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Bivariate Weibull-G Family Based on Copula Function: Properties, Bayesian and non-Bayesian Estimation and Applications

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Abstract This paper aims to obtain a new flexible bivariate generalized family of distributions based on FGM copula, which is called bivariate FGM Weibull-G family. Some of its statistical properties are studied as marginal distributions, product moments, and moment generating functions. Some dependence measures as Kendall's tau and median regression model are discussed. After introducing the general class, four special sub models of the new family are introduced by taking the baseline distributions as Pareto, inverted Topp-Leone, exponential, and Rayleigh distributions. Maximum likelihood and Bayesian approaches are used to estimate the model unknown parameters. Further, percentile bootstrap confidence interval and bootstrap-t confidence interval are estimated for the model's parameters. A Monte-Carlo simulation study is carried out of the maximum likelihood and Bayesian estimators. Finally, we illustrate the importance of the proposed bivariate family using two real data sets in medical field.

Keywords Weibull-G family, FGM copula, Median Regression, Goodness of Fit, Bayesian Estimation, Bootstrap

AMS 2010 subject classifications 62N01, 60E05, 62E15, 62H10, 62E10

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1. Introduction

Many authors proposed and studied bivariate distributions, which have wide applications in the fields of reliability, life testing, censoring, competing risks models, stress-strength model, sports, medical, engineering, drought, finance, weather, and among others. In the last few years, several ways of generating new bivariate distributions based on different copula functions and Marshall-Olkin methodology were discussed as follows:

By using copula function, Flores [6] presented six different bivariate Weibull distributions derived from copula functions, a bivariate survival model also based on the (Farlie-Gumbel-Morgenstern) FGM, Clayton, Ali-Mikhail-Haq (AMH), Gumbel-Hougaard, and Gumbel-Barnett copulas. Verrill et al. [4] introduced a bivariate Gaussian¨CWeibull distribution and the associated pseudo-truncated Weibull, and they also obtained an asymptotically efficient estimator of the parameter vector of the bivariate Gaussian¨C Weibull. El-Sherpieny and Almetwally [27] introduced a new extension of bivariate generalized Rayleigh distribution by using the Clayton copula function. Almetwally and Muhammed [18] proposed a new bivariate Fréchet distribution based on FGM and AMH copula functions and discussed their statistical properties. Almetwally et al. [7] introduced bivariate Weibull distribution by using the FGM copula function and some properties of this distribution are obtained. Samanthi and Sepanski [11]

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proposed families of bivariate copulas based on the Kumaraswamy distribution of existing copulas as Gumbel, Clayton, Frank, and Galambos.
By using the Marshall-Olkin method, Muhammed [19] proposed bivariate inverse Weibull distribution whose marginals are inverse Weibull distributions. El-Morshedy et al.[20] introduced a bivariate Burr X-G (BBX-G) family of distributions based on the Marshall-Olkin method, and they also discussed maximum likelihood and Bayesian approaches to estimate the model parameters. Eliwa and El-Morshedy [21] introduced bivariate odd Weibull-G family of distributions based on the Marshall-Olkin method. Bivariate generalized inverted Kumaraswamy distribution is presented by Muhammed [9]. Alotaibi et al. [10] proposed a new bivariate exponentiated half logistic distribution. In [38, 39] introduced bivariate gamma distribution, whose both the marginals are finite mixtures of gamma distributions and study its properties. The bivariate Lindley distribution using FGM copula has been obtained by [40] and study some of its properties.

However, there are many important problems in many practical situations, classical bivariate distributions do not provide adequate fits to real data. Therefore, there is an increased interest in developing more flexible bivariate distributions. A copula is a convenient approach to describe a multivariate distribution with a dependence structure. Nelsen [1] introduced copulas as follows; a copula is a function that joins multivariate distribution functions with uniform [0, 1] margins. Sklar [2] introduced the pdf and cdf for the two-dimension copula as follows: the two random variables X and Y with distribution functions $F(x)$ and $F(y)$ respectively, then the cdf and pdf for bivariate copula were respectively given as

$$F(x,y) = C(F(x;\Omega_1),\ F(y;\Omega_2)), \tag{1}$$

and

$$f(x,y) = f(x;\Omega_1)\ f(y;\Omega_2)\ c(F(x;\Omega_1),\ F(y;\Omega_2)). \tag{2}$$

Many copulas had been defined based on Equations (1) and (2) such as Farlie-Gumbel-Morgenstern (FGM). FGM copula is one of the most popular parametric families of copulas, the family was firstly introduced by Gumbel [3]. Almetwally [28] discussed FGM copula to introduce bivariate Weibull distribution. The joint cdf of FGM copula and pdf of FGM copula are shown as follows:

$$C\left(F(x;\Omega_1),\ F(y;\Omega_2)\right) = F(x;\Omega_1)\ F(y;\Omega_2)\ \left\{1+\theta\ \left[(1-F(x;\Omega_1))\ (1-F(y;\Omega_2))\right]\right\}, \tag{3}$$

$$c(F(x;\Omega_1),\ F(y;\Omega_2)) = 1+\theta(1-2\ F(x;\Omega_1))(1-2\ F(y;\Omega_2)), \tag{4}$$

where Ω_1 is a vector of parameter for the first variable X, Ω_2 is a vector of parameter for the second variable Y. The range of Spearman correlation coefficient for FGM copula is [-0.333,0.333] and the range of Kendall correlation coefficient for FGM copula is [-0.222,0.222].

Alzaatreh et al. [17] discussed a general method of generalized families by using the transformed-transformer (T-X) approach. Based on this approach, Bourguignon et al. [12] proposed a flexible family called the Weibull-G family, which can be called the odd Weibull-G (WG) family. The Weibull-G family has received increased attention, which accommodates all five major hazard shapes: increasing, decreasing, constant, unimodal, and bathtub failure rates. The cdf of the WG family is given by

$$F_{WG}(x;\Omega) = 1-exp\left(-\alpha\left[\frac{G(x;\Omega)}{1-G(x;\Omega)}\right]^{\beta}\right);\quad x>0,\ \alpha,\ \beta>0 \tag{5}$$

where Ω is a vector of parameters. The odds ratio $\frac{G(x;\Omega)}{1-G(x;\Omega)}$ means that for each baseline distribution $G(x;\Omega)$ we have a different distributions $F(x;\ \alpha,\beta,\Omega)$. Also, the generator $\frac{G(x;\Omega)}{1-G(x;\Omega)}$ satisfies the following conditions:

1. $\frac{G(x;\Omega)}{1-G(x;\Omega)} \in [a,b]$ for $0 < a < b < \infty$.

2. $\frac{G(x;\Omega)}{1-G(x;\Omega)}$ is differentiable and monotonically non-decreasing.

3. $\frac{G(x;\Omega)}{1-G(x;\Omega)} \implies a$ as $x \implies 0$ but $\frac{G(x;\Omega)}{1-G(x;\Omega)} \implies b$ as $x \implies \infty$.

The pdf of WG family is given by

$$f_{WG}(x;\Omega) = \alpha\beta g(x;\Omega)\frac{[G(x;\Omega)]^{(\beta-1)}}{[1-G(x;\Omega)]^{(\beta+1)}} exp\left(-\alpha\left[\frac{G(x;\Omega)}{1-G(x;\Omega)}\right]^{\beta}\right); \quad x>0,\ \alpha,\ \beta>0 \tag{6}$$

Because of the flexibility of the WG family, there are several extensions are presented, see Almarashi et al. [13], Eghwerido et al. [14], Elgarhy [15], Chesneau and Achiand [16], and others.

In this paper, we proposed a new bivariate family based on FGM copula function and WG family, which will be denoted as the BFGMWG family. Statistical properties for this family have been discussed. The bivariate exponential WG distribution, bivariate inverted Topp-Leone WG distribution, bivariate Rayleigh WG distribution, and bivariate Pareto WG distribution based on the FGM copula function will be introduced. Parameter estimation has been discussed by the maximum likelihood estimation method and Bayesian estimation method by using Metropolis-Hastings. Three types of confidence intervals (CI) are considered here namely asymptomatic, bootstrap, and Bayesian credible interval to discuss the interval estimation of the unknown parameters. Some dependence measures as Kendall's tau correlation and median regression model will be discussed for the BFGMWG family. Some measures of goodness of fit will be shown for the bivariate model as Kolmogorov-Smirnov statistic (KSS), Anderson-Darling, Anderson-Darling, and goodness of fit test for copulas. The application of real data shows that the proposed distributions are very competitive to some traditional distributions by using different data as medical data of kidney patients, and diabetic nephropathy data.

The rest of this paper is organized as follows: bivariate BFGMWG family is obtained in Section 2. Some statistical properties of the BFGMWG family are given in Section 3. A special model is obtained in Section 5. Parameter estimation methods are discussed in Section 6. In Section 7, asymptotic and bootstrap confidence intervals are discussed. In Section 8, the potentiality of the new model is illustrated by a simulation study. The goodness of Fit methodology is discussed in Section 9. In Section 10, application of two real data sets is discussed. Finally, the conclusion of some remarks for the bivariate BFGMWG family is addressed in Section 11.

2. BFGMWG Family

According to Sklar theorem in Equations (1 and 2), and WG family Equations (5 and 6), we get the joint cdf and pdf of bivariate WG family based on any copula function as follows

$$F(x,y) = C\left\{1-exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right), 1-exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}, \tag{7}$$

and

$$\begin{aligned} f(x,y) =&\alpha_1\beta_1 g(x;\Omega_1)\frac{[G(x;\Omega_1)]^{(\beta_1-1)}}{[1-G(x;\Omega_1)]^{(\beta_1+1)}} exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\\ &\alpha_2\beta_2 g(y;\Omega_2)\frac{[G(y;\Omega_2)]^{(\beta_2-1)}}{[1-G(y;\Omega_2)]^{(\beta_2+1)}} exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\\ &c\left\{1-exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right), 1-exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}. \end{aligned} \tag{8}$$

By using FGM copula function that is given in Equations (3, 4) and bivariate WG family based on any copula function in Equations (7, 8), we get the joint cdf and pdf of BFGMWG family as follows:

$$\begin{aligned} F_{BFGMWG}(x,y) = &\left\{1-exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\right\}\left\{1-exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}\\ &\left(1+\theta\left\{exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right) exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} f_{BFGMWG}(x,y) = &\alpha_1\beta_1 g(x;\Omega_1)\frac{[G(x;\Omega_1)]^{(\beta_1-1)}}{[1-G(x;\Omega_1)]^{(\beta_1+1)}}\ exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\\ &\alpha_2\beta_2 g(y;\Omega_2)\frac{[G(y;\Omega_2)]^{(\beta_2-1)}}{[1-G(y;\Omega_2)]^{(\beta_2+1)}}\ exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\\ &\left(1+\theta\left\{2\ exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)-1\right\}\left\{2\ exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)-1\right\}\right), \end{aligned} \tag{10}$$

In the sub-models of the BFGMWG family, the BFGMWG family is a very flexible model that approaches different bivariate distributions when its parameters are changed.

