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A New Version of the Exponentiated Exponential Distribution: Copula, Properties and Application to Relief and Survival Times

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Abstract In this paper, we introduce a new generalization of the Exponentiated Exponential distribution. Various structural mathematical properties are derived. Numerical analysis for mean, variance, skewness and kurtosis and the dispersion index is performed. The new density can be right skewed and symmetric with "unimodal" and "bimodal" shapes. The new hazard function can be "constant", "decreasing", "increasing", "increasing-constant", "upsidedown-constant", "decreasing-constant". Many bivariate and multivariate type model have been also derived. We assess the performance of the maximum likelihood method graphically via the biases and mean squared errors. The usefulness and flexibility of the new distribution is illustrated by means of two real data sets.

Keywords Exponentiated Exponential; Morgenstern family; Clayton Copula; Real Data Modeling; Hazard Function.

AMS 2010 subject classifications 62N01; 62N02; 62E10

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1. Introduction and motivation

A random variable (RV) W is said to have the Exponentiated Exponential (EE) distribution if its probability density function (PDF) is given by

$$\pi_{a,b}(\mathbf{w})|_{(\mathbf{w}>0,a>0\text{ and }b>0)} = ab\,\mathrm{e}^{-b\mathbf{w}}\,\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a-1}. \tag{1}$$

The corresponding cumulative distribution function (CDF) can be written as

$$\mathbb{W}_{a,b}(\mathbf{w})|_{(\mathbf{w}>0,a>0\text{ and }b>0)} = \left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a}. \tag{2}$$

Clearly, for $a=1$, the EE reduces to the standard exponential (E) model. If $1>a$, the function $\pi_{a,b}(\mathbf{w})$ monotonically decreases with $\mathbf{w}$. If $a>1$, the function $\pi_{a,b}(\mathbf{w})$ attains a mode at $\mathbf{w}=\frac{1}{b}\log(a)$. The statistical properties of the EE model have been studied by many authors. Many authors have derived and studied the EE model, see Zheng [66], Zheng and Park [67], Kundu and Pradhan [48], Aslam et al. [15], Aryal et al. [13], Khalil et al. [42], Abouelmagd et al. ([1],[2]), Ibrahim et al. [35] and Bhatti et al. [16] among others. Recently, Alizadeh et al. [6] defined a new family based on the exponential model called the generalized odd generalized exponential family of distributions. Analogously, Hamedani et al. ([33] and [34]) defined the the type I and type I general exponential class of distributions. Other works can be cited such as Korkmaz et al. [46] (exponential Lindley odd log-logistic family) and Yadav et al. [62] (Burr-Hatke exponential model). In the work, we introduce a new version of the EE model using the Odd-Burr generalized (OB-G) family called the OBEE (OBEE). On the other hand, some new bivariate type OBEE are derived. Due to Alizadeh et al. [6], the CDF of the the OB-G family is given by

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$$F_{\alpha,\beta,\underline{\mathbf{\Phi}}}(\mathbf{w}) = 1 - \frac{\overline{\mathbb{W}}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha\beta}}{\left[\mathbb{W}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha} + \overline{\mathbb{W}}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha}\right]^{\beta}}, \tag{3}$$

where $\overline{\mathbb{W}}_{\underline{\mathbf{\Phi}}}(\mathbf{w}) = 1 - \mathbb{W}_{\underline{\mathbf{\Phi}}}(\mathbf{w})$ and $\underline{\mathbf{\Phi}}$ refers to the parameter vector of the base line model. The PDF corresponding to (3) is given by

$$f_{\alpha,\beta,\underline{\mathbf{\Phi}}}(\mathbf{w}) = \frac{\alpha\beta\pi_{\underline{\mathbf{\Phi}}}(\mathbf{w})\,\mathbb{W}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha-1}\,\overline{\mathbb{W}}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha\beta-1}}{\left[\mathbb{W}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha} + \overline{\mathbb{W}}_{\underline{\mathbf{\Phi}}}(\mathbf{w})^{\alpha}\right]^{1+\beta}}, \tag{4}$$

where $\pi_{\underline{\mathbf{\Phi}}}(\mathbf{w}) = d\mathbb{W}_{\underline{\mathbf{\Phi}}}(\mathbf{w})/dx$. For $\beta = 1$, we get the Odd G (O-G) family. For $\alpha = 1$, we have the proportional reversed hazard rate family (PRHR). The OBEE CDF is given by

$$F_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) = 1 - \frac{\left\{1 - \left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a}\right\}^{\alpha\beta}}{\left(\left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a\alpha} + \left\{1 - \left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a}\right\}^{\alpha}\right)^{\beta}}, \tag{5}$$

where $\underline{\mathbf{\Lambda}}$ refers to the parameter vector of the new OBEE model. For $\beta = 1$, the OBEE reduces to the OEE. For $\alpha = 1$, the OBEE reduces to the PRHREE. The PDF corresponding to (5) is given by

$$f_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) = \alpha\beta ab\,\mathrm{e}^{-b\mathbf{w}}\frac{\left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{\dot{a}}\left\{1 - \left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a}\right\}^{\alpha\beta-1}}{\left(\left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a\alpha} + \left\{1 - \left[1 - \mathrm{e}^{-b\mathbf{w}}\right]^{a}\right\}^{\alpha}\right)^{1+\beta}}, \tag{6}$$

where $\dot{a} = a\alpha - 1$. The hazard function (HRF) can be derived from $f_{\underline{\mathbf{\Lambda}}}(\mathbf{w})/S_{\underline{\mathbf{\Lambda}}}(\mathbf{w})$. For simulation of this new model, we obtain the quantile function (QF) of $\mathbf{w}$ (by inverting (5)), say $\mathbf{w}_u = F^{-1}(u)$, as

$$\mathbf{w}_u = -b^{-1}\ln\left(1 - \left\{\frac{\left[1 - (1-u)^{\beta^{-1}}\right]^{\alpha^{-1}}}{(1-u)^{\frac{1}{\alpha\beta}} + \left[1 - (1-u)^{\beta^{-1}}\right]^{\alpha^{-1}}}\right\}^{a^{-1}}\right). \tag{7}$$

Equation (7) is used for simulating the OBEE model. Figure 1 gives some PDF plots for some selected parameters value. Figure 2 gives some HRF plots for some selected parameters value. Based on Figure 1 the OBEE density can be right skewed and symmetric with unimodal and bimodal PDFs. Based on Figure 2 the OBEE HRF can be”constant”, ”decreasing”, ”increasing”, ”increasing-constant”, ”upside-downconstant” and ”decreasing-constant”.

2. Mathematical properties

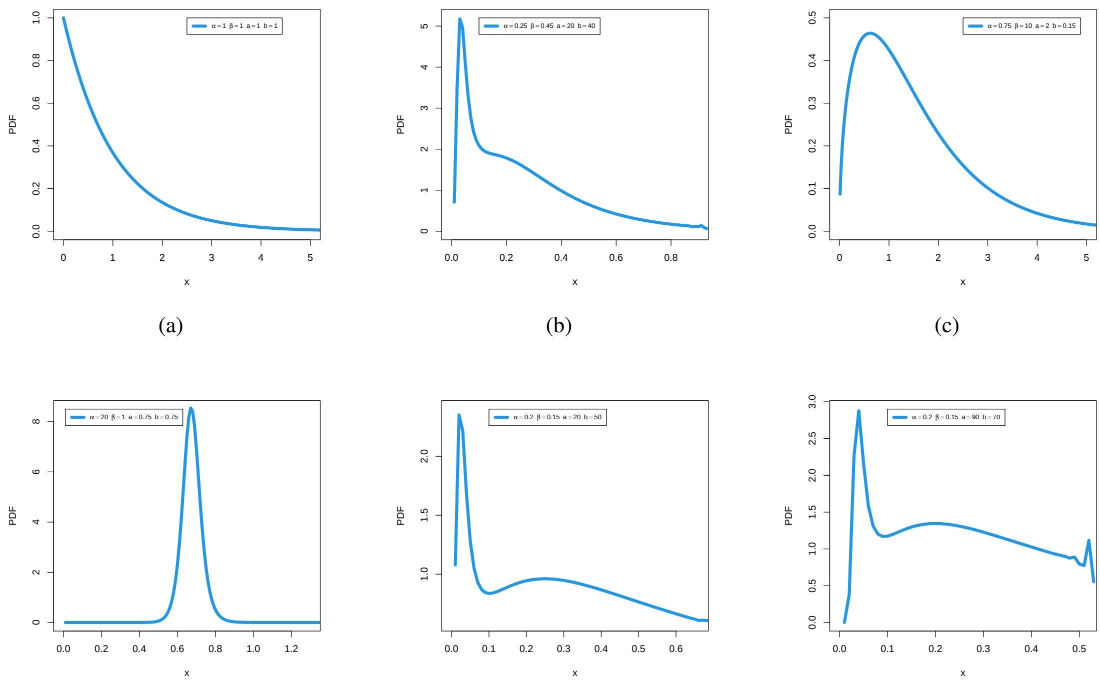
2.1. Useful representations

Due to Alizadeh et al. (2016), the PDF in (6) can be expressed as

$$f(\mathbf{w}) = \sum_{\kappa=0}^{\infty} \nabla_{\kappa}\,\pi_{a^*,b}(\mathbf{w}), \tag{8}$$

where $a^* = a(1+\kappa)$ and

$$\begin{aligned}\nabla_{\kappa} \quad = \quad & \frac{\alpha\beta}{1+\kappa}\sum_{i_1,i_2=0}^{\infty}\sum_{i_3=\kappa}^{\infty}(-1)^{i_2+k+\kappa}\binom{-(1+\beta)}{i_1} \\ & \times\binom{-[\alpha(1+i_1)+1]}{i_2}\binom{\alpha(1+i_1)+i_2+1}{i_3}\binom{i_3}{\kappa}\end{aligned}$$



(a) (b) (c)

(d) (e) (f)

Figure 1. PDF plots for some selected parameters value.

and $\pi_{a^*,b}(\mathbf{w})$ refers to the density of the exponentiated exponential (EE) model with power parameter a^*. By integrating (8), the CDF of W becomes

$$F(\mathbf{w}) = \sum_{\kappa=0}^{\infty} \nabla_{\kappa} \, \Pi_{a^*,b}(\mathbf{w}), \tag{9}$$

where $\Pi_{a^*,b}(\mathbf{w})$ refers to the EE distribution with power parameter a^*.

