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Extended Exponentiated Chen Distribution: Mathematical Properties and Applications

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Abstract In this paper, we introduce a new four-parameter distribution which is called Extended Exponentiated Chen (EE-C) distribution. Theoretical properties of this model including the hazard function, moments, conditional moments, mean residual life, mean past lifetime, coefficients of skewness and kurtosis, order statistics and asymptotic properties are derived and studied. The maximum likelihood estimation technique is used to estimate the parameters of this model. The estimation of the model parameters by Least squares, Weighted Least Squares, Crammer-von-Mises, Anderson-Darling and right-tailed Anderson-Darling methods are also briefly introduced and numerically investigated. Moreover, simulation schemes are derived. At the end, three applications of the model with three real data sets are presented for the illustration of the flexibility of the proposed distribution.

Keywords Chen distribution, Exponentiated Chen distribution, Moments, Mean residual life, Mean past lifetime, Maximum likelihood estimation

AMS 2010 subject classifications 60E05, 62F10, 62G05

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1. Introduction

Modeling and analyzing lifetime data are important aspects of statistical research in many applied sciences such as engineering, medicine and economics.

In many applied areas such as lifetime analysis, finance and insurance, we need extended forms of distributions. So, several methods for generating new families of distributions have been studied. Some attempts have been made to define new families of probability distributions that extend well-known families of distributions and with great flexibility in modeling data in practice.

In literature, there exist many generalized (G-) classes of distributions where one or more parameter(s) are added to the baseline distribution. Chen [6] proposed a new two parameter lifetime distribution with bathtub shaped or increasing hazard rate function.

Three generalizations of Chen distribution are Marshall-Olkin class (Marshall and Olkin, [6]), exponentiated-G class (Gupta *et al.*, [14]) and transmuted G-class (Aryal and Tsokos, [4]), where one shape parameter is added to the baseline model. Recently, Khan *et al.* [18] proposed the transmuted Chen distribution and investigated various structural properties and their applications. Chaubey and Zhang [5] introduced an extension of the Chen's family,

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called the Extended Chen (EC) family with the cumulative distribution function (cdf) given by the following:

$$F(x;a,b,\alpha)=\left(1-e^{a(1-e^{x^b})}\right)^{\alpha}, \tag{1}$$

where $\alpha, a, b > 0$ and $x > 0$. The probability density function (pdf) of EC distribution is

$$f(x;a,b,\alpha)=ab\alpha x^{b-1}e^{x^b}e^{a(1-e^{x^b})}\left(1-e^{a(1-e^{x^b})}\right)^{\alpha-1}. \tag{2}$$

Dey *et al.* [10] studied further various properties and estimation methods for the EC distribution. In this paper, we propose a new lifetime distribution family, an extension of the EC distribution. We provide different statistical properties of this new distribution such as moments, mean deviations, Lorenz and Bonferroni curves and order statistics. The maximum likelihood estimates (MLE) of the model parameters are determined. The estimation of the model parameters is also approached by Least squares, Weighted Least Squares, Crammer-von-Mises, Anderson-Darling and right-tailed Anderson-Darling estimation methods. Finally the performance of our distribution is examined with data sets. Other articles in this field include Mozafari *et al.* [21], Karamikabir *et al.* [16] and Karamikabir *et al.* [17].

The paper is organized as follows: In Section 2, we construct the new family of distributions. Shape characteristics of the probability density and hazard functions of the family are investigated. Various properties of the proposed distribution are explored in Section 3. These properties include moments, conditional moments, mean residual (past) lifetime, mean deviations, Lorenz and Bonferroni curves and order statistics. Estimation of the model parameters by maximum likelihood is performed in Section 4. Simulation study and applications to real data sets illustrate the performance of the new family in Section 5 and Section 6. In Section 7, we offer some concluding remarks.

1.1. Probability density and cumulative distribution functions

Alizadeh *et al.* [1] introduced a new family of distributions which is called Extended Exponentiated family of distribution (EE-G). The cdf and pdf of this family for any baseline cdf $G(x;\boldsymbol{\theta})$ are given by

$$F(x;\alpha,\beta,\boldsymbol{\theta})=\int_0^{\frac{G(x;\boldsymbol{\theta})^{\alpha}}{1-G(x;\boldsymbol{\theta})^{\beta}}}\frac{dt}{(1+t)^2}=\frac{G(x;\boldsymbol{\theta})^{\alpha}}{G(x;\boldsymbol{\theta})^{\alpha}+1-G(x;\boldsymbol{\theta})^{\beta}}, \tag{3}$$

$$f(x;\alpha,\beta,\boldsymbol{\theta})=\frac{g(x)G(x;\boldsymbol{\theta})^{\alpha-1}\left[\alpha+(\beta-\alpha)G(x;\boldsymbol{\theta})^{\beta}\right]}{\left[G(x;\boldsymbol{\theta})^{\alpha}+1-G(x;\boldsymbol{\theta})^{\beta}\right]^2}, \tag{4}$$

where α, β are two shape parameters and $\boldsymbol{\theta}$ is the vector of parameters for baseline cdf G and $g(x;\boldsymbol{\theta})=\frac{dG(x;\boldsymbol{\theta})}{dx}$. By inserting $G(x;\boldsymbol{\theta})=1-e^{a(1-e^{x^b})}$ as Chen cdf for any $x>0$ and $a,b>0$ in Eq. (3), we obtain a new extension of Exponentiated Chen distribution whose cdf is given by

$$F(x;\alpha,\beta,a,b)=\frac{\left[1-e^{a(1-e^{x^b})}\right]^{\alpha}}{\left[1-e^{a(1-e^{x^b})}\right]^{\alpha}+1-\left[1-e^{a(1-e^{x^b})}\right]^{\beta}}, \tag{5}$$

where $x>0$ and $a,b,\alpha,\beta>0$. A random variable X with the cdf (5), is called Extended Exponentiated Chen distribution and denoted by $X\sim$ EE-C(α,β,a,b).

The pdf and hazard function (hrf) of this distribution are given by the following:

$$f(x;\alpha,\beta,a,b)=\frac{abx^{b-1}e^{x^b}e^{a(1-e^{x^b})}\left[1-e^{a(1-e^{x^b})}\right]^{\alpha-1}\left\{\alpha+(\beta-\alpha)\left[1-e^{a(1-e^{x^b})}\right]^{\beta}\right\}}{\left\{\left[1-e^{a(1-e^{x^b})}\right]^{\alpha}+1-\left[1-e^{a(1-e^{x^b})}\right]^{\beta}\right\}^2}, \tag{6}$$

and

$$h(x;\alpha,\beta,a,b)=\frac{abx^{b-1}e^{x^b}e^{a(1-e^{x^b})}\left[1-e^{a(1-e^{x^b})}\right]^{\alpha-1}\left\{\alpha+(\beta-\alpha)\left[1-e^{a(1-e^{x^b})}\right]^{\beta}\right\}}{\left\{\left[1-e^{a(1-e^{x^b})}\right]^{\alpha}+1-\left[1-e^{a(1-e^{x^b})}\right]^{\beta}\right\}\left\{1-\left[1-e^{a(1-e^{x^b})}\right]^{\beta}\right\}}. \tag{7}$$

Figure 1. provide the pdf and the hrf of EE-C(α,β,a,b) for different parameter values.

