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Validation of Xgamma Exponential Model via Nikulin-Rao-Robson Goodness-of- Fit-Test under Complete and Censored Sample with Different Methods of Estimation

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Abstract In this article, a new extension of the one parameter Xgamma distribution has been proposed. Also the associated different statistical properties are derived. The unknown parameter of the proposed distribution is estimated by using different classical estimation methods and by using Bayesian estimation method. Under classical methods of estimation, we briefly describe the method of moment estimators, maximum likelihood estimators, maximum product of spacing estimators, least squares and weighted least squares estimators and Cramer-von-Mises estimators. The Bayesian estimation using gamma prior under squared error loss function has been discussed and computed via Lindley's approximation and Markov Chain Monte Carlo techniques. Furthermore, the $100(1-\alpha)\%$ asymptotic confidence interval and credible interval along with the coverage probability are also discussed. The obtained classical and the Bayesian estimators are compared through Monte Carlo simulations. Next, we construct a modified Chi-squared goodness of fit test based on the Nikulin-Rao-Robson (NRR) statistic in presence of censored and complete data. The applicability of our proposed model has been illustrated for both complete data and right censored data by using two real data sets for each.

Keywords Bayesian and Classical Estimation; Censored Data; Monte Carlo Simulations; Nikulin-Rao-Robson goodness-of-fit test.

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1. Introduction and genesis

In statistics, many probability distributions are available to model the time to event data. Statistical distributions are often employed to characterize real world phenomena. For characterization,, one parameter exponential distribution is one of the widely used model. The probability density function (PDF) and the cumulative distribution function (CDF) of a random variable (RV) X with exponential (E) distribution is

$$g_\lambda(x) = \lambda \mathrm{e}^{-\lambda\, x} \mid_{(x>0,\lambda>0)} \text{ and } G_\lambda(x) = 1 - \mathrm{e}^{-\lambda\, x}.$$

The features of one parameter exponential distribution has been broadly described on the basis of hazard rate and lack of memory property and purposely used by several researchers of the medical sciences/engineering sciences/actuarial sciences to model the data and validate their findings. However, the use of exponential

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distribution is restricted to the constant hazard rate thus, statisticians strive continuously for exploring more flexible models. Therefore, many generalized classes of distributions are proposed to illustrate various lifetime phenomena, namely, Weibull, gamma, generalized exponential distribution and many more. The generalization using mixture of two probability distributions is also of deep interest for applied statisticians. In this paper, a new alternative of one parameter E model has been proposed using the approach suggested by Alzaatreh et al. [4].

Let $R(t)$ be the CDF of a rv $T \in [a, b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a function of a baseline CDF $G(x)$ of a rv X, which satisfies the following conditions:

(i) $W[G(x)] \in [a, b]$;
(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing;
(iii) $\lim_{x\to-\infty} W[G(x)] = a$ and $\lim_{x\to\infty} W[G(x)] = b$.
Alzaatreh et al. [4] defined the CDF of the T–X family by

$$F(x) = R\left(W[G(x)]\right), \tag{1}$$

where $W[G(x)]$ satisfies the conditions (i), (ii) and (iii). Let $\overline{G}(x) = 1 - G(x)$.

To implement the approach in our study, Xgamma (Xg) distribution, a mixture of exponential and gamma distribution, proposed and studied the Xg distribution by Sen et al. [42], is used. Xg distribution possess many interesting properties, hence it might be a better alternative choice of E model. The CDF and the PDF are, respectively, given as

$$F_\theta\left(x\right) = 1 - \frac{1+\theta+\theta x+\frac{1}{2}\theta^2x^2}{1+\theta}\,\mathrm{e}^{-\theta x} \mid_{(x>0,\theta>0)}, \tag{2}$$

and

$$f_\theta\left(x\right) = \frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)\mathrm{e}^{-\theta x}, \tag{3}$$

By taking $R(x) = F_\theta\left(x\right)$ and $W[G(x)] = [-\log \overline{G}_\lambda(x)]$ in (3), where $\overline{G}_\lambda(x) = 1 - G_\lambda(x) = e^{-\lambda\, x}$, we define the CDF of the Xgamma-E (Xg-E) model by

$$F_{\text{Xg-E}}(x;\lambda) = 1 - \frac{1}{2}\mathrm{e}^{-\lambda\, x}\left[2+\lambda\, x+\frac{1}{2}\left(\lambda\, x\right)^2\right] \mid_{(x>0,\lambda>0)}, \tag{4}$$

The PDF corresponding to CDF, given in Equation (4), reduces to

$$f_{\text{Xg-E}}(x;\lambda) = \frac{1}{2}\lambda\mathrm{e}^{-\lambda\, x}\left[1+\frac{1}{2}\left(\lambda\, x\right)^2\right]. \tag{5}$$

Equations (4) and (5) can be also derived according to Cordeiro et al. [13]. Many useful probability distributions were presented based on Cordeiro et al. [13] such as Yousof et al. [50]) and Ibrahim et al. ([25]). The hazard function for the proposed distribution is given as

$$H_{\text{Xg-E}}(x;\lambda) = \frac{\lambda\left[1+\frac{1}{2}\left(\lambda\, x\right)^2\right]}{2+\lambda\, x+\frac{1}{2}\left(\lambda\, x\right)^2}.$$

The shape of the density function and hazard function for different choices of the parameter λ are presented in Figure 1. From Figure 1, we note that the proposed model is positively skewed and the hazard rate function $H\left(x,\lambda\right)$ exhibit many important shapes such as the increasing, decreasing, bathtub and the approximately constant shapes. The shape of the hazard rate can also be traced mathematically [see, Glaser [15]]. For this purpose, let us define the function as; $g(t) = \frac{1}{h(t)}$, $\eta(t) = \frac{-f'(t)}{f(t)}$. Now, in case of the Xg-E model

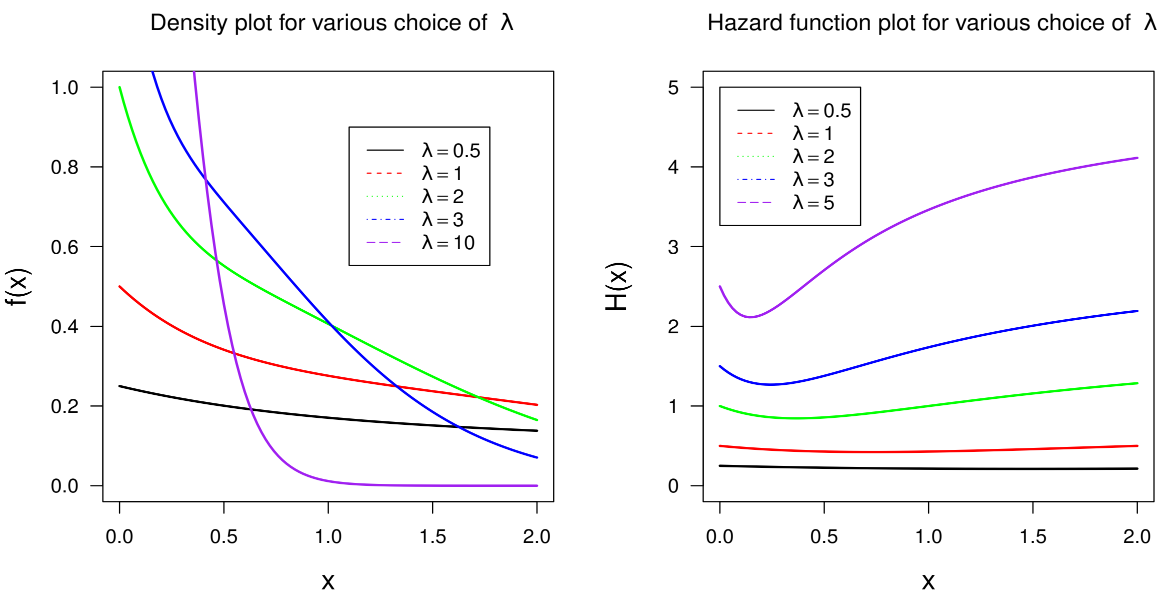

Figure 1. Density and hazard functions plot.

$$g(t) = \frac{2 + \lambda t + \frac{1}{2}(\lambda t)^2}{1 + \frac{1}{2}(\lambda t)^2}, \ \eta(t) = \lambda - \frac{(\lambda t)^2}{1 + \frac{1}{2}(\lambda t)^2}.$$

the, it may be easily verified that

$$\eta'(t) = \begin{cases} = 0 & t = \frac{\sqrt{2}}{\lambda} \\ < 0 & t \, \epsilon \left(0, \frac{\sqrt{2}}{\lambda}\right) \\ > 0 & t > \frac{\sqrt{2}}{\lambda} \end{cases}, \tag{6}$$

where

$$\eta'(t) = \frac{-\lambda^2 \left[1 - \frac{1}{2}(\lambda t)^2\right]}{\left[1 + \frac{1}{2}(\lambda t)^2\right]^2}.$$

From Equation (5), it can be evidently stated that the shape of hazard rate is either bathtub or increasing. Further, to get a more clear picture of the hazard rate we go on analyzing the behaviour of $g(t)\,\eta(t)$ by defining

$$\delta = \lim_{t \to 0} g(t)\,\eta(t) = 2\lambda$$

and obtained that;

$$\delta = \begin{cases} < 1 \, |_{(\lambda < \frac{1}{2})} \\ = 1 \, |_{(\lambda = \frac{1}{2})} \\ > 1 \, |_{(\lambda > \frac{1}{2})} \end{cases}, \tag{7}$$

the above expression clearly elucidate that for $\lambda < \frac{1}{2}$ we have increasing hazard rate and for $\lambda > \frac{1}{2}$ it is bathtub. Now, to comment on the shape of hazard at $\lambda = \frac{1}{2}$, we studied the nature of second derivative of $g(t)$ near zero, i.e,

$$\lim_{t \to 0} g''(t) = -2\lambda^2(1 + \lambda) < 0,$$

indicating that at $\lambda = \frac{1}{2}$ hazard rate initially decreases.

The objective of the article is four fold: First, we have introduced a new extension of Xg distribution named as one parameter Xg-E distribution and studied its related distributional properties such as moments, mean deviation, generating functions, mean residual life, reliability curve, entropy, stochastic ordering, stress-strength reliability and order statistics. The proposed distribution admits the shape of increasing ($\lambda \leq 0.5$) and bathtub ($\lambda > 0.5$) hazard rate. Also, for $\lambda = 1$, mean is lesser than variance and it is vice versa for all choices of $\lambda \geq 2$. All these characteristics of the newly developed model provide a flexible approach to analyze several reliability/survival data sets. Second, we consider different classical methods of point estimation, namely method of moment estimator (MME), maximum likelihood estimator (MLE), maximum product spacing estimator (MPSE), ordinary least and weighted least squares estimator (LSE & WLSE) and Carmer-Von Mises estimator (CVME) to estimate the unknown parameter λ based on complete sample information. Next, Bayesian estimation method for the unknown parameter under gamma prior has also been discussed using two Bayesian computational techniques. Third, the asymptotic confidence interval (ACI) and Bayesian credible interval (BCI) of the parameter λ based on asymptotic theory of MLE and posterior distribution have also been constructed. The performances of these methods of point estimation are assessed on the basis of average mean square error (MSE) by using Monte Carlo simulations. However, the interval estimation are compared in terms of average widths and corresponding coverage probabilities. Fourth, the MLE for right censored sample for the proposed model is also discussed for different variation of sample size. Also, the validity of Xg-E model for complete and censored reliability/ survival data has been explained through goodness of fit test. We have also constructed a modified Chi-square goodness-of-fit test based on the Nikulin-Rao-Robson (NRR) statistic for censored and complete data. The theory and the mechanism of the Y_n^2 test statistic is discussed as well. To the best of our knowledge, many literature are available to introduce new probability distribution but no attempt has been made to introduce an alternative of exponential distribution by using Xg distribution as base line distribution. Therefore, the present work aims to fill the gap in the light of this model.

