpha = 1$, if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - \beta x^{\beta-1} h_F(x) = \alpha\beta(\beta-1)\lambda x^{\beta-2} e^{x^\beta}, \quad x > 0.$$

Proof
The proof is straightforward and hence omitted. □

3.2. Characterization results, $\gamma \neq 0$

In this subsection, we assume $\gamma < 0$ and characterize the EE-Ch distribution in the directions (i) and (ii) mentioned before. The characterizations for the case $\gamma > 0$, will be similar and hence omitted.

3.2.1. Characterization based on two truncated moments In this sub-subsection, we assume that $\gamma < 0$ and will characterize the EE-Ch distribution based on a simple relationship between two truncated moments.

Proposition 5

Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 - \left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right]^{1/\gamma}\right\}^{1-\alpha}$ and $q_2(x) = q_1(x)\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right]$ for $x > 0$. The random variable X has pdf (2.1), for $\gamma < 0$, if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{\gamma}{\gamma+1}\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right], \quad x > 0.$$

Proof

Suppose the random variable X has pdf (2.1), for $\gamma < 0$, then

$$(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] = \alpha\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right]^{1/\gamma}, \quad x > 0,$$

and

$$(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] = \frac{\alpha}{2}\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right]^{\frac{1}{\gamma}+1}, \quad x > 0.$$

Further,

$$\xi(x)\, q_1(x) - q_2(x) = -\frac{1}{\gamma+1} q_1(x)\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right] < 0, \quad for \quad x > 0.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \frac{\gamma^2\beta\lambda x^{\beta-1} e^{x^\beta}}{1 - \gamma\lambda\left(e^{x^\beta} - 1\right)}, \quad x > 0,$$

and consequently

$$s(x) = -\gamma \log\left[1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right], \quad x > 0.$$

Now, in view of Theorem 1, X has pdf (2.1), for $\gamma < 0$. □

Corollary 3

Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3. The random variable X has pdf (2.1), for $\gamma < 0$, if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \frac{\gamma^2\beta\lambda x^{\beta-1} e^{x^\beta}}{1 - \gamma\lambda\left(e^{x^\beta} - 1\right)}, \quad x > 0.$$

Corollary 4

The general solution of the differential equation in Corollary 3 is

$$\xi(x) = \left\{1 - \gamma\lambda\left(e^{x^\beta} - 1\right)\right\}^{-1}\left[-\int \gamma^2\beta\lambda x^{\beta-1} e^{x^\beta} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 3 with $D = 0$. Clearly, there are other triplets (q_1, q_2, ξ) which satisfy conditions of Theorem 1.

3.2.2. Characterization in terms of hazard function In this sub-subsection, we present a non-trivial characterization of EE-Ch distribution for $\gamma < 0, \alpha = 1$, in terms of the hazard function.

Proposition 6
Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (2.1), for $\gamma < 0, \alpha = 1$, if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - \beta x^{\beta-1} h_F(x) = \alpha\beta e^{x^\beta} \frac{d}{dx}\left\{ \frac{x^{\beta-1}}{1-\gamma\lambda\left(e^{x^\beta}-1\right)} \right\}, \quad x > 0.$$

4. Estimation

Let $X_1, \ldots, X_n$ be a random sample from the EE-Ch$(\alpha, \beta, \lambda, \gamma)$ with the associated observed values $x_1, \ldots, x_n$. In this section, we pay attention to estimating the unknown parameters of EE-Ch model. First, the MLEs of the parameters and their properties are discussed. Then, five approaches based on the concept of minimum spacing distance estimator (MSDE) and the method of percentile estimation (MPE) are proposed. We let $\boldsymbol{\theta} = (\alpha, \beta, \lambda, \gamma)^T$ and $w_i = \lambda(e^{x_i^\beta} - 1)$.