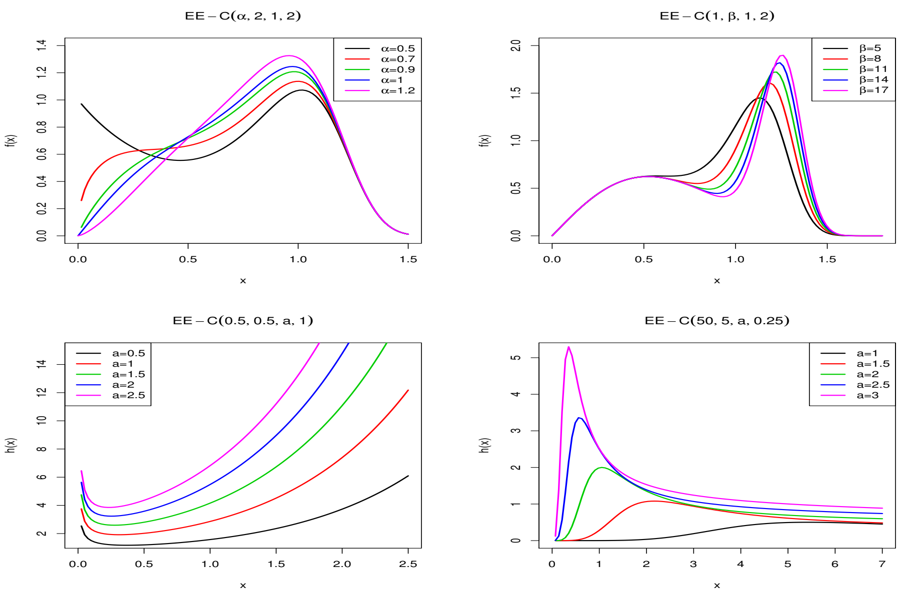

Figure 1. The sample curves of density and hazard function of EE-C.

Special cases: The $EE-C(\alpha,\beta,a,b)$ distribution contains as special sub-models the following well-known distributions:
• For $\alpha=\beta$, we obtain Exponentiated Chen distribution.
• For $\alpha=\beta=1$, we obtain Chen distribution.
In Figure 2 some pdfs for above special cases of EE-C have been drawn.

2. Main properties

Some mathematical properties of the new model such as moments, moment generating function, mean residual and mean past lifetime are derived in this section. Moreover, mean deviations, Lorenz and Bonferroni curves and order statistics are presented. First, we investigate asymptotic properties of this model and give mixture representations for cdf and pdf.

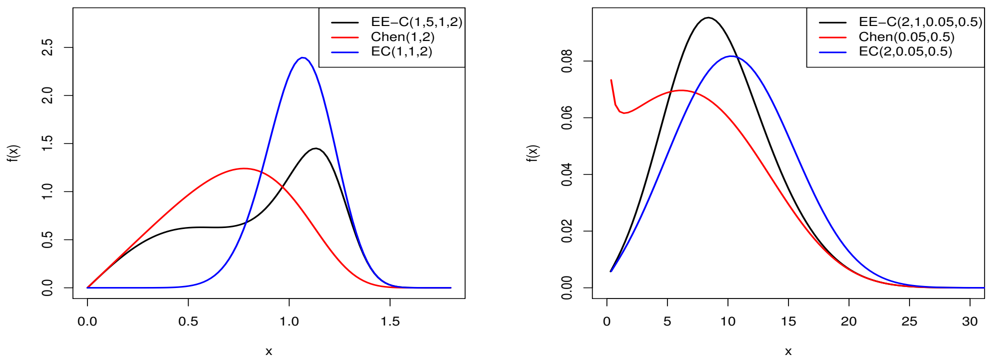

Figure 2. The sample curves of hazard function of EE-C.

2.1. Asymptotic

One of the main usage of the idea of an asymptotic distribution is in providing approximations to the cdfs of the statistical estimators. The asymptotic cdf, pdf and hrf of EE-C distribution as $x \to 0$ are given by

$$\begin{aligned} F(x) &\sim ax^b && as\;\; x \to 0, \\ f(x) &\sim abx^{b-1} && as\;\; x \to 0, \\ h(x) &\sim abx^{b-1} && as\;\; x \to 0. \end{aligned}$$

The asymptotic cdf, pdf, and hrf of EE-C distribution as $x \to \infty$ are given as follows:

$$\begin{aligned} 1 - F(x) &\sim \beta e^{-ae^{x^b}} && as\;\; x \to \infty, \\ f(x) &\sim ab\beta x^{b-1} e^{x^b} e^{-ae^{x^b}} && as\;\; x \to \infty, \\ h(x) &\sim abx^{b-1} e^{x^b} && as\;\; x \to \infty. \end{aligned}$$

2.2. Mixture representations for the cdf and pdf

In the following, we show that the EE-C distribution can be obtained as a mixture of EC distributions. Using the geometric series and binomial expansion and after changing the indices we can write as follows:

$$\begin{aligned}
F(x) &= \frac{G(x)^{\alpha}}{1-[G(x)^{\beta}-G(x)^{\alpha}]} = G(x)^{\alpha}\sum_{i=0}^{\infty}[G(x)^{\beta}-G(x)^{\alpha}]^{i} \\
&= G(x)^{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}\,(G(x)^{\alpha})^{j}\left(G(x)^{\beta}\right)^{(i-j)} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}G(x)^{\alpha(j+1)+\beta(i-j)} \\
&= \sum_{i,l=0}^{\infty}\sum_{j=0}^{i}(-1)^{j+l}\binom{i}{j}\binom{\alpha(j+1)+\beta(i-j)}{l}\bar{G}(x)^{l} \\
&= \sum_{i,l=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{l}(-1)^{j+l+k}\binom{i}{j}\binom{\alpha(j+1)+\beta(i-j)}{l}\binom{l}{k}G(x)^{k} \\
&= \sum_{i,k=0}^{\infty}\sum_{j=0}^{i}\sum_{l=k}^{\infty}(-1)^{j+l+k}\binom{i}{j}\binom{\alpha(j+1)+\beta(i-j)}{l}\binom{l}{k}G(x)^{k} \\
&= \sum_{k=0}^{\infty}c_k\,G(x)^{k}, \qquad (8)
\end{aligned}$$

where

$$c_k=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=k}^{\infty}(-1)^{j+l+k}\binom{i}{j}\binom{\alpha(i+1)+\beta(i-j)}{l}\binom{l}{k}.$$

With replacing $G(x)=1-e^{a(1-e^{x^{b}})}$ as Chen distribution, we have

$$F(x)=\sum_{k=0}^{\infty}c_kF_{EC(a,b,k)}(x), \qquad (9)$$

and

$$f(x)=\sum_{k=0}^{\infty}c_{k+1}f_{EC(a,b,k+1)}(x), \qquad (10)$$

where $F_{EC(a,b,k)}$ and $f_{EC(a,b,k)}$ denote the cdf and the pdf of EC distribution with parameters a, b and k.

2.3. Moments

We need the moments in statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., central tendency, dispersion, skewness and kurtosis). Here, we give lemma, which will be used later.

Lemma 1
For $a_1, a_2 > 0$ and $a_3 > -1$, let

$$L_1(a_1,a_2,a_3,r;\beta)=\int_0^{\infty}x^{r}x^{\beta-1}e^{a_1x^{\beta}}e^{a_2(1-e^{a_1x^{\beta}})}\left(1-e^{a_2(1-e^{a_1x^{\beta}})}\right)^{a_3}dx.$$

Then,

$$L_1(a_1,a_2,a_3,r;\beta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_p\left(\frac{r}{\beta}\right)a_q\left(\frac{r}{\beta}+p\right)\frac{(-1)^{\frac{2r}{\beta}+p}}{a_1^{\frac{r}{\beta}+1}a_2^{\frac{r}{\beta}+p+1}[(q+p+a_3+1)\beta+r]},$$

where $a_p\left(\frac{r}{\beta}\right)$ is the coefficient of $\left[\frac{1}{a_2}\log(1-u)\right]^{\frac{r}{\beta}+p}$ in the expansion of $\left[\sum_{i=1}^{\infty}\frac{(\frac{1}{a_2}\log(1-u))^i}{i}\right]^{\frac{r}{\beta}}$ and $a_q\left(\frac{r}{\beta}+p\right)$ is the coefficient of $u^{p+q+\frac{r}{\beta}}$ in the expansion of $\left(\sum_{j=1}^{\infty}\frac{u^j}{j}\right)^{\frac{r}{\beta}+p}$.