2.2. Asymptotics

Let $c = \inf\left\{F(\mathbf{w})|_{\mathbb{W}_{a,b}(\mathbf{w})>0}\right\}$, then

$$\begin{aligned} F_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) &\sim \beta\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a\alpha} \text{ as } \mathbf{w} \to 0, \\ f_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) &\sim \alpha\beta ab\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{\dot{a}} \mathrm{e}^{-b\mathbf{w}} \text{ as } \mathbf{w} \to 0, \end{aligned}$$

and

$$h_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) \sim \alpha\beta ab\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{\dot{a}} \mathrm{e}^{-b\mathbf{w}} \text{ as } \mathbf{w} \to 0.$$

The asymptotics of CDF, PDF and HRF as $\mathbf{w} \to \infty$ are given by

$$1 - F_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) \sim \left\{\alpha\left[1-\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a}\right]\right\}^{\beta} \text{ as } \mathbf{w} \to \infty,$$

$$f_{\underline{\mathbf{\Lambda}}}(\mathbf{w}) \sim \frac{\alpha^{\beta}\beta ab\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a-1}\mathrm{e}^{-b\mathbf{w}}}{\left[1-\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a}\right]^{1-\beta}} \text{ as } \mathbf{w} \to \infty,$$

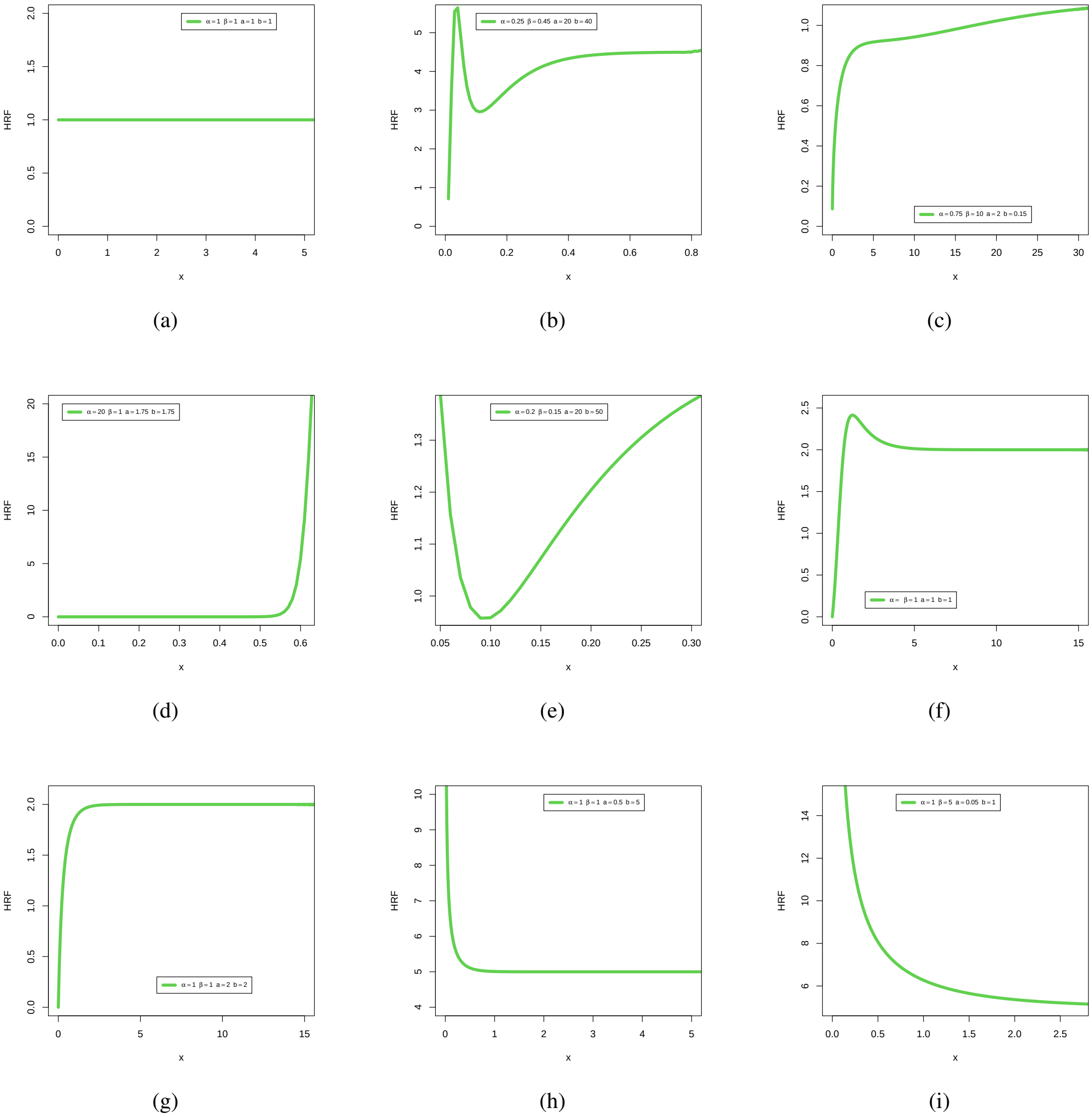


Figure 2. HDF plots for some selected parameters value.

and

$$h_{\underline{\mathbf{\Lambda}}}\left(\mathbf{w}\right) \sim \frac{\beta ab\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a-1}\mathrm{e}^{-b\mathbf{w}}}{1-\left(1-\mathrm{e}^{-b\mathbf{w}}\right)^{a}} \text{ as } \mathbf{w} \to \infty.$$

2.3. Moments and incomplete moments

The ς^{th} ordinary moment of W is given by

$$\mu'_{\varsigma} = \mathbf{E}(W^{\varsigma}) = \int_{-\infty}^{\infty} \mathbf{w}^{\varsigma}\, f\left(\mathbf{w}\right) d\mathbf{w},$$

then we obtain

$$\mu'_{\varsigma}|_{(\varsigma>-\mathbf{1})} = b^{-\varsigma}\Gamma\left(1+\varsigma\right)\sum_{\kappa,h=0}^{\infty}\nabla_{\kappa,h}^{(a^*,\varsigma)}, \tag{10}$$

where

$$\nabla_{\kappa,h}^{(a^*,\varsigma)} = \nabla_{\kappa}\,\frac{a^*\left(-1\right)^h}{\left(h+1\right)^{\varsigma+1}}\binom{a^*-1}{h}$$

and

$$\Gamma(1+\zeta)|_{(\zeta\in\mathbf{R}^+)} = \zeta! =_{\varsigma=0}^{\zeta-1}\left(\zeta-\varsigma\right).$$

where $\mathbf{E}(\mathbf{w}) = \mu'_1$ is the mean of $\mathbf{w}$. The ς^{th} incomplete moment, say $\varphi_{\varsigma}\left(t\right)$, of $\mathbf{w}$ can be expressed, from (9), as

$$\varphi_{\varsigma}(t) = \int_{-\infty}^{t}\mathbf{w}^{\varsigma}f(\mathbf{w})d\mathbf{w} = \sum_{\kappa=0}^{\infty}\nabla_{\kappa}\int_{-\infty}^{t}\mathbf{w}^{\varsigma}\pi_{a^*,b}(\mathbf{w})d\mathbf{w}$$

then

$$\varphi_{\varsigma}\left(t\right)|_{(\varsigma>-\delta)} = b^{-\varsigma}\gamma\left(\varsigma+1,bt\right)\sum_{\kappa,h=0}^{\infty}\nabla_{\kappa,h}^{(a^*,\varsigma)}, \tag{11}$$

where $\gamma\left(\zeta,\vartheta\right)$ is the incomplete gamma function.

$$\begin{aligned}
\gamma\left(\zeta,\vartheta\right)|_{(\zeta\neq0,-1,-2,\ldots)} &= \int_{0}^{\vartheta}\exp\left(-\mathbf{w}\right)d\mathbf{w}\\
&= \frac{1}{\zeta}\vartheta^{\zeta}\left\{{}_1\mathbf{F}_1\left[\zeta;\zeta+1;-\vartheta\right]\right\}\\
&= \sum_{\kappa=0}^{\infty}\frac{\left(-1\right)^{\kappa}}{\kappa!\left(\zeta+\kappa\right)}\vartheta^{\zeta+\kappa},
\end{aligned}$$

and ${}_1\mathbf{F}_1\left[\cdot,\cdot,\cdot\right]$ is a confluent hypergeometric function. The first incomplete moment given by (11) with $\varsigma=1$ as

$$\varphi_1\left(t\right) = b\gamma\left(2,\frac{1}{t}\right)\sum_{\kappa,h=0}^{\infty}\nabla_{\kappa,h}^{(1,a^*)}.$$