- When $\alpha_1 = \alpha_2 = 1$, we get a new bivariate FGM modified Kies family, which is a new univariate family of modified Kies has been introduced by Al-Babtain et al. [35].
- When $\beta_1 = \beta_2 = 1$, we get a new bivariate FGM odd exponential family, which is an odd generalized exponential family has been introduced in the univariate case by Tahir et al. [36].
- When $\beta_1 = \beta_2 = 2$, we get a new bivarite FGM odd Rayleigh family.
- When $\alpha_1 = \alpha_2 = 1,\ \beta_1 = \beta_2 = 1$, we get a new bivarite FGM odd stander exponential family.

3. Properties of BFGMWG Family

In this section, we get some statistical properties of the BFGMWG family such as marginal distributions, product moments, and moment generating function.

3.1. The Marginal Distributions

The joint pdf of BFGMWG family given in Equation (10) has WG family marginals. The marginal density functions for X and Y respectively are,

$$f(x;\alpha_1,\beta_1,\Omega_1) = \alpha_1\beta_1 g(x;\Omega_1)\frac{[G(x;\Omega_1)]^{(\beta_1-1)}}{[1-G(x;\Omega_1)]^{(\beta_1+1)}}\ exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right), \tag{11}$$

and

$$f(y;\alpha_2,\beta_2,\Omega_2)=\alpha_2\beta_2 g(y;\Omega_2)\frac{[G(y;\Omega_2)]^{(\beta_2-1)}}{[1-G(y;\Omega_2)]^{(\beta_2+1)}}\ exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right), \tag{12}$$

which are WG family as shown in Equations (11, 12).

3.2. Conditional Distribution

The conditional probability distribution of X given Y is given as follows:

$$\begin{aligned} f(x|y)=&\alpha_1\beta_1 g(x;\Omega_1)\frac{[G(x;\Omega_1)]^{(\beta_1-1)}}{[1-G(x;\Omega_1)]^{(\beta_1+1)}}\ \zeta_1(x;\alpha_1,\beta_1,\Omega_1)\\ &\{[1+\theta-2\theta(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))]+2\theta(1-\zeta_1(x;\alpha_1,\beta_1,\Omega_1))[2(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))-1]\}. \end{aligned} \tag{13}$$

where $\zeta_j(z_j;\alpha_j,\beta_j,\Omega_j)=exp\left(-\alpha_j\left[\frac{G(z_j;\Omega_j)}{1-G(z_j;\Omega_j)}\right]^{\beta_j}\right)$
and $j=1,2,z$ is vector of x,y. The conditional cdf of X given Y is as follows:

$$\begin{aligned} F(x|y)=&[1-\zeta_1(x;\alpha_1,\beta_1,\Omega_1)]\ \{[1+\theta-2\theta(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))]+\\ &\theta(1-\zeta_1(x;\alpha_1,\beta_1,\Omega_1))[2(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))-1]\ \}. \end{aligned} \tag{14}$$

The conditional probability distribution of Y given X as follows:

$$\begin{aligned} f(y|x)=&\alpha_2\beta_2 g(y;\Omega_2)\frac{[G(y;\Omega_2)]^{(\beta_2-1)}}{[1-G(y;\Omega_2)]^{(\beta_2+1)}}\ \zeta_2(y;\alpha_2,\beta_2,\Omega_2)\\ &\{[1+\theta-2\theta(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))]+2\theta(1-\zeta_1(x;\alpha_1,\beta_1,\Omega_1))[2(1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2))-1]\}, \end{aligned} \tag{15}$$

The conditional cdf of Y given X is as follows:

$$F(y|x)=[1-\zeta_2(y;\alpha_2,\beta_2,\Omega_2)]\ \{1+\zeta_2(y;\alpha_2,\beta_2,\Omega_2)[\theta-2\theta(1-\zeta_1(x;\alpha_1,\beta_1,\Omega_1))]. \tag{16}$$

By (16), we can generate a bivariate sample of the WG family by using the conditional approach:

- Generate U and V as independently from uniform (0, 1) distribution.
- Set $x=Q_{WG}(u)=G^{-1}\left\{1+\left[\frac{-1}{\alpha_1}\ln(1-U)\right]^{\frac{1}{\beta_1}}\right\}$.
- Set $F(y|x)=V$ in (16) to find y by numerical analysis as Newton Raphson, and etc.
- Repeat above steps (n) items to obtain (x_i,y_i), $i=1,...,n$.

3.3. Product Moments, Moment Generating Function

Let (X,Y) denote a random variable with the pdf of BFGMWG family as in (10). Then its r^{th} and s^{th} joint moments around zero denoted by μ'_{rs} can be expressed as follows

$$\begin{aligned} \mu'_{rs}=&E(x^r y^s)=\\ &\sum_{j_1,k_1=0}^{\infty}\sum_{j_2,k_2=0}^{\infty}\mu'_r\mu'_s\left\{1+\theta-2\theta\left[(1-2^{k_1})+(1-2^{k_2})-4(1-2^{k_1})(1-2^{k_2})\right]\right\}. \end{aligned} \tag{17}$$

where $\mu'_r=\xi_{j_1,k_1}\int_0^\infty x^r g(x;\Omega_1)G(x;\Omega_1)^{\beta_1(k_1+1)+j_1-1}\,dx$, $\xi_{j_l,k_l}=\frac{(-1)^{k_l+j_l}}{k_l!}\beta_l\alpha_l^{k_l+1}\binom{\beta_l(k_l+1)+j_l}{j_l}$; $l=1,\ 2$ and $\mu'_s=\xi_{j_2,k_2}\int_0^\infty y^s g(y;\Omega_2)G(y;\Omega_2)^{\beta_2(k_2+1)+j_2-1}\,dy$. The moment generating function of BFGMWG family is given as

$$M_{x,y}(t_1,t_2) = E(e^{t_1 x}\ e^{t_2 y})$$
$$= \sum_{j_1,k_1,d_1=0}^{\infty} \sum_{j_2,k_2,d_2=0}^{\infty} \frac{t_1^{d_1}}{d_1!}\frac{t_2^{d_2}}{d_2!}\mu_{d_1}^{'}\mu_{d_2}^{'} \left\{1+\theta-2\theta\left[(1-2^{k_1})+(1-2^{k_2})-4(1-2^{k_1})(1-2^{k_2})\right]\right\}. \quad (18)$$

3.4. Survival Function

Osmetti and Chiodini (2011) discussed that the reliability function for bivariate distribution based on copula function, which is more convenient to express a joint survival function as a copula of its marginal survival functions, where X and Y be a random variable with survival functions $1-F(x)$ and $1-F(y)$. The expression of the joint survival function for copula is as following

$$s(x,y) = F(x) + F(y) - 1 + C(1-F(x),\ 1-F(y)).$$

Then the reliability function of BFGMWG distribution is

$$s(x,y) = 1 - exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right) - exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)$$
$$\left\{exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\right\}\left\{exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}$$
$$\left(1+\theta\left\{\left[1-exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\right]\left[1-exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right]\right\}\right).$$

For further details, see Nelsen [1].

4. Some Dependence Measures

As it's before indicated FGM copula function serves for discussing dependence between random variables. Also, copula may be a tool for dependence measuring. Here we will discuss Kendall's tau and median regression model.

4.1. Kendall's Tau Crrelation

Kendall's tau defines as the probability of concordance minus the probability of discordance of two pairs of random variables. In terms of copula function

$$\tau = 1 - 4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}dudv. \quad (19)$$

In case of FGM copula function $\frac{\partial C(u,v)}{\partial u} = v+\theta v-\theta v^2-2\theta uv+2\theta uv^2$ and $\frac{\partial C(u,v)}{\partial v} = u+\theta u-\theta u^2-2\theta uv+2\theta u^2 v$. Then $\tau = \frac{2\theta}{9}$.