4.1. The maximum likelihood estimators

For $\gamma \neq 0$, the log-likelihood function of the EE-Ch model can be written as

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= n\log(\alpha\lambda\beta) + (\beta-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n} x_i^\beta + \left(\frac{1}{\gamma}-1\right)\sum_{i=1}^{n}\log(1-\gamma w_i) \\ &\quad + (\alpha-1)\sum_{i=1}^{n}\log\left[1-(1-\gamma w_i)^{\frac{1}{\gamma}}\right]. \end{aligned}$$

Therefore, the components of the score vector can be obtained by taking the first derivative of $\ell(\boldsymbol{\theta})$ with respect to each parameter and equating them to zero. Unfortunately, the MLEs of the parameters do not have closed forms, but they can be approximated using a numerical method like the Newton method. Under the regularity conditions that are fulfilled for case $\gamma \leq 0$, the asymptotic distribution of the MLE of vector parameter $\boldsymbol{\theta}$ is multivariate normal distribution, i.e.,

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{d} N_4\left(\mathbf{0}, I(\boldsymbol{\theta})^{-1}\right),$$

where $I(\boldsymbol{\theta})$ is the Fisher information matrix. For $\gamma > 0$, the regularity conditions are not satisfied because the support of the distribution depends on unknown parameters. In this case, we refer the readers to [10] for the asymptotic distributions of the MLEs of the parameters.

4.2. Minimum spacing distance estimators

Here, the parameters of the EE-Ch distribution are estimated using the minimum spacing distance method introduced by [17]. Let $h(x, y)$ be an appropriate distance function. For the ordered statistics $Y_1, \ldots, Y_n$ associated with the random sample $X_1, \ldots, X_n$. Define

$$D_i(\boldsymbol{\theta}) = F_{\boldsymbol{\theta}}(Y_i) - F_{\boldsymbol{\theta}}(Y_{i-1}), \quad i = 1, \ldots, n+1, \tag{8}$$

where $F_{\boldsymbol{\theta}}(Y_0) = 0$ and $F_{\boldsymbol{\theta}}(Y_{n+1}) = 1$. The minimum spacing distance estimators (MSDEs) for the parameters can be obtained by minimizing the following function

$$T(\boldsymbol{\theta}) = \sum_{i=1}^{n+1} h\left(D_i(\boldsymbol{\theta}), 1/(n+1)\right). \tag{9}$$

Table 1. The biases and MSEs of the methods for $\boldsymbol{\theta} = (0.5, 0.5, 4, -1.5)$.

		n													
		10		30		50		70		90		200		500	
Parameters	Methods	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
α	MLE	0.807	4.485	0.224	0.501	0.130	0.128	0.102	0.085	0.086	0.054	0.044	0.035	0.032	0.029
	MSDE1	1.007	8.280	0.616	3.847	0.333	1.337	0.238	0.550	0.235	0.541	0.139	0.110	0.112	0.042
	MSDE2	1.188	8.788	0.336	0.878	0.210	0.296	0.163	0.142	0.154	0.110	0.108	0.047	0.093	0.033
	MSDE3	0.218	0.776	0.092	0.231	0.041	0.085	0.031	0.042	0.022	0.026	0.014	0.011	0.016	0.007
	MSDE4	0.309	1.582	0.139	0.362	0.085	0.136	0.078	0.072	0.079	0.061	0.064	0.030	0.069	0.022
	MSDE5	1.005	8.274	0.607	3.765	0.332	1.340	0.236	0.550	0.233	0.535	0.138	0.110	0.108	0.041
λ	MLE	2.462	21.87	0.797	5.721	0.361	1.754	0.232	0.813	0.148	0.303	0.062	0.162	0.034	0.022
	MSDE1	1.657	14.47	0.847	6.353	0.511	3.472	0.341	1.794	0.293	1.357	0.115	0.223	0.069	0.033
	MSDE2	1.720	13.80	0.711	4.640	0.318	1.412	0.222	0.738	0.165	0.244	0.076	0.052	0.053	0.021
	MSDE3	0.804	5.099	0.387	2.134	0.264	1.418	0.163	0.554	0.113	0.290	0.045	0.063	0.026	0.021
	MSDE4	0.940	6.941	0.411	2.443	0.256	1.231	0.160	0.488	0.130	0.254	0.060	0.054	0.046	0.022
	MSDE5	1.665	14.57	0.849	6.385	0.516	3.529	0.343	1.834	0.295	1.391	0.115	0.225	0.067	0.033
β	MLE	0.077	4.850	-0.378	3.683	-0.400	3.807	-0.330	3.603	-0.433	3.594	0.361	2.047	0.154	1.365
	MSDE1	-1.127	8.558	-1.517	8.628	-1.453	7.677	-1.384	6.958	-1.402	6.806	-1.350	5.625	-1.238	4.573
	MSDE2	-1.177	8.551	-1.287	6.894	-1.337	6.665	-1.182	5.913	-1.225	5.782	-1.076	4.757	-0.998	4.159
	MSDE3	-0.140	0.523	-0.154	0.485	-0.106	0.360	-0.097	0.351	-0.126	0.406	-0.123	0.391	-0.068	0.579
	MSDE4	-0.424	1.584	-0.572	2.055	-0.568	2.164	-0.609	2.331	-0.697	2.606	-0.708	2.474	-0.801	2.817
	MSDE5	-1.109	8.429	-1.501	8.575	-1.443	7.598	-1.369	6.892	-1.388	6.726	-1.343	5.581	-1.202	4.394
γ	MLE	-0.391	1.616	-0.184	1.016	-0.139	0.753	-0.089	0.620	-0.109	0.542	0.015	0.427	0.094	0.018
	MSDE1	0.196	5.253	-0.155	1.744	-0.222	1.281	-0.201	1.026	-0.226	0.873	-0.297	0.651	-0.313	0.498
	MSDE2	-0.387	1.975	-0.329	1.084	-0.339	0.934	-0.290	0.790	-0.302	0.694	-0.285	0.563	-0.276	0.462
	MSDE3	0.156	1.091	0.052	0.376	0.044	0.298	0.043	0.220	0.035	0.189	0.004	0.111	-0.032	0.089
	MSDE4	0.096	1.118	-0.026	0.460	-0.065	0.386	-0.098	0.341	-0.118	0.324	-0.170	0.290	-0.222	0.315
	MSDE5	0.230	5.497	-0.145	1.758	-0.218	1.281	-0.198	1.028	-0.221	0.869	-0.295	0.646	-0.303	0.479