The remainder of the present article is unified as follows. In Section 2, we describe the different distributional properties such as moments, generating functions, reliability curve, stochastic ordering, entropy, stress-strength reliability, order statistics of the new distribution. Different methods of estimation, including classical and Bayesian for the complete sample have been discussed in Section 3 and compared in Section 4. MLE for the censored sample is discussed in Section 4 and corresponding simulation result is presented in its subsection. Section 5, describes MLE and corresponding simulation result for the right censored case. Section 6, 7, describe the goodness of fit test using NRR statistic for complete sample. Application based on censored data using modified NRR statistic has been discussed in Section 8. Finally, concluding remarks are given in Section 9.

2. Distributional properties

2.1. Moments and related measures

Let a RV X follows Xg-E(λ). The r^{th} raw moments about origin is $\mu_r^{'} = \mathbf{E}(x^r)$ and is obtained as

$$\begin{aligned} \mu_r^{'} &= \frac{\lambda}{2}\int_{x=0}^{\infty} x^r e^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right] dx \\ &= \frac{1}{2}\lambda^{-r}\left[\Gamma(r+1)+\frac{1}{2}\Gamma(r+3)\right]. \end{aligned}$$

In particular, the first four moments are

$$\mu_1^{'} = \frac{2}{\lambda},\ \mu_2^{'} = \frac{7}{\lambda^2},\ \mu_3^{'} = \frac{33}{\lambda^3} \text{ and } \mu_4^{'} = \frac{192}{\lambda^4}.$$

Making use of these raw moments, the first four central moments are obtained as follows

$$\mu_2 = \frac{3}{\lambda^2},\ \mu_3 = \frac{7}{\lambda^3} \text{ and } \mu_4 = \frac{48}{\lambda^4}.$$

The coefficent of variation ($\mathbf{CV}$) is calculated as

$$\mathbf{CV} = \frac{\sqrt{\mu_2}}{\mu_1^{'}} = 0.86.$$

The coefficient of skewness (β_1) and kurtosis (β_2) based on central moments are computed by using the following relations;

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 1.81 \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = 5.33.$$

From above, it is clear that $\mu_3,\ \beta_1 > 0$ and $\beta_2 > 3$ which indicates that the proposed distribution is positively skewed and nature of curve is leptokurtic. Hence, the proposed model may adequately fit the lifetime data and sometimes taken as an alternative to one parameter family of distributions.

2.2. Mean deviations

Here, we have derived the expressions of mean deviations about mean (μ) and median ($\mathbf{M}$) for the proposed model. The mean deviation about mean is defined as;

$$\begin{aligned} \mathbf{MD}_{(\mu)} &= \int_{x=0}^{\infty} |x-\mu| f(x)\, dx = \int_{\mu}^{\infty} (x-\mu) f(x) dx - \int_{0}^{\mu} (x-\mu) f(x) dx \\ &= 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} x f(x) dx. \end{aligned} \tag{8}$$

Now, we evaluated the value of integral

$$\int_{\mu}^{\infty} \frac{\lambda}{2} x e^{-x\lambda} \left[1 + \frac{1}{2}(\lambda x)^2\right] dx = 11\lambda^{-1}\mathrm{e}^{-2},$$

using the CDF at the point μ and integral value, we get

$$\mathbf{MD}(\mu) = 10\lambda^{-1}\mathrm{e}^{-2}.$$

The mean deviation about median (M) is calculated by;

$$\begin{aligned} \mathbf{MD}_{(M)} &= \int_{x=0}^{\infty} |x-M| f(x)\, dx = \int_{M}^{\infty} (x-M) f(x) dx - \int_{0}^{\mu} (x-\mu) f(x) dx \\ &= -\mu + 2\int_{M}^{\infty} x f(x) dx \\ &= -mu + 2\int_{M}^{\infty} \frac{x\lambda}{2} \mathrm{e}^{-x\lambda} \left[1 + \frac{1}{2}(\lambda x)^2\right] dx \\ &= -\mu + \mathrm{e}^{-\lambda M} \left[4M + \frac{4}{\lambda} + \frac{\lambda^2 M^3}{2} + \frac{3}{2}\lambda M^2\right]. \end{aligned} \tag{9}$$

2.3. Generating functions

The moment generating function $M_X(t)$ for the Xg-E(λ) is calculated by

$$M_X(t) = \mathbf{E}(\mathrm{e}^{tx}) = \int_{x=0}^{\infty} \mathrm{e}^{tx} f(x,\lambda)\, dx = \frac{\lambda}{2} \int_{x=0}^{\infty} \mathrm{e}^{-x(\lambda-t)} \left[1 + \frac{1}{2}(\lambda x)^2\right] dx, \tag{10}$$

after simplification

$$M_X(t) = \frac{\lambda}{2(\lambda - t)}\left[1 + \frac{\lambda^2}{(\lambda - t)^2}\right].$$

The cumulant generating function (CGF) can be expressed as

$$K_X(t) = \log M(t) = \log\lambda - \log(2) - \log(\lambda - t) + \log\left[1 + \frac{\lambda^2}{(\lambda - t)^2}\right].$$

Proceeding on same lines as in mgf and replacing t by it we get characteristic function as follows

$$\phi_X(t) = \mathbf{E}(\mathrm{e}^{itx}) = \frac{\lambda}{2(\lambda - it)}\left[1 + \frac{\lambda^2}{(\lambda - it)^2}\right],$$

where $i^2 = -1$.

2.4. Mean residual life

Mean residual life (MRL) $m(x)$ is an important characteristic of any lifetime distribution to study the expected remaining life. Mathematically, it is obtained by

$$m(x) = \frac{1}{1 - F(x)}\int_{t=x}^{\infty}[1 - F(t)]\,dt, \tag{11}$$

using the CDF of the Xg-E in above equation

$$\begin{aligned} m(x) &= \frac{\int_{t=x}^{\infty}\mathrm{e}^{-\lambda\,t}\left[2 + \lambda\,t + \frac{1}{2}\,(\lambda\,x)^2\right]\,dt}{\mathrm{e}^{-\lambda\,x}\left[2 + \lambda\,x + \frac{1}{2}\,(\lambda\,x)^2\right]} \\ &= \frac{6 + \lambda^2\,x^2 + 4\lambda\,x}{4\lambda + 2\lambda^2\,x + \lambda^3\,x^2}. \end{aligned} \tag{12}$$

2.5. Reliability curves

Reliability curve is also called the Bonferroni and Lorenz curves. These curves have vital application in economics and are used to study the income and poverty level, but now a days these are frequently used in reliability, demography, insurance, and medical sciences. The Bonferroni and Lorenz curves are defined by

$$B_c(p) = \frac{1}{p\mu}\int_{x=0}^{q} xf(x)dx, \tag{13}$$

and

$$\mathbf{L}_c(q) = \frac{1}{\mu}\int_{x=0}^{q} xf(x)dx, \tag{14}$$

where μ is mean of the distribution. After putting the $f(x)$ in above two equations, the Bonferroni and Lorentz curves are computed as follows

$$\begin{aligned} B_c(p) &= \frac{1}{p\mu}\int_{x=0}^{q}\frac{x}{2}\lambda\mathrm{e}^{-\lambda\,x}\left[1 + \frac{1}{2}\,(\lambda\,x)^2\right]dx \\ &= \frac{1}{p} - \frac{\lambda\mathrm{e}^{-q\lambda}}{2p}\left(2q + \frac{2}{\lambda} + \frac{\lambda^2 p^3}{4} + 3\lambda p^2\right), \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathbf{L}_c(q) &= \frac{1}{\mu}\int_{x=0}^{q}\frac{x}{2}\lambda\mathrm{e}^{-\lambda\,x}\left[1 + \frac{1}{2}\,(\lambda\,x)^2\right]dx \\ &= 1 - \frac{\lambda\mathrm{e}^{-q\lambda}}{2}\left(2q + \frac{2}{\lambda} + \frac{\lambda^2 q^3}{4} + 3\lambda q^2\right). \end{aligned} \tag{16}$$

2.6. Entropies

The entropy of a RV X measures the level of uncertainty associated with the distribution of X. A popular entropy measure is Renyi entropy. If RV X has the probability density function $f(x)$, then Renyi entropy is defined by

$$R_{\mathrm{e}}(\tau)=\frac{1}{1-\tau}\log\left[\int_x f(x)^\tau\,dx\right], \tag{17}$$

where $\tau>0$ and $\tau\neq 1$. Thus the Renyi entropy for the proposed model is calculated as;

$$\begin{aligned} R_{\mathrm{e}}(\tau)&=\frac{1}{1-\tau}\log\left[\int_x\left\{\frac{1}{2}\lambda\mathrm{e}^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right]\right\}^\tau dx\right]\\ &=\frac{1}{1-\tau}\log\left[\left(\frac{\lambda}{2}\right)^\tau\int_{x=0}^{\infty}\mathrm{e}^{-\tau\lambda x}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left(\frac{\lambda^2x^2}{2}\right)^\kappa dx\right], \end{aligned} \tag{18}$$

after simplifying the above expression, we get

$$R_{\mathrm{e}}(\tau)=\frac{1}{1-\tau}\log\left[\sum_{\kappa=0}^{\tau}\frac{\lambda^{\tau-1}}{2^{\tau+\kappa}}\binom{\tau}{\kappa}\frac{\Gamma(2\kappa+1)}{\tau^{2\kappa+1}}\right].$$

Another particular form of stated entropy, obtained through simple algebraic manipulation is called as Shannon entropy and is defined by

$$S_{\mathrm{e}}=\mathrm{e}(-\log f(x))=-\mathbf{E}(\log f(x))=-\int_{x=0}^{\infty}\log f(x)\,f(x)\,dx,$$

after putting the value of density function,

$$\begin{aligned} S_{\mathrm{e}}&=-\int_{x=0}^{\infty}\begin{pmatrix}\log\left\{\left[\frac{1}{2}\lambda\mathrm{e}^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right]\right]\right\}\\ \times\frac{1}{2}\lambda\mathrm{e}^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right]\end{pmatrix}dx\\ &=-\int_{x=0}^{\infty}\log\left(\frac{\lambda}{2}\right)\frac{\lambda\mathrm{e}^{-\lambda x}}{2}\left[1+\frac{1}{2}(\lambda x)^2\right]dx\\ &+\frac{\lambda^2}{2}\int_{x=0}^{\infty}x\mathrm{e}^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right]dx\\ &-\frac{\lambda}{2}\int_{x=0}^{\infty}\log\left[1+\frac{1}{2}(\lambda x)^2\right]\mathrm{e}^{-\lambda x}\left[1+\frac{1}{2}(\lambda x)^2\right]dx, \end{aligned} \tag{19}$$

after simplifying the above integral, the Shanon entropy is given by

$$S_{\mathrm{e}}=2-\log\left(\frac{\lambda}{2}\right)-\sum_{n=1}^{\infty}\frac{\Gamma(1+2n)}{2^{n+1}n}\left[1+\frac{(1+2n)(2+2n)}{2}\right].$$

2.7. Stochastic ordering

Stochastic ordering is very useful property of a rv to study the ordering relations between them. Let X_1 and X_2 are the two rv having CDF $F_1(x)$ and $F_2(x)$ respectively. Then X_1 is said to be stochastically greater than X_2 iff $F_1(x)\geq F_2(x)$ for all x. At first this criterion was used by Shaked and Shanthikumar [41] and latter-on Mann et al. [30] proposed it for estimator comparison criterion.