2.4. Moment generating function (MGF)

The MGF $M_W\left(t\right) = \mathbf{E}\left(\exp\left(t\,W\right)\right)$ of W can be derived from equation (8) as

$$M_W\left(t\right) = \sum_{\kappa=0}^{\infty}\nabla_{\kappa}\,M_{a^*,b}\left(T\right),$$

where $M_{a^*,b}\left(T\right)$ is the MGF of the EW model with power parameter a^*.

$$M_W\left(t\right)|_{(\varsigma>-\mathbf{1})} = \sum_{\varsigma=0}^{\infty}\sum_{\kappa,h=0}^{\infty}\frac{t^{\varsigma}}{\varsigma!}b^{-\varsigma}\Gamma\left(1+\varsigma\right)\nabla_{\kappa,h}^{(a^*,\varsigma)}.$$

2.5. Residual life and reversed residual life functions

The ρ^{th} moment of the residual life

$$a_\rho(t) = \mathbf{E}\left[(Z-t)^\rho \mid_{\mathbf{w}>t,\ \rho=1,2,\ldots}\right].$$

The ρ^{th} moment of the residual life of W is given by

$$a_\rho(t) = \frac{1}{1-F_{\underline{\mathbf{\Lambda}}}(t)} \int_t^\infty (\mathbf{w}-t)^\rho dF(\mathbf{w}).$$

Therefore,

$$a_\rho(t) = \frac{1}{1-F_{\underline{\mathbf{\Lambda}}}(t)} \sum_{\kappa,h=0}^{\infty} \mathbf{c}_{\kappa,h}^{(a^*,\rho)} b^\rho \Gamma\left(\rho+1, bt\right)\mid_{(\rho>-1)},$$

where

$$\mathbf{c}_{\kappa,h}^{(a^*,\rho)} = \nabla_\kappa \sum_{r=0}^{\rho} \binom{\rho}{r} (-t)^{\rho-\varsigma},$$

$$\Gamma\left(\zeta,\varsigma\right)\mid_{\varsigma>0} = \int_\varsigma^\infty \mathbf{w}^{\zeta-1} \exp\left(-\mathbf{w}\right) d\mathbf{w}$$

and

$$\Gamma\left(\zeta,\varsigma\right) = \Gamma\left(\zeta\right) - \gamma\left(\zeta,\varsigma\right).$$

The ρ^{th} moment of the reversed residual life, say

$$A_\rho(t) = \mathbf{E}\left[(t-Z)^\rho \mid_{\mathbf{w}\leq t,\ t>0 \text{ and } \rho=1,2,\ldots}\right]$$

uniquely determines $F_{\underline{\mathbf{\Lambda}}}(\mathbf{w})$. We obtain

$$A_\rho(t) = \frac{1}{F_{\underline{\mathbf{\Lambda}}}(t)} \int_0^t (t-\mathbf{w})^\rho dF(\mathbf{w}).$$

Then, the ρ^{th} moment of the reversed residual life of W becomes

$$A_\rho(t) = \frac{1}{F_{\underline{\mathbf{\Lambda}}}(t)} \sum_{\kappa,h=0}^{\infty} \mathbf{C}_{\kappa,h}^{(a^*,\rho)} b^\rho \gamma\left(\rho+1, bt\right)\mid_{(\varsigma>-\mathbf{1})},$$

where

$$\mathbf{C}_{\kappa,h}^{(a^*,\rho)} = \nabla_\kappa \sum_{r=0}^{\rho} (-1)^{\mathbf{r}} \binom{\rho}{r} t^{\rho-\mathbf{r}}.$$

2.6. Numerical analysis

Table 1 gives Numericals results for the variance (V(Z)), mean ($\mathbf{E}\left(Z\right)$), kurtosis (K(Z)), skewness (S(Z)) and dispersion index (DisIx(Z)). Based on Table 1, we note that: 1-The skewness of the OBEE distribution can range in the interval (-2.7792, 8.2978). 2-The spread for the OBEE kurtosis is much larger ranging from -46.275 to 35.526. 3-DisIx(Z) can be "between 0 and 1" or "equal 1" or more than 1.

Table 1: Mean, variance, skewness, kurtosis and dispersion index.

α	β	a	b	$\mathbf{E}(Z)$	$\mathrm{V}(Z)$	$\mathrm{S}(Z)$	$\mathrm{K}(Z)$	$\mathrm{DisIx}(Z)$
0.5	2	1.5	1.5	0.502266	0.4852427	2.478655	11.13074	0.9661068
1				0.4849408	0.1495599	1.598563	6.885345	0.3084085
5				0.5970219	0.0104331	-0.0226027	3.506458	0.0174753
20				0.6444730	0.0007732	-0.4386726	4.015127	0.0011998
50				0.6552973	0.0001281	-0.5222116	4.205749	0.0001955
100				0.6590055	3.2397×10^{-5}	-0.5496712	4.250314	4.9161×10^{-5}
200				0.6608784	8.1462×10^{-6}	-0.5636244	4.324122	1.2326×10^{-5}
5	0.5	0.25	0.25	0.586532	0.40893890	4.073737	32.78264	0.6972151
	1			0.3147107	0.05833588	2.792337	20.03788	0.1853635
	10			0.08191879	0.00203066	0.665321	3.385543	0.0247887
	50			0.03443559	0.00039684	0.678116	3.479092	0.0115240
	100			0.02328808	0.00019240	0.515537	3.49165	0.0082615
	200			0.01554562	9.0857×10^{-5}	1.290022	3.473409	0.0058446
2	5	0.5	5	0.019935	0.000259295	1.4310290	6.028403	0.01300695
		1		0.068523	0.001137761	0.6051527	3.431216	0.01660408
		20		0.553863	0.004671701	-0.1685215	3.095139	0.00843472
		50		0.733010	0.004878618	-0.1930617	3.120336	0.00665559
		100		0.870260	0.004949493	-0.2013219	3.130186	0.00568737
		200		1.008199	0.004985267	-0.2054172	3.134844	0.00494473
		500		1.191042	0.005006845	-0.2078738	3.13767	0.00420375
		1000		1.329533	0.005013977	-0.2083907	3.132891	0.00377123
		2000		1.468093	0.005017667	-0.2091019	3.139091	0.00341781
1.5	1.5	1.5	0.5	1.7901830	1.055172	1.228803	5.744852	0.5894217
			1	0.8950913	0.263793	1.228803	5.744886	0.2947109
			5	0.1790183	0.010552	1.228803	5.744886	0.0589422
			10	0.0895091	0.002638	1.228803	5.744886	0.0294711
			50	0.0179018	0.0001055	1.113265	6.978685	0.0058942
			100	0.0089509	2.638×10^{-5}	2.447838	-2.87494	0.0029472
			150	0.0059673	7.114×10^{-6}	9.999243	-7.45278	0.0011922
1	1	1	1	1	1	2	9	1
2	2	2	100	0.0100978	1.3090×10^{-5}	22.320	-168.8085	0.00129637
2	2	2	500	0.0020196	1.0182×10^{-5}	-1.2126	1.136435	0.00504159
1	1	200	500	0.0117561	6.5598×10^{-6}	128.59	-1726.776	0.00055799

3. Copula under the OBEE model

In this section, we derive some new bivariate type OBEE (BvOBEE) model using FGM-copula, Clayton copula, modified FGM-copula and Renyi's entropy. The Multivariate OBEE (MvOBEE) type is also presented. Recently, many authors used and applied many different copulas in distribution theory such as Mansour et al. ([49], [50], [51], [52], [53],[54]), Elgohari and Yousof ([20], [21]), Salah et al. [?], Al-Babtain [3], Yousof et al. [64], Ibrahim et al. [39], Ali et al. ([4], [5]) and Yousof et al. [60].