4.2. Median Regression Model of BFGMWG Famliy

In addition to measures of association and dependence properties, regression is a method for describing the dependence of one random variable on another. For random variables X and Y, the regression curve $y = E(Y|x)$ [Nelsen [1] P. 217]. Let X and Y be a random variables from BFGMWG family. The median regression curve of Y on X is $P(Y \leq y|X=x) = \frac{1}{2}$. In BFGMWG family,

$$P(Y \leq y|X=x) = P(F(Y) \leq F(y)|F(X)=F(x)) = \frac{\partial F_{BFGMWG}(x,y)}{\partial F(x)}, \tag{20}$$

By using cdf of BFGMWG family in Equation (9), the partial drvatives of Y on X

$$\frac{\partial F_{BFGMWG}(x,y)}{\partial F(x)} = \left\{1 - exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}$$
$$\left\{1+\theta\left[-1+2exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)\right]exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)\right\}, \tag{21}$$

by setting $\frac{\partial F_{BFGMWG}(x,y)}{\partial F(x)} = \frac{1}{2}$ and simplifying yields

$$(4\theta u - 2\theta)\, v = \sqrt{4\theta u - 2\theta + (1+\theta-2\theta u)^2} - (1+\theta-2\theta u), \tag{22}$$

where $u = F(x) = 1 - exp\left(-\alpha_1\left[\frac{G(x;\Omega_1)}{1-G(x;\Omega_1)}\right]^{\beta_1}\right)$ and $v = F(y) = 1 - exp\left(-\alpha_2\left[\frac{G(y;\Omega_2)}{1-G(y;\Omega_2)}\right]^{\beta_2}\right)$ Thus the median regression curve of V on U is the line in $\boldsymbol{I}^2$ connecting the points $\left(0,\ \frac{\theta+1-\sqrt{\theta^2+1}}{2\theta}\right)$ and $\left(1,\ \frac{\theta-1+\sqrt{\theta^2+1}}{2\theta}\right)$. Note the special cases: when $\theta=-1$ then the median regression line is $v = \frac{\sqrt{4u^2-4u+2}-2u}{2-4u}$, and when $\theta = 1$ then the median regression line is $v = \frac{\sqrt{4u^2-4u+2}+2u-2}{4u-2}$. The slope of the median regression line is $\frac{\sqrt{\theta^2+1}-1}{\theta}$, see Figure 1 as a kind of illustration. Then the median regression curve is linear in $F(x)$ and $F(y)$:

$$F(Y) = \frac{\sqrt{4\theta F(x) - 2\theta + (1+\theta-2\theta F(x))^2} - (1+\theta-2\theta F(x))}{4\theta F(x) - 2\theta} \tag{23}$$

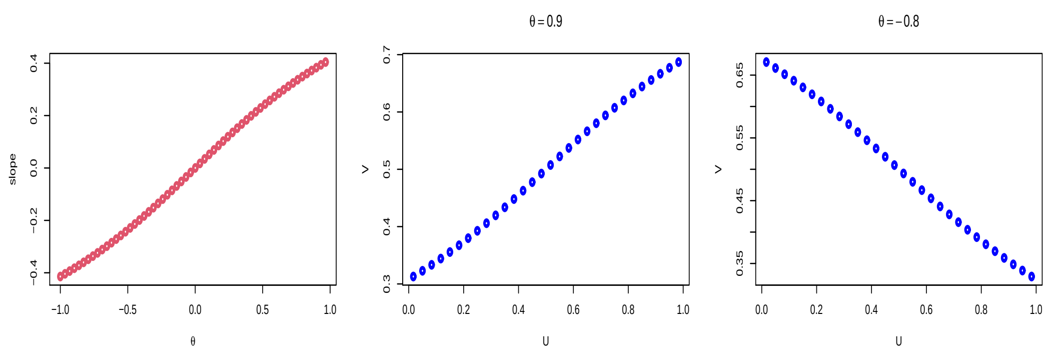

Figure 1. Plots of slope and regression curve of V on U of BFGMWG family

5. Special Model

In this section, we introduced four special models of the BFGMWG family of distributions, the pdf (10) will be most tractable when the cdf $G(x)$ and pdf $g(x)$ have simple analytic expressions. We provide four sub-models of this family by taking the baseline distributions: Pareto, inverted Topp-Leone (ITL), exponential, and Rayleigh distributions. The cdf and pdf of these baseline models are listed in the following Table 1.

Table 1. cdf and pdf as baseline models

$Model$	$Cdf : G(x;\delta)$	$pdf : g(x;\delta)$	$\frac{G(x;\delta)}{1-G(x;\delta)}$
$Pareto$	$1-\frac{1}{x^{\delta}}; x>1$	$\frac{\delta}{x^{\delta+1}}$	$\frac{\delta}{x}$
ITL	$1-\frac{(1+2x)^{\delta}}{(1+x)^{2\delta}}; x>0$	$2\delta x\frac{(1+2x)^{\delta-1}}{(1+x)^{2\delta+1}}$	$\frac{2\delta x}{(1+x)(1+2x)}$
$Exponential$	$1-e^{-\delta x}; x>0$	$\delta e^{-\delta x}$	$e^{\delta x}-1$
$Rayleigh$	$1-e^{-\frac{\delta}{2}x^2}; x>0$	$\delta x e^{-\frac{\delta}{2}x^2}$	$e^{\frac{\delta}{2}x^2}-1$

5.1. BFGMWG-Pareto (BFGMWGP) Distribution

By using WG family and Pareto distribution to obtain WG-Pareto (WP) distribution, the cdf and pdf of BFGMWGP distribution are

$$F_{BFGMWGP}(x,y)=\left\{1-exp\left(-\alpha_1\left[\frac{\delta}{x}\right]^{\beta_1}\right)\right\}\left\{1-exp\left(-\alpha_2\left[\frac{\delta_2}{y}\right]^{\beta_2}\right)\right\}$$
$$1+\theta\left\{exp\left(-\alpha_1\left[\frac{\delta}{x}\right]^{\beta_1}\right)exp\left(-\alpha_2\left[\frac{\delta_2}{y}\right]^{\beta_2}\right)\right\}, \tag{24}$$

and

$$f_{BFGMWGP}(x,y)=\alpha_1\beta_1\alpha_2\beta_2\frac{\delta_1}{x^{\delta_1+1}}\frac{\delta_2}{y^{\delta_2+1}}\frac{\left[1-\frac{1}{x^{\delta_1}}\right]^{(\beta_1-1)}}{\left[\frac{1}{x^{\delta_1}}\right]^{(\beta_1+1)}}\;exp\left(-\alpha_1\left[\frac{\delta_1}{x}\right]^{\beta_1}-\alpha_2\left[\frac{\delta_2}{y}\right]^{\beta_2}\right)$$
$$\frac{\left[1-\frac{1}{y^{\delta_2}}\right]^{(\beta_2-1)}}{\left[\frac{1}{y^{\delta_2}}\right]^{(\beta_2+1)}}\;\left(1+\theta\left\{2\;exp\left(-\alpha_1\left[\frac{\delta_1}{x}\right]^{\beta_1}\right)-1\right\}\left\{2\;exp\left(-\alpha_2\left[\frac{\delta_2}{y}\right]^{\beta_2}\right)-1\right\}\right), \tag{25}$$

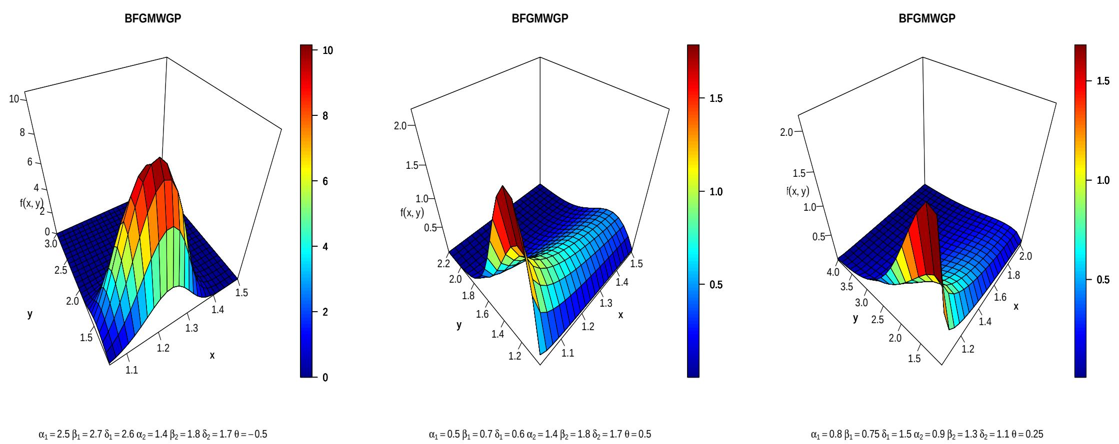

Figure 2. Plots of joint pdf of BFGMWGP distribution

5.2. BFGMWG-ITL (BFGMWGITL) Distribution

By using WG family and ITL distribution to obtain WG-Pareto (WITL) distribution, the cdf and pdf of BFGMWGITL distribution are

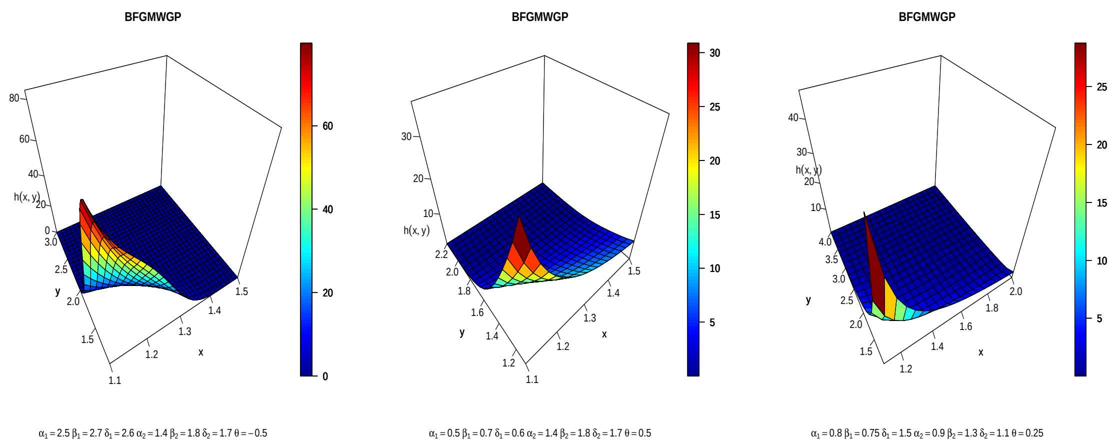

Figure 3. Plots of joint hazard function of BFGMWGP distribution

$$F_{BFGMWGITL}(x,y) = \left\{1 - exp\left(-\alpha_1\left[\frac{2\delta_1 x}{(1+x)(1+2x)}\right]^{\beta_1}\right)\right\}\left\{1 - exp\left(-\alpha_2\left[\frac{2\delta_2 y}{(1+y)(1+2y)}\right]^{\beta_2}\right)\right\}$$
$$1+\theta\left\{exp\left(-\alpha_1\left[\frac{2\delta_1 x}{(1+x)(1+2x)}\right]^{\beta_1}\right)exp\left(-\alpha_2\left[\frac{2\delta_2 y}{(1+y)(1+2y)}\right]^{\beta_2}\right)\right\}, \tag{26}$$

and

$$f_{BFGMWGITL}(x,y) = \alpha_1\beta_1\alpha_2\beta_2 4\delta_1 x(1+2x)^{\delta_1-1}(1+x)^{-2\delta_1-1}\delta_2 x(1+2y)^{\delta_2-1}(1+y)^{-2\delta_2-1}$$
$$\frac{\left[(1+x)^{2\delta_1}-(1+2x)^{\delta_1}\right]^{(\beta_1-1)}}{(1+x)^{-4\delta_1}(1+2x)^{\delta_1(\beta_1+1)}}\;\frac{\left[(1+y)^{2\delta_2}-(1+2y)^{\delta_2}\right]^{(\beta_2-1)}}{(1+y)^{-4\delta_2}(1+2y)^{\delta_2(\beta_2+1)}}$$
$$exp\left(-\alpha_1\left[\frac{2\delta_1 x(1+x)^{-1}}{(1+2x)}\right]^{\beta_1}-\alpha_2\left[\frac{2\delta_2 y(1+y)^{-1}}{(1+2y)}\right]^{\beta_2}\right)$$
$$\left(1+\theta\left\{4exp\left(-\alpha_1 2\delta_1\left[\frac{x(1+x)^{-1}}{(1+2x)}\right]^{\beta_1}\right)-1\right\}\left\{exp\left(-\alpha_2 2\delta_2\left[\frac{y(1+y)^{-1}}{(1+2y)}\right]^{\beta_2}\right)-1\right\}\right), \tag{27}$$