Theorem 1
Let X_1 and X_2 be the two RVs from the Xg-E distribution, with parameter λ_1 and λ_2 respectively, then X_1 is stochastically greater than X_2 iff $\lambda_1>\lambda_2$.

Proof
Form Equation (3), we have

$$\frac{F_1(x,\lambda_1)}{F_2(x,\lambda_2)} = \frac{\left[1-\frac{1}{2}\mathrm{e}^{-\lambda_1 x}\left(2+\lambda_1 x+\frac{1}{2}(\lambda_1 x)^2\right)\right]}{\left[1-\frac{1}{2}\mathrm{e}^{-\lambda_2 x}\left(2+\lambda_2 x+\frac{1}{2}(\lambda_2 x)^2\right)\right]}. \tag{20}$$

which will be always greater than 1, showing that X_1 is stochastically greater than X_2 for $\lambda_1 > \lambda_2$. ∎

Theorem 2
Let X_1 and X_2 are the two continuous rv with densitie functions $f(\lambda_1)$ and $g(\lambda_2)$ respectively. Then $X_1 \leq_{lr} X_2$, iff $\Psi = \left[\frac{f(x)}{g(x)}\right]$ is decreasing function in x. Shaked and Santhikumar [41] have mentioned that the ordering in likelihood ratio implies ordering in hazard rate and stochastic ordering i.e.

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

Proof
Since, X_1 ~Xg-E(λ_1) and X_2 ~Xg-E(λ_2), then

$$\Psi = \frac{\lambda_1 \mathrm{e}^{(\lambda_2-\lambda_1)x}\left[1+\frac{1}{2}(\lambda_1 x)^2\right]}{\lambda_2\left[1+\frac{1}{2}(\lambda_2 x)^2\right]}, \tag{21}$$

and

$$\frac{d\Psi}{dx} = \left(\frac{\lambda_1}{\lambda_2}\right)(\lambda_2-\lambda_1)\mathrm{e}^{(\lambda_2-\lambda_1)x}\left[\frac{2+\frac{1}{2}\lambda_1^2x^2}{2+\frac{1}{2}\lambda_2^2x^2} - \frac{4(\lambda_2+\lambda_1)x}{\left(2+\frac{1}{2}\lambda_2^2x^2\right)^2}\right], \tag{22}$$

if $\lambda_1 > \lambda_2$, $\frac{d\Psi}{dx} < 0$. Hence Ψ is a decreasing function of x which implies $X_1 \leq_{lr} X_2$. Similarly, the other ordering relations can be prove.

$$X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2.$$

∎

2.8. Stress-strength reliability

In mechanical engineering, stress-strength reliability is very frequently used to measure the performances of the equipment in use. Let X and Y denote the strength-stress RVs observed from the population Xg-E(λ_1) and Xg-E(λ_2), respectively. Then the probability $P[Y < X]$ is called as stress-strength reliability parameter. It is denoted by R. The same is evaluated for the proposed model and is given by;

$$\begin{aligned} R = \Pr[Y < X] &= \int_{x=0}^{\infty}\int_{y=0}^{x} f(x,\lambda_1)\, f(y,\lambda_2)dx\,dy \\ &= \int_{x=0}^{\infty} f(x,\lambda_1)\, F(x,\lambda_2)dx, \end{aligned} \tag{23}$$

using the PDF and CDF of the proposed model, R is calculated as;

$$\begin{aligned} R &= \frac{1}{2}\lambda_1 \int_{x=0}^{\infty} \left\{ \begin{array}{c} \mathrm{e}^{-\lambda_1 x}\left[1+\frac{1}{2}(\lambda_1 x)^2\right] \\ \times\left[1-\frac{1}{2}\mathrm{e}^{-\lambda_2 x}\left(\begin{array}{c} 2+\lambda_2 x \\ +\frac{1}{2}(\lambda_2 x)^2 \end{array}\right)\right] \end{array} \right\} dx \\ &= 1 - \frac{\lambda_1}{2(\lambda_1+\lambda_2)}\left[\begin{array}{c} 1+\frac{\lambda_2}{2(\lambda_1+\lambda_2)}+\frac{2\lambda_1^2+\lambda_2^2}{2(\lambda_1+\lambda_2)^2} \\ +\frac{3\lambda_1^2\lambda_2}{2(\lambda_1+\lambda_2)^3}+\frac{3\lambda_1^2\lambda_2^2}{(\lambda_1+\lambda_2)^4} \end{array}\right]. \end{aligned} \tag{24}$$

2.9. Order statistics

Let $X_1, X_2, \ldots, X_n$ are the random sample of size n taken from the Xg-E(λ). Then, the observations $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ constitute the order statistics. Let $f_r(x)$, $F_r(x)$ be the PDF and CDF of r^{th} order statistics $X_{r:n}$ and are given as

$$f_r(X_r = t) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(t)[1 - F(t)]^{n-r} f(t). \tag{25}$$

$$F_r(x) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^j F^{i+j}(x).$$

respectively. Now by using equations (2) and (3) in above expression, we get

$$\begin{aligned} f_r(X_r = t) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{1}{2}\right)^{n-r+1} \left[1 - \frac{1}{2} e^{-\lambda t} \left(\begin{array}{c} 2 + \lambda t \\ + \frac{1}{2} (\lambda t)^2 \end{array} \right)\right]^{r-1} \\ \times \left[\left(\begin{array}{c} 2 + \lambda t \\ + \frac{1}{2} (\lambda t)^2 \end{array} \right)\right]^{n-r} \frac{1}{2} \lambda e^{-\lambda t(n-r+1)} \left[1 + \frac{1}{2} (\lambda t)^2\right], \end{aligned} \tag{26}$$

and

$$F_r(X_r = t) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^j \left\{1 - \frac{1}{2} e^{-\lambda t} \left[\begin{array}{c} 2 + \lambda t \\ + \frac{1}{2} (\lambda t)^2 \end{array} \right]\right\}^{i+j} (t),$$

respectively. Further the density and distribution function of $\min(X_1, X_2, \cdots, X_n)$ order statistics and largest order statistics $\max(X_1, X_2, \cdots, X_n)$ are obtained by putting $r = 1$ & $r = n$ in above equations. Also, the joint distribution of r^{th} and s^{th} order statistics is computed by using the following relation.

$$\begin{aligned} f_{r,s}(X_r = t_1, X_s = t_2) = \frac{n! F^{r-1}(t_1)[f(t_2) - F(t_1)]^{s-r-1}}{(s-r-1)!(n-s)!} \\ \times [1 - F(t_2)]^{n-s} f(t_1) f(t_2), \end{aligned} \tag{27}$$

now by putting the value of PDF and CDF, we get

$$\begin{aligned} f_{r,s}[X_r = t_1, X_s = t_2] = \frac{n! \left(\frac{1}{2}\right)^{n-s+2}}{(s-r-1)!(n-s)!} \left\{1 - \frac{1}{2} e^{-\lambda t} \left[\begin{array}{c} 2 + \lambda t \\ + \frac{1}{2} (\lambda t)^2 \end{array} \right]\right\}^{r-1} \\ \times \left\{\left(2 + \lambda t_1 + \frac{1}{2} (\lambda t_1)^2\right) - \left[\begin{array}{c} 2 + \lambda t_2 \\ + \frac{1}{2} (\lambda t_2)^2 \end{array} \right]\right\}^{s-r-1} \\ \times \left[\left(2 + \lambda t_2 + \frac{1}{2} (\lambda t_2)^2\right)\right]^{n-s} \lambda^2 e^{-\lambda (t_1 + t_2 + (n-s) t_2)} \\ \times \left[1 + \frac{1}{2} (\lambda t_1)^2\right] \left[1 + \frac{1}{2} (\lambda t_2)^2\right]. \end{aligned} \tag{28}$$

The joint distribution of (X_1, X_n) is obtained by putiing $r = 1, s = n$ in above equation.

3. Different method of estimation

In this section, the different classical methods of estimation, namely, method of moment, the method of maximum likelihood, the method of product spacing, least squares, Crammer Von-Mises method of estimation and the Bayesian estimation have been discussed. A brief description of these methods is detailed in the following sub-sections.

3.1. Method of moment

Method of moment estimate for the proposed one parameter model can be obtained by equating the first theoretical moment with the sample moment $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$. The theoretical moment for (1) is obtained as $\mu = \frac{2}{\lambda}$. Therefore, the moment estimate is obtained by

$$\hat{\lambda}_m = \frac{2}{m_1}.$$

3.2. Method of maximum likelihood estimation

The most efficient and widely used method of estimation for the parameter is the method of maximum likelihood. The estimator obtained by this method possess many desirable properties such as consistency, asymptotic efficiency, and invariance. Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from Equation (5), then it's the log-likelihood function of Equation (5) without constant term is given by;

$$\log \mathbf{L} = n \log \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log \left[1 + \frac{1}{2}(\lambda x)^2\right]. \tag{29}$$

The maximum likelihood estimate of the parameter λ is obtained by solving the following non-linear equation.

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\lambda x^2}{1 + \frac{1}{2}(\lambda x)^2} = 0, \tag{30}$$

from above equation it is clear that the direct solution for λ cannot be obtained; thus here we use nlm() function to extract the solution. Also, due to the implicit form of the likelihood equation, the exact distribution of MLE is not obtainable. Therefore the asymptotic theory of MLE is used to compute $100(1-\alpha)\%$ confidence interval for the parameter λ are obtained by following;

$$[\lambda_L, \lambda_U] \in \hat{\lambda}_{ml} \mp Z_{\alpha/2}\sqrt{\mathbf{Var}(\hat{\lambda})},$$

where; λ_{ml} is the MLE of λ, $\mathbf{Var}(\hat{\lambda}) = I^{-1}(\hat{\lambda})$ and $Z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ quantile of standard normal variates.

3.3. Method of maximum product spacing estimation

An alternative method of estimation to method of maximum likelihood estimation is the MPS method which possess similar property as former. The MPS method was discussed by Cheng and Amin [9]. It has been proven by Coolen and Newby [12] under certain regularity conditions MPS estimators seem to be as efficient as MLE. Recently, Singh et al. [43] used this method and illustrated its beauty. In this method, the likelihood function is defined as the differences between two consecutive CDFs and is given by;

$$\mathbf{L}^{'}(\lambda) = \sqrt[n+1]{\prod_{i=1}^{n+1} \Delta_i} \,\Big|_{\left(\sum_{i=1}^{n} \Delta_i = 1\right)}, \tag{31}$$

and

$$\begin{aligned} \ln \mathbf{L}^{'}(\lambda) &= \frac{1}{n+1}\sum_{i=1}^{n+1} \ln \Delta_i \\ &= \frac{1}{n+1}\left[\ln \Delta_1 + \sum_{i=2}^{n} \ln \Delta_i + \ln \Delta_{n+1}\right], \end{aligned} \tag{32}$$

where

$$\Delta_1 = 1 - \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(1)}}\left[2 + \lambda\, x_{(1)} + \frac{1}{2}\left(\lambda\, x_{(1)}\right)^2\right],$$

$$\Delta_i = \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(i-1)}}\left[\begin{array}{c} 2 + \lambda\, x_{(i-1)} \\ +\frac{1}{2}\left(\lambda\, x_{(i-1)}\right)^2 \end{array}\right] - \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(i)}}\left[\begin{array}{c} 2 + \lambda\, x_{(i)} \\ +\frac{1}{2}\left(\lambda\, x_{(i)}\right)^2 \end{array}\right]$$

and

$$\Delta_{n+1} = \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(n)}}\left[2 + \lambda\, x_{(n)} + \frac{1}{2}\left(\lambda\, x_{(n)}\right)^2\right].$$

The MPSE of λ is obtained by maximizing the above Equation with respect to the parameter.