Five well-known distance functions are defined as $h(x, y) = (x - y)^2$ (square distance), $h(x, y) = (\log x - \log y)^2$ (square - log distance), $h(x, y) = |x - y|$ (absolute distance), $h(x, y) = |\log x - \log y|$ (absolute - log distance), and $h(x, y) = \exp(x - y) - (x - y) - 1$ (linex distance). Simulation results show that the MSDE performs well for estimating the parameters of presented model in this paper (see the results in Subsection 4.3).

4.3. Simulation study

In this section, we assess the behavior of the proposed estimators given in Section 4 for the parameters of EE-Ch distribution. To verify the validity of these estimators, the bias and the mean square error (MSE) of them have been checked. We consider the sample sizes $n = 10, 30, \ldots, 90$. The *nlminb* function in R software has been used to estimate the model parameters for all proposed estimators. We replicate this experiment 5000 times to compare the biases and MSEs of six approaches: MLE and five MSDE methods (denoted by MSDE1,..., MSDE5). The biases and MSEs are computed as

$$Bias(\hat{\theta}^j) = \frac{1}{r}\sum_{i=1}^{r}(\hat{\theta}_i^j - \theta^j) \quad \text{and} \quad MSE(\hat{\theta}^j) = \frac{1}{r}\sum_{i=1}^{r}(\hat{\theta}_i^j - \theta^j)^2,$$

where $r = 50000$, and $\hat{\theta}_i^j$ is the estimation of the parameter θ^j, the jth element of the vector parameter $\boldsymbol{\theta} = (\alpha, \lambda, \beta, \gamma)$ in the ith replication. The results for $\boldsymbol{\theta} = (0.5, 0.5, 4, -1.5)$ and $\boldsymbol{\theta} = (2, 0.5, 1, 0.5)$ are given in Tables 1 and 2, respectively. We can conclude that
1. The biases and MSEs of all methods decrease as the sample size increases.
2. When $\gamma < 0$, the MSDE3 and MLE methods perform better than the other methods.
3. When $\gamma > 0$, the MSDE3, MSDE4, and MLE methods perform better than the other methods.