Proof
Substituting $x^r=\left(x^{\beta}\right)^{\frac{r}{\beta}}$ and $u=1-e^{a_2(1-e^{a_1x^{\beta}})}$, we get

$$L_1(a_1,a_2,a_3,r;\beta)=\frac{1}{\beta a_1^{\frac{r}{\beta}+1}a_2}\int_0^1\left\{\log\left[1-\frac{1}{a_2}\log(1-u)\right]\right\}^{\frac{r}{\beta}}u^{a_3}du.$$

The desired result is obtained by expanding the function $\{\log[1-\frac{1}{a_2}\log(1-u)]\}$. (For more details see Dey *et al.* [10]). □

Next, the n-th moment of the EE-C distribution is given as follows:

$$E(X^n)=ab\sum_{k=0}^{\infty}(k+1)c_{k+1}L_1(1,a,k,n;b). \tag{11}$$

For integer values of n, Let $\mu_n^{'}=E(X^n)$ and $\mu=\mu_1^{'}=E(X)$, then one can also find the n-th central moment of the EE-C distribution as the following:

$$\mu_n=E(X-\mu)^n=\sum_{i=0}^{n}\binom{n}{i}\mu_i^{'}(-\mu)^{n-i}. \tag{12}$$

Using (12), the measures of skewness and kurtosis of the EE-C distribution can be obtained as follows:

$$Skewness(X)=\frac{\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3}{(\mu_2'-(\mu_1')^2)^{\frac{3}{2}}}, \tag{13}$$

$$Kurtosis(X)=\frac{\mu_4'-4\mu_1'\mu_3'+6(\mu_1')^2\mu_3'-3(\mu_1')^4}{\mu_2'-(\mu_1')^2}, \tag{14}$$

respectively. Additionally, the moment generating function of EE-C distribution can be written as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=ab\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^r}{r!}(k+1)c_{k+1}L_1(1,a,k,r;b). \tag{15}$$

Figure 3 shows the behaviour of skewness and kurtosis of EE-C distribution.

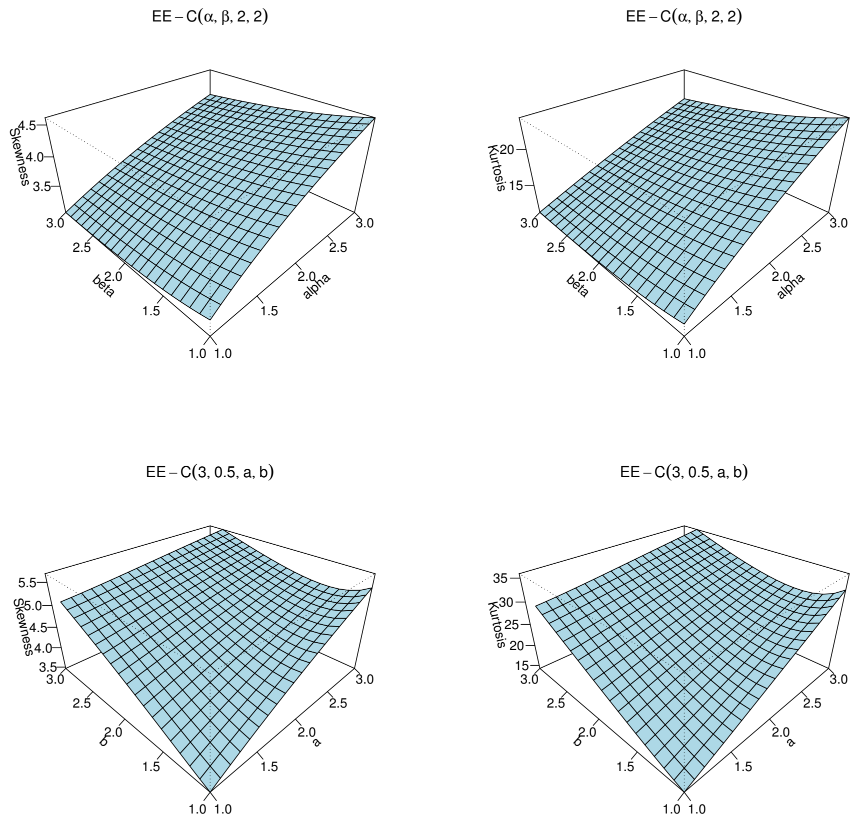

Figure 3. Skewness and Kurtosis for EE-C.

2.4. Conditional moments:

Here, we intend to determine the conditional moments of the new family. Let

$$L_2(a_1,a_2,a_3,r,t;\beta)=\int_t^{\infty} x^r x^{\beta-1} e^{a_1 x^{\beta}} e^{a_2(1-e^{a_1 x^{\beta}})}\left(1-e^{a_2(1-e^{a_1 x^{\beta}})}\right)^{a_3} dx,$$

for $a_1, a_2 > 0$ and $a_3 > -1$. Then, we obtain

$$L_2(a_1,a_2,a_3,r,t;\beta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_p\left(\frac{r}{\beta}\right) a_q\left(\frac{r}{\beta}+p\right)\frac{(-1)^{\frac{2r}{\beta}+p}\left\{1-\left[1-\left(e^{a_2(1-e^{a_1 t^{\beta}})}\right)\right]^{\frac{(q+p+a_3+1)\beta+r}{\beta}}\right\}}{a_1^{\frac{r}{\beta}+1} a_2^{\frac{r}{\beta}+p+1}\left[(q+p+a_3+1)\beta+r\right]}.$$

So, the n-th conditional moments of X can be expressed as

$$E(X^n|X>x)=\frac{ab\sum_{k=0}^{\infty}(k+1)c_{k+1}L_2(1,a,k,n,x;b)}{1-\sum_{k=0}^{\infty}c_k V^k(x)}, \tag{16}$$

where $V(x) = 1 - e^{a(1-e^{x^b})}$. In the following for $a_1, a_2 > 0$ and $a_3 > -1$, we define and compute

$$L_3(a_1, a_2, a_3, r, t; \beta) = \int_0^t x^r x^{\beta-1} e^{a_1 x^\beta} e^{a_2(1-e^{a_1 x^\beta})} \left(1 - e^{a_2(1-e^{a_1 x^\beta})}\right)^{a_3} dx.$$

We can write

$$L_3(a_1, a_2, a_3, r, t; \beta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_p \left(\frac{r}{\beta}\right) a_q \left(\frac{r}{\beta} + p\right) \frac{(-1)^{\frac{2r}{\beta}+p} \left\{1 - [e^{a_2(1-e^{a_1 t^\beta})}]\right\}^{\frac{(q+p+a_3+1)\beta+r}{\beta}}}{a_1^{\frac{r}{\beta}+1} a_2^{\frac{r}{\beta}+p+1} [(q+p+a_3+1)\beta + r]}. \tag{17}$$

Therefore

$$E(X^n | X \leq x) = \frac{ab \sum_{k=0}^{\infty} (k+1) c_{k+1} L_3(1, a, k, n, x; b)}{\sum_{k=0}^{\infty} c_k V^k(x)}. \tag{18}$$