3.1. FGM copula

First, we start with the joint CDF for Morgenstern family (Morgenstern (1956)) of two RVs (W_1, W_2) which has the following form $C_\lambda(\varsigma,\omega) = (1+\lambda\overline{\varsigma}\overline{\omega})\,\varsigma\omega$ where $\lambda \in \mathbf{I}_{(-1,1)}$ and $\varsigma,\omega \in (0,1)$. Setting $\overline{\varsigma} = 1-\varsigma$ and $\overline{\omega} = 1-\omega$. Then,

$$\overline{\varsigma}|_{\left(\overline{\wp}_{(\mathbf{w}_1)}=1-e^{-b\mathbf{w}_1}\right)} = \frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1}\right)^{\beta_1}},$$

where $a_i = a|_{i=1,2}$, $b_i = b|_{i=1,2}$. and

$$\overline{\omega}|_{\left(\overline{\wp}_{(\mathbf{w}_2)}=1-e^{-b\mathbf{w}_2}\right)} = \frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2\beta_2}}{\left(\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a\alpha_2}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2}\right)^{\beta_2}},$$

where $a_i = a|_{i=1,2}$, $b_i = b|_{i=1,2}$. Then, we have

$$\begin{aligned} C_\lambda(\mathbf{w}_1,\mathbf{w}_2) &= \left(1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1}\right)^{\beta_1}}\right) \\ &\times\left(1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2\beta_2}}{\left(\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a\alpha_2}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2}\right)^{\beta_2}}\right) \\ &\times\left(1+\lambda\left\{\begin{array}{c} \frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^a\right\}^{\alpha_1}\right)^{\beta_1}} \\ \times\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2\beta_2}}{\left(\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a\alpha_2}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^a\right\}^{\alpha_2}\right)^{\beta_2}} \end{array}\right\}\right). \end{aligned}$$

3.2. Modified FGM copula

Consider the following modified version of the bivariate FGM copula defined as (see Farlie [26], Gumbel [31], Gumbel [32] and Morgenstern [57])

$$C_{\mathbf{\Delta}}(\varsigma,\omega)|_{\mathbf{\Delta}\in(-1,1)} = \varsigma\omega\left[1+\mathbf{\Delta}\vartheta(\varsigma)\mho(\omega)\right] = \varsigma\omega+\mathbf{\Delta}\dot{\vartheta}(\varsigma)\dot{\mho}(\omega),$$

where $\dot{\vartheta}(\varsigma) = \varsigma\vartheta(\varsigma)$, and $\dot{\mho}(\omega) = \omega\mho(\omega)$. Where $\vartheta(\varsigma)$ and $\mho(\omega)$ are on $\mathbf{I}_{(0,1)}$ where $\vartheta(0) = \vartheta(1) = \mho(0) = \mho(1) = 0$. Let

$$\alpha = \inf\left\{\frac{\partial}{\partial\varsigma}\dot{\vartheta}(\varsigma) : d_1(\varsigma)\right\} < 0, \xi = \inf\left\{\frac{\partial}{\partial\omega}\dot{\mho}(\omega) : d_2(\omega)\right\} > 0,$$

$$\beta = \sup\left\{\frac{\partial}{\partial\varsigma}\dot{\vartheta}(\varsigma) : d_1(\varsigma)\right\} < 0, \eta = \sup\left\{\frac{\partial}{\partial\omega}\dot{\mho}(\omega) : d_2(\omega)\right\} > 0,$$

and $\min(\alpha\beta,\xi\eta) \geq 1$ where

$$\frac{\partial}{\partial\varsigma}\dot{\vartheta}(\varsigma) = \vartheta(\varsigma)+\varsigma\frac{\partial}{\partial\varsigma}\vartheta(\varsigma),$$

$$d_1(\varsigma) = \left\{\varsigma : \varsigma\in\mathbf{I}_{(0,1)}|\frac{\partial}{\partial\varsigma}\dot{\vartheta}(\varsigma)\quad\text{exists}\right\},$$

and

$$d_2(\omega) = \left\{\omega : \omega\in\mathbf{I}_{(0,1)}|\frac{\partial}{\partial\omega}\dot{\mho}(\omega)\quad\text{exists}\right\}.$$

*3.2.1. Bivariate OBEE-**FGM (Type-I) model*** The bivariate OBEE-FGM (Type-**I**) model

$$C_{\mathbf{\Delta}}(\varsigma,\omega) = \mathbf{\Delta}\left[\dot{\vartheta}(\varsigma)\,\dot{\mho}(\omega)\right] + \left\{\begin{array}{c}\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\varsigma)}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\varsigma)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\varsigma)}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}}\right] \\ \times\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\omega)}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\omega)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\omega)}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}}\right]\end{array}\right\},$$

where

$$\dot{\vartheta}(\varsigma) = \varsigma\frac{\left\{1-\left[\overline{\wp}_{(\varsigma)}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\varsigma)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\varsigma)}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}},$$

and

$$\dot{\mho}(\omega) = \omega\frac{\left\{1-\left[\overline{\wp}_{(\omega)}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\omega)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\omega)}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}}.$$

*3.2.2. BivariateOBEE-**FGM (Type II) model*** Consider $\vartheta(\varsigma)$ and $\mho(\omega)$ where

$$\vartheta^{*}(\varsigma)\,|_{(\mathbf{\Delta}_1>0)} = \varsigma^{\mathbf{\Delta}_1}(1-\varsigma)^{1-\mathbf{\Delta}_1} \text{ and } \mho^{*}(\omega)\,|_{(\mathbf{\Delta}_2>0)} = \omega^{\mathbf{\Delta}_2}(1-\omega)^{1-\mathbf{\Delta}_2}.$$

The bivariate OBEE-FGM **(**Type-**II)** copula can be derived from

$$C_{\mathbf{\Delta},\mathbf{\Delta}_1,\mathbf{\Delta}_2}(\varsigma,\omega) = \varsigma\omega\left[1+\mathbf{\Delta}\vartheta^{*}(\varsigma)\,\mho^{*}(\omega)\right].$$

3.3. The bivariate OBEE via Renyi's entropy

Following Pougaza and Djafari [58], the joint CDF of the bivariate OBEE via Renyi's entropy can be written as

$$C(\varsigma,\omega) = \mathbf{w}_2\varsigma + \mathbf{w}_1\omega - \mathbf{w}_1\mathbf{w}_2,$$

then, the associated bivariate OBEE will be

$$C(\mathbf{w}_1,\mathbf{w}_2) = \mathbf{w}_2\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_1)}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}}\right] + \mathbf{w}_1\left[\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a}\right\}^{\alpha_2\beta_2}}{\left(\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a\alpha_2}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_2)}\right]^{a}\right\}^{\alpha_2}\right)^{\beta_2}}\right] - \mathbf{w}_1\mathbf{w}_2,$$

where $a_1 = a_2 = a$, $b_1 = b_2 = b$. Then, we get the BOBEE type distribution via Renyi's entropy.

*3.3.1. **The bivariate OBEE extension Via** Clayton copula* The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

$$C(\varsigma,\omega)|_{[\eta\geq 0]} = \left[\varsigma^{-\eta}+\omega^{-\eta}-1\right]^{-\frac{1}{\eta}}.$$

Next, setting $\varsigma = 1-\overline{\varsigma} = \varsigma(x) \in \mathbf{I}_{(0,1)}$ and $\omega = 1-\overline{\omega} = \omega(y) \in \mathbf{I}_{(0,1)}$. Then, the associated CDF bivariate OBEE type distribution will be

$$C(\mathbf{w},y)=\left\{\begin{array}{c}\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w})}\right]^{a}\right\}^{\alpha_1\beta_1}}{\left(\left[\overline{\wp}_{(\mathbf{w})}\right]^{a\alpha_1}+\left\{1-\left[\overline{\wp}_{(\mathbf{w})}\right]^{a}\right\}^{\alpha_1}\right)^{\beta_1}}\right]^{-\eta}\\+\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{y})}\right]^{a}\right\}^{\alpha_2\beta_2}}{\left(\left[\overline{\wp}_{(\mathbf{y})}\right]^{a\alpha_2}+\left\{1-\left[\overline{\wp}_{(\mathbf{y})}\right]^{a}\right\}^{\alpha_2}\right)^{\beta_2}}\right]^{-\eta}-1\end{array}\right\}^{-\frac{1}{\eta}}.$$

3.3.2. ***The Multivariate OBEE extension*** A straightforward d-dimensional extension from the above will be

$$C(\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_d)=\left\{\begin{array}{c}\sum_{i=1}^{d}\left[1-\frac{\left\{1-\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a}\right\}^{\alpha_i\beta_i}}{\left(\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a\alpha_i}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a}\right\}^{\alpha_i}\right)^{\beta_i}}\right]^{-\eta}\\+1-d\end{array}\right\}^{-\frac{1}{\eta}}.$$

4. Maximum likelihood method

For getting the maximum likelihood estimates (MLE) of the vector $\underline{\mathbf{\Lambda}}$, we have the log-likelihood ($\ell(\underline{\mathbf{\Lambda}})$) function

$$\begin{aligned}\ell(\underline{\mathbf{\Lambda}}) \quad &= n\log(\alpha\beta ab)-b_{i=1}^{n}\mathbf{w}_i+(\dot{a})_{i=1}^{n}\log\left[\overline{\wp}_{(\mathbf{w}_i)}\right]\\&+(\alpha\beta-1)_{i=1}^{n}\log\left\{1-\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a}\right\}\\&-(1+\beta)_{i=1}^{n}\log\left(\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a\alpha}+\left\{1-\left[\overline{\wp}_{(\mathbf{w}_i)}\right]^{a}\right\}^{\alpha}\right)\end{aligned}$$

The components of the score vector

$$\partial\ell/\partial\underline{\mathbf{\Lambda}}=\mathbf{U}\left(\underline{\mathbf{\Lambda}}\right)=\left(\partial\ell/\partial\alpha,\partial\ell/\partial\beta,\partial\ell/\partial a,\partial\ell/\partial b\right)$$

are available if needed. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the OBEE distribution.