3.4. Method of ordinary least squares estimation

In the theory of classical estimation method, LSE and WLSE are also a conventional estimator to obtain the estimate of the parameter and was introduced by Swain et al. [45]. They used LSE and WLSE to estimate the parameters of a Beta distribution. The LSEs of the unknown parameter of Xg-E has been obtained by minimizing the residual sum of the square; where residual is defined as the differences of theoretical CDF and empirical CDF.

$$\mathbf{L}^{''} = \sum_{i=1}^{n}\left[F_{\text{Xg-E}}(x_i, \theta) - \frac{i}{n+1}\right]^2, \tag{33}$$

substituting Equation (4) above, we get

$$\mathbf{L}^{''} = \sum_{i=1}^{n}\left(1 - \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(1)}}\left\{\begin{array}{c} 2 + \lambda\, x_{(1)} \\ +\frac{1}{2}\left[\lambda\, x_{(1)}\right]^2 \end{array}\right\} - \frac{i}{n+1}\right)^2. \tag{34}$$

The LSE of the parameter λ is obtained by minimizing above with respect to λ and WLSE is obtained by minimizing the following;

$$\mathbf{L}^{'''} = \sum_{i=1}^{n} W_i^{'}\left(1 - \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(1)}}\left\{\begin{array}{c} 2 + \lambda\, x_{(1)} \\ +\frac{1}{2}\left[\lambda\, x_{(1)}\right]^2 \end{array}\right\} - \frac{i}{n+1}\right)^2, \tag{35}$$

where, $W_i^{'}$ is the weight function at the point i and is taken as

$$W_i^{'} = \mathbf{Var}^{-1}\left[(F(x_i)\right] = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

3.5. Cramer-von-Mises estimation

CVME was proposed and used by MacDonald [29]. This method is based on the minimum difference between empirical and cumulative distribution functions. For λ it is obtained by minimizing

$$M^{'} = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2. \tag{36}$$

Hence, from equation (4) and (36), we get

$$M^{'} = \frac{1}{12n} + \sum_{i=1}^{n}\left[1 - \frac{1}{2}\mathrm{e}^{-\lambda\, x_{(1)}}\left[2 + \lambda\, x_{(1)} + \frac{1}{2}\left(\lambda\, x_{(1)}\right)^2\right] - \frac{2i-1}{2n}\right]^2. \tag{37}$$

Now, differentiating Equation (36) respect to λ and equating to zero we get;

$$\sum_{i=1}^{n}\left[W_i(\theta) - \frac{2i-1}{2n}\right] W_i^{'}(\theta) = 0. \tag{38}$$

Since this equation cannot be solved analytically hence we use some numerical technique.

3.6. Bayesian estimation

In this subsection, we discuss the Bayesian procedure to estimate the unknown parameter of the proposed distribution. It is to be noted that the Bayesian estimation is posterior based inference and hence for the parameter λ it is derived under the assumption of gamma prior. Since no conjugate prior exist for Xg-E distribution; thus gamma prior is taken under consideration. The considered prior is very flexible and also converted to the non-informative prior. The prior density for λ is given by;

$$g(\lambda) \propto \lambda^{r-1} \mathrm{e}^{-s\lambda} \quad |_{(\lambda>0)}, \tag{39}$$

where r, s are the hyper-parameters of the considered prior and are assumed to be known. Now, using likelihood equation and prior, the posterior distribution is obtained by

$$\begin{aligned} p(\lambda|\underline{\mathbf{x}}) &= \frac{\mathbf{L}(x|\lambda)\, g(\lambda)}{\int_{\lambda=0}^{\infty} \mathbf{L}(x|\lambda)\, g(\lambda)\, d\lambda} \\ &= \frac{\lambda^{n+r-1} \mathrm{e}^{-\lambda(s+\sum_{i=1}^{n} x_i)} \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x)^2\right]}{\int_{\lambda=0}^{\infty} \lambda^{n+r-1} \mathrm{e}^{-\lambda(s+\sum_{i=1}^{n} x_i)} \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x)^2\right] d\lambda}. \end{aligned} \tag{40}$$

Here, we assume the squared error loss function (SELF) to obtain the Bayesian estimate of the parameter λ. Let $\hat{\theta}$ is the estimate of θ, then SELF is defined as

$$\mathbf{L}_f(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2.$$

It is mentioned that under SELF, posterior mean is the Bayesian estimate of the parameter.

$$\hat{\lambda}_b = E_\lambda(\lambda|\underline{\mathbf{x}}) = \frac{\int_{\lambda=0}^{\infty} \lambda^{n+r} \mathrm{e}^{-\lambda(s+\sum_{i=1}^{n} x_i)} \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x)^2\right] d\lambda}{\int_{\lambda=0}^{\infty} \lambda^{n+r-1} \mathrm{e}^{-\lambda(s+\sum_{i=1}^{n} x_i)} \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x)^2\right] d\lambda}, \tag{41}$$

provided the above expectation exist. The above expression involves the ratio of two integrals; thus the explicit solution is not possible. Therefore, we use Lindley's approximation method to obtain the Bayesian estimates of the parameter λ.

3.6.1. Lindley's approximation method Lindley suggested one of the most efficient technique to extract the Bayesian estimate from the ratio of the two integral in the year of 1988 [see, Lindley [28]]. Applying this approximation, the Bayesian estimator of λ is obtained by

$$\hat{\lambda}_{bl} = \hat{\lambda}_{ml} + \hat{\tau}_\lambda \hat{\sigma}_{\lambda\lambda} + \frac{1}{2} \hat{\sigma}_{\lambda\lambda}^2 \widehat{\mathbf{L}}_{\lambda\lambda\lambda}, \tag{42}$$

where

$$\frac{\partial^2}{\partial\lambda^2}\mathbf{L} = \mathbf{L}_{\lambda\lambda} = -\frac{n}{\lambda^2} + \sum_{i=1}^{n} \frac{x^2}{[1+\frac{1}{2}(\lambda x)^2]} - \lambda^2 \sum_{i=1}^{n} \frac{x^4}{[1+\frac{1}{2}(\lambda x)^2]^2},$$

$$\frac{\partial^3}{\partial\lambda^3}\mathbf{L} = \mathbf{L}_{\lambda\lambda\lambda} = \frac{2n}{\lambda^3} - 3\lambda \sum_{i=1}^{n} \frac{x^4}{[1+\frac{1}{2}(\lambda x)^2]^2} + 2\lambda^3 \sum_{i=1}^{n} \frac{x^6}{[1+\frac{1}{2}(\lambda x)^2]^3},$$

$$\sigma_{\lambda\lambda} = -\left(\frac{1}{\mathbf{L}_{\lambda\lambda}}\right),$$

and

$$\tau_\lambda = \frac{a-1}{\lambda} - b,$$

all the above derivatives are evaluated at the point $\hat{\lambda}_{ml}$. If n is sufficiently large then $\hat{\lambda}_{bl} \to \hat{\lambda}_{ml}$. One of the biggest drawback of this approximation method is that we can not construct the credible interval using it. Therefore, here we also consider Markov Chain Monte Carlo (MCMC) method to extract the sample from the respective posterior distribution.

3.6.2. Markov Chain Monte Carlo method MCMC method is the most widely used technique to draw the posterior samples whenever the marginal posterior distributions do not yield explicit form. Since the proposed distribution belongs to one parameter exponential family of distributions; thus the implementation of the stated method is straightforward. Several application of this technique is available in every area of applied sciences. The Bayesian estimate of the parameter is obtained using generated posterior samples. Further, the credible intervals is also reported based on the same sample. The posterior distribution by ignoring the constant terms is given by;

$$\begin{aligned} p_1(\lambda|\underline{x}) &\propto \lambda^{n+r-1} e^{-\lambda(s+\sum_{i=1}^{n} x_i)} \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x_i)^2\right] \\ &\propto G_{\lambda|x}\left(n+r, s+\sum_{i=1}^{n} x_i\right) \phi(\lambda), \end{aligned} \tag{43}$$

where

$$\phi(\lambda) = \prod_{i=1}^{n} \left[1+\frac{1}{2}(\lambda x_i)^2\right].$$

The following steps may be considered to extract the sample from above equation.

- Draw λ from $G_{\lambda|x}(\bullet, \bullet)$
- Repeat step 1, κ times to obtain $\lambda_1, \lambda_2, \cdots, \lambda_\kappa$.
- Now, the Bayesian estimate of λ under SELF is obtained by

$$\hat{\lambda}_{mc} = \frac{\sum_{j=1}^{\kappa} \lambda_j \phi(\lambda_j)}{\sum_{j=1}^{\kappa} \phi(\lambda_j)},$$

- Using the idea of Chen and Shao (1999), we can obtain the BCI for the unknown parameter.

4. Simulation study: case of complete data

In this section, Monte Carlo simulation study has been perform to assess the performances of the proposed classical and Bayesian estimators. The study is carried out for the different variation of sample size and parameter. In particular, we have taken $n = 10, 20, 30, 50, 100, 200$ for different variation of the parameter value $\lambda = 0.5, 0.85, 1.0, 2.0, 3.0$. In classical method of estimation, MME ($\hat{\lambda}_m$), MLE ($\hat{\lambda}_{ml}$), MPSE ($\hat{\lambda}_{mp}$), LSE ($\hat{\lambda}_{ls}$), CVME ($\hat{\lambda}_{cv}$) are considered and under Bayesian estimation the estimate is obtained using Lindley's method ($\hat{\lambda}_{bs1}$) and MCMC method ($\hat{\lambda}_{bs2}$) using informative prior and non-informative prior ($\hat{\lambda}_{bs3}$). The values of hyper-parameters are such chosen that the prior mean accurately matches with true beliefs with minimum variability. The average estimates of the parameter and corresponding mean square errors (MSEs) are reported based on 3000 replications in Table 1. Further, the ACI and BCI are also computed for the same variation of the parameters and sample size. The coverage probabilities and average width of the interval based on ACIs (CP_A, ACL) and BCIs (CP_B, BCL) for the considered design are also computed and reported in Table 2. From this extensive simulation study, it is examined that under informative gamma prior the Bayesian estimators are less penalize as compared to the other classical estimators, while the performance of classical and Bayesian estimators are almost same under non-informative. The obtained estimators also ensure the consistency of the estimators, i.e., the MSE of each estimator decreases when n increases. In case of large sample all estimators are more or less the same. Further, among the classical estimation methods (MOM, MLE, MPS, LSE & CVME), the MPS estimation method provides the more efficient result as compared to others. Further, the width of the ACIs and BCIs are reported for same setup and observed that the width of the intervals decreases by increasing the sample size while no specific trend has been obtained in coverage probabilities. The width of BCIs are smaller than the ACIs for all setup and consequently the coverage probability of Bayesian interval estimation is lesser than the ACIs.