2.5. Mean residual life

In life testing situations, the expected additional lifetime given that a component has survived until time x is a function of x, called the mean residual life. More specifically, if the random variable X represents the life of a component, then the mean residual life is given by $M_X(x) = E(X - x | X > x)$ and can be expressed as

$$M_X(x) = ab \frac{\sum_{k=0}^{\infty} (k+1) c_{k+1} L_2(1, a, k, 1, x; b)}{1 - \sum_{k=0}^{\infty} c_k V^k(x)} - x.$$

2.6. Mean past lifetime

In many realistic situations, the random variables are not necessarily related only to the future, but they can also refer to the past. In fact, in many reliability problems, it is of interest to consider variables of the kind $(x - X | X \leq x)$ for fixed x, called the past lifetime, which denotes the time elapsed after failure till time x given that the item has already failed by time x defined for a nonnegative random variable X. The mean past lifetime of nonnegative random variable X is defined as $m_X(x) = E(x - X | X \leq x)$ and is equal to

$$m_X(x) = x - ab \frac{\sum_{k=0}^{\infty} (k+1) c_{k+1} L_3(1, a, k, 1, x; b)}{\sum_{k=0}^{\infty} c_k V^k(x)}.$$

2.7. Mean deviations

The mean deviations can be used as a measure of spread in a population. The mean deviations about the mean and about the median are given by the following:

$$\delta_1(X) = \int_0^{\infty} |x - \mu| f(x) dx,$$

$$\delta_2(X) = \int_0^{\infty} |x - m| f(x) dx,$$

respectively, where $\mu = E(X)$ and $m = median(X)$. These quantities can be calculated as

$$\delta_1(X) = 2\mu F(\mu) - 2 \int_0^{\mu} x f(x) dx$$

$$\delta_2(X) = \mu - 2 \int_0^{m} x f(x) dx.$$

Using Eqs. (17) and (18), it follows that

$$\delta_1(X) = 2\mu F(\mu) - 2ab\sum_{k=0}^{\infty}(k+1)c_{k+1}L_3(1,a,k,1,\mu;b)$$

$$\delta_2(X) = \mu - 2ab\sum_{k=0}^{\infty}(k+1)c_{k+1}L_3(1,a,k,1,m;b).$$

2.8. Bonferroni and Lorenz curves

We can construct Bonferroni and Lorenz curves, which are important in several fields such as economics, reliability, demography, insurance and medicine. They are defined as the following:

$$B(F(x)) = \frac{1}{\mu F(x)}\int_0^x tf(t)dt, \tag{19}$$

$$L(F(x)) = \frac{1}{\mu}\int_0^x tf(t)dt. \tag{20}$$

By Eqs. (18) and (6), one can obtain

$$B(F(x)) = \frac{ab}{\mu F(x)}\sum_{k=0}^{\infty}(k+1)c_{k+1}L_3(1,a,k,1,x;b),$$

$$L(F(x)) = \frac{ab}{\mu}\sum_{k=0}^{\infty}(k+1)c_{k+1}L_3(1,a,k,1,x;b).$$

2.9. Order statistics

Order statistics are among the most fundamental tools in nonparametric statistics and inference. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from Eq. (6). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. It is well known that the pdf of the i-th order statistic, is given by the following:

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}F^{i-1}(x)(1-F(x))^{n-i}f(x) \\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F^{j+i-1}(x)f(x), \end{aligned}$$

where $B(i, n-i+1)$ is the Beta function. Here and henceforth, we use an equation by Gradshteyn and Ryzhik [12], page 17, for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty}a_iu^i\right)^n = \sum_{i=0}^{\infty}d_{n,i}u^i, \tag{21}$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \cdots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (ia_0)^{-1}\sum_{m=1}^{i}\left[m(n+1)-i\right]a_m d_{n,i-m}.$$

From Eq. (21), we can show that the density function of the i-th order statistic of any EE-C distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} e_{r,k} f_{EC}(x; a, b, r+k+1),$$

where $f_{EC}(x; a, b, r+k+1)$ denotes the density function of EC distribution with parameters a, b and $r+k+1$ and

$$e_{r,k} = \frac{n!(r+1)(i-1)!c_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}.$$

Here the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = c_0^{j+i-1}$ and recursively we have

$$f_{j+i-1,k} = (kc_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, c_m f_{j+i-1,k-m},\ k \geq 1,$$

and c_r is given by (9). The m-th moments of $X_{i:n}$ is equal to

$$E(X_{i:n}^m) = ab \sum_{r,k=0}^{\infty} (r+k+1) e_{r,k} L_1(1, a, r+k, m; b).$$

2.10. Entropy

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are due to Rényi [22] and Shannon [23]. The Rényi entropy of a random variable with pdf $f(x)$ is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_0^{\infty} f^{\gamma}(x) dx\right),$$

for $\gamma > 0$ and $\gamma \neq 1$. In Figure 4 one can see some curves of the Rényi entropy function of the EE-C distribution for some parameters.

The Shannon entropy of a random variable X is defined by $E\left\{-\log\left[f(X)\right]\right\}$. It is the special case of the Rényi entropy when $\gamma \uparrow 1$.

We tend to derive an expression for the Shannon entropy of the EE-C distribution. The Shannon entropy of a random variable with pdf $f(x)$ is defined by $H(X) = E\{-\log f(X)]\}$. For the pdf in Eq. (4), we have

$$\begin{aligned} H(X) \quad &= \quad -E\{\log g(X)]\} + (1-\alpha)E\{\log G(X)]\} - E\{\log[\alpha + (\beta-\alpha)G^{\beta}(X)]\} \\ & \quad\quad -2E\{\log[G^{\alpha}(X) + 1 - G^{\beta}(X)]\}. \end{aligned}$$

Let

$$L_4(a_1, a_2, a_3; \alpha, \beta) = \int_0^1 \frac{u^{a_1}[\alpha + (\beta-\alpha)u^{\beta}]^{a_2}}{[u^{\alpha} + 1 - u^{\beta}]^{a_3}} du,$$

by using binomial expansion one can obtain

$$L_4(a_1, a_2, a_3; \alpha, \beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} \binom{-a_3}{i}\binom{a_2}{k} \frac{(-1)^j \alpha^{a_2-k}(\beta-\alpha)^k}{a_1 + \beta(j+k) + \alpha(i-j) + 1}.$$

Therefore, we obtain the following proposition.