Table 2. The biases and MSEs of the methods for $\boldsymbol{\theta} = (2, 0.5, 1, 0.5)$.

		n													
		10		30		50		70		90		200		500	
Parameters	Methods	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
α	MLE	0.625	0.327	0.562	0.755	0.527	0.889	0.531	0.903	0.464	0.861	0.421	0.688	0.071	0.561
	MSDE1	3.086	27.45	2.028	20.38	1.797	18.47	1.626	16.55	1.647	15.96	0.359	4.51	0.071	1.49
	MSDE2	3.169	28.15	2.390	22.75	2.097	19.97	1.931	18.37	1.753	16.07	0.352	2.475	0.193	1.157
	MSDE3	0.306	1.913	0.052	0.504	-0.010	0.232	0.017	0.378	-0.015	0.285	0.140	0.274	0.031	0.266
	MSDE4	0.426	2.715	0.082	0.954	0.014	0.788	-0.013	0.772	-0.018	0.607	0.063	0.582	0.012	0.532
	MSDE5	3.023	27.07	2.022	20.29	1.791	18.36	1.621	16.51	1.658	16.07	0.363	4.526	0.070	1.483
λ	MLE	0.029	0.055	0.012	0.023	0.011	0.018	0.002	0.016	0.007	0.014	0.008	0.010	0.005	0.001
	MSDE1	0.081	0.269	-0.013	0.114	-0.044	0.071	-0.044	0.054	-0.030	0.047	0.027	0.040	0.030	0.027
	MSDE2	0.114	0.262	-0.005	0.075	-0.026	0.050	-0.030	0.045	-0.022	0.037	0.052	0.031	0.032	0.026
	MSDE3	0.072	0.067	0.020	0.021	0.008	0.012	0.004	0.010	-0.004	0.007	0.003	0.006	0.000	0.001
	MSDE4	0.049	0.070	0.007	0.028	-0.007	0.018	-0.015	0.016	-0.017	0.015	0.010	0.012	0.008	0.008
	MSDE5	0.078	0.267	-0.013	0.114	-0.043	0.071	-0.045	0.053	-0.029	0.047	0.025	0.044	0.015	0.037
β	MLE	0.171	0.402	0.142	0.371	0.068	0.409	0.058	0.358	0.038	0.337	0.009	0.128	0.005	0.036
	MSDE1	0.315	1.127	0.309	1.134	0.325	1.085	0.290	0.916	0.232	0.810	0.179	0.244	0.111	0.104
	MSDE2	0.278	1.664	0.333	1.187	0.292	1.037	0.283	0.961	0.222	0.797	0.107	0.150	0.056	0.088
	MSDE3	0.001	0.096	0.045	0.095	0.041	0.085	0.053	0.100	0.053	0.101	0.032	0.036	0.026	0.026
	MSDE4	-0.034	0.288	0.106	0.270	0.125	0.265	0.157	0.268	0.144	0.265	0.064	0.061	0.052	0.048
	MSDE5	0.023	1.138	0.304	1.120	0.322	1.074	0.288	0.915	0.232	0.809	0.178	0.242	0.111	0.103
γ	MLE	0.363	0.277	0.191	0.148	0.109	0.096	0.075	0.065	0.062	0.054	0.058	0.055	0.035	0.010
	MSDE1	0.333	1.101	-0.063	0.226	-0.087	0.151	-0.089	0.124	-0.084	0.105	0.075	0.092	0.072	0.032
	MSDE2	0.037	0.428	-0.093	0.169	-0.083	0.131	-0.082	0.108	-0.070	0.088	0.056	0.083	0.050	0.076
	MSDE3	-0.034	0.080	-0.028	0.050	-0.018	0.038	-0.023	0.032	-0.017	0.027	0.014	0.051	0.010	0.008
	MSDE4	0.036	0.148	-0.045	0.069	-0.048	0.054	-0.061	0.049	-0.053	0.041	0.059	0.044	0.042	0.045
	MSDE5	0.339	1.131	-0.060	0.227	-0.085	0.151	-0.087	0.124	-0.084	0.105	0.071	0.074	0.061	0.041

5. Real data set

The following data set is given in [16] which represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England.
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89

The mean and standard deviation are 1.50 and 0.32, respectively. Since the coefficient of skewness is -0.87 which indicates that the data is left skewed. In this example we illustrate how the EE-Ch distribution can be used to model such a left skewed data. Beside the EE-Ch distribution, we also consider the following distributions as the competing models:
i. Chen (Ch) distribution [3]

$$F_{Ch}(x) = 1 - \exp\left\{-\lambda\left[\exp\left(x^{\beta}\right) - 1\right]\right\}.$$

ii. Exponentiated Chen (ECh) distribution [2]

$$F_{ECh}(x) = \left[F_{Ch}(x)\right]^{\alpha}.$$

iii. Transmuted-Chen (TCh) distribution [12]

$$F_{TCh}(x) = (1+\alpha)\left[F_{Ch}(x)\right] - \alpha\left[F_{Ch}(x)\right]^{2}.$$

iv. Beta-Chen (BCh) distribution

$$F_{BCh}(x) = F_B\left(F_{Ch}(x), \alpha, \gamma\right),$$

where $F_B(., \alpha, \gamma)$ is the cdf of beta distribution with parameters α and γ.