Table 1. Average estimates of the parameter and corresponding MSEs (in each second row) for small sample.

n	λ	$\hat{\lambda}_m$	$\hat{\lambda}_{ml}$	$\hat{\lambda}_{mp}$	$\hat{\lambda}_{ls}$	$\hat{\lambda}_{cv}$	$\hat{\lambda}_{bs1}$	$\hat{\lambda}_{bs2}$	$\hat{\lambda}_{bs3}$
10	0.5	0.5484	0.5578	0.4969	0.5399	0.5506	0.4242	0.4262	0.5434
		0.0297	0.0287	0.0199	0.0545	0.0556	0.0196	0.0204	0.0249
	0.85	0.9292	0.9453	0.8424	0.9094	0.9288	0.7130	0.7224	0.9176
		0.0827	0.0780	0.0541	0.1364	0.1427	0.0515	0.0572	0.0624
	1	1.0917	1.1086	0.9876	1.0780	1.0995	0.8310	0.8471	1.0732
		0.1268	0.1240	0.0883	0.2193	0.2259	0.0738	0.0893	0.0918
	2	2.2178	2.2506	2.0057	2.1989	2.2435	1.5832	1.7209	2.1329
		0.5019	0.4970	0.3440	0.9186	0.9449	0.2245	0.3327	0.2347
	3	3.3168	3.3571	2.9871	3.2972	3.3918	3.1172	3.1264	2.5677
		1.1382	1.0794	0.7435	2.1084	2.4500	0.2692	0.2796	0.7475
20	0.5	0.5292	0.5358	0.4995	0.5187	0.5245	0.4721	0.4726	0.5297
		0.0128	0.0118	0.0089	0.0192	0.0197	0.0088	0.0090	0.0110
	0.85	0.9028	0.9134	0.8514	0.8846	0.8946	0.8031	0.8056	0.9019
		0.0331	0.0327	0.0247	0.0449	0.0464	0.0230	0.0243	0.0293
	1	1.0440	1.0552	0.9834	1.0230	1.0346	0.9273	0.9308	1.0417
		0.0463	0.0437	0.0349	0.0655	0.0673	0.0339	0.0365	0.0388
	2	2.0994	2.1144	1.9675	2.0786	2.1015	1.8375	1.8654	2.0764
		0.1778	0.1682	0.1320	0.2725	0.2802	0.1030	0.1189	0.1211
	3	3.1752	3.2031	2.9798	3.1343	3.1693	2.7160	2.8257	3.1157
		0.4621	0.4401	0.3392	0.6760	0.6982	0.1719	0.2161	0.2306
30	0.5	0.5213	0.5267	0.5003	0.5102	0.5142	0.4851	0.4853	0.5228
		0.0069	0.0066	0.0053	0.0089	0.0091	0.0052	0.0052	0.0063
	0.85	0.8854	0.8917	0.8468	0.8733	0.8800	0.8207	0.8218	0.8847
		0.0227	0.0207	0.0168	0.0338	0.0346	0.0163	0.0170	0.0193
	1	1.0449	1.0539	1.0008	1.0251	1.0331	0.9693	0.9712	1.0452
		0.0297	0.0283	0.0226	0.0396	0.0406	0.0212	0.0224	0.0259
	2	2.0782	2.0958	1.9891	2.0472	2.0629	1.9174	1.9312	2.0724
		0.1142	0.1055	0.0850	0.1748	0.1781	0.0701	0.0827	0.0847
	3	3.1237	3.1472	2.9857	3.0818	3.1053	2.8505	2.9001	3.0979
		0.2901	0.2680	0.2177	0.4229	0.4326	0.1342	0.1599	0.1767

5. Estimation with censored data

5.1. Maximum likelihood estimation

Here, we consider the case of right censored data and obtained MLE of the parameter. Let T be a rv distributed according to a Xg-E distribution with $\boldsymbol{\theta} = \lambda$. For i^{th} (individual); T_i is the lifetime and C_i is the censorship time, where T_i and C_i are independent rvs. Suppose the data consists of n independent observations

$$t_i = \min(T_i, C_i) \text{ for } i = 1, ..., n.$$

Censorship is assumed to be non-informative (the distribution of C_i does not depend on the unknown parameters of T_i). The likelihood function in the case of censored data can be given by:

$$\mathbf{L}(t, \boldsymbol{\theta}) = \prod_{i=1}^{n} \lambda^{\delta_i}(t_i, \boldsymbol{\theta}) S(t_i, \boldsymbol{\theta});\ \boldsymbol{\theta} = \lambda,\, \delta_i = 1_{\{T_i \leq C_i\}}.$$

In our case, let T_i be a rv distributed with the parameter $\boldsymbol{\theta} = \lambda$, so the likelihood function reduces to

Table 2. Average estimates of the parameter and corresponding MSE's (in each second row) for large sample.

n	λ	$\hat{\lambda}_m$	$\hat{\lambda}_{ml}$	$\hat{\lambda}_{mp}$	$\hat{\lambda}_{ls}$	$\hat{\lambda}_{cv}$	$\hat{\lambda}_{bs1}$	$\hat{\lambda}_{bs2}$	$\hat{\lambda}_{bs3}$
50	0.5	0.5140	0.5182	0.5009	0.5045	0.5069	0.4937	0.4938	0.5160
		0.0043	0.0042	0.0035	0.0055	0.0055	0.0035	0.0035	0.0040
	0.85	0.8805	0.8878	0.8580	0.8654	0.8695	0.8454	0.8459	0.8836
		0.0127	0.0122	0.0100	0.0170	0.0173	0.0096	0.0098	0.0116
	1	1.0288	1.0370	1.0021	1.0112	1.0159	0.9873	0.9880	1.0321
		0.0164	0.0158	0.0133	0.0213	0.0216	0.0128	0.0132	0.0150
	2	2.0633	2.0758	2.0043	2.0361	2.0457	1.9721	1.9781	2.0628
		0.0660	0.0637	0.0530	0.0909	0.0924	0.0466	0.0531	0.0559
	3	3.1111	3.1303	3.0232	3.0642	3.0786	2.9600	2.9829	3.1025
		0.1662	0.1556	0.1271	0.2201	0.2241	0.0921	0.1263	0.1303
100	0.5	0.5115	0.5149	0.5053	0.5034	0.5046	0.5027	0.5028	0.5138
		0.0023	0.0023	0.0020	0.0029	0.0030	0.0019	0.0020	0.0022
	0.85	0.8686	0.8740	0.8578	0.8560	0.8581	0.8533	0.8534	0.8720
		0.0057	0.0055	0.0048	0.0074	0.0075	0.0046	0.0047	0.0053
	1	1.0217	1.0293	1.0101	1.0049	1.0073	1.0048	1.0051	1.0270
		0.0082	0.0081	0.0071	0.0105	0.0106	0.0069	0.0070	0.0079
	2	2.0465	2.0614	2.0222	2.0138	2.0186	2.0107	2.0128	2.0554
		0.0311	0.0314	0.0268	0.0382	0.0386	0.0248	0.0265	0.0293
	3	3.0565	3.0758	3.0138	3.0132	3.0204	2.9971	3.0034	3.0650
		0.0746	0.0719	0.0610	0.0974	0.0980	0.0546	0.0631	0.0632
200	0.5	0.5117	0.5149	0.5097	0.5039	0.5046	0.5088	0.5088	0.5143
		0.0013	0.0013	0.0012	0.0015	0.0015	0.0012	0.0012	0.0013
	0.85	0.8649	0.8709	0.8622	0.8512	0.8522	0.8606	0.8606	0.8699
		0.0033	0.0035	0.0031	0.0039	0.0039	0.0031	0.0031	0.0034
	1	1.0194	1.0263	1.0160	1.0044	1.0056	1.0141	1.0142	1.0251
		0.0046	0.0048	0.0043	0.0058	0.0058	0.0042	0.0042	0.0047
	2	2.0406	2.0528	2.0301	2.0122	2.0146	2.0278	2.0286	2.0500
		0.0171	0.0168	0.0142	0.0216	0.0217	0.0140	0.0145	0.0161
	3	3.0494	3.0644	3.0294	3.0104	3.0140	3.0259	3.0283	3.0593
		0.0377	0.0375	0.0321	0.0466	0.0468	0.0311	0.0334	0.0350

$$\begin{aligned}\mathbf{L}(t,\boldsymbol{\theta}) &= \prod_{i=1}^{n}\left[\frac{-\left(\lambda^2 t+\lambda\right)+\lambda\left(\frac{\lambda^2 t^2}{2}+\lambda t+2\right)}{\left[2+\lambda\, t+\frac{1}{2}\left(\lambda\, t\right)^2\right]}\right]^{\delta_i} \\ &\quad\times\left[\frac{1}{2}\mathrm{e}^{-\lambda\, t}\left[2+\lambda\, t+\frac{1}{2}\left(\lambda\, t\right)^2\right]\right],\end{aligned}$$

and the log likelihood function is given by

$$\begin{aligned}\ell(t,\boldsymbol{\theta}) &= \ln\left(\frac{1}{2}\mathrm{e}^{-\lambda\, t}\left[2+\lambda\, t+\frac{1}{2}\left(\lambda\, t\right)^2\right]\right) \\ &\quad+\sum_{i=1}^{n}\left[\delta_i\left(\begin{array}{c}\ln\left[-\left(\lambda^2 t+\lambda\right)+\lambda\left(\frac{\lambda^2 t^2}{2}+\lambda t+2\right)\right] \\ -\ln\left[2+\lambda\, t+\frac{1}{2}\left(\lambda\, t\right)^2\right]\end{array}\right)\right],\end{aligned}$$

Table 3. Interval estimation of the parameter along with its coverage probabilities (CPs).

n	λ	$\hat{\lambda}_L$	$\hat{\lambda}_U$	CP_A	ACL	$\hat{\lambda}^L$	$\hat{\lambda}^U$	CP_B	BCL
	0.5	0.2718	0.8438	0.9690	0.5720	0.2957	0.8024	0.9480	0.5066
	0.85	0.4608	1.4298	0.9630	0.9690	0.5046	1.3494	0.9460	0.8447
10	1	0.5394	1.6778	0.9670	1.1384	0.5926	1.5752	0.9480	0.9826
	2	1.0966	3.4046	0.9680	2.3079	1.2469	3.0568	0.9670	1.8099
	3	1.6337	5.0804	0.9650	3.4467	1.9613	4.3433	0.9840	2.3819
	0.5	0.3409	0.7307	0.9690	0.3897	0.3582	0.7071	0.9390	0.3488
	0.85	0.5811	1.2457	0.9690	0.6646	0.6118	1.2019	0.9530	0.5900
20	1	0.6711	1.4394	0.9650	0.7683	0.7070	1.3871	0.9350	0.6801
	2	1.3432	2.8856	0.9620	1.5423	1.4359	2.7380	0.9550	1.3021
	3	2.0359	4.3702	0.9540	2.3343	2.2101	4.0504	0.9600	1.8403
	0.5	0.3701	0.6832	0.9610	0.3131	0.3839	0.6654	0.9370	0.2815
	0.85	0.6264	1.1570	0.9650	0.5306	0.6501	1.1246	0.9440	0.4744
30	1	0.7406	1.3672	0.9680	0.6266	0.7697	1.3283	0.9380	0.5586
	2	1.4723	2.7193	0.9740	1.2470	1.5399	2.6175	0.9570	1.0776
	3	2.2104	4.0839	0.9580	1.8736	2.3343	3.8797	0.9480	1.5454
	0.5	0.3988	0.6377	0.9530	0.2389	0.4093	0.6246	0.9370	0.2152
	0.85	0.6832	1.0924	0.9520	0.4092	0.7015	1.0691	0.9350	0.3676
50	1	0.7978	1.2761	0.9580	0.4784	0.8198	1.2486	0.9370	0.4288
	2	1.5968	2.5549	0.9580	0.9581	1.6451	2.4882	0.9440	0.8430
	3	2.4082	3.8525	0.9560	1.4443	2.4909	3.7265	0.9490	1.2355
	0.5	0.4309	0.5989	0.9340	0.1680	0.4384	0.5902	0.9060	0.1518
	0.85	0.7314	1.0167	0.9470	0.2853	0.7444	1.0014	0.9350	0.2570
100	1	0.8615	1.1972	0.9510	0.3357	0.8767	1.1792	0.9250	0.3025
	2	1.7251	2.3977	0.9490	0.6726	1.7570	2.3575	0.9330	0.6005
	3	2.5739	3.5778	0.9570	1.0039	2.6267	3.5105	0.9360	0.8838
	0.5	0.4555	0.5743	0.9120	0.1188	0.4610	0.5683	0.8820	0.1073
	0.85	0.7704	0.9713	0.9320	0.2009	0.7798	0.9610	0.9030	0.1812
200	1	0.9079	1.1448	0.9240	0.2369	0.9186	1.1325	0.8990	0.2139
	2	1.8159	2.2898	0.9490	0.4739	1.8382	2.2643	0.9210	0.4260
	3	2.7106	3.4182	0.9330	0.7076	2.7452	3.3766	0.9120	0.6314