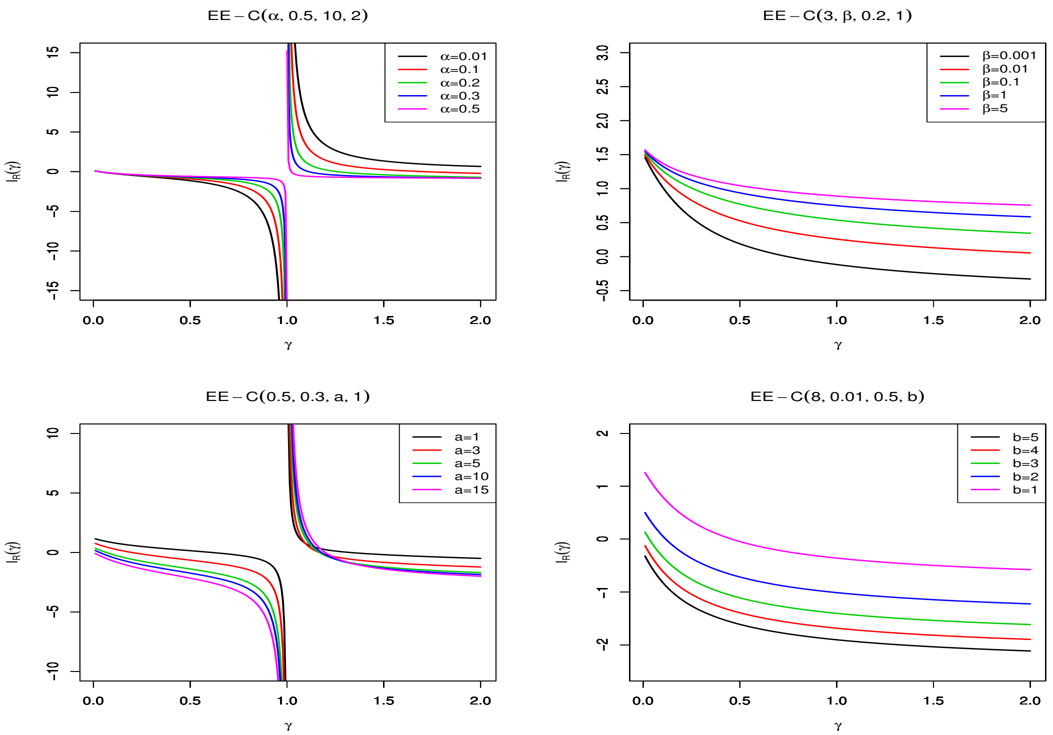

Figure 4. Plots of Rényi entropy of EE-C distribution for parameters.

Proposition 1
Suppose X be a random variable with pdf (4). Then

$$\begin{aligned} E\{\log G(X)]\} &= \frac{\partial}{\partial t} L_4(\alpha-1+t, 1, 2; \alpha, \beta)\Big|_{t=0}, \\ E\{\log[\alpha+(\beta-\alpha)G^{\beta}]\} &= \frac{\partial}{\partial t} L_4(\alpha-1, t+1, 2; \alpha, \beta)\Big|_{t=0}, \\ E\{\log[G^{\alpha}+1-G^{\beta}]\} &= \frac{\partial}{\partial t} L_4(\alpha-1, 1, -t+2; \alpha, \beta)\Big|_{t=0}, \end{aligned}$$

$$E\{\log g(X)]\} = \log(ab) + (b-1)E\{\log X\} + E(X^b) + aE(1-e^{X^b})$$

where

$$\begin{aligned} E\{\log X\} &= \frac{1}{b}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k(j)a_l(k+j)\frac{(-1)^{3j+k+1}L_4(j+k+l+\alpha-1, 1, 2; \alpha, \beta)}{ia^i}, \\ E(X^b) &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_j(i)\frac{(-1)^{i+1}L_4(i+j+\alpha-1, 1, 2; \alpha, \beta)}{ia^i}, \\ E(1-e^{X^b}) &= -\frac{1}{a}\sum_{i=1}^{\infty}\frac{L_4(i+\alpha-1, 1, 2; \alpha, \beta)}{i}. \end{aligned}$$

So, we can obtain the Shannon entropy as

$$\begin{aligned} H(X) &= -\log(ab) + \frac{1-b}{b}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k(j)a_l(k+j)\frac{(-1)^{3j+k+1}L_4(j+k+l+\alpha-1,1,2;\alpha,\beta)}{ia^i} \\ &- \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_j(i)\frac{(-1)^{i+1}L_4(i+j+\alpha-1,1,2;\alpha,\beta)}{ia^i} + \sum_{i=1}^{\infty}\frac{L_4(i+\alpha-1,1,2;\alpha,\beta)}{i} \\ &+ (1-\alpha)\frac{\partial}{\partial t}L_4(\alpha-1+t,1,2;\alpha,\beta)\Big|_{t=0} - \frac{\partial}{\partial t}L_4(\alpha-1,t+1,2;\alpha,\beta)\Big|_{t=0} \\ &- 2\frac{\partial}{\partial t}L_4(\alpha-1,1,-t+2;\alpha,\beta)\Big|_{t=0}. \end{aligned} \tag{22}$$

3. Maximum-likelihood estimation

In order to estimate the parameters of the proposed EE-C density function as defined in Eq. (6), the loglikelihood of the sample is maximized with respect to the parameters. Let $X_1,\cdots,X_n$ be a random sample from EE-C model with unknown parameters α, β, a and b and observed values $x_1,\cdots,x_n$. The log-likelihood function for the parameters of this distribution is given as follows:

$$\begin{aligned} l(\alpha,\beta,a,b;x) &= n\log(ab) + (b-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}x_i^b + a\left[n - \sum_{i=1}^{n}e^{x_i^b}\right] + (\alpha-1) \\ & \sum_{i=1}^{n}\log t_i + \sum_{i=1}^{n}\log\left[\alpha+(\beta-\alpha)t_i^\beta\right] - 2\sum_{i=1}^{n}\log\left[t_i^\alpha+1-t_i^\beta\right], \end{aligned}$$

where $t_i = G(x_i) = 1 - e^{a(1-e^{x_i^b})}$. Therefore we can write

$$\frac{\partial l}{\partial\alpha} = \sum_{i=1}^{n}\log t_i + \sum_{i=1}^{n}\frac{1-t_i^\beta}{\alpha+(\beta-\alpha)t_i^\beta} - 2\sum_{i=1}^{n}\frac{t_i^\alpha(\log t_i)}{t_i^\alpha+1-t_i^\beta} = 0,$$

$$\frac{\partial l}{\partial\beta} = \sum_{i=1}^{n}\frac{t_i^\beta\left[1+(\beta-\alpha)\log t_i\right]}{\alpha+(\beta-\alpha)t_i^\beta} + 2\sum_{i=1}^{n}\frac{t_i^\beta(\log t_i)}{t_i^\alpha+1-t_i^\beta} = 0,$$

$$\begin{aligned} \frac{\partial l}{\partial a} &= \frac{n}{a} + \left(n - \sum_{i=1}^{n}e^{x_i^b}\right) + (\alpha-1)\sum_{i=1}^{n}\frac{t_i^{(a)}}{t_i} + \beta(\beta-\alpha) \\ & \sum_{i=1}^{n}\frac{t_i^{(a)}t_i^{\beta-1}}{\alpha+(\beta-\alpha)t_i^\beta} - 2\sum_{i=1}^{n}\frac{t_i^{(a)}\left[\alpha t_i^{\alpha-1}-\beta t_i^{\beta-1}\right]}{t_i^\alpha+1-t_i^\beta} = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial l}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n}\log x_i(1+x_i^b) - a\sum_{i=1}^{n}(\log x_i)x_i^b e^{x_i^b} + (\alpha-1)\sum_{i=1}^{n}\frac{t_i^{(b)}}{t_i} \\ &+ \beta(\beta-\alpha)\sum_{i=1}^{n}\frac{t_i^{(b)}t_i^{\beta-1}}{\alpha+(\beta-\alpha)t_i^\beta} - 2\sum_{i=1}^{n}\frac{t_i^{(b)}\left[\alpha t_i^{\alpha-1}-\beta t_i^{\beta-1}\right]}{t_i^\alpha+1-t_i^\beta} = 0, \end{aligned}$$

where,

$$t_i^{(a)} = \frac{\partial t_i}{\partial a} = -\left(1 - e^{x_i^b}\right) e^{a(1-e^{x_i^b})}$$

and

$$t_i^{(b)} = \frac{\partial t_i}{\partial b} = a(\log x_i) x_i^b e^{x_i^b} e^{a(1-e^{x_i^b})}.$$

The maximum likelihood estimates $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$ and $\hat{b}$ of α, β, a and b are obtained by solving these nonlinear system of equations.

4. Simulation study

In this section we have presented various methods of parameter estimation such as maximum likelihood, least square, weighted least square, Cramér–von–Mises, Anderson-Darling and right-tailed Anderson-Darling using simulation study.