v. Kumaraswamy Chen (KwCh) distribution [4]

$$F_{KwCh}(x) = F_{Kw}(F_{Ch}(x), \alpha, \gamma),$$

where $F_{Kw}(., \alpha, \gamma)$ is the cdf of Kumaraswamy distribution with parameters α and γ.
vi. Gamma-Chen (GCh) distribution

$$F_{GaCh}(x) = F_{Ga}(-\log(1 - F_{Ch}(x)), \alpha, \gamma),$$

where $F_{Ga}(., \alpha, \gamma)$ is the cdf of gamma distribution with parameters α and γ.

The MLEs of parameters of the above candidate models as well as the goodness of fit statistics such as minus of log-likelihood function ($-\log(\mathrm{L})$), Anderson-Darling (AD), Cramér-von Mises (CVM), Kolmogorov-Smirnov (K-S) statistic with its p-value, Akaike information criterion (AIC), Akaike information criterion corrected (AICC), and Bayesian information criterion (BIC), are given in Table 3.

Table 3. MLEs of the model parameters and the goodness of fit statistics for data set.

	Distribution						
	Ch	TCh	ECh	EECh	BCh	KwCh	GaCh
$\hat{\alpha}$	-	-0.9427	1.9489	0.5635	0.1539	0.0904	0.1743
(S.E.)	(-)	(0.2536)	(0.6635)	(0.1467)	(0.2068)	(0.1322)	(1.8832)
$\hat{\lambda}$	0.0720	0.1678	0.1725	0.0056	1.9690	2.1187	2.0513
(S.E.)	(0.0162)	(0.0513)	(0.0760)	(0.0053)	(0.7109)	(0.9938)	(0.7900)
$\hat{\beta}$	1.9604	1.6979	1.6831	3.3212	1.1818	4.3307	1.5349
(S.E.)	(0.0939)	(0.1459)	(0.1771)	(0.3409)	(2.1489)	(15.718)	(16.6107)
$\hat{\gamma}$	-	-	-	1.6769	1.6699	1.4905	1.6084
(S.E.)	(-)	(-)	(-)	(0.8202)	(0.2278)	(0.5427)	(0.2204)
$-\log(L)$	16.4613	14.2487	14.2732	11.1176	14.2685	14.1277	14.3669
CVM	0.2335	0.1576	0.1646	0.0474	0.1647	0.1600	0.1692
AD	1.3195	0.8859	0.9324	0.3085	0.9323	0.9068	0.9519
K-S	0.1376	0.1302	0.1334	0.0808	0.1336	0.1331	0.1350
p-value (K-S)	0.1835	0.2354	0.2120	0.8049	0.2104	0.2138	0.2009
AIC	36.9226	34.4975	34.5465	30.2353	36.5370	36.2555	36.7338
AICC	37.1226	34.9043	34.9533	30.9249	37.2267	36.9451	37.4235
BIC	41.2089	40.9269	40.9759	38.8078	45.1096	44.8280	45.3064

From Table 3, based on the p-values of the K-S statistic, we can conclude that all considered distributions can be fitted to this data set. Also, the new proposed model has the smallest $-\log(\mathrm{L})$, AIC, AICC, and BIC values among other models. Also, from AD and CVM criterion, it is seen that the EE-Ch model is the best candidate for this real data set. Therefore, the EE-Ch distribution provides a better fit than the other considered models.

In Figure 2, we plot the histogram, empirical distribution and empirical hrf of the data and the estimated pdf, cdfs and hrfs of all considered models. Also, the P-P plots of all models are provided in Figure 3. Furthermore, It is clear from this figure that the EE-Ch distribution yields a better fit than the other competing models. From the above discussion, we recommend to use EE-Ch for modeling this data set. Table 4 provides the MSDEs for parameters of considered distributions.