then

$$\begin{aligned}\ell(t,\boldsymbol{\theta}) \quad = \quad & r\left[\ln\left(\frac{1}{2}\right)\right] - \lambda\sum_{i\in C} t_i + \sum_{i\in C}\ln\left[2+\lambda\, t_i+\frac{1}{2}\left(\lambda\, t_i\right)^2\right] \\ & + \sum_{i\in F}\ln\left[-\left(\lambda^2 t_i+\lambda\right)+\lambda\left(\frac{\lambda^2 t_i^2}{2}+\lambda t_i+2\right)\right] \\ & - \sum_{i\in F}\ln\left[2+\lambda\, t_i+\frac{1}{2}\left(\lambda\, t_i\right)^2\right],\end{aligned}$$

where r is the number of failures, F and C denote the sets of uncensored and censored observations, respectively. The maximum likelihood estimator $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$ can be find by solving the system formed by equalizing the following score functions to zero

$$\begin{aligned}\frac{\partial \ell(t,\boldsymbol{\theta})}{\partial \lambda} &= -\sum_{i\in C} t_i + \sum_{i\in C} \frac{2+t_i+\lambda\, t_i^2}{2+\lambda\, t_i + \frac{1}{2}(\lambda\, t_i)^2} \\ &+ \sum_{i\in F} \frac{-2\lambda\, t_i + \frac{3\lambda^2 t_i^2}{2} + t_i}{-(\lambda^2 t_i + \lambda) + \lambda\left(\frac{\lambda^2 t_i^2}{2} + \lambda t_i + 2\right)} \\ &- \sum_{i\in F} \frac{2+t_i+\lambda\, t_i^2}{2+\lambda\, t_i + \frac{1}{2}(\lambda\, t_i)^2}.\end{aligned}$$

To solve the system of score functions which is quite complicated we use numerical methods such as the Monte Carlo method, the Barzilai-Borwein (BB) algorithm or others.

5.2. *Simulations: case of censored data*

We consider the Xg-E model. The data were simulated $N = 10000$ times (with sample sizes $n = 30$, $n = 100$, $n = 200$, $n = 500$) and parameter value $\lambda = 0.6$. The averages of the simulated values of MLE $\widehat{\lambda}$ of the parameter, and it's MSE are calculated and presented in Table 4. From Table 4, we can notice that the mean squared errors are very small, which confirms the convergence of MLE.

Table 4: Maximum likelihood estimators $\widehat{\lambda}$ of the parameter and its mean squared errors (censored data)

$N = 10000$	$n = 30$	$n = 100$	$n = 200$	$n = 500$
$\widehat{\lambda}$	0.6143	0.6127	0.6112	0.6029
MSE	4.28×10^{-03}	3.51×10^{-03}	2.09×10^{-03}	1.58×10^{-03}

6. Goodness-of-fit test

In case of complete data, various techniques are used to verify the adequacy of mathematical models to data from observation. The most common tests are those based on Pearson's Chi-square statistics. Nevertheless, these can not be applied in all situations, especially when the parameters of the model are unknown or when the data is censored. Since the middle of the last century, researchers have begun to propose modifications of existing statistics to take into account unknown parameters on the one hand and censorship on the other. For the complete data, Nikulin [33], Nikulin [34], Nikulin [35] and Rao and Robson [39] separately proposed a statistic known today as the Nikulin-Rao-Robson (NRR) statistic. This statistical test, which follows a Chi-square distribution, is a natural modification of the Pearson statistic.

If, in addition to the unknown parameters, the data are censored, the classical tests are inadequate to verify a hypothesis H_0 according to which a series of observations comes from a parametric family $F(t)$. Habib and Thomas [23] considered the natural modifications of the NRR statistic. These tests are based on the differences between two probability estimators, one based on the Kaplan-Meier estimator, the other based on the MLE of the unknown parameters of the cumulative distribution function of the Kaplan-Meier estimator. Bagdonavicius and Nikulin [7] and Bagdonavicius et al. [5] proposed a modification of the NRR statistic that takes into account random right censorship. This statistic, based on the MLLE on the initial data, also follows a Chi-square distribution at the limit. For more details on the construction of these statistics, we can see Voinov et al. [46]. These techniques were used to adjust observations to the generalized inverse Weibull model (see Goul and Seddik [16]), the distribution of Birbaurm Saunders (Nikulin et al. [37]), the kumaraswamy generalized inverse Weibull distribution (see Gaul and Seddik [17]), Bertholon model (see Chouia and Seddik [11]), a new Burr type XII distribution (see Ibrahim et al. ([24] and [26])), the odd Lindley exponentiated exponential distribution (see Goual et al. [19]), the Topp-Leone-Lomax model (see Yadav et al. [48]), Lomax inverse Weibull model (see Goual et al. [20]), Burr XII inverse

Rayleigh model (see Goual and Yousof [18]), in some new G families and its applications (see Abouelmagd et al. ([1],[2] and [3])) and finally xgamma reciprocal Rayleigh extension (see Yousof et al. [49]). In this work we construct a modified chi-square type tests for the Xg-E model in case of complete and censored data. The NRR statistic is used in case of complete data. In presence of censorship, we work with the modification of the N.R.R. statistic proposed by Bagdonavicius and Nikulin [7].

6.1. NRR statistic test

To test the hypothesis H_0 according to which $T_1, T_2, \cdots, T_n$, an n-sample comes from a parametric family $F(t; \boldsymbol{\theta})$

$$H_0 : \Pr\{T_i \leq t\} = F(t, \boldsymbol{\theta}), \;\; t \in \mathbb{R}, \;\; \boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots, \boldsymbol{\theta}_s)^T,$$

where $\boldsymbol{\theta}$ represents the vector of unknown parameters, Nikulin [33], Nikulin [34], Nikulin [35] and Rao and Robson [39] proposed Y^2 the N.R.R. statistic defined as following:

Observations $T_1, T_2, \cdots, T_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_r$ mutually disjoint $\mathbf{I}_{\mathbf{j}_2} =]a_{\mathbf{j}_2} - 1; a_{\mathbf{j}_2}]$; where $\mathbf{j}_2 = \overline{1; r}$.

The limits $a_{\mathbf{j}_2}$ of the intervals $\mathbf{I}_{\mathbf{j}_2}$ are obtained such that

$$p_{\mathbf{j}_2}(\boldsymbol{\theta}) = \int_{a_{\mathbf{j}_2 - 1}}^{a_{\mathbf{j}_2}} f(t, \boldsymbol{\theta}) dt \; |_{(\,\mathbf{j}_2 = 1, 2, \cdots, r)},$$

so

$$a_{\mathbf{j}_2} = F^{-1}\left(\frac{\mathbf{j}_2}{r}\right) |_{(\mathbf{j}_2 = 1, \cdots, r-1)}.$$

If

$$\nu_{\mathbf{j}_2} = (\nu_1, \nu_2, \cdots, \nu_r)^T$$

is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_{\mathbf{j}_2}$ intervals

$$\nu_{\mathbf{j}_2} = \sum_{i=1}^{n} 1_{\left\{t_i \in \mathbf{I}_{\mathbf{j}_2}\right\}} \; |_{(\mathbf{j}_2 = 1, \ldots, r)}.$$

The N.R.R. statistic is given by

$$Y^2(\widehat{\boldsymbol{\theta}}_n) = X_n^2(\widehat{\boldsymbol{\theta}}_n) + \frac{1}{n} \mathbf{L}^T(\widehat{\boldsymbol{\theta}}_n)(\mathbf{I}(\widehat{\boldsymbol{\theta}}_n) - \;\mathbf{J}(\widehat{\boldsymbol{\theta}}_n))^{-1} \mathbf{L}(\widehat{\boldsymbol{\theta}}_n),$$

where

$$X_n^2(\boldsymbol{\theta}) = \left(\frac{\nu_1 - np_1(\boldsymbol{\theta})}{\sqrt{np_1(\boldsymbol{\theta})}}, \frac{\nu_2 - np_2(\boldsymbol{\theta})}{\sqrt{np_2(\boldsymbol{\theta})}}, \cdots, \frac{\nu_r - np_r(\boldsymbol{\theta})}{\sqrt{np_r(\boldsymbol{\theta})}}\right)^T,$$

and $\mathbf{J}(\boldsymbol{\theta})$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\boldsymbol{\theta}) = B(\boldsymbol{\theta})^T B(\boldsymbol{\theta}),$$

with

$$B(\boldsymbol{\theta}) = \left[\frac{1}{\sqrt{p}_i} \frac{\partial}{\partial \mu} p_i(\boldsymbol{\theta})\right]_{r \times s} |_{(i = 1, 2, \cdots, r \text{ and } \kappa = 1, \cdots, s)},$$

then

$$\mathbf{L}(\boldsymbol{\theta}) = (\mathbf{L}_1(\boldsymbol{\theta}), ..., \mathbf{L}_s(\boldsymbol{\theta}))^T \text{ with } \; \mathbf{L}_\kappa(\boldsymbol{\theta}) = \sum_{i=1}^{r} \frac{\nu_i}{p_i} \frac{\partial}{\partial \boldsymbol{\theta}_\kappa} p_i(\boldsymbol{\theta}),$$

where $\mathbf{I}_n(\widehat{\boldsymbol{\theta}_n})$ represents the estimated Fisher information matrix and $\widehat{\boldsymbol{\theta}_n}$ is the MLE of the parameter vector. The Y^2 statistic follows a distribution of chi-square χ^2_{r-1} with $(r-1)$ degrees of freedom.

6.2. NRR statistic for the Xg-E model

Consider a sample $T = (T_1, T_2, \cdots, T_n)^T$. To verify if these data are distributed according to the Xg-E model, $P\{T_i \le t\} = F_{\text{Xg-E}}(t, \boldsymbol{\theta})$; with unknown parameters $\boldsymbol{\theta} = \lambda$, a chi-square goodness-of-fit test is constructed by fitting the N.R.R. statistic developed in the previous section. The maximum likelihood estimator $\widehat{\boldsymbol{\theta}_n}$ of the unknown parameter of the Xg-E distribution is computed on the initial data. The statistic Y^2 does not depend on the parameter, we can therefore use the Fisher information matrix estimated $\mathbf{I}_n(\widehat{\boldsymbol{\theta}_n})$. All the components of the statistic Y^2, for the distribution Xg-E are provided, therefore Y^2 can be deduced easily.

6.3. Simulation studies (N.R.R. statistics Y^2)

To support the results obtained in this work, we conduct an intensive study by numerical simulations. Thus, to test the null hypothesis H_0 that a sample belongs to the Xg-E model, we calculate Y^2 the NRR statistic of 10000 simulated samples with sizes $n = 30, n = 50, n = 100, n = 200$ and $n = 500$, respectively. For different theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$); we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \le \chi^2_\epsilon(r-1)$ then, we present the results of the corresponding empirical and theoretical levels in Table 5. As can be seen, the values of the empirical levels calculated are very close to those of their corresponding theoretical levels. Thus, we conclude that the proposed test is well suited to the Xg-E distribution.