4.1. The maximum likelihood estimator

It is impossible to find the closed form for the maximum likelihood estimators. In this subsection, the maximum likelihood estimators for parameters of proposed density function has been assessed by simulating at the point $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$. The density function has been indicated in Figure 5.

To verify the validity of the maximum likelihood estimators, the bias and the mean square error of MLE have been used. For example, as described in Section 3, for $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$, $r = 1000$ times for samples of size $n = 20, 21, ..., 80$ of EE-C$(3, 1, 0.2, 1)$. To estimate the numerical value of the maximum likelihood, the $optim$ function (in the $stat$ package) and Nelder-Mead method in R software have been used. If $\boldsymbol{\xi} = (\alpha, \beta, a, b)$, for any simulation by n size and $i = 1, 2, ..., r$, the maximum likelihood estimates are obtained as $\widehat{\boldsymbol{\xi}}_i = (\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{a}_i, \widehat{b}_i)$.

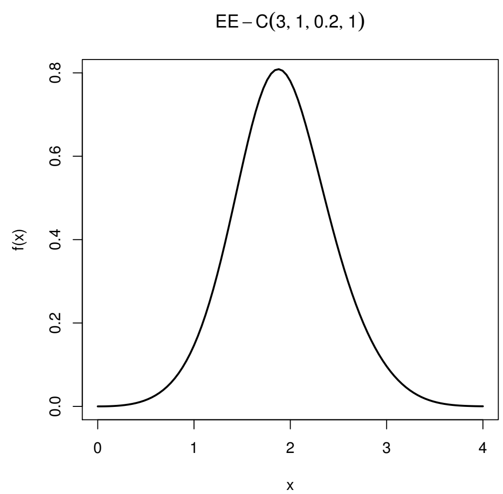

Figure 5. The density function for simulation study.

To examine the performance of the MLE's for the EE-C distribution, we perform a simulation study:

1. Generate r samples of size n from Eq. (6).

2. Compute the MLE's for the r samples, say $(\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})$ for $i = 1, 2, \ldots, r$.
3. Compute the standard errors of the MLE's for r samples, say $(s_{\hat{\alpha}}, s_{\hat{\beta}}, s_{\hat{a}}, s_{\hat{b}})$ for $i = 1, 2, \ldots, r$.
4. Compute the biases and mean squared errors given by

$$Bias_{\widehat{\boldsymbol{\xi}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\xi}}_i - \boldsymbol{\xi}_i)$$

and

$$MSE_{\widehat{\boldsymbol{\xi}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\xi}}_i - \boldsymbol{\xi}_i)^2,$$

for $\boldsymbol{\xi} = (\alpha, \beta, a, b)$.

We repeat these steps for $r = 1000$ and $n = 20, 21, \ldots, n^*$ (n^* is different in each repetition) with different values of (α, β, a, b), so computing $Bias_{\widehat{\boldsymbol{\xi}}}(n)$, $MSE_{\widehat{\boldsymbol{\xi}}}(n)$. By finding a vector of Bias and MSE values, we can understand the changes of these values with increasing n. Figures 6, 7 respectively reveal how the four biases, mean squared errors vary with respect to n. As expected, the Biases and MSEs of the estimated parameters converge to zero as n increases.

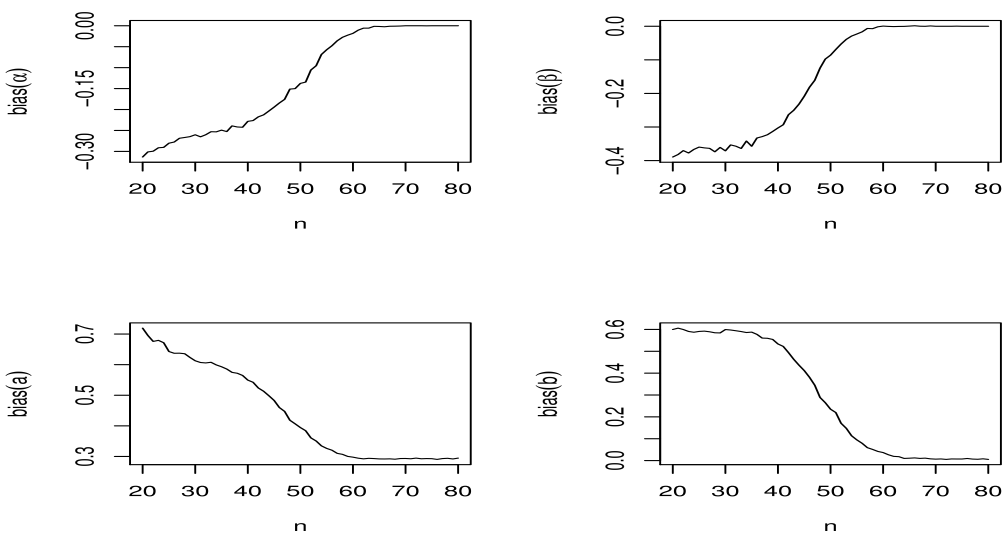

Figure 6. Bias of $\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}$ versus n when $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$.

4.2. The other estimation methods

There are several approaches to estimate the parameters of distributions where each of them has its characteristic features and benefits. In this subsection, five of these methods are briefly introduced and will be numerically investigated in the simulation study (Figure 5). A useful summary of these methods can be seen in Dey *et al.* [10]. Here $\{t_{i:n}; i = 1, 2, ..., n\}$ is the associated order statistics and F is the distribution function of EE-C.

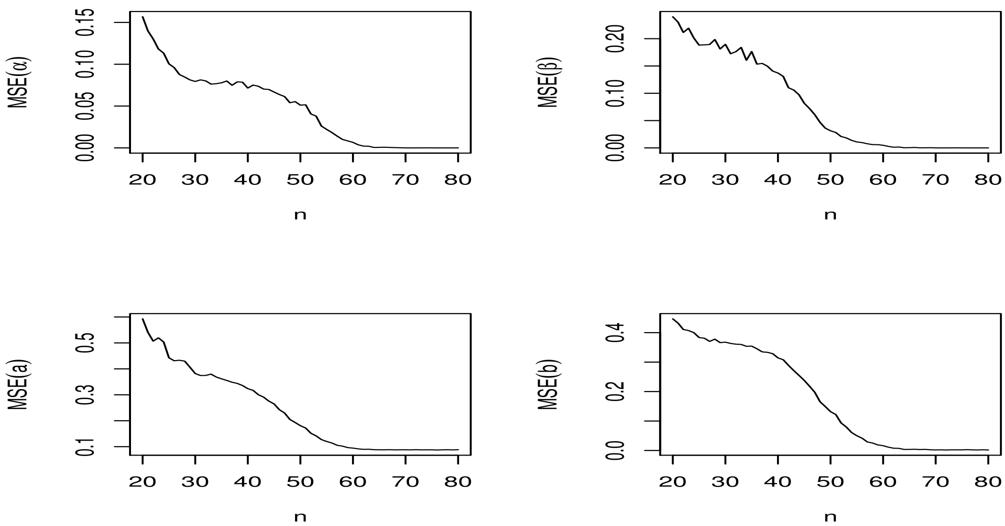

Figure 7. MSE of $\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}$ versus n when $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$.

4.2.1. Least square and weighted least square estimators The Least square (LSE) and Weighted Least Square estimators (WLSE) are introduced by Swain *et al.* [24]. The LSE's and WLSE's are obtained by minimizing

$$S_{\text{LSE}}(\alpha, \beta, a, b) = \sum_{i=1}^{n} \left(F(t_{i:n}; \alpha, \beta, a, b) - \frac{i}{n+1} \right)^2$$

and

$$S_{\text{WLSE}}(\alpha, \beta, a, b) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F(t_{i:n}; \alpha, \beta, a, b) - \frac{i}{n+1} \right)^2$$

with respect to α, β, a and b.