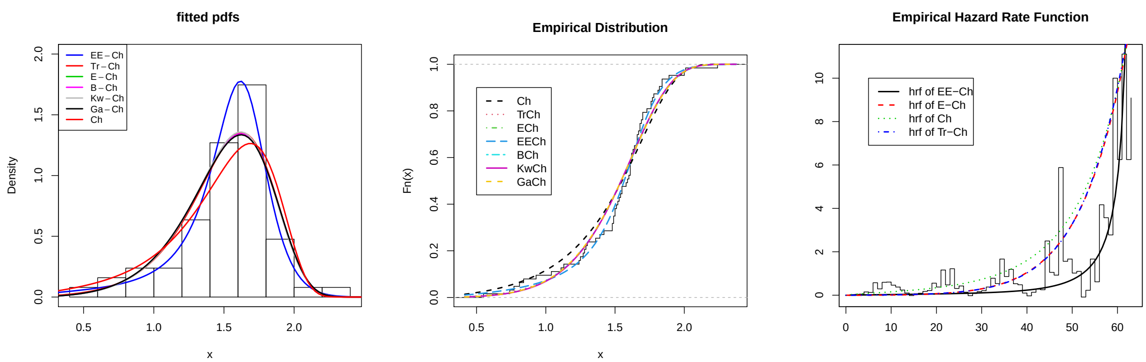

Figure 2. Fitting estimated pdfs, cdfs and hrfs of underlying models to the histogram, empirical distribution and empirical hrf of the data set.

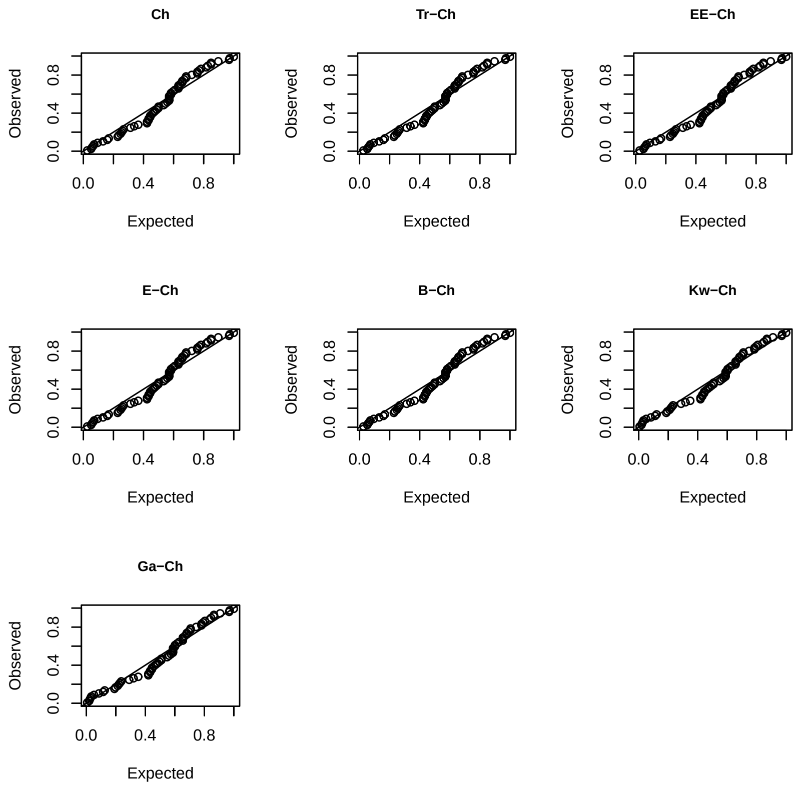

Figure 3. The P-P plots of all considered models.

6. Shifted log EE-Ch regression model

Let X be a random variable with the $\mathrm{EE}-\mathrm{Ch}\,(\alpha,\beta,\lambda,\gamma)$ distribution. Now, define $Y=\log(X)+\mu$. Then, taking $\beta=1/\sigma$, the pdf of the random variable Y is

$$f_Y\left(y\right)=\frac{\alpha\lambda}{\sigma}\left(d+1\right)\log\left(d+1\right)\,q^{\frac{1}{\gamma}-1}\left(1-q^{\frac{1}{\gamma}}\right)^{\alpha-1},$$

Table 4. The estimations of the parameters of the models based on the five MSDE approaches.