Table 5: Empirical levels and corresponding theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$)

$N = 10000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9841	0.9521	0.9946	0.9032
$n = 50$	0.9830	0.9517	0.9930	0.9024
$n = 100$	0.9819	0.9515	0.9915	0.9019
$n = 250$	0.9806	0.9508	0.9906	0.9008
$n = 500$	0.9802	0.9503	0.9902	0.9001

7. Simulated distribution of $\mathbf{Y^2}$ statistic for Xg-E model

For demonstrating that the Y^2 statistic follows in the limit; a chi-squared distribution with $\kappa = r - 1$ degrees of freedom; we compute $N = 10000$ times, the simulated distribution of $Y^2(\widehat{\theta})$ under the null hypothesis H_0 with different values of parameters Xg-E(λ), and $r = 12$ intervals, versus the chi-squared distribution with $\kappa = 11$ degree of freedom. Their histograms are represented in Figure 3 versus the chi-squared distribution with κ degree of freedom.

From Figure 3, we can observe that the statistical distribution of Y^2 with different values of parameter and different numbers κ of grouping cells; in the limit follows a chi-squared with κ degrees of freedom within the statistical errors of simulation. The same results is obtained for different number of equiprobable grouping intervals and different value of parameter. It is means that the limiting distribution of the generalized chi-squared Y^2 statistic is distribution free.

7.1. Applications to real data

7.1.1. Breaking stress of carbon fibres (in Gba) data To test the null hypothesis H_0 that these data are adjusted by a Xg-E distribution, we use the NRR statistic obtained previously. Using the R software and the BB algorithm (see Ravi and Gilbert [40]), we compute the maximum likelihood estimators (MLE) $\widehat{\lambda} = 7.1985$. Then we deduce the value of

$$Y^2 = 19.167408.$$

For significance level $\epsilon = 0.01$, the critical value is

$$\chi^2_{0.01}(10-1) = 21.66599,$$

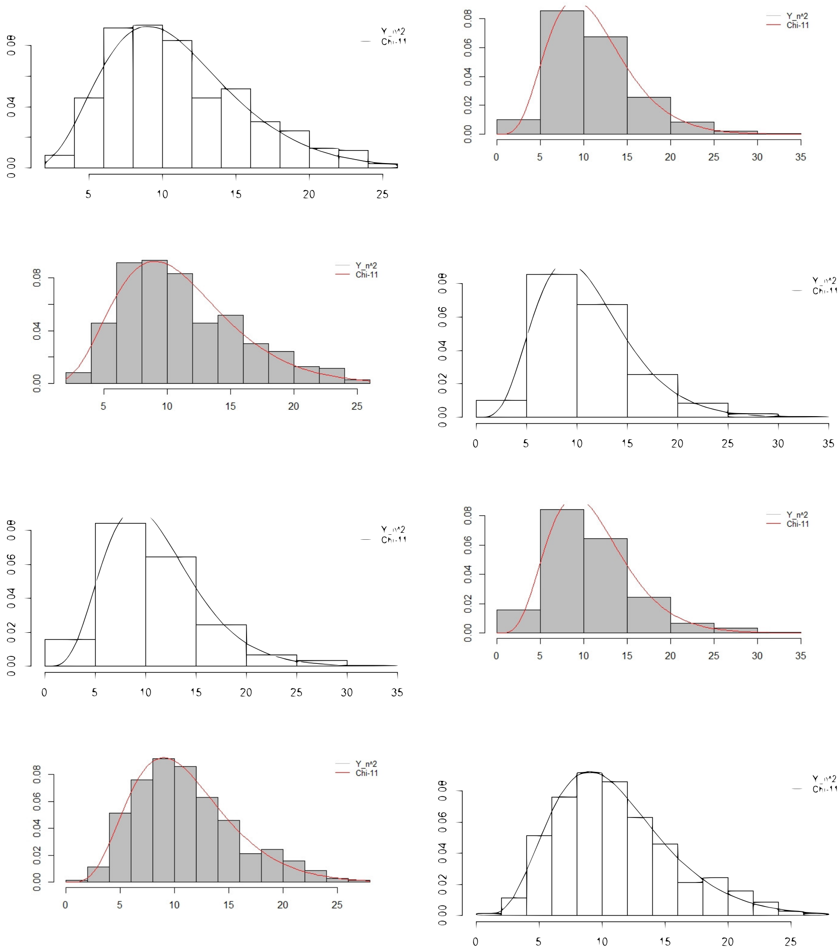

Figure 2. Simulated distribution of the Y^2 statistic under the null hypothesis H_0,with different parameters of $\widehat{\theta}$ versus the chi-squared distribution with 11 degrees of freedom, with $n = 150$, $N = 10000$.

then, the NRR Y^2 statistic is less than the critical value, this allows us to say that these data correspond appropriately to the Xg-E model.

7.1.2. Strengths of 1.5 cm glass fibres Assuming that the Strengths of 1.5 cm glass fibres data (see [44]) can be fitted by our Xg-E model, we can find (using the BB algorithm) the MLE's of the $\boldsymbol{\theta}$ parameter as:

$$\widehat{\boldsymbol{\theta}} = \widehat{\lambda} = 1.884931.$$

After calculate, we give the NRR statistic test and the critical value as

$$Y^2 = 12.27954$$

and

$$\chi^2_{0.05}(7-1) = 12.59159,$$

respectively. We can affirm that data of 1.5 cm glass fibres can be modeled by our Xg-E model with a satisfactory manner.

8. Goodness-of-fit test for right censored data

To verify the adequacy of the Xg-E model when the parameters are unknown and the data censored, we use the approach proposed by Bagdonavicius and Nikulin [7] and Bagdonavicius et al. [5] that we develop in this paragraph. It is a chi-square type test based on a modification of the NRR statistic. We adapt this test for a Xg-E model. Let us consider the composite hypothesis

$$H_0 : F(t) \in F_0 = \left\{ F_0(t, \boldsymbol{\theta}), t \in R^1, \ \boldsymbol{\theta} \in \Theta \subset R^s \right\},$$

where

$$\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_s)^T \in \Theta \subset R^s,$$

is an unknown m-dimensional parameter and F_0 is a differentiated completely specified CDF with the support $(0, \infty)$. Let us consider a finite time interval only say $[0, \tau]$, where τ is the maximum time of the study, and divide it into $\kappa > s$ smaller intervals $\mathbf{I}_{\mathbf{j}_2} = (a_{\mathbf{j}_2 - 1}, a_{\mathbf{j}_2}]$, where

$$0 =< a_0 < a_1 ... < a_{\kappa-1} < a_\kappa = +\infty.$$

In this case the estimated $\widehat{a}_{\mathbf{j}_2}$ is given by

$$\widehat{a}_{\mathbf{j}_2} = \boldsymbol{\Lambda}^{-1} \left((E_{\mathbf{j}_2} - \sum_{\mathbf{j}_1=1}^{i-1} \boldsymbol{\Lambda}(T_{(\mathbf{j}_1)}, \widehat{\boldsymbol{\theta}}))/(n - i + 1), \widehat{\theta} \right), \quad \widehat{a_\kappa} = T_{(n)}|_{(\mathbf{j}_2=1,\ldots,\kappa)},$$

where $\widehat{\boldsymbol{\theta}}$ is the MLE of the parameter $\boldsymbol{\theta}$, $\boldsymbol{\Lambda}^{-1}$ is the inverse of cumulative hazard function $\boldsymbol{\Lambda}$, $T_{(i)}$ is the i^{th} element in the ordered statistics $(T_{(1)}, T_{(2)}, ..., T_{(n)})$ and

$$E_{\mathbf{j}_2} = (n - i + 1)\boldsymbol{\Lambda}(\widehat{a_{\mathbf{j}_2}}, \widehat{\boldsymbol{\theta}}) + \sum_{\mathbf{j}_1=1}^{i-1} \boldsymbol{\Lambda}(T_{(\mathbf{j}_1)}, \widehat{\boldsymbol{\theta}}),$$

and $a_{\mathbf{j}_2}$ are random data functions such as the κ intervals chosen have equal expected numbers of failures $\mathrm{e}_{\mathbf{j}_2}$. Usually in real application we fix κ. Bagdonavicus et al. [6], Greenwood and Nikulin [21], Gupta et al. [22] and Habib and Thomas [23] give some recommendations for the choice of intervals. The test is based on the vector

$$\mathbf{Z} = (Z_1, Z_2, ..., Z_\kappa)^T, \ \ Z_{\mathbf{j}_2} = \frac{1}{\sqrt{n}}(\mathbf{U}_{\mathbf{j}_2} - \mathrm{e}_{\mathbf{j}_2})|_{(\mathbf{j}_2=1,2,\ldots,\kappa)},$$

where $\mathbf{U}_{\mathbf{j}_2}$ represent the numbers of observed failures in these intervals. The test for hypothesis H_0 can be based on the statistic

$$Y_n^2 = \mathbf{Z}^T \widehat{\boldsymbol{\Sigma}}^{-1} \mathbf{Z},$$

where

$$\widehat{\boldsymbol{\Sigma}}^{-1} = \widehat{\mathbf{A}}^{-1} + \widehat{\mathbf{C}}^{-1} \widehat{\mathbf{A}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{C}} \widehat{\mathbf{A}}^{-1}$$

and

$$\widehat{\mathbf{G}} = \widehat{i} - \widehat{\mathbf{C}} \widehat{\mathbf{A}}^{-1} \widehat{\mathbf{C}}^T.$$

The test statistic can be written in the following form

$$Y_n^2 = \sum_{\mathbf{j}_2=1}^{\kappa} \frac{(\mathbf{U}_{\mathbf{j}_2} - \mathrm{e}_{\mathbf{j}_2})^2}{\mathbf{U}_{\mathbf{j}_2}} + \mathbf{Q},$$

where

$$\begin{aligned}
\widehat{\mathbf{A}}_{\mathbf{j}_2} &= n^{-1}\mathbf{U}_{\mathbf{j}_2}, \mathbf{U}_{\mathbf{j}_2} = \sum_{\left(i:X_i\in\mathbf{I}_{\mathbf{j}_2}\right)} \delta_i, \\
\widehat{\mathbf{G}} &= [\widehat{g}_{\mathbf{j}_1\mathbf{j}_1'}]_{s\times s}, \mathbf{Q} = \widehat{\mathbf{W}}^T\widehat{\mathbf{G}}^{-1}\widehat{\mathbf{W}}, \\
\widehat{\mathbf{C}}_{\mathbf{j}_1\mathbf{j}_2} &= \frac{1}{n}\sum_{i:X_i\in\mathbf{I}_{\mathbf{j}_2}} \delta_i \frac{\partial}{\partial\boldsymbol{\theta}} \ln\left[\lambda_i(t_i,\widehat{\boldsymbol{\theta}})\right], \\
\widehat{\mathbf{W}}_{\mathbf{j}_1} &= \sum_{\mathbf{j}_2=1}^{\kappa} \widehat{\mathbf{C}}_{\mathbf{j}_1\mathbf{j}_2}\widehat{\mathbf{A}}_{\mathbf{j}_2}^{-1}\mathbf{Z}_{\mathbf{j}_2}, \quad \mathbf{j}_1,\mathbf{j}_1' = 1,...,s, \\
\widehat{\mathbf{W}} &= (\widehat{\mathbf{W}}_1, \widehat{\mathbf{W}}_2, ..., \widehat{\mathbf{W}}_s)^T, \\
\widehat{g}_{\mathbf{j}_1\mathbf{j}_1'} &= \widehat{i}_{\mathbf{j}_1\mathbf{j}_1'} - \sum_{\mathbf{j}_2=1}^{\kappa} \widehat{\mathbf{C}}_{\mathbf{j}_1\mathbf{j}_2}\widehat{\mathbf{C}}_{\mathbf{j}_1'\mathbf{j}_2}\hat{A}_{\mathbf{j}_2}^{-1}, \\
\widehat{i}_{\mathbf{j}_1\mathbf{j}_1'} &= n^{-1}\sum_{i=1}^{n} \delta_i \frac{\partial}{\partial\boldsymbol{\theta}_{\mathbf{j}_1}} \ln\left[\lambda_i(t_i,\widehat{\boldsymbol{\theta}})\right] \frac{\partial}{\partial\boldsymbol{\theta}_{\mathbf{j}_1'}} \ln\left[\lambda_i(t_i,\widehat{\boldsymbol{\theta}})\right]
\end{aligned}$$

and

$$\widehat{\mathbf{C}}_{\mathbf{j}_1\mathbf{j}_2} = \frac{1}{n}\sum_{i:X_i\in\mathbf{I}_{\mathbf{j}_2}} \delta_i \frac{\partial}{\partial\boldsymbol{\theta}} \ln\lambda_i(t_i,\widehat{\boldsymbol{\theta}}),$$

calculation of the matrices $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{I}}$ are given in the Appendix C. The limit distribution of the statistic Y_n^2 is chi-square with $r = rank(\mathbf{\Sigma}) = tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma})$ degrees of freedom. If $\mathbf{G}$ is non-degenerate then $r = \kappa$. The hypothesis is rejected with approximate significance level ϵ if $Y_n^2 > \chi_\epsilon^2(r)$ where $\chi_\epsilon^2(r)$ is the quantile of chi-square with r degrees of freedom.