4.2.2. Cramér–von–Mises estimator Cramér–von–Mises Estimator (CME) is introduced by Choi and Bulgren [7]. The CME is obtained by minimizing the following function

$$S_{\text{CME}}(\alpha, \beta, a, b) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(t_{i:n}; \alpha, \beta, a, b) - \frac{2i-1}{2n} \right)^2.$$

4.2.3. Anderson-Darling and right-tailed Anderson-Darling estimators The Anderson Darling (ADE) and Right-tailed Anderson Darling estimators (RTADE) are introduced by Anderson and Darling [3]. The ADE's and RTADE's are obtained by minimizing with respect to α, β, a and b, the functions

$$S_{\text{ADE}}(\alpha, \beta, a, b) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \{ \log F(t_{i:n}; \alpha, \beta, a, b) + \log \overline{F}(t_{i:n+1-i}; \alpha, \beta, a, b) \}$$

and

$$S_{\text{RTADE}}(\alpha, \beta, a, b) = \frac{n}{2} - 2 \sum_{i=1}^{n} F(t_{i:n}; \alpha, \beta, a, b) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \overline{F}(t_{i:n+1-i}; \alpha, \beta, a, b),$$

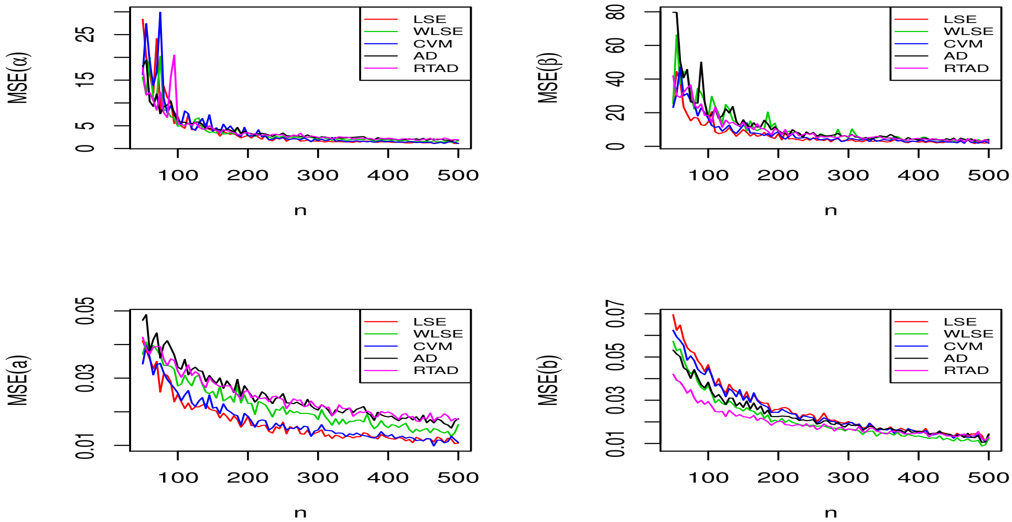

Figure 8. MSE of $\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}$ versus n when $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$.

where $\overline{F}(\cdot) = 1 - F(\cdot)$.
Now, we consider the one model that has been used in this section and investigate MSE of estimators for different samples. For $(\alpha, \beta, a, b) = (3, 1, 0.2, 1)$, we consider the sample of sizes $n = 50, 55, 60, \cdots 500$ with $r = 1000$ replications.

The result of the simulations of this subsection is shown in Figure 8. As it is clear from the MSE plot for two parameters with the increase in the size of the sample all methods will approach to zero.

5. Applications

In this section, we present three applications by fitting the EE-C model and some famous models. The Akaike information criterion (AIC), Bayesian information criterion (BIC), Cramér–von Mises (W*), Anderson-Darling (A*), Kolmogorov Smirnov (K.S) and the P-Value of K.S test, have been chosen for comparison of models for examples.

The Gamma-Chen distribution (GaC) (Alzaatreh *et al.*, [2]), the Beta-Chen distribution (BC) (Eugene emphet al., [11]), Marshall-Olkin Normal distribution (MOC) (Jose, [15]), the Kumaraswamy Chen distribution (KwC) (Cordeiro and de Castro, [8]), the Transmuted Chen (TC) (Khan *et al.*, [18]), the Transmuted Exponentiated Chen (TEC) (Khan *et al.*, [19]), the Extended Chen (EC) and Chen distribution have been selected for comparison in three examples. The parameters of models have been estimated by the MLE method.

5.1. The relief times of twenty patients data

In this section we have examined the data set Gross and Clark [13] on the relief times of twenty patients receiving an analgesic. This data is as follows:
1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.
In Tables 1 and 2, a summary of the fitted information criteria and estimated MLE's for this data with different models have been given, respectively. Models have been sorted from the lowest to the highest value of A*. As you see, the EE-C is selected as the best model with more criteria. The histogram of the relief times of twenty patients

data and the plots of fitted pdf are displayed in Figure 9.

Table 1. The relief times of twenty patients data.

Model	AIC	BIC	W^*	A^*	K.S	P-Value
EE-C	39.41	43.39	0.03	0.18	0.09	0.994
GaC	46.35	50.33	0.03	0.20	0.99	0
TEC	39.56	43.55	0.04	0.23	0.12	0.949
EC	38.14	41.13	0.05	0.30	0.13	0.864
KwC	40.02	44.00	0.05	0.30	0.14	0.820
BC	40.51	44.49	0.06	0.34	0.15	0.769
MOC	44.88	47.87	0.14	0.84	0.15	0.774
TC	53.63	56.62	0.27	1.57	0.23	0.243
Cehn	53.14	55.13	0.29	1.66	0.24	0.206

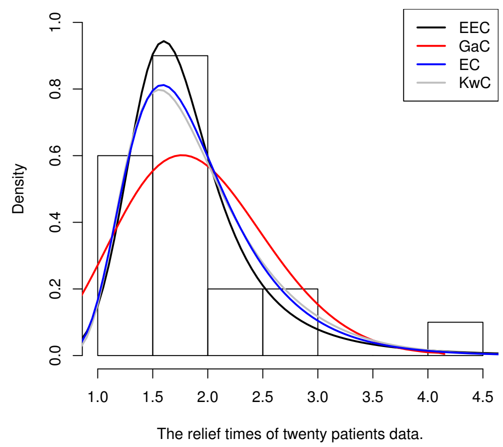

Figure 9. Histogram for the relief times of twenty patients data.

5.2. *Example of Chen (2000)*

In this subsection we study the Example of Chen [6]. The data set is:

29, 1.44, 8.38, 8.66, 10.2, 11.04, 13.44, 14.37, 17.05, 17.13, 18.35.

As is clear, in Tables 3 and 4, the EE-C is selected as the best model with more criteria. The histogram of the Example of Chen [6] data and the plots of fitted pdf are displayed in Figure 10.

5.3. *Minimum Flow*

This subsection is related to study of Minimum Flow data which was presented by Cordeiro *et al.* [9] that include 38 observations. The data set is the following: 43.86, 44.97, 46.27, 51.29, 61.19, 61.20, 67.80, 69.00, 71.84,

Table 2. Estimated MLE's and Standard errors for the relief times of twenty patients data.