		Distribution						
Method		Ch	TCh	ECh	EECh	B-Ch	KwCh	GaCh
MSDE1	$\hat{\alpha}$	-	-0.2841	0.6432	0.3642	0.6323	0.6123	0.6320
	$\hat{\lambda}$	0.0592	0.0803	0.0219	0.0011	0.0020	0.1287	0.0177
	$\hat{\beta}$	2.1598	2.0621	2.5198	3.6154	2.5488	2.3959	2.5493
	$\hat{\gamma}$	-	-	-	2.5167	8.8309	0.2697	0.9832
MSDE2	$\hat{\alpha}$	-	-0.9909	2.0919	0.7838	2.1129	2.3473	2.2298
	$\hat{\lambda}$	0.0978	0.2416	0.2570	0.0392	0.2329	0.1220	0.2627
	$\hat{\beta}$	1.7100	1.4259	1.4029	2.6877	1.3906	1.1687	-1.3245
	$\hat{\gamma}$	-	-	-	1.3629	1.1587	6.0495	1.5785
MSDE3	$\hat{\alpha}$	-	-0.5467	1.0224	0.5880	1.0163	1.0205	0.9952
	$\hat{\lambda}$	0.0548	0.1005	0.0591	0.0083	0.0580	0.0597	0.0554
	$\hat{\beta}$	2.3034	2.0754	2.2781	3.2370	2.2849	2.2801	2.3119
	$\hat{\gamma}$	-	-	-	1.4501	0.9982	0.9838	0.9635
MSDE4	$\hat{\alpha}$	-	-0.8930	2.0672	0.6603	2.0821	2.4708	2.1432
	$\hat{\lambda}$	0.0804	0.1931	0.2407	0.0119	0.2301	0.1298	0.2486
	$\hat{\beta}$	1.8274	1.6202	1.4572	3.2464	1.4510	1.2122	1.3827
	$\hat{\gamma}$	-	-	-	1.9754	1.0732	5.5928	1.5119
MSDE5	$\hat{\alpha}$	-	-0.2828	0.6415	0.3636	0.6306	0.6077	0.6304
	$\hat{\lambda}$	0.0591	0.0800	0.0217	0.0011	0.0022	0.1420	0.0176
	$\hat{\beta}$	2.1604	2.0634	2.5211	3.6153	2.5501	2.3898	2.5507
	$\hat{\gamma}$	-	-	-	2.5154	7.8703	0.2490	1.4534

where $q = 1 - \gamma\lambda d$, $d = \exp\left[\exp\left(z\right)\right] - 1$ and $z = (y - \mu)/\sigma$. We call the corresponding distribution shifted log-EE-Ch (SLEE-Ch) distribution with parameters α, λ, γ, μ and σ, where μ is the location parameter and σ is the scale parameter. It is denoted by $\mathrm{SLEE-Ch}\left(\alpha,\gamma,\lambda,\mu,\sigma\right)$. We consider the following multiple linear regression for dependent variable y_i and independent variables $x_{i1},\ldots,x_{ip}$ as $y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i, \quad i = 1,2,\ldots,n$, where $(\beta_0,\beta_1,\ldots,\beta_p)$ is the vector of unknown parameters and ε_i has $\mathrm{SLEE{-}Ch}\left(\alpha,\gamma,\lambda,0,\sigma\right)$ distribution. Assuming non-informative censoring $\min\left\{y_i,c_i\right\}$, one might want to perform a linear regression model linking y_i and explanatory variables $x_{i1},\ldots,x_{ip}$. Let L and C be the sets of individual for which y_i is the lifetime or censoring, respectively. As we know the log-likelihood function for this case can be written as

$$\iota\left(\boldsymbol{\eta}\right) = \sum_{i\in L}\iota_i\left(\boldsymbol{\eta}\right) + \sum_{i\in C}\iota_i^{(c)}\left(\boldsymbol{\eta}\right),$$

where $\boldsymbol{\eta} = (\beta_0,\beta_1,\ldots,\beta_p,\alpha,\gamma,\lambda,\sigma)$, $\iota_i^{(c)}\left(\boldsymbol{\eta}\right) = \log\left[S\left(y_i\right)\right]$ and $S\left(y_i\right)$ is the survival function of the SLEE-Ch distribution which has the form $S\left(y\right) = 1 - \left[1 - \left(1 - \gamma\lambda d\right)^{1/\gamma}\right]^{\alpha}$. So, the log-likelihood function can be written as