5.3. BFGMWG-Exponential (BFGMWGE) Distribution

By using WG family and exponential distribution to obtain WG-exponential (WE) distribution, the cdf and pdf of BFGMWGE distribution are

$$F_{BFGMWGE}(x,y) = \left\{1 - exp\left(-\alpha_1\left[e^{\delta_1 x}-1\right]^{\beta_1}\right)\right\}\left\{1 - exp\left(-\alpha_2\left[e^{\delta_2 y}-1\right]^{\beta_2}\right)\right\}$$
$$\left(1+\theta\left\{exp\left(-\alpha_1\left[e^{\delta_1 x}-1\right]^{\beta_1}\right)exp\left(-\alpha_2\left[e^{\delta_2 y}-1\right]^{\beta_2}\right)\right\}\right), \tag{28}$$

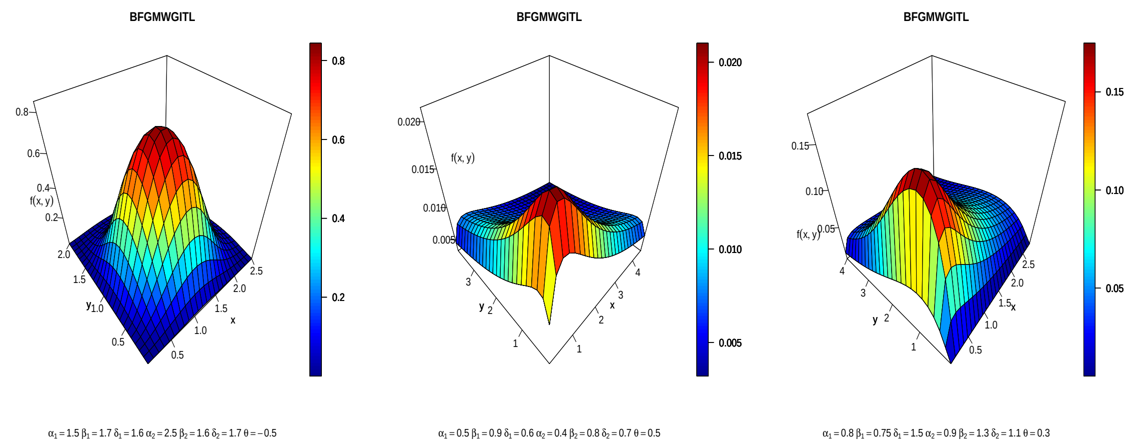

Figure 4. Plots of joint pdf of BFGMWGITL distribution

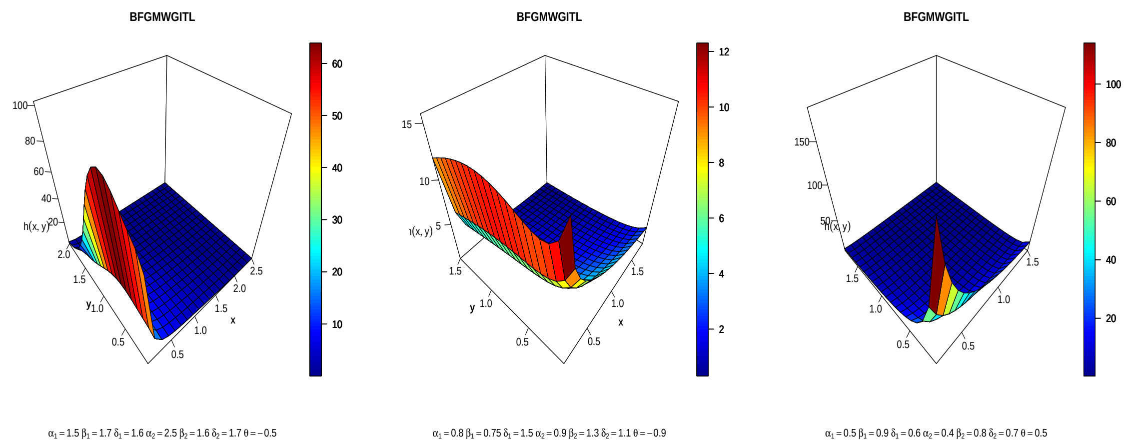

Figure 5. Plots of joint hazard function of BFGMWGITL distribution

and

$$f_{BFGMWGE}(x,y) = \alpha_1\beta_1\alpha_2\beta_2\delta_2\delta \frac{\left[1-e^{-\delta_1 x}\right]^{(\beta_1-1)}}{\left[e^{-\delta_1 x}\right]^{(\beta_1)}}\ exp\left(-\alpha_1\left[e^{\delta_1 x}-1\right]^{\beta_1}-\alpha_2\left[e^{\delta_2 y}-1\right]^{\beta_2}\right)$$
$$\frac{\left[1-e^{-\delta_2 y}\right]^{(\beta_2-1)}}{\left[e^{-\delta_2 y}\right]^{(\beta_2)}}\ \left(1+\theta\left\{2\ exp\left(-\alpha_1\left[e^{\delta_1 x}-1\right]^{\beta_1}\right)-1\right\}\left\{2\ exp\left(-\alpha_2\left[e^{\delta_2 y}-1\right]^{\beta_2}\right)-1\right\}\right), \tag{29}$$

The r^{th} and s^{th} joint moments around zero for BFGMWGE can be expressed as follows

$$\mu'_{rs} = \sum_{j_1,k_1,q_1=0}^{\infty} \xi^*{}_{(j_1,k_1,q_1)} \frac{r!}{\delta_1^r(q_1+1)^{r+1}} \sum_{j_2,k_2,q_2=0}^{\infty} \xi^*{}_{(j_2,k_2,q_2)} \Xi_{k_1,k_2} \frac{s!}{\delta_2^s(q_2+1)^{s+1}} \tag{30}$$

where $\Xi_{k_1,k_2} = 1+\theta\left[1-2\left(1-2^{k_1}\right)-2\left(1-2^{k_2}\right)+4\left(1-2^{k_1}\right)\left(1-2^{k_2}\right)\right]$ and $\xi^*{}_{(j_l,k_l,q_l)} = \xi_{(j_l,k_l)}\binom{\beta_l(k_l+1+1k_l-1)}{q_l}(-1)^{q_l}$. The moment generating function of BFGMWGE distribution is given

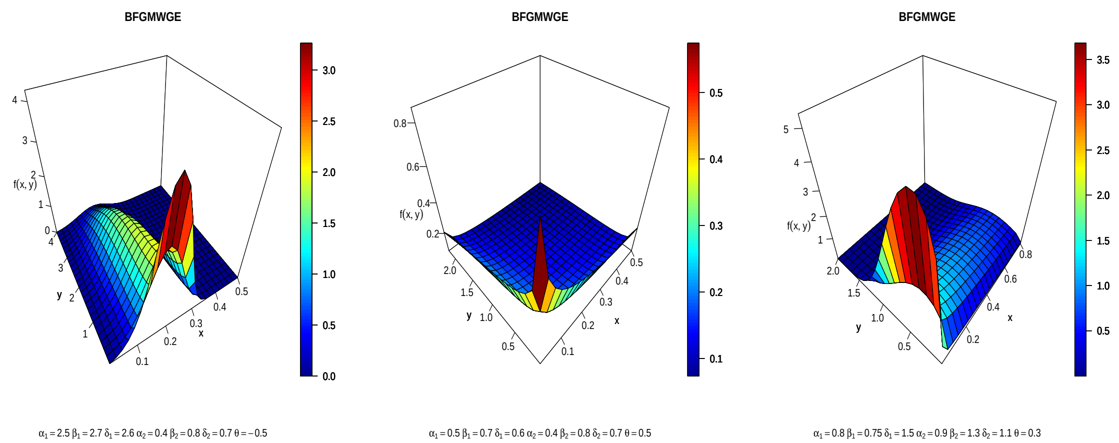

Figure 6. Plots of joint pdf of BFGMWGE distribution

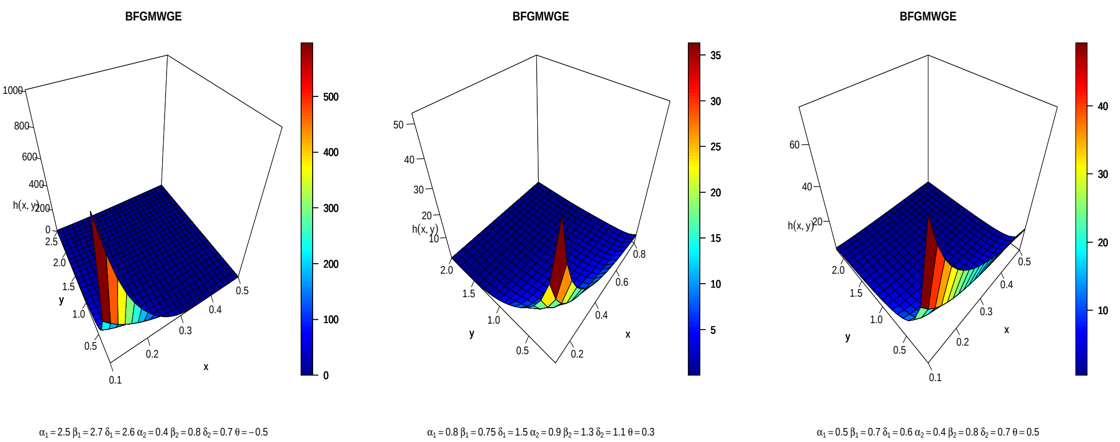

Figure 7. Plots of joint hazard function of BFGMWGE distribution

as

$$M_{x,y}(t_1,t_2) = \sum_{j_1,k_1,q_1,d_1=0}^{\infty} \sum_{j_2,k_2,q_2,d_2=0}^{\infty} \frac{t_1^{d_1}}{d_1!} \frac{t_2^{d_2}}{d_2!} \xi^*{}_{(j_1,k_1,q_1)} \xi^*{}_{(j_2,k_2,q_2)} \Xi_{k_1,k_2} \frac{d_1!}{\delta_1^{d_1}(q_1+1)^{d_1+1}} \frac{d_2!}{\delta_2^{d_2}(q_2+1)^{d_2+1}}. \tag{31}$$