8.1. Goodness-of-fit test for the Xg-E model in case of censored data

In this section, we study the validity of the Xg-E model, by a goodness-of-fit test based on Y_n^2, the modified NRR statistic presented in the previous section. Suppose H_0 is checked, that is, the failure rate T_i follows an Xg-E distribution, the survival function is:

$$S(t,\boldsymbol{\theta}) = 1 - F(t;\alpha,\beta,a) = \frac{1}{2}\mathrm{e}^{-\lambda\,t}\left[2 + \lambda\,t + \frac{1}{2}\left(\lambda\,t\right)^2\right].$$

The choice of $\widehat{a}_{\mathbf{j}_2}$ when the baseline distribution is the Xg-E model, is obtained as follow:

First, we have

$$\mathbf{\Lambda}_{Xg-E}(t,\boldsymbol{\theta}) = -\ln S(t,\boldsymbol{\theta}) = \lambda\,t - \ln\left(\frac{1}{2}\right) - \ln\left[2 + \lambda\,t + \frac{1}{2}\left(\lambda\,t\right)^2\right]$$

$$E_{\mathbf{j}_2} = \sum_{i:X_i>a_{\mathbf{j}_2}} (\mathbf{\Lambda}(a_{\mathbf{j}_2}\wedge t_i,\widehat{\boldsymbol{\theta}}) - \mathbf{\Lambda}(a_{\mathbf{j}_2-1},\widehat{\boldsymbol{\theta}}) \text{ and } E_\kappa = \sum_{i=1}^{n}\mathbf{\Lambda}(t_i,\widehat{\boldsymbol{\theta}}).$$

Under such choice of intervals we have a constant value of $\mathrm{e}_{\mathbf{j}_2} = E_\kappa/\kappa$ for any $\mathbf{j}_2$. There is no explicit form of the inverse hazard function of Xg-E distribution, so we can estimate intervals by iterative method.

8.2. Simulation study

To test the null hypothesis H_0 that a sample comes from a Xg-E model, we calculate Y_n^2 the NRR statistic of 10000 simulated samples with sizes $n = 30, n = 150, n = 200, n = 500$, respectively. For different levels of meaning ($\epsilon = 0.02, 0.05, 0.01, 0.1$); we calculate the mean of the number of no rejections of the null hypothesis when $Y_n^2 \leq \chi_\epsilon^2(r)$, then we present the results of the empirical values and the corresponding theoretical values in Table 6.

Table 6: Empirical levels and corresponding theoretical levels ($\epsilon = 0.02; 0.05; 0.01; 0.1$).

$N = 10000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9840	0.9521	0.9919	0.9024
$n = 150$	0.9822	0.9513	0.9911	0.9015
$n = 200$	0.9810	0.9509	0.9906	0.9008
$n = 500$	0.9804	0.9503	0.9901	0.9002

According to this results, we find that the empirical signification levels of the Y_n^2 statistic coincide with those corresponding to the theoretical levels of the chi-square distributions at r degrees of freedom. Therefore, we can say that the proposed test can properly fit censored data from the Xg-E distribution.

8.3. Application to real data

8.3.1. Aluminum reduction cells data The data of Whitmore [47], who considered the times of failures for 20 aluminum reduction cells, and the numbers of failures in $1,000$ days units are : 0.468, 0.725, 0.838, 0.853, 0.965-1.139, 1.142, 1.304, 1.317, 1.427, 1.554, 1.658, 1.764, 1.776, 1.990, 2.010, 2.224, 2.279*, 2.244*, 2.286*. (* censoring). Assuming that these data are distributed according to the Xg-E distribution, the maximum likelihood estimator $\widehat{\boldsymbol{\theta}}$ of the parameter $\boldsymbol{\theta}$ is:

$$\widehat{\boldsymbol{\theta}} = 0.99574$$

We choose $r = 4$ as number of classes. The element of the statistic test Y_n^2 are presented as:

$\widehat{a_{\mathbf{j}_2}}$	0.9430	1.2106	1.6682	2.2949
$\widehat{\mathbf{U}_{\mathbf{j}_2}}$	4	3	5	8
$\mathrm{e}_{\mathbf{j}_2}$	1.9455	1.9455	1.9455	1.9455
$\widehat{\mathbf{C}_{1\mathbf{j}_2}}$	0.6241	0.34577	0.31589	0.02846

Then, we can calculate the value of the statistic test

$$Y_n^2 = 9.4435.$$

The critical value is

$$\chi_{0.05}^2(4) = 9.4877 > Y_n^2,$$

we conclude that the data of Aluminum reduction cells is in concordance with the Xg-E model.

8.3.2. Arm-A head and neck cancer data The data considered below (was conducted by northern California oncology group) was used by Efron [14] for logistic distribution. Nikulin and Haghighi [36] reanalyzed the same data and give the acceptable fit (chi-square type test) to the generalized Weibull distribution model. The survival times in days for the patients ($n = 51$) were as below ($\delta = 42$). 7, 34, 42, 63, 64, 74*, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185*, 218, 225, 241, 248, 273, 277, 279*, 297, 319*, 405, 417, 420, 440, 523*, 523, 583, 594, 1101, 1116*, 1146, 1226*, 1349*, 1412*, 1417. * censoring We use the data after transforming the survival times in months (1 month=30.438 days). The maximum likelihood estimator $\widehat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}$ is, if we suppose that this data are distributed according to the Xg-E distribution :

$$\widehat{\boldsymbol{\theta}} = 2.15246$$

We choose $r = 7$ as a number of classes. The elements of the test statistic Y_n^2 was presented as follow :

$\widehat{a_{\mathbf{j_2}}}$	2.739	4.548	9.512	21.015	37.0095	44.491	47.021
$\widehat{\mathbf{U_{j_2}}}$	7	7	20	10	2	3	2
$\mathrm{e}_{\mathbf{j_2}}$	2.9475	2.9475	2.9475	2.9475	2.9475	2.9475	2.9475
$\widehat{\mathbf{C_{1j_2}}}$	0.2709	0.31963	0.34751	0.0413	0.19245	0.4084	0.19125

after calculate, we find

$$Y_n^2 = 14.00945.$$

The critical value

$$\chi^2_{0.05}(7) = 14.06714 > Y_n^2 = 14.0094,$$

we can say that this data can be well modelised by the our Xg-E model.

9. Concluding remarks

In this article, a new version of the exponential distribution is proposed, studied, estimated and validated. Different statistical properties for the new model are derived. The unknown parameter of the proposed distribution has been estimated using different classical estimation method and Bayesian estimation method. Under classical estimation method, we briefly describe the method of moment, maximum likelihood estimators, maximum product of spacings estimators, least squares and weighted least squares estimators, Cramer-von-Mises estimators. The Bayesian estimation using gamma prior using squared error loss function has been discussed and computed using Lindley's and Markov Chain Monte Carlo techniques. The $100(1-\alpha)\%$ asymptotic confidence interval and credible interval along with the coverage probability are also discussed. The obtained classical and Bayesian estimators are compared through Monte Carlo simulations and noted that the Bayesian procedure is more efficient than the corresponding classical estimators. We describe the theory and the mechanism of the test statistic. The maximum likelihood estimators is employed based on the initial non grouped data sets. Then, A numerical simulations is performed to validate the results. Next, we construct a modified Chi-squared goodness-of-fit test based on the NRR statistic in presence of two censored and two complete data sets to illustrate the applicability of our model in various fields.

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Appendix

Calculation of the matrix $\widehat{\mathbf{W}}$

The elements of the estimated matrix $\widehat{\mathbf{W}}$ defined by

$$\widehat{\mathbf{W}}_{\mathbf{j}_1} = \sum_{\mathbf{j}_2=1}^{\kappa} \widehat{\mathbf{C}}_{\mathbf{j}_1\mathbf{j}_2} \ \widehat{\mathbf{A}}_{\mathbf{j}_2}^{-1} \mathbf{Z}_{\mathbf{j}_2} \mid_{(\ \mathbf{j}_1=1,2,3. \text{ and } \mathbf{j}_2=1,\ldots,\kappa)},$$

are obtained as follow

$$\widehat{\mathbf{C}_{\mathbf{j}_1\mathbf{j}_2}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_{\mathbf{j}_2}} \delta_i \frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(t_i, \widehat{\boldsymbol{\theta}}),$$

$$\ln \lambda(t_i, \widehat{\boldsymbol{\theta}}) = \ln \left[-\left(\lambda^2 t_i + \lambda\right) + \lambda \left(\frac{\lambda^2 t_i^2}{2} + \lambda t_i + 2 \right) \right] - \ln \left[2 + \lambda\, t_i + \frac{1}{2} \left(\lambda\, t_i\right)^2 \right]$$

The expressions of the element of the matrix $\widehat{\mathbf{C}_{\mathbf{j}_1\mathbf{j}_2}}$ is given as follows

$$\widehat{\mathbf{C}_{1\mathbf{j}_2}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_{\mathbf{j}_2}} \delta_i \left(3\lambda^2 \frac{t_i^2}{2} + 1 - t_i - \lambda\, t_i^2 \right),$$

Calculation of the matrix $\widehat{\mathbf{I}}$

The formulas of the element of the Fisher's information matrix $\widehat{\mathbf{I}} = (\widehat{i}_{\mathbf{j}_1\mathbf{j}_1'})_{1\times 1}$ is

$$\widehat{i_{\mathbf{j}_1\mathbf{j}_1'}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_{\mathbf{j}_2}} \delta_i \frac{\partial \ln \lambda(t_i, \widehat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}_{\mathbf{j}_1}} \frac{\partial \ln \lambda(t_i, \widehat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}_{\mathbf{j}_1'}}.$$

In our case we have:

$$\widehat{i_{11}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_{\mathbf{j}_2}} \delta_i \left(3\lambda^2 \frac{t_i^2}{2} + 1 - t_i - \lambda\, t_i^2 \right)^2,$$

Notice that, the components of the information matrix $\widehat{\mathbf{I}}$ are required for computation of the statistic Y_n^2.

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