5. Simulations

In statistics, simulation is usually used for assessing the performance of a method, typically when there is a lack of theoretical background. In this section, we assess the performance of the maximum likelihood (ML) method. The assessment can be performed numerically or graphically. Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the ML estimators (MLEs) via the biases and mean squared errors (MSEs). The following algorithm is considered for the assessment:

1. Using the inversion method, we we generate N=1000 samples of size n from the OBEE distribution using (7).
2. Compute the MLEs for the 1000 samples, say

$$\left(\widehat{\alpha_\hbar},\widehat{\beta_\hbar},\widehat{a_\hbar},\widehat{b_\hbar}\right)|_{(\hbar=1,2,\ldots,2000)},$$

3. Compute the standard errors (StErs) of the MLEs for the 1000 samples, say

$$\left(S_{\widehat{\alpha_\hbar}},S_{\widehat{\beta_\hbar}},S_{\widehat{a_\hbar}},S_{\widehat{b_\hbar}}\right)|_{(\hbar=1,2,\ldots,2000)}.$$

   The StErs were computed by inverting the observed information matrix.

4. Compute the biases and MSEs given for $\underline{\mathbf{V}} = (\alpha, \beta, a, b)$.
5. Repeated these steps for $n = 50, 100, \ldots, 300$ with $\alpha = 1, 2, .., 100, \beta = 1, 2, .., 100, a = 1, 2, .., 100$ and $b = 1, 2, .., 100$, so computing biases$\left(\text{Bias}_{\underline{\mathbf{V}}}(n)\right)$, $MSEs$ $\left(\text{MSE}_{\underline{\mathbf{V}}}(n)\right)$ for α, β, a, b $\forall$ $n = 50, 100, \ldots, 300$ where

$$\text{Bias}_{\underline{\mathbf{\Lambda}}}(n)|_{(\underline{\mathbf{\Lambda}}=\nu,\theta,c_2,c_1)} = \frac{1}{2000}\sum_{\hbar=1}^{2000}\left(\widehat{\underline{\mathbf{\Lambda}}}_{\hbar} - \underline{\mathbf{V}}\right),$$

and

$$\text{MSE}_{\underline{\mathbf{\Lambda}}}(n)|_{(\mathbf{\Lambda}=\nu,\theta,c_2,c_1)} = \frac{1}{2000}\sum_{\hbar=1}^{2000}\left(\widehat{\underline{\mathbf{\Lambda}}}_{\hbar} - \underline{\mathbf{V}}\right)^2$$

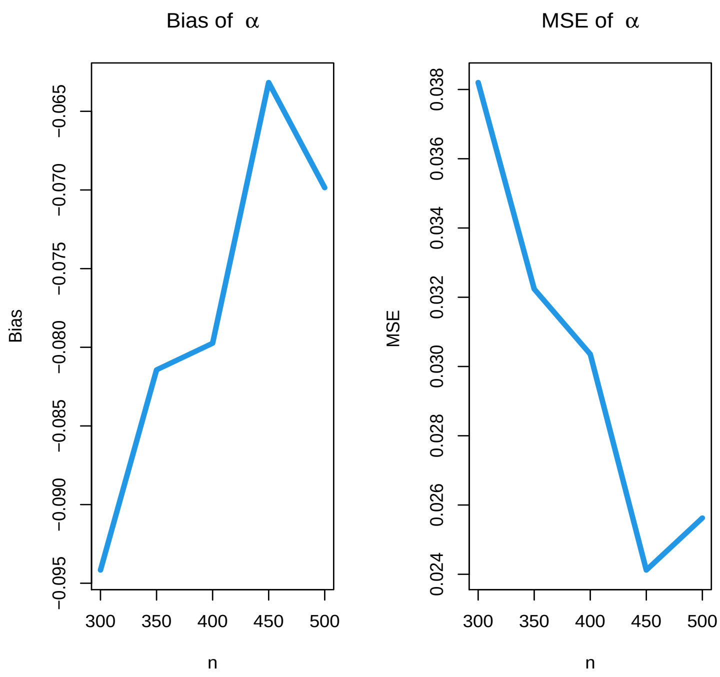


Figure 3. Biases (left) and MSEs (right) for the parameter .

Figures 3, 4, 5 and 6 gives the biases (left) and MSEs (right) for the parameters α, β, a and b respectively. These figures (lefts) shows how the four biases vary with respect to n and also shows how the four MSEs vary with respect to n. From Figures 3, 4, 5 and 6, the biases for each parameter are generally negative and getting close to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$.

6. Real data applications

We shall compare thefits of the OBEE distribution with those of other competitive models, namely: Exponential (E), Odd Lindley Exponential (OLE), Marshall-Olkin Exponential (MOE), Moment Exponential (ME), The Logarithmic Burr-Hatke Exponential (LBHE), Generalized Marshall-Olkin Exponential (GMOE), Beta Exponential (BE), Marshall-Olkin Kumaraswamy Exponential (MOKwE), Kumaraswamy Exponential (KwE), the Burr X Exponential (BrXE) and Kumaraswamy Marshall-Olkin Exponential (KwMOE). Some other competitive model are can be derived based on Aryal, G. and Yousof

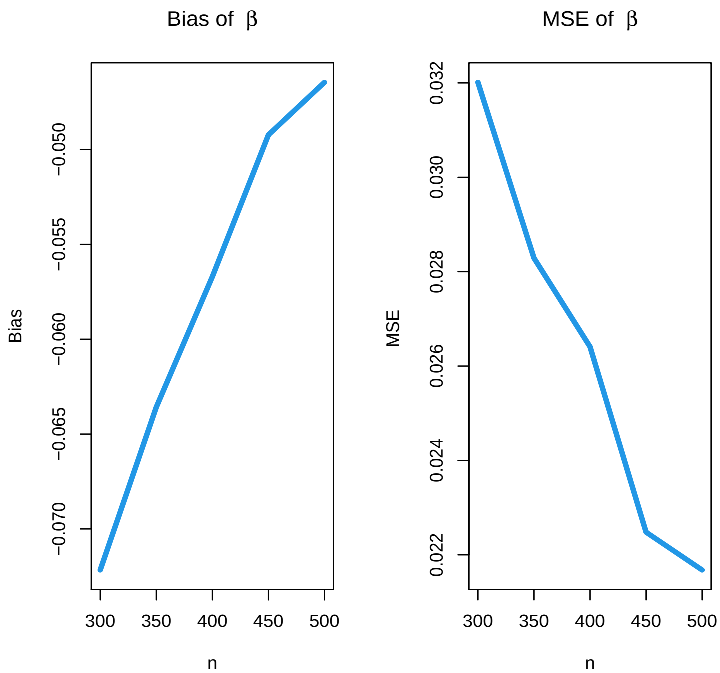


Figure 4. Biases (left) and MSEs (right) for the parameter .

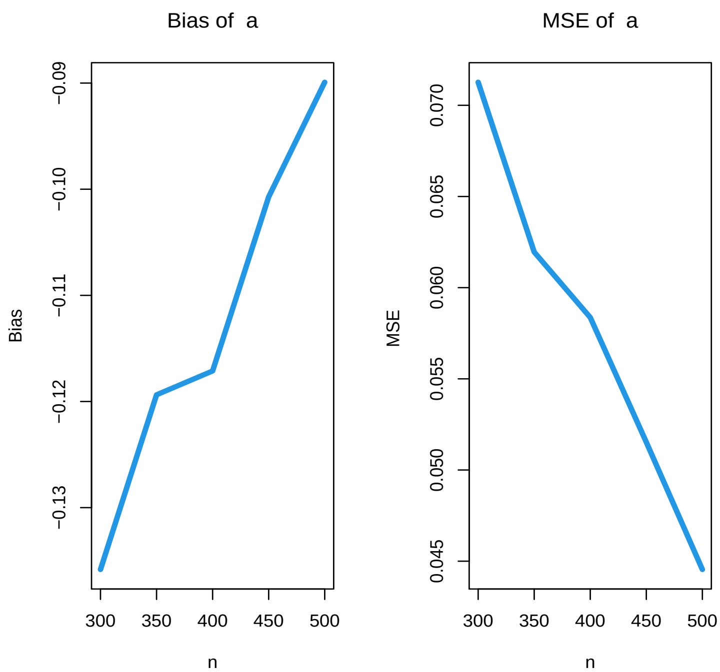


Figure 5. Biases (left) and MSEs (right) for the parameter a.

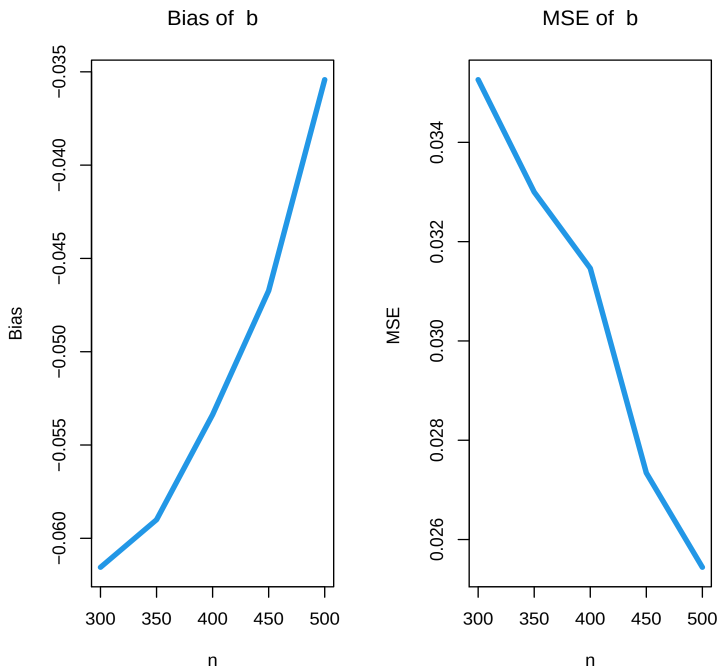


Figure 6. Biases (left) and MSEs (right) for the parameter b.