5.4. BFGMWG-Rayleigh (BFGMWGR) Distribution

By using WG family and Rayleigh distribution to obtain WG-Rayleigh (WR) distribution, the cdf and pdf of BFGMWGR distribution are

$$\begin{aligned} F_{BFGMWGR}(x,y) = &\left\{1 - exp\left(-\alpha_1 \left[e^{\frac{\delta_1}{2}x^2} - 1\right]^{\beta_1}\right)\right\}\left\{1 - exp\left(-\alpha_2 \left[e^{\frac{\delta_2}{2}y^2} - 1\right]^{\beta_2}\right)\right\} \\ &\left(1 + \theta\left\{exp\left(-\alpha_1 \left[e^{\frac{\delta_1}{2}x^2} - 1\right]^{\beta_1}\right) exp\left(-\alpha_2 \left[e^{\frac{\delta_2}{2}y^2} - 1\right]^{\beta_2}\right)\right\}\right), \end{aligned} \tag{32}$$

and

$$f_{BFGMWGR}(x,y) = \alpha_1\beta_1\delta_1\alpha_2\beta_2\delta_2 xy \frac{\left[1-e^{-\frac{\delta_1}{2}x^2}\right]^{(\beta_1-1)}}{\left[e^{-\frac{\delta_1}{2}x^2}\right]^{(\beta_1)}} \, exp\left(-\alpha_1\left[e^{\frac{\delta_1}{2}x^2}-1\right]^{\beta_1} - \alpha_2\left[e^{\frac{\delta_2}{2}y^2}-1\right]^{\beta_2}\right)$$
$$\frac{\left[1-e^{-\frac{\delta_2}{2}y^2}\right]^{(\beta_2-1)}}{\left[e^{-\frac{\delta_2}{2}y^2}\right]^{(\beta_2)}} \left(1+\theta\left\{2\,exp\left(-\alpha_1\left[e^{\frac{\delta_1}{2}x^2}-1\right]^{\beta_1}\right)-1\right\}\left\{2\,exp\left(-\alpha_2\left[e^{\frac{\delta_2}{2}y^2}-1\right]^{\beta_2}\right)-1\right\}\right), \tag{33}$$

The r^{th} and s^{th} joint moments around zero for BFGMWGR can be expressed as follows

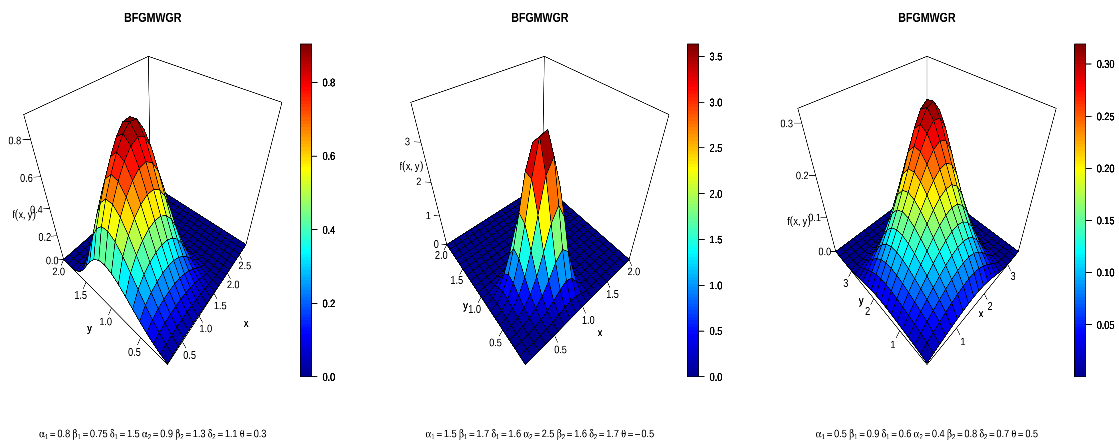

Figure 8. Plots of joint pdf of BFGMWGR distribution

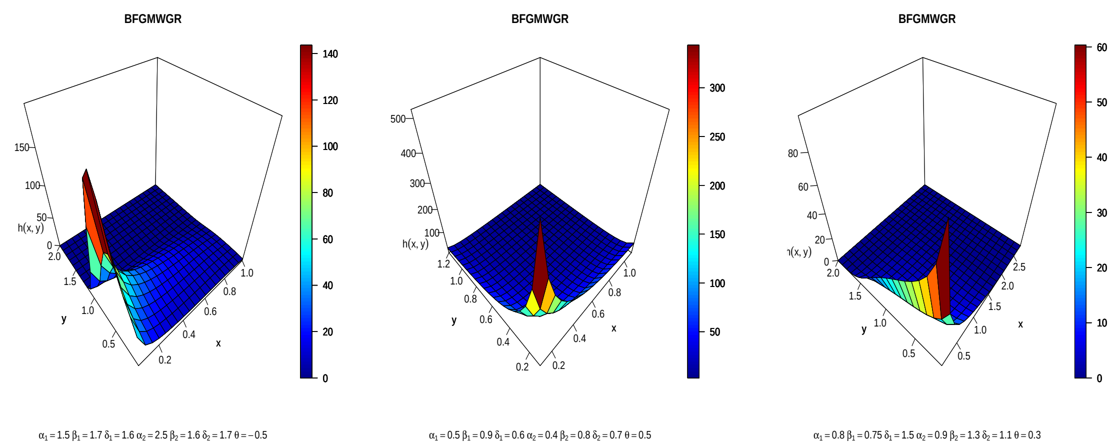

Figure 9. Plots of joint hazard function of BFGMWGR distribution

$$\mu'_{rs} = \sum_{j_1,k_1,q_1=0}^{\infty} \xi^*{}_{(j_1,k_1,q_1)}\Gamma\left(\frac{r}{2}+1\right)2^{\frac{r}{2}}\delta_1^{2-\frac{r}{2}}\frac{1}{1-q_1}^{\frac{r}{2}-1} \sum_{j_2,k_2,q_2=0}^{\infty} \xi^*{}_{(j_2,k_2,q_2)}\Xi_{k_1,k_2}\Gamma\left(\frac{s}{2}+1\right)2^{\frac{s}{2}}\delta_2^{2-\frac{s}{2}}\frac{1}{1-q_2}^{\frac{s}{2}-1}. \tag{34}$$

The moment generating function of BFGMWGR distribution is given as

$$M_{x,y}(t_1,t_2) = \sum_{j_1,k_1,q_1,d_1=0}^{\infty} \sum_{j_2,k_2,q_2,d_2=0}^{\infty} \frac{t_1^{d_1}}{d_1!}\frac{t_2^{d_2}}{d_2!} \xi^*{}_{(j_1,k_1,q_1)} \xi^*{}_{(j_2,k_2,q_2)} \Xi_{k_1,k_2} \Gamma\left(\frac{d_1}{2}+1\right) 2^{\frac{d_1}{2}} \delta_1^{2-\frac{d_1}{2}} \frac{1}{1-q_1}^{\frac{d_1}{2}-1} \Gamma\left(\frac{d_2}{2}+1\right) 2^{\frac{d_2}{2}} \delta_2^{2-\frac{d_2}{2}} \frac{1}{1-q_2}^{\frac{d_2}{2}-1}. \tag{35}$$

Figures (2, 4, 6, 8) show that the 3-dimension plots for the joint pdf for different distributions of BFGMWG family, and Figures (3, 5 7, 9) joint hazard rate functions for different distributions of BFGMWG family with different values of parameters. The hazard function behavior has more than one direction, where it takes an increasing and decreasing, which will have many applications in life testing.

6. Parameter Estimation Methods

In this section, we introduce two estimation methods that are used to estimate the unknown parameters of the BFGMWG family, such as maximum likelihood estimation (MLE) and Bayesian estimation.