Model	MLE	Standard errors
EE-C(α, β, a, b)	$(43.94, 4.44, 1.30, 0.39)$	$(79.16, 16.34, 1.03, 0.18)$
GaC(α, β, a, b)	$(7.59, 1.99, 5.00, 0.53)$	$(2.09, 0.46, 1.07, 0.003)$
TEC(α, β, a, b)	$(300.01, 0.50, 2.43, 0.34)$	$(587.04, 0.56, 1.08, 0.11)$
EC(α, β, c, a, b)	$(250.01, 2.40, 0.37)$	$(407.52, 0.89, 0.10)$
KwC(α, β, a, b)	$(160.07, 0.49, 2.21, 0.52)$	$(222.41, 0.51, 0.75, 0.21)$
BC(α, β, a, b)	$(85.87, 0.48, 2.01, 0.55)$	$(103.13, 0.51, 0.69, 0.20)$
MOC(α, a, b)	$(400.01, 2.32, 0.43)$	$(488.06, 0.64, 0.08)$
TC(α, a, b)	$(0.75, 0.07, 1.02)$	$(0.28, 0.03, 0.09)$
Chen(a, b)	$(0.14, 0.95)$	$(0.05, 0.09)$

Table 3. Example of Chen (2000) data.

Model	AIC	BIC	W^*	A^*	K.S	P-Value
MOC	78.20	79.40	0.03	0.25	0.13	0.976
EE-C	80.27	81.87	0.04	0.29	0.13	0.973
TEC	80.55	82.14	0.04	0.33	0.15	0.930
TC	78.71	79.90	0.05	0.34	0.15	0.920
KwC	80.74	82.33	0.05	0.35	0.16	0.905
BC	80.73	82.32	0.05	0.35	0.16	0.910
EC	78.74	79.93	0.05	0.36	0.16	0.907
Chen	77.17	77.96	0.06	0.40	0.17	0.846
GaC	81.86	83.45	0.27	1.61	0.99	0

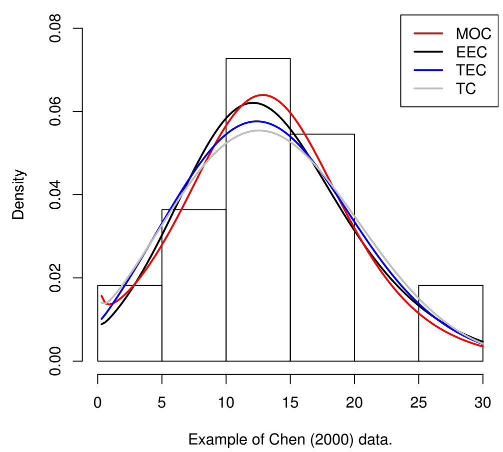

Figure 10. Histogram for Example of Chen (2000) data.

77.31,85.39, 86.59, 86.66, 88.16, 96.03, 102.00, 108.29, 113.00, 115.14, 116.71, 126.86, 127.00, 127.14, 127.29, 128.00, 134.14, 136.14, 140.43, 146.43, 146.43,148.00, 148.43, 150.86, 151.29, 151.43, 156.14, 163.00, 186.43. It is obvious that, in Tables 5 and 6, the EE-C is selected as the best model with all criteria. The histogram of the

Table 4. Estimated MLE's and Standard errors for the Example of Chen (2000) data.

Model	MLE	Standard errors
MOC(α, a, b)	$(100.01, 0.61, 0.30)$	$(340.34, 0.91, 0.12)$
EE-C(α, β, a, b)	$(1.21, 0.22, 0.01, 0.52)$	$(0.68, 0.45, 0.01, 0.08)$
TEC(α, β, a, b)	$(1.60, 0.62, 0.03, 0.47)$	$(1.36, 0.72, 0.05, 0.11)$
TC(α, a, b)	$(-0.88, 0.05, 0.45)$	$(0.63, 0.05, 0.08)$
KwC(α, β, a, b)	$(1.96, 3.21, 0.03, 0.41)$	$(3.38, 44.54, 0.13, 0.47)$
BC(α, β, a, b)	$(1.76, 0.64, 0.07, 0.46)$	$(1.62, 2.36, 0.17, 0.12)$
EC(α, β, c, a, b)	$(1.81, 0.05, 0.45)$	$(1.67, 0.08, 0.10)$
Chen(a, b)	$(0.02, 0.51)$	$(0.01, 0.05)$
GaC(α, β, a, b)	$(6.74, 2.82, 2.83, 0.28)$	$(2.62, 1.18, 0.01, 0.01)$

Minimum Flow data and the plots of fitted pdf are displayed in Figure 11.

Table 5. The Minimum Flow data.

Model	AIC	BIC	W^*	A^*	K.S	P-Value
EE-C	383.81	390.36	0.02	0.21	0.08	0.963
MOC	390.56	395.47	0.09	0.61	0.12	0.689
Cehn	398.62	401.90	0.10	0.64	0.16	0.262
KwC	391.24	397.79	0.11	0.66	0.14	0.409
TC	389.53	394.44	0.11	0.66	0.15	0.383
TEC	391.74	398.29	0.11	0.70	0.15	0.377
EC	389.91	394.82	0.13	0.72	0.15	0.348
BC	392.21	398.76	0.12	0.75	0.15	0.343
GaC	391.48	398.03	0.28	1.71	0.57	0

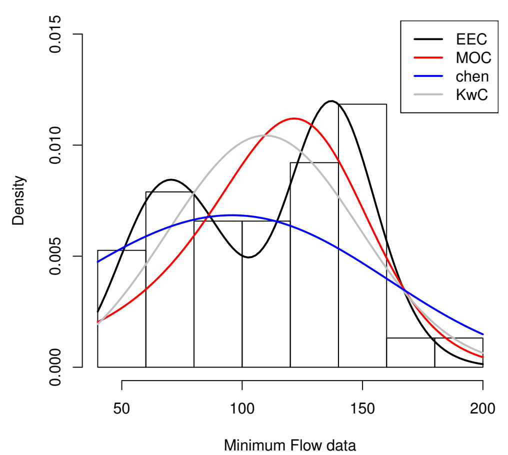

Figure 11. Histogram for Minimum Flow data.

Table 6. Estimated MLE's and Standard Errors for the Minimum Flow data.

Model	**MLE**	**Standard errors**
EE-C(α, β, a, b)	$(5.17, 65.001, 0.02, 0.35)$	$(2.91, 64.92, 0.01, 0.02)$
MOC(α, a, b)	$(13.00, 0.02, 0.34)$	$(18.66, 0.02, 0.04)$
Chen(a, b)	$(0.003, 0.36)$	$(0.001, 0.01)$
KwC(α, β, a, b)	$(4.51, 21.11, 0.02, 0.27)$	$(2.02, 42.85, 0.02, 0.05)$
TC(α, a, b)	$(-1.00, 0.004, 0.37)$	$(0.70, 0.002, 0.01)$
TEC(α, β, a, b)	$(2.74, -0.25, 0.01, 0.35)$	$(1.21, 0.47, 0.01, 0.02)$
EC(α, β, c, a, b)	$(2.86, 0.01, 0.36)$	$(0.98, 0.004, 0.02)$
BC(α, β, a, b)	$(3.01, 0.77, 0.01, 0.35)$	$(1.90, 1.24, 0.01, 0.05)$
GaC(α, β, a, b)	$(3.13, 4.36, 0.09, 0.34)$	$(1.14, 4.43, 0.02, 0.02)$

6. Conclusions

We introduce a new class of distributions called the Extended Exponentiated Chen (EE-C) family. Some characteristics of the new family, such as moments, mean past lifetime, coefficients of skewness and kurtosis, order statistics and asymptotic properties are obtained. We estimate the parameters using maximum likelihood and other different methods. The Bias and MSE plots of parameters for all methods, will approach to zero with the increase in the size of the sample which verifies the validity of the these estimation methods. The flexibility of this distribution is assessed by applying it to real data sets and comparing proposed distribution with others. The results of tables and figures illustrate that the new model provides consistently better fits than other competitive models for these data sets.

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