[14], Ibrahim et al. [36], Alizadeh et al. [8], Merovci et al. ([55], [59]), Korkmaz et al. [44], Karamikabir et al. [43] and Al-Babtain et al. [3]. For comparing models, we consider the Cramér-Von Mises ($C^{\cdot}$) and the Anderson-Darling ($A^{\cdot}$) and the Kolmogorov-Smirnov (KS) statistic. Moreover and for more accuracy, we consider another five goodness-of-fit measures: the Akaike Information Criterion (AIC) (C_1), Bayesian IC (C_2), Consistent AIC (C_3), Hannan-Quinn IC (C_4).

6.1. Modeling lfaiure (relief) times

The first data set $\{1.1, 0.7, 1.9, 3.0, 1.7, 1.0, 1.8, 1.5, 1.2, 1.8, 1.6, 2.7, 4.1, 1.4, 1.3, 1.7, 2.2, 1.4, 2.3, 1.6, 2\}$ called the failure time data: The data represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic (see Gross and Clark [30]). This data was recently analyzed by Ibrahim et al. [40] and Al-Babtain et al. [3]. Table 2 lists the MLEs, StErs confidence intervals (CIs). Table 3 lists the C_1, C_2, C_3, C_4, $A^{\cdot}$, $C^{\cdot}$, K.S. and p-value. Figure 7 gives the E-PDF, E-CDF, E-HRF and P-P plot for relief times data. Figure 8 below gives Kaplan-Meier survival plot for relief times data.

Table 2: MLEs, StErs, CIs for the relief times data.

Models		MLEs, StErs and CIs
$E_{(b)}$	MLE	0.526
	StEr	(0.117)
	L_{CI},U_{CI}	(0.29, 0.75)
$OLE_{(b)}$	MLE	0.6044
	StEr	(0.0535)
	L_{CI},U_{CI}	(0.5, 0.7)
$ME_{(b)}$	MLE	0.950
	StEr	(0.150)
	L_{CI},U_{CI}	(0.66, 1.24)
$LBHE_{(b)}$	MLE	0.5263
	StEr	(0.118)
	L_{CI},U_{CI}	(0.43, 0.63)
$MOE_{(\alpha,b)}$	MLE	54.474, 2.316
	StEr	(35.582), (0.374)
	L_{CI},U_{CI}	(0, 124.2), (1.58, 3.05)
$GMOE_{(\lambda,\alpha,b)}$	MLE	0.519, 89.462, 3.169
	StEr	(0.26), (66.28), (0.77)
	L_{CI},U_{CI}	(0.02, 1), (0, 219.4), (1.66, 4.7)
$KwE_{(\alpha,\beta,b)}$	MLE	83.756, 0.568, 3.330
	StEr	(42.361), (0.326), (1.188)
	L_{CI},U_{CI}	(0.73, 166.78), (0, 1.21), (1.00, 5.66)
$BE(\alpha,\beta,b)$	MLE	81.633, 0.542, 3.514
	StEr	(120.41), (0.33), (1.410)
	L_{CI},U_{CI}	(0, 317.6), (0, 1.2), (0.75, 6.3)
$MOKwE_{(\alpha,\beta,\lambda,b)}$	MLE	0.133, 33.24, 0.571, 1.67
	StEr	(0.332), (57.85), (0.7), (1.8)
	L_{CI},U_{CI}	(0, 0.8), (0, 146.59), (0, 1.98), (0, 5.22)
$KwMOE_{(\alpha,\beta,\lambda,b)}$	MLE	8.868, 34.826, 0.299, 4.899
	StEr	(9.146), (22.312), (0.239), (3.176)
	L_{CI},U_{CI}	(0, 28.8), (0, 78.6), (0, 0.8), (0, 11)
$BrXE_{(a,b)}$	MLE	1.1635, 0.3207
	StEr	(0.33), (0.03)
	L_{CI},U_{CI}	(0.5, 1.8), (0.26,0.4)
OBEE$_{(\alpha,\beta,a,b)}$	**MLE**	**3.74, 0.27, 4.183, 1.366**
	StEr	**(6.89), (0.19), (18.3), (2.94)**
	L_{CI},U_{CI}	**(0, 16.9), (0, 0.65), (0, 40), (0, 7.2)**

Table 3: C_1, C_2, C_3, C_4, $A^{\cdot}$, $C^{\cdot}$, K.S. and (p-value) for the relief times data.

Models	C_1, C_2, C_3, C_4	$A^{\cdot}$	$C^{\cdot}$	K.S. and (p-value)
E	67.70, 68.70, 67.89, 68.90	4.60	0.96	0.44 (< 0.01)
OLE	49.12, 50.14, 49.33, 49.34	1.3	0.22	0.85 (< 0.001)
ME	54.32, 55.31, 54.54, 54.50	2.76	0.53	0.32 (0.1)
LBHE	67.70, 68.70, 67.89, 67.90	0.62	0.105	0.44 (< 0.001)
MOE	43.51, 45.51, 44.22, 43.90	0.8	0.14	0.18 (0.55)
GMOE	42.75, 45.74, 44.25, 43.34	0.51	0.08	0.15 (0.78)
KwE	41.78, 44.75, 43.28, 42.32	0.45	0.07	0.14 (0.86)
BE	43.48, 46.45, 44.98, 44.02	0.70	0.12	0.16 (0.80)
MOKE	41.58, 45.54, 44.25, 42.30	0.60,	0.11	0.14 (0.87)
KMOE	42.82, 46.84, 45.55, 43.60	1.08	0.19	0.15 (0.86)
BrXE	48.13, 50.15, 48.83, 48.52	1.39	0.24	0.248 (0.171)
OBEE	**38.95, 42.93, 41.62, 39.73**	**0.155**	**0.0268**	**0.0902 (0.9969)**

Based on Table 3, we conclude that the proposed lifetime OBEE model is much better than all other mentioned models with C_1 =38.95, C_2 =42.93, C_3 =41.62, C_4 =39.73, $A^{\cdot}$ =0.155, $C^{\cdot}$ =0.0268, K.S=0.09016 and p-value=0.9969 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 8 and 9, the OBEE distribution provides adequate fits to the empirical functions.

6.2. Modeling survival times

The second data set $\{$0.10, 0.92, 0.93, 0.96, 0.33, 0.44, 0.56, 0.72, 0.74, 0.77, 1.0, 1.0, 1.02, 1.05, 1.07, 07, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 2.22, 2.3, 2.31, 2.4, 0.59, 1.08, 1.08, 1.08, 1.2, 1.21, 1.6, 1.09, 1.12, 1.13, 1.22, 1.22, 1.24, 1.30, 1.34, 1.36, 1.39, 1.44, 1.83, 1.95, 1.96, 1.97, 2.02, 1.15, 1.16, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 2.13, 2.15, 2.16, 1.46, 1.53, 1.59, 3.61, 4.02, 4.32, 4.58, 5.55$\}$ called

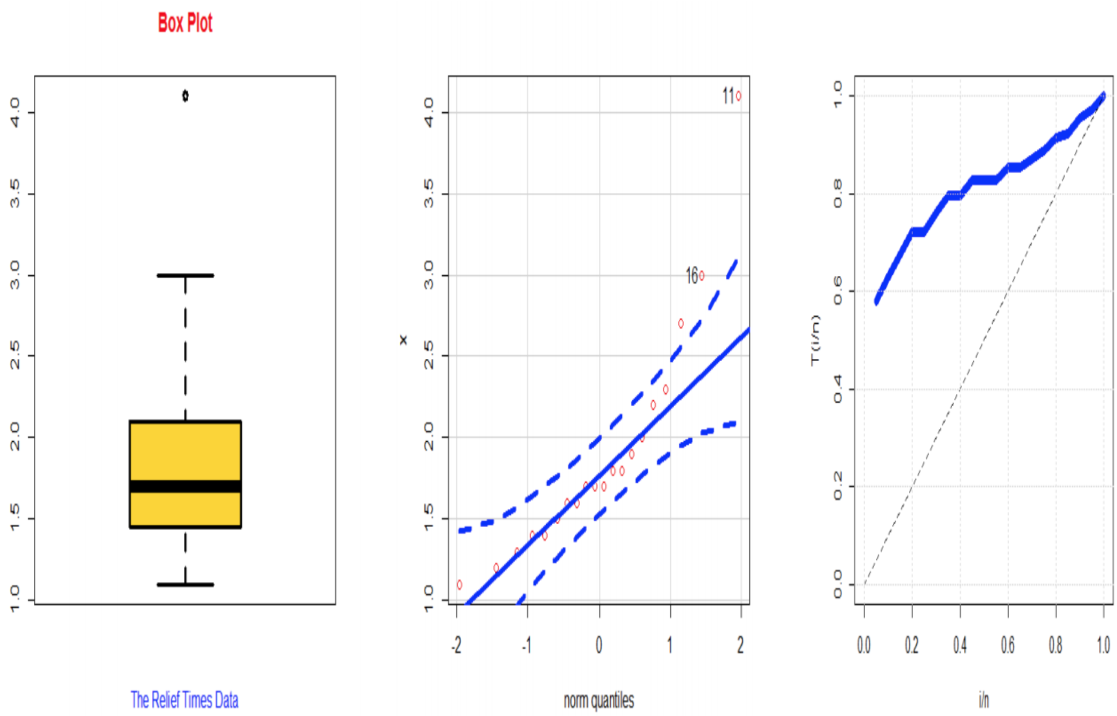


Figure 7. The box plot, Q-Q plot and TTT plot for the relief times data.