6.1. Maximum Likelihood Estimation (MLE)

In this subsection, we estimate the unknown parameters of the BFGMWG family using the maximum likelihood method. Suppose that (x_1, y_1), $(x_2, y_2), ..., (x_n, y_n)$ is a sample of size n, from the BFGMWG family. For more information about this method see Kim et al. [22]. The likelihood function of the BFGMWG family is as follows

$$(\Theta) = (\alpha_1\beta_1\alpha_2\beta_2)^n \, exp\left(-\alpha_1 \sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1}\right) \prod_{i=1}^{n} g(x_i;\Omega_1)\frac{[G(x_i;\Omega_1)]^{(\beta_1-1)}}{[1-G(x_i;\Omega_1)]^{(\beta_1+1)}} \, exp\left(-\alpha_2 \sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2}\right) \prod_{i=1}^{n} g(y_i;\Omega_2)\frac{[G(y_i;\Omega_2)]^{(\beta_2-1)}}{[1-G(y_i;\Omega_2)]^{(\beta_2+1)}} \prod_{i=1}^{n}\left(1+\theta\left\{2\;\zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\;\zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right), \tag{36}$$

where Θ is a vector of parameters of BFGMWG family,
$\zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right) = exp\left(-\alpha_1 \sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1}\right)$,
$\zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right) = exp\left(-\alpha_2 \sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2}\right)$ and the log-likelihood function of BFGMWG family can be written as

$$(\Theta) = n\left[\log(\alpha_1)+\log(\beta_1)\right] - \alpha_1 \sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1} + \sum_{i=1}^{n}\log\left(\frac{g(x_i;\Omega_1)}{G(x_i;\Omega_1)2}\right) + (\beta_1+1)\sum_{i=1}^{n}\log\left\{\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]\right\} + n\left[\log(\alpha_2)+\log(\beta_2)\right] - \alpha_2 \sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2} + \sum_{i=1}^{n}\log\left(\frac{g(y_i;\Omega_2)}{G(y_i;\Omega_2)2}\right) + (\beta_2+1)\sum_{i=1}^{n}\log\left\{\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]\right\} + \sum_{i=1}^{n}\log\left(1+\theta\left\{2\;\zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\;\zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right). \tag{37}$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (37) concerning each parameter separately and equating to zero, as following

$$\frac{\partial l(\Theta)}{\partial \alpha_j} = -2\theta \sum_{i=1}^{n} \frac{\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j} \zeta_j\left(z_j;\alpha_j,\beta_j,\Omega_j\right)\left\{2\ \zeta_l\left(z_l;\alpha_l,\beta_l,\Omega_l\right)-1\right\}}{1+\theta\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}} + \frac{n}{\alpha_j} - \sum_{i=1}^{n}\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j},$$

$$\begin{aligned}\frac{\partial l(\Theta)}{\partial \beta_j} =& -2\theta\alpha_j \sum_{i=1}^{n} \frac{\sum_{i=1}^{n}\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j} \log\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right] \zeta_j\left(z_j;\alpha_j,\beta_j,\Omega_j\right)\left\{2\ \zeta_l\left(z_l;\alpha_l,\beta_l,\Omega_l\right)-1\right\}}{1+\theta\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}} \\ &+ \frac{n}{\beta_j} - \alpha_j \sum_{i=1}^{n}\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j} \log\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right] + \sum_{i=1}^{n}\log\left\{\left[\frac{G(z_ji;\Omega_j)}{1-G(z_ji;\Omega_j)}\right]\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial l(\Theta)}{\partial \Omega_j} =& (\beta_j+1)\sum_{i=1}^{n}\frac{\Xi(z_{ji};\Omega_j)}{G(z_{ji};\Omega_j)\left[1-G(z_{ji};\Omega_j)\right]} - \alpha_j\beta_j\sum_{i=1}^{n}\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j-1}\frac{\Xi(z_{ji};\Omega_j)}{\left[1-G(z_{ji};\Omega_j)\right]^2} \\ &+ \sum_{i=1}^{n}\frac{\xi(z_{ji};\Omega_j)G(z_{ji};\Omega_j)-2g(z_{ji};\Omega_j)\Xi(z_{ji};\Omega_j)}{g(z_{ji};\Omega_j)G(z_{ji};\Omega_j)} \\ &- 2\alpha_j\beta_j\theta\sum_{i=1}^{n}\frac{\left[\frac{G(z_{ji};\Omega_j)}{1-G(z_{ji};\Omega_j)}\right]^{\beta_j-1}\frac{\Xi(z_{ji};\Omega_j)}{[1-G(z_{ji};\Omega_j)]^2}\zeta_j\left(z_j;\alpha_j,\beta_j,\Omega_j\right)\left(2\zeta_l\left(z_l;\alpha_l,\beta_l,\Omega_l\right)-1\right)}{1+\theta\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}},\end{aligned}$$

and

$$\frac{\partial l(\Theta)}{\partial \theta} = \sum_{i=1}^{n}\frac{\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}}{1+\theta\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}},$$

where $\Xi(z_{ji};\Omega_j) = \frac{\partial G(z_{ji};\theta)}{\partial \Omega_j}$, $\xi(z_{ji};\Omega_j) = \frac{\partial g(z_{ji};\Omega_j)}{\partial \Omega_j}$ and $j = l = 1,2; j \neq$l

6.2. Bayesian Estimation

The Bayesian estimation method is an alternative statistical estimation method that allows the incorporation of prior knowledge of parameters through an informative prior distribution. When there is not much prior knowledge one can consider a non-informative prior structure. In this section, we consider Bayesian estimation of the BFGMWG family parameters assuming that random variables $\Theta = (\alpha_1,\beta_1,\Omega_1,\alpha_2,\beta_2,\Omega_2,\theta)$. The Bayesian approach of Bivariate model based on FGM copula, in inference, is usually carried out in the following steps:

1. Choose the independent prior distributions $\pi_j(\Theta_j);\ j = 1,...,p-1$ for all parameters, where $p = length(\Theta)$, the joint independent prior distribution is $\Pi(\Theta) = \prod_{j=1}^{p-1}\pi_j(\Theta_j)$.
2. Choose the independent prior distribution for copula parameter $\pi_j(\theta)$.
3. Obtain the likelihood function for joint statistical model $L(x,y|\Theta)$ that reflects our beliefs about X and Y given Θ.
4. Compute the joint posterior distribution $(\Theta|x,y)$ by applying Bayes law of conditional probabilities

$$\Pi(\Theta|x,y) = \frac{\Pi(\Theta)\ L(x,y|\Theta)}{\int_{\Theta_1}\cdots\int_{\Theta_j}\Pi(\Theta)\ L(x,y|\Theta)d\Theta_1...d\Theta_j}$$

5. Obtaining the proportional posterior distribution for $\Theta_j;\ j = 1,...,p$.

6. Using numerical analysis of Bayesian estimation as Markov chain Monte Carlo (MCMC) by using Gibbs-sampling or Metropolis-Hastings (MH) Algorithm.
7. Choosing symmetric and asymmetric loss functions.

For more information see Suzuki et al. [32] and Louzada et al. [33].
In parameters $(\alpha_1, \beta_1, \Omega_1, \alpha_2, \beta_2, \Omega_2)$, we have use informative prior as independent gamma distributions. In copula parameter, we use non informative prior distribution such as $uniform(a, b)$; $-1 < \theta < 1$. In case of BFGMWGE distribution, the independent joint prior density function of Θ can be written as follows:

$$\Pi(\Theta) \propto \alpha_1^{a_1-1}\beta_1^{a_2-1}\Omega_1^{a_3-1}\alpha_2^{a_4-1}\beta_2^{a_5-1}\Omega_2^{a_6-1}\frac{1}{b_7-a_7}e^{-(b_1\alpha_1+b_2\beta_1+b_3\Omega_1+b_4\alpha_2+b_5\beta_2+b_6\Omega_2)} \tag{38}$$

To determine elicit hyper-parameters of the independent joint prior, we can use estimate and variance-covariance matrix of MLE method. By equating mean and variance of gamma priors, the estimated hyper-parameters can be written as

$$a_j = \frac{\left[\frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i\right]^2}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1,$$

$$b_j = \frac{\frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1,$$

where, L is the number of Iteration. For copula parameter, the estimated hyper-parameter can be written as

$$a_7 = \sqrt{\frac{3}{L-1}\sum_{i=1}^{L}\left[\hat{\theta}^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i\right]^2} - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i,$$

$$b_7 = \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i - \sqrt{\frac{3}{L-1}\sum_{i=1}^{L}\left[\hat{\theta}^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i\right]^2},$$

The joint posterior density function of Θ is obtained from the likelihood function and joint prior function. Then the joint posterior of the BFGMWG family can be written as

$$\begin{aligned}\Pi(\Theta|x, y) \propto & \alpha_1^{n+a_1-1}\beta_1^{n+a_2-1}\Omega_1^{a_3-1}\alpha_2^{n+a_4-1}\beta_2^{n+a_5-1}\Omega_2^{a_6-1}e^{-(b_2\beta_1+b_3\Omega_1+b_5\beta_2+b_6\Omega_2)} \\ & exp\left\{-\alpha_1\left(\sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1} + b_1\right)\right\} \prod_{i=1}^{n} g(x_i;\Omega_1)\frac{[G(x_i;\Omega_1)]^{(\beta_1-1)}}{[1-G(x_i;\Omega_1)]^{(\beta_1+1)}} \\ & exp\left\{-\alpha_2\left(\sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2} + b_4\right)\right\} \prod_{i=1}^{n} g(y_i;\Omega_2)\frac{[G(y_i;\Omega_2)]^{(\beta_2-1)}}{[1-G(y_i;\Omega_2)]^{(\beta_2+1)}} \\ & \prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),\end{aligned} \tag{39}$$

By using the most common symmetric loss function, which is a squared error loss function. The Bayes estimators of $\hat{\Theta}$ based on squared error loss function is given by

$$\begin{aligned}S(\tilde{\Theta}) = & E\left(\tilde{\Theta}-\Theta\right)^2 \\ & \int_0^\infty ... \int_0^\infty \int_{-1}^{1}\left(\tilde{\Theta}-\Theta\right)^2 \Pi(\Theta|x, y)d\Theta_1...d\Theta_7\end{aligned} \tag{40}$$