$$\iota(\boldsymbol{\eta}) = n_L \log\left(\frac{\alpha\lambda}{\sigma}\right) + \sum_{i\in L}\left(\frac{y_i-\mu_i}{\sigma}\right) + \sum_{i\in L}\log(d_i+1) + \left(\frac{1}{\gamma}-1\right)\sum_{i\in L}\log(q_i)$$
$$+ (\alpha-1)\sum_{i\in L}\log\left(1-q_i^{1/\gamma}\right) + (\alpha-1)\sum_{i\in L}\log\left(1-q_i^{1/\gamma}\right) + \sum_{i\in C}\log\left[1-\left(1-q_i^{1/\gamma}\right)^{\alpha}\right],$$

where n_L is the number of uncensored observations, $q_i = 1-\gamma\lambda d_i$, $d_i = \exp\left[\exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right]-1$ and $\mu_i = \beta_0+\beta_1 x_{i1}+\cdots+\beta_p x_{ip}$. For this regression equation, the model parameters need to be estimated. We use maximum likelihood approach to estimate the model parameters by maximizing the log-likelihood function with respect to parameters $\boldsymbol{\eta}$. The optim package in R software can be used to obtain the MLEs of the parameters in the regression model.

Here, we illustrate how to fit the SLEE-Ch multiple linear regression to a real data set. Consider the data set known as myelogenous leukemia data that is presented in [7]. There are two groups of patients based on the presence or absence of a morphologic characteristic of white cells. Presence of Auer rods and/or significant granulative of the leukemic cells in the bone marrow at diagnosis indicates AG positive and absence of these factors indicates AG negative. There are 33 patients of whom 17 have AG positive and 16 have AG negative. We would like to fit the following model, $y_i = \beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}+\varepsilon_i$, where ε_i follows $\mathrm{SLEE-Ch}(\alpha,\gamma,\lambda,0,\sigma)$, and the variables are, y_i: logarithm of survival time as the response variable, x_{i1}: white blood count and x_{i2}: presence or absence of AG (0 = AG positive, 1 = AG negative). Recently, [1] fitted log-odd log-logistic exponentiated Weibull (LOLLEW) regression model to this data set based on the above multiple regression model. They showed that their model fits better than log-beta generalized half-normal geometric [15], log-beta generalized half-normal [14], and log-generalized half-normal [5] regression models. Therefore, we consider LOLLEW introduced in [1] as a competing model to compare our proposed regression model (SLEE-Ch). The selection criteria for the best model is based on the AIC, AICC, and BIC. In Table 5, the MLEs of parameters, and values of AIC, AICC, and BIC are given for the models. We can claim that our proposed model is one of the better candidate adopted to the data. Further, we use the likelihood ratio statistic for testing $H_0:\beta_l=0$ against $H_1:\beta_l\neq 0$. From the p-values of likelihood ratio test (LRT), it can be concluded that the null hypothesis H_0 is rejected for all three cases at 1% significant level. Therefore, we conclude that there is a significant difference between patients with AG positive and AG negative. This result is also stated in [1].

Table 5. The MLEs of parameters (p-value of LRT) and the AIC, CAIC and BIC statistics.

	MLE									
Model	β_0	β_1	β_2	α	γ	λ	σ	AIC	AICC	BIC
SLEE-Ch	3.984 (0.002)	-10^{-5} (0.002)	-0.826 (<0.001)	1.000	7.648	30.442	0.410	109.2	112.5	118.2
LOLLEW	3.666 (<0.001)	- 0.005 (0.001)	0.712 (<0.001)	–	0.817	0.210	0.240	115.5	118.8	123.9

7. Conclusion

In this paper, we introduce a new family of continuous distributions called the exponentiated extended Chen (EE-Ch) family. Some mathematical properties of the proposed family are obtained. We also discuss the maximum likelihood estimation of the EE-Ch parameters from one observed sample. One application to a real data set is given to illustrate empirically the flexibility of the proposed model. A new shifted log EE-Ch regression model is introduced based on the new generated lifetime distribution. Empirical findings show that the proposed models provide better fits than other competitive models.

Appendix A

Theorem 1

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ ($a = -\infty,\ b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that $\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right] \xi(x), x \in H$, is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ, particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u) q_1(u) - q_2(u)} \right| \exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi'\ q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

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