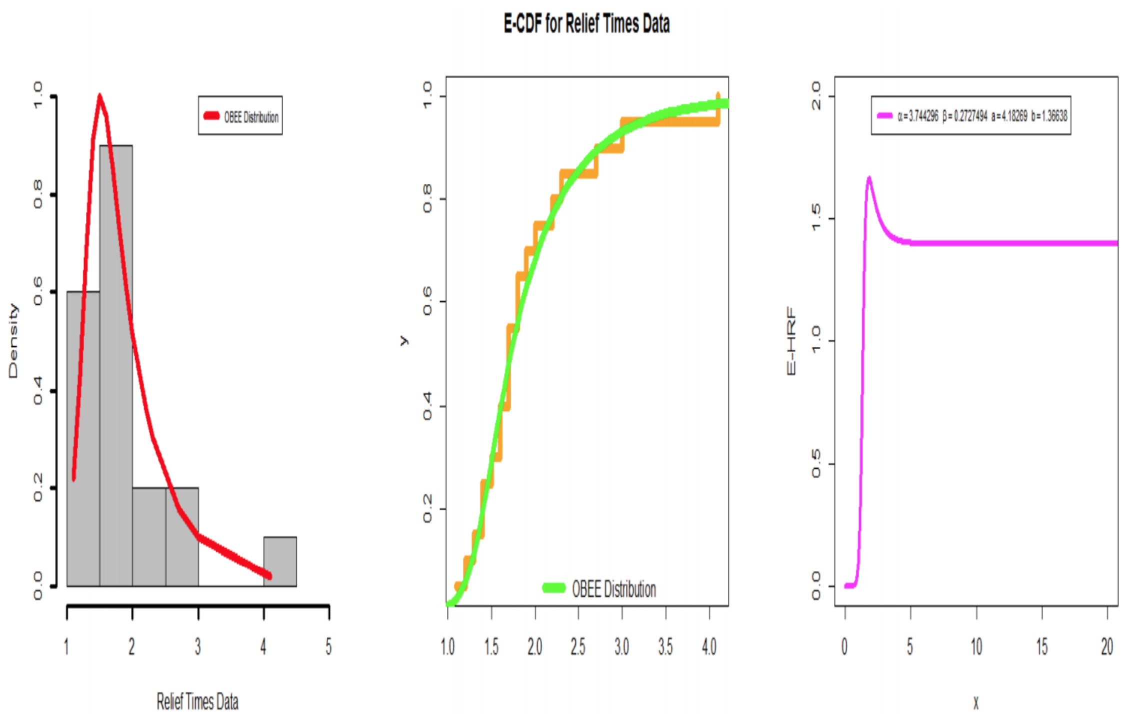


Figure 8. E-PDF, E-CDF and E-HRF for relief times data.

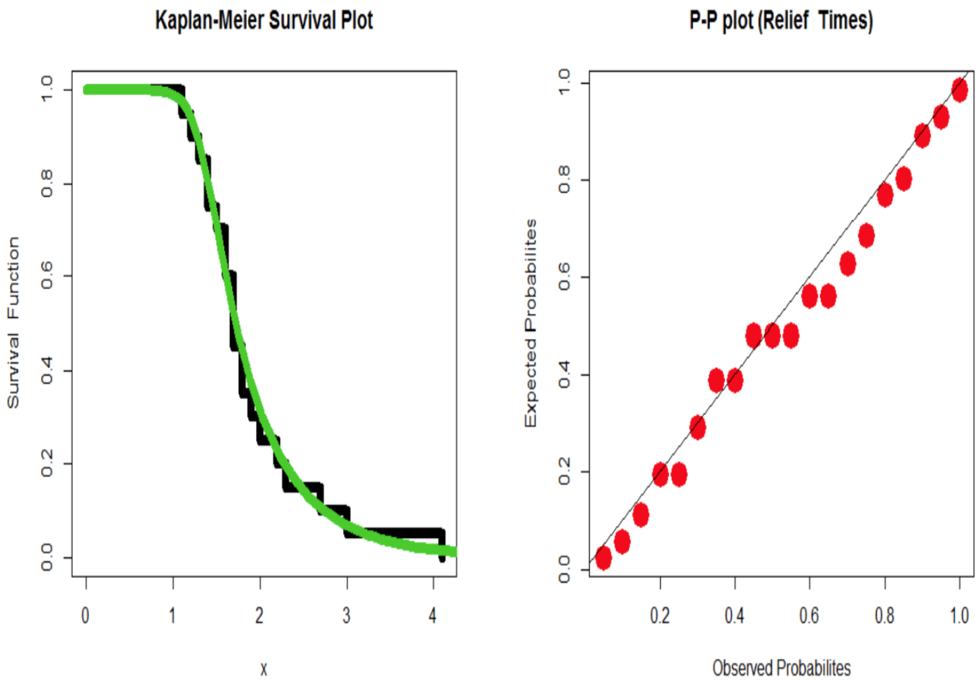


Figure 9. Kaplan-Meier survival plot and P-P plot for relief times data.

the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [17]. Table 4 lists the MLEs, StErs confidence intervals (CIs). Table 5 lists the C_1, C_2, C_3, C_4, $A^{\cdot}$, $C^{\cdot}$, K.S. and p-value. For many other real-life data sets see [19], [22], [18], [47], [45], [63], [23], [24], [25], [65], [38], [61], [37], [7]. Figure 9 gives the estimated PDF (E-PDF), E-CDF, E-HRF and P-P plot survival times data. Figure 10 gives the Kaplan-Meier survival plot survival times data. Based on Table 5, we conclude that the OBEE model is much better than all other competitive models with C_1 =203.84, C_2 =212.94, C_3 =204.44, C_4 =207.46, $A^{\cdot}$ =0.297, $C^{\cdot}$ = 0.046, K.S=0.0675 and p-value=0.8987 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 11 and 12, the OBEE distribution provides adequate fits to the empirical functions.

Table 4: MLEs, StErs, CIs for the survival times data.

Models		MLEs, StErs and CIs
$E_{(b)}$	MLE	0.540
	StEr	(0.063)
	L_{CI},U_{CI}	(0.4, 0.7)
$OLE_{(b)}$	MLE	0.3815
	StEr	(0.021)
	L_{CI},U_{CI}	(0.3, 0.4)
$ME_{(b)}$	MLE	0.925
	StEr	(0.077)
	L_{CI},U_{CI}	(0.6, 1.1)
$LBHE_{(b)}$	MLE	0.54
	StEr	(0.064)
	L_{CI},U_{CI}	(0.4, 0.67)
$MOE_{(\alpha,b)}$	MLE	8.78, 1.38
	StEr	(3.56), (0.19)
	L_{CI},U_{CI}	(1.8,15.7), (1.00,1.8)
$GMOE_{(\lambda,\alpha,b)}$	MLE	0.18, 47.64, 4.47
	StEr	(0.07), (44.9), (1.33)
	L_{CI},U_{CI}	(0.04, 0.3), (0, 136), (1.8. 7)
$KwE_{(a,\beta,b)}$	MLE	3.304, 1.100, 1.037
	StEr	(1.106), (0.764), (0.614)
	L_{CI},U_{CI}	(1.13, 5.47), (0, 2.59), (0, 2.24)
$BE_{(a,\beta,b)}$	MLE	0.807, 3.461, 1.331
	StEr	(0.696), (1.003), (0.860)
	L_{CI},U_{CI}	(0, 2.2), (1.49, 5.42), (0, 3)
$MOKwE_{(\alpha,\beta,\lambda,b)}$	MLE	0.008, 2.716, 1.986, 0.099
	StEr	(0.002), (1.316), (0.784), (0.05)
	L_{CI},U_{CI}	(0.004, 0.01), (0.14, 5), (0.5, 4), (0, 0.2)
$KwMOE_{(\alpha,\beta,\lambda,b)}$	MLE	0.373, 3.478, 3.306, 0.299
	StEr	(0.136), (0.861), (0.779), (1.112)
	L_{CI},U_{CI}	(0.11, 0.6), (1.79, 5), (1.78, 5), (0, 3)
$BrXE_{(a,b)}$	MLE	0.48, 0.21
	StEr	(0.060), (0.012)
	L_{CI},U_{CI}	(0.36, 0.49), (0.18, 0.23)
$OBEE_{(\alpha,\beta,a,b)}$	**MLE**	**3.5, 0.59, 0.56, 0.28**
	StEr	**3.138, 0.306, 0.711, 0.557**
	L_{CI},U_{CI}	**(0, 7.9), (0, 1.21), (0, 1.96), (0, 1.48)**

Table 5: C_1, C_2, C_3, C_4, $A^{\cdot}$, $C^{\cdot}$, K.S. and (p-value) for survival times data.