It is noticed that the integrals given by (40) can't be obtained explicitly. Because of that, we use the MCMC to find an approximate value of integrals. An important sub-class of the MCMC techniques is Gibbs sampling and more general Metropolis within Gibbs samplers. The MH algorithm together with the Gibbs sampling is the two most popular examples of an MCMC method. It's similar to acceptance-rejection sampling, the MH algorithm considers that, to each iteration of the algorithm, a candidate value can be generated from a proposal distribution. We use the MH within Gibbs sampling steps to generate random samples from conditional posterior densities of the BFGMWG family are as follows:

$$\Pi(\alpha_1|\beta_1,\Omega_1,\alpha_2,\beta_2,\Omega_2,\theta,x,y) \propto \alpha_1^{n+a_1-1} exp\left\{-\alpha_1\left(\sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1}+b_1\right)\right\}$$
$$\prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

$$\Pi(\beta_1|\alpha_1,\Omega_1,\alpha_2,\beta_2,\Omega_2,\theta,x,y) \propto \beta_1^{n+a_2-1}e^{-b_2\beta_1} exp\left\{-\alpha_1\left(\sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1}+b_1\right)\right\}$$
$$\prod_{i=1}^{n}\frac{[G(x_i;\Omega_1)]^{(\beta_1-1)}}{[1-G(x_i;\Omega_1)]^{(\beta_1+1)}}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

$$\Pi(\Omega_1|\alpha_1,\beta_1,\alpha_2,\beta_2,\Omega_2,\theta,x,y) \propto \Omega_1^{a_3-1}e^{-b_3\Omega_1} exp\left\{-\alpha_1\left(\sum_{i=1}^{n}\left[\frac{G(x_i;\Omega_1)}{1-G(x_i;\Omega_1)}\right]^{\beta_1}+b_1\right)\right\}$$
$$\prod_{i=1}^{n} g(x_i;\Omega_1)\frac{[G(x_i;\Omega_1)]^{(\beta_1-1)}}{[1-G(x_i;\Omega_1)]^{(\beta_1+1)}}$$
$$\prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

$$\Pi(\alpha_2|\alpha_1,\beta_1,\Omega_1,\beta_2,\Omega_2,\theta,x,y) \propto \alpha_2^{n+a_4-1} exp\left\{-\alpha_2\left(\sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2}+b_4\right)\right\}$$
$$\prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

$$\Pi(\beta_2|\alpha_1,\beta_1,\Omega_1,\alpha_2,\Omega_2,\theta,x,y) \propto \beta_2^{n+a_5-1}e^{-b_5\beta_2} exp\left\{-\alpha_2\left(\sum_{i=1}^{n}\left[\frac{G(y_i;\Omega_2)}{1-G(y_i;\Omega_2)}\right]^{\beta_2}\right)\right\}$$
$$\prod_{i=1}^{n}\frac{[G(y_i;\Omega_2)]^{(\beta_2-1)}}{[1-G(y_i;\Omega_2)]^{(\beta_2+1)}}\left(1+\theta\left\{2\ \zeta_1\left(x;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

$$\Pi(\Omega_2|\alpha_1,\beta_1,\Omega_1,\alpha_2,\beta_2,\theta,x,y) \propto \Omega_2^{a_6-1}e^{-b_6\delta_2}\prod_{i=1}^{n} g(y_i;\Omega_2)\frac{[G(y_i;\Omega_2)]^{(\beta_2-1)}}{[1-G(y_i;\Omega_2)]^{(\beta_2+1)}}$$
$$\zeta_2\left(y;\alpha_2,\beta_2,\Omega_2\right)\prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

and

$$\Pi(\theta|\alpha_1,\beta_1,\Omega_1,\alpha_2,\beta_2,\Omega_2,x,y) \propto \prod_{i=1}^{n}\left(1+\theta\left\{2\ \zeta_1\left(x_i;\alpha_1,\beta_1,\Omega_1\right)-1\right\}\left\{2\ \zeta_2\left(y_i;\alpha_2,\beta_2,\Omega_2\right)-1\right\}\right),$$

In Highest Posterior Density (HPD) Intervals: Chen and Shao [34] technique was used extensively to generate the HPD intervals of unknown parameters of the benefit distribution. In this study, samples drawn with the proposed MH algorithm should be used to generate time-lapse estimates. From the percentile tail points, for instance, a 90% HPD interval can be obtained with two endpoints being the 7^th and 95% percentiles respectively from the MCMC sampling outputs. It is sometimes useful to present the posterior median to informally check on possible asymmetry in the posterior density of a parameter.

7. Confidence Intervals

In this section, we propose two different methods to construct confidence intervals (CI) for the unknown parameters of the BFGMW-G family, which are asymptotic confidence interval (ACI) and bootstrap confidence interval of $\alpha_j, \beta_j, \delta_j$ where $j = 1, 2$ and θ. The bootstrap approach is subdivided into percentile bootstrap and bootstrap-t.

7.1. Asymptotic Confidence Intervals

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLE. In relation to the asymptotic variance-covariance matrix of the MLE of the parameters, Fisher information matrix $I(\Theta)$, where it is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at $\hat{\Theta} = (\hat{\alpha_1}, \hat{\beta_1}, \hat{\delta_1}, \hat{\alpha_2}, \hat{\beta_2}, \hat{\delta_2}, \hat{\theta})$. Suppose the asymptotic variance-covariance matrix of the parameter vector Θ is

$$I(\hat{\Theta}) = \begin{bmatrix} I_{\hat{\alpha_1}\hat{\alpha_1}} & & & & & & \\ I_{\hat{\beta_1}\hat{\alpha_1}} & I_{\hat{\beta_1}\hat{\beta_1}} & & & & & \\ I_{\hat{\delta_1}\hat{\alpha_1}} & I_{\hat{\delta_1}\hat{\beta_1}} & I_{\hat{\delta_1}\hat{\delta_1}} & & & & \\ I_{\hat{\alpha_2}\hat{\alpha_1}} & I_{\hat{\alpha_2}\hat{\beta_1}} & I_{\hat{\alpha_2}\hat{\delta_1}} & I_{\hat{\alpha_2}\hat{\alpha_2}} & & & \\ I_{\hat{\beta_2}\hat{\alpha_1}} & I_{\hat{\beta_2}\hat{\beta_1}} & I_{\hat{\beta_2}\hat{\delta_1}} & I_{\hat{\beta_2}\hat{\alpha_2}} & I_{\hat{\beta_2}\hat{\beta_2}} & & \\ I_{\hat{\delta_2}\hat{\alpha_1}} & I_{\hat{\delta_2}\hat{\beta_1}} & I_{\hat{\delta_2}\hat{\delta_1}} & I_{\hat{\delta_2}\hat{\alpha_2}} & I_{\hat{\delta_2}\hat{\beta_3}} & I_{\hat{\delta_2}\hat{\delta_2}} & \\ I_{\hat{\theta}\hat{\alpha_1}} & I_{\hat{\theta}\hat{\beta_1}} & I_{\hat{\theta}\hat{\delta_1}} & I_{\hat{\theta}\hat{\alpha_2}} & I_{\hat{\theta}\hat{\beta_2}} & I_{\hat{\theta}\hat{\delta_2}} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}$$

where $V(\hat{\Theta}) = I^{-1}(\hat{\Theta})$
A $100(1-\gamma)\%$ confidence interval for parameter Θ can be constructed based on the asymptotic normality of the MLE.

$\hat{\alpha}_j \pm Z_{0.025}\sqrt{I_{\hat{\alpha}_j\hat{\alpha}_j}}$, $\hat{\beta}_j \pm Z_{0.025}\sqrt{I_{\hat{\beta}_j\hat{\beta}_j}}$, $\hat{\delta}_j \pm Z_{0.025}\sqrt{I_{\hat{\delta}_j\hat{\delta}_j}}$ and $\hat{\theta} \pm Z_{0.025}\sqrt{I_{\hat{\theta}\hat{\theta}}}$,

where $Z_{0.025}$ is the percentile of the standard normal distribution with right tail probability $\frac{\gamma}{2}$.

7.2. Bootstrap Confidence Interval

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals. For more information see Efron [24]. In this subsection, we use the parametric bootstrap method to construct confidence intervals for the unknown parameters $\alpha_k, \beta_k, \Omega_k$ where $k = 1, 2$ and θ. We introduced two parametric bootstrap methods, percentile bootstrap (B-P) and bootstrap-t (B-t) CI.

7.2.1. Percentile Bootstrap Confidence Interval

1. Compute the MLE or Bayesian estimators of BFGMWG distribution.
2. Generate a bootstrap samples using $\alpha_k, \beta_k$and θ to obtain the bootstrap estimate of α_k say α_k^b , β_k say β_k^b and θ say θ^b using the bootstrap sample.
3. Repeat step (2) $\mathcal{B}$ times to have $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(\mathcal{B})}), (\beta_k^{b(1)}, \beta_k^{b(2)}, \ldots \beta_k^{b(\mathcal{B})})$and $(\theta_k^{b(1)}, \theta_k^{b(2)}, \ldots, \theta_k^{b(\mathcal{B})})$.

4. Arrange $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(\mathcal{B})}), (\beta_k^{b(1)}, \beta_k^{b(2)}, \ldots \beta_k^{b(\mathcal{B})}), (\delta_k^{b(1)}, \delta_k^{b(2)}, \ldots \delta_k^{b(\mathcal{B})})$ and $(\theta_k^{b(1)}, \theta_k^{b(2)}, \ldots, \theta_k^{b(\mathcal{B})})$ in ascending order as $(\alpha_k^{b[1]}, \alpha_k^{b[2]}, \ldots, \alpha_k^{b([\mathcal{B}])}), (\beta_k^{b[1]}, \beta_k^{b[2]}, \ldots \beta_k^{b[\mathcal{B}]}), (\delta_k^{b[1]}, \delta_k^{b[2]}, \ldots \delta_k^{b[\mathcal{B}]})$ and $(\theta_k^{b[1]}, \theta_k^{b[2]}, \ldots, \theta_k^{b[\mathcal{B}]})$.
5. Two side $100(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters $\alpha_k, \beta_k, \delta_k$ where $k = 1, 2$ and θ are given by $[\alpha_k^{b([\mathcal{B}\frac{\gamma}{2}])}, \alpha_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}], [\beta_k^{b([\mathcal{B}\frac{\gamma}{2}])}, \beta_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}], [\delta_k^{b([\mathcal{B}\frac{\gamma}{2}])}, \delta_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$ and $[\theta^{b([\mathcal{B}\frac{\gamma}{2}])}, \theta^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$.