Models	C_1, C_2, C_3, C_4	$A^{\cdot}$	$C^{\cdot}$	K.S. and (p-value)
E	234.60, 236.90, 234.68, 235.55	6.53	1.25	0.3 (0.06)
OLE	229.13, 231.43, 229.21, 230.11	1.94	0.33	0.5 ($<$ 0.001)
ME	210.40, 212.68, 210.45, 211.30	1.52	0.25	0.15 (0.13)
LBHE	234.63, 236.92, 234.71, 235.51	0.71	0.115	0.28 ($<$ 0.001)
MOE	210.37, 214.93, 210.52, 212.17	1.18	0.17	0.10 (0.43)
GMOE	210.54, 217.38, 210.89, 213.24	1.02	0.16	0.09 (0.5)
KwE	209.42, 216.24, 209.77, 212.12	0.74	0.11	0.09 (0.5)
BE	207.37, 214.21, 207.73, 210.09,	0.98	0.15	0.11 (0.34)
MOKwE	209.44, 218.56, 210.04, 213.04,	0.79	0.12	0.10 (0.44)
KwMOE	207.82, 216.94, 208.42, 211.42	0.61	0.11	0.09 (0.5)
BrXE	235.31, 239.92, 235.53, 237.14	2.9	0.52	0.22 (0.002)
OBEE	**203.84, 212.94, 204.44, 207.46**	**0.297**	**0.0460**	**0.0675 (0.8987)**

7. conclusions

In this article, we introduced and studied a new flexible version of the exponentiated exponential model called the odd Burr exponentiated exponential (OBEE) model. The new density can be right skewed and symmetric with unimodal and bimodal shapes. The new HRF can be ”constant”, ”decreasing”, ”increasing”, ”increasing-constant”, ”upside-down-constant”, ”decreasing-constant”. Some of its mathematical properties including the ordinary moments, incomplete moment, moment generating function are derived. Numerical calculations for the expected value, skewness, variance, kurtosis and the index of dispersion is presented. The skewness of the OBEE distribution can range in the interval $(-2.779, 8.2978)$. The spread for the OBEE kurtosis is much larger ranging from -46.275 to 35.526. The index of dispersion can be ”between 0 and 1” or ”equal

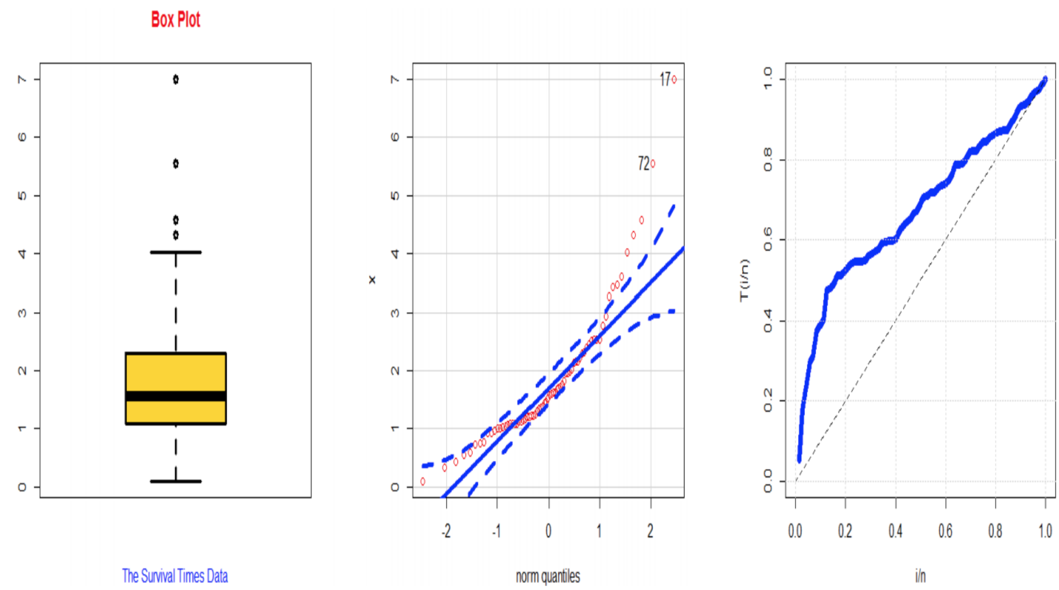


Figure 10. The box plot, Q-Q plot and TTT plot for the survival times data.

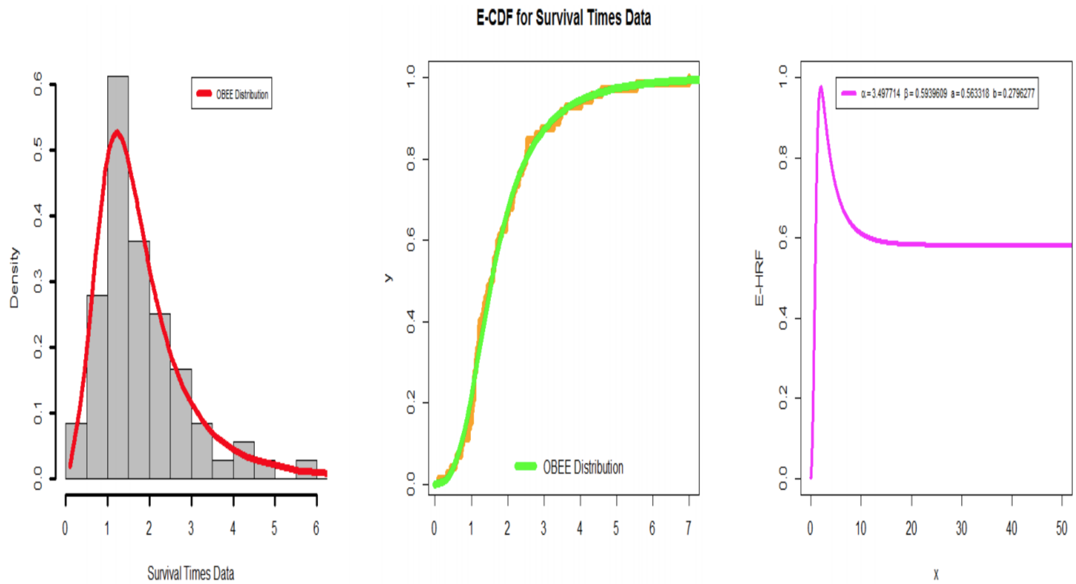


Figure 11. E-PDF, E-CDF and E-HRF for survival times data.

1" or more than 1. Some bivariate and multivariate OBEE type model have been also derived. Estimation of OBEE parameters is performed by maximum likelihood estimation method. We assessed the performance of the maximum likelihood method. The assessment can be performed graphically via the biases and mean squared errors. The usefulness and flexibility of the new distribution is illustrated by means of two real data sets. The new model is much better than many other competitive models in modeling relief times and survival times data sets according to the Akaike Information Criterion, the Consistent Akaike Information Criterion, the Hannan-Quinn Information Criterion, the Bayesian Information Criterion, the Cramér-Von Mises, the Anderson-Darling statistics. As a future work, we can apply the Bagdonavičius–Nikulin goodness-of-fit test,

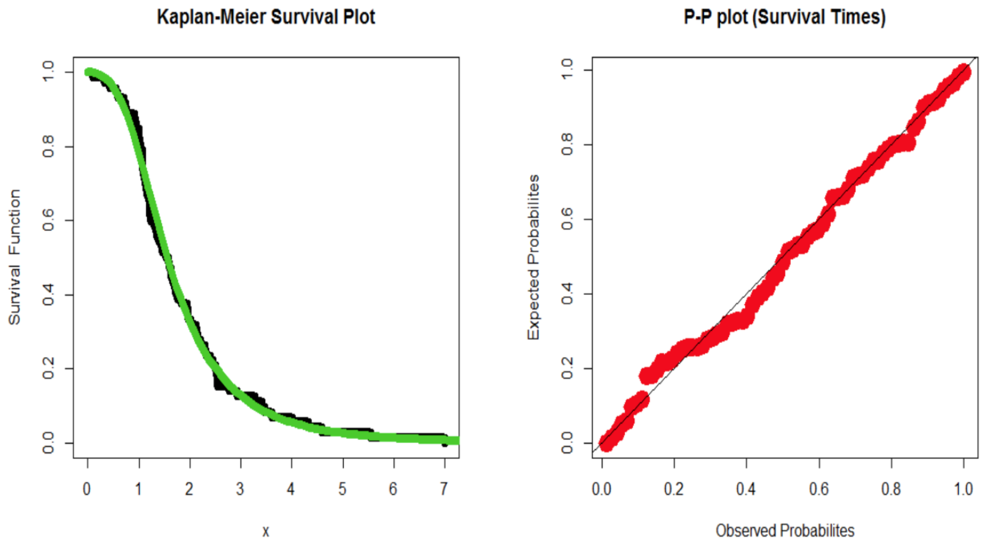


Figure 12. Kaplan-Meier survival plot and P-P plot for survival times data.

modified Bagdonavičius–Nikulin goodness-of-fit test, Nikulin-Rao-Robson goodness-of-fit test and modified Nikulin-Rao-Robson goodness-of-fit test to our new model (see Goual et al. ([27], [28] and [29]) and Yadav et al. [62] for more details). Characterization results and regression modeling can be derived based on OBEE model (see [9], [10], [11] and [12]for more details).

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