7.2.2. Bootstrap-t Confidence Interval

1. Same as the steps (1,2) in Boot-p.
2. Compute the t-statistic of Θ as $T = \frac{\hat{\Theta}^b - \hat{\Theta}}{\sqrt{V(\hat{\Theta}^b)}}$ where $V(\hat{\Theta}^b)$ is asymptotic variances of $\hat{\Theta}^b$ and it can be obtained using the Fisher information matrix.
3. Repeat steps II-III $\mathcal{B}$ times and obtain $(T^{(1)}, T^{(2)}, \ldots, T^{(\mathcal{B})})$.
4. Arrange $(T^{(1)}, T^{(2)}, \ldots, T^{(\mathcal{B})})$ in ascending order as $(T^{[1]}, T^{[2]}, \ldots, T^{[\mathcal{B}]})$.
5. A two side $100(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters α_k, β_k where $k = 1, 2$ and θ are given by $[\alpha_k + T_k^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\alpha_k^b)}, \alpha_k + \sqrt{V(\alpha_k^b)}T_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$, $[\beta_k + T_k^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\beta_k^b)}, \beta_k + \sqrt{V(\beta_k^b)}T_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}], [\delta_k + T_k^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\delta_k^b)}, \delta_k + \sqrt{V(\delta_k^b)}T_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$ and $[\theta + T_k^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\theta^b)}, \theta + \sqrt{V(\theta^b)}T_k^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$

8. Simulation Study

In this section; A Monte Carlo simulation is done based on copula function. For estimating BFGMWGE distribution parameters, R program is used.

To generate random variables: Nelsen [1] discussed generating a sample from a specified joint distribution. By using the following steps, we can generate a bivariate sample by using the conditional approach.

- Generate U and V as independently from uniform (0, 1) distribution.
- Set $x = Q_{WE}(u) = \frac{1}{\delta_1} \ln \left\{ 1 + \left[\frac{-1}{\alpha_1} ln\,(1-U) \right]^{\frac{1}{\beta_1}} \right\}$.
- Set $F(y|x) = V$ in (16) to find y by numerical analysis.
- Repeat above steps (n) items to obtain (x_i, y_i), $i = 1, ..., n$.

A simulation algorithm: Simulation experiments were carried out based on the following data generated form BFGMWGE distribution, where X, Y are distributed as WG-Exponential with α_j, λ_j and δ_j as a parameters WG-Exponential distribution, $j = 1, 2$ the values of the parameters $\alpha_1, \lambda_1, \delta_1, \alpha_2, \lambda_2, \delta_2$ and θ are chosen as the following cases for the random variables generating:
Case 1: $(\alpha_1 = 1.3, \beta_1 = 1.7, \delta_1 = 1.2, \alpha_2 = 1.4, \beta_2 = 1.5, \delta_2 = 1.3)$,
Case 2: $(\alpha_1 = 2.3, \beta_1 = 2.5, \delta_1 = 2.2, \alpha_2 = 1.4, \beta_2 = 1.5, \delta_2 = 1.8)$,
For different sample size $n = 35, 60, 100, 150$ and 200. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculating the Bias, the mean of square error (MSE), the length of asymptotic and bootstrap confidence intervals (L.CI) for each method of estimation as following:
$Bias = (\hat{\Theta} - \Theta)$ where $\hat{\Theta}$ is the estimated value of Θ. $MSE = Mean(\hat{\Theta} - \Theta)^2$. and L.CI=Upper.CI-Lower.CI.
We restricted the number of repeated-samples to 1000.

From these tables 2, 3, 4, and 5, we conclude the following, as the sample size n increases, the Bias, MSE,

Table 2. MLE of the Parameters of BFGMWGE Distributions: Case 1

θ		0.25					0.8				
n		Bias	MSE	L.ACI	L.BT	L.BP	Bias	MSE	L.ACI	L.BT	L.BP
35	α_1	0.0957	0.9238	3.7509	0.4075	0.4120	0.1298	1.0762	4.0366	0.5967	0.5900
	β_1	0.0268	0.0780	1.0902	0.1927	0.1905	0.0390	0.0806	1.1029	0.1954	0.1932
	δ_1	0.1414	0.1934	1.6333	0.2865	0.2850	0.1260	0.1894	1.6338	0.2099	0.2114
	α_2	0.1545	1.0776	4.0260	0.7017	0.7038	0.1631	1.3589	4.5269	0.6848	0.6882
	β_2	0.0240	0.0690	1.0260	0.1468	0.1470	0.0303	0.0662	1.0023	0.1507	0.1518
	δ_2	0.1836	0.3553	2.2242	0.3440	0.3476	0.1724	0.3280	2.1418	0.3534	0.3510
	θ	0.0694	0.6004	3.0268	0.3954	0.3948	0.2783	2.0273	5.4765	0.5455	0.5401
60	α_1	0.1400	0.7948	3.4532	0.5499	0.5511	0.1207	0.6489	3.1237	0.5001	0.5061
	β_1	0.0174	0.0461	0.8397	0.1081	0.1088	0.0323	0.0471	0.8413	0.1037	0.1031
	δ_1	0.0950	0.1645	1.5464	0.3086	0.3099	0.0822	0.1431	1.4484	0.2254	0.2277
	α_2	0.1886	1.0588	3.9672	0.3960	0.3945	0.2039	0.9068	3.6482	0.5075	0.5089
	β_2	0.0031	0.0397	0.7816	0.1163	0.1163	0.0247	0.0363	0.7410	0.1017	0.1032
	δ_2	0.1184	0.2329	1.8349	0.2490	0.2469	0.0735	0.1759	1.6193	0.2040	0.2030
	θ	0.0166	0.1747	1.6379	0.1883	0.1902	0.0425	0.2258	1.8561	0.2012	0.2010
100	α_1	0.0817	0.5095	2.7811	0.2720	0.2737	0.1319	0.6129	3.0265	0.2851	0.2886
	β_1	0.0019	0.0278	0.6543	0.0584	0.0591	0.0245	0.0291	0.6623	0.0661	0.0659
	δ_1	0.0751	0.0953	1.1747	0.1137	0.1139	0.0472	0.0838	1.1203	0.1061	0.1053
	α_2	0.1122	0.7006	3.2531	0.3009	0.3004	0.1958	0.7411	3.2877	0.2945	0.2960
	β_2	0.0006	0.0234	0.6005	0.0619	0.0613	0.0242	0.0226	0.5824	0.0542	0.0539
	δ_2	0.0939	0.1543	1.4961	0.1434	0.1450	0.0437	0.1296	1.4014	0.1643	0.1649
	θ	-0.0032	0.0969	1.2208	0.1229	0.1223	-0.0051	0.0952	1.2099	0.1375	0.1394
150	α_1	0.0919	0.4277	2.5395	0.2027	0.2006	0.1328	0.4721	2.6441	0.2004	0.2008
	β_1	0.0056	0.0186	0.5344	0.0442	0.0441	0.0239	0.0196	0.5406	0.0418	0.0416
	δ_1	0.0435	0.0661	0.9940	0.0906	0.0905	0.0333	0.0742	1.0605	0.0796	0.0794
	α_2	0.1286	0.6421	3.1019	0.2464	0.2465	0.2168	0.6799	3.1202	0.2751	0.2720
	β_2	0.0002	0.0180	0.5268	0.0454	0.0455	0.0297	0.0168	0.4953	0.0389	0.0391
	δ_2	0.0693	0.1262	1.3667	0.1166	0.1164	0.0139	0.0939	1.2005	0.0991	0.1000
	θ	0.0069	0.0624	0.9792	0.0817	0.0811	-0.0082	0.0562	0.9296	0.0713	0.0711
200	α_1	0.0696	0.3726	2.3783	0.1697	0.1702	0.0978	0.3915	2.4237	0.1582	0.1601
	β_1	-0.0051	0.0147	0.4749	0.0338	0.0337	0.0192	0.0153	0.4792	0.0335	0.0335
	δ_1	0.0435	0.0572	0.9225	0.0627	0.0627	0.0291	0.0525	0.8916	0.0677	0.0683
	α_2	0.1091	0.4678	2.6482	0.1966	0.1960	0.1790	0.5580	2.8442	0.2060	0.2063
	β_2	0.0022	0.0133	0.4516	0.0311	0.0305	0.0217	0.0131	0.4413	0.0295	0.0294
	δ_2	0.0484	0.0930	1.1808	0.0840	0.0833	0.0223	0.0881	1.1607	0.0799	0.0805
	θ	-0.0019	0.0502	0.8788	0.0601	0.0597	-0.0091	0.0397	0.7810	0.0610	0.0607

and length of CI decreases. As the value of θ increases, the Bias, MSE, and length of CI increases. It is noted that the Bayesian method gives better results than the MLE method. The Bayesian method is the best because has the lower values for MSE, Bias, and length of CI.

9. Goodness of Fit

The three most important classes of tests of goodness of fit based on the empirical distribution function of a random sample are the Kolmogorov-Smirnov test, Anderson¨CDarling test, and the Cramer-von Mises

Table 3. Bayesian Estimation of the Parameters of BFGMWGE Distributions: Case 1

θ		0.25					0.8				
n		Bias	MSE	L.ACI	L.BT	L.BP	Bias	MSE	L.ACI	L.BT	L.BP
35	α_1	-0.0665	0.1216	1.3430	0.2342	0.2338	-0.0577	0.1379	1.4394	0.2155	0.2162
	β_1	0.0223	0.0011	0.0996	0.0155	0.0153	0.0348	0.0020	0.1093	0.0176	0.0176
	δ_1	0.0889	0.0181	0.3957	0.0721	0.0721	0.0721	0.0156	0.4005	0.0623	0.0624
	α_2	-0.0244	0.2167	1.8241	0.3513	0.3489	-0.0257	0.1981	1.7434	0.3663	0.3657
	β_2	0.0204	0.0009	0.0856	0.0159	0.0157	0.0281	0.0012	0.0783	0.0110	0.0111
	δ_2	0.1145	0.0550	0.8026	0.1006	0.1002	0.1052	0.0439	0.7108	0.1073	0.1077
	θ	0.0160	0.0423	0.8049	0.1390	0.1395	-0.0831	0.2171	1.7991	0.3217	0.3228
60	α_1	-0.0226	0.0869	1.1532	0.1591	0.1593	-0.0319	0.0738	1.0583	0.1157	0.1158
	β_1	0.0160	0.0004	0.0547	0.0070	0.0070	0.0295	0.0011	0.0561	0.0077	0.0078
	δ_1	0.0533	0.0121	0.3766	0.0396	0.0397	0.0380	0.0084	0.3283	0.0422	0.0422
	α_2	-0.0118	0.1182	1.3485	0.1792	0.1793	0.0129	0.0955	1.2119	0.1444	0.1428
	β_2	0.0033	0.0002	0.0477	0.0063	0.0064	0.0229	0.0006	0.0424	0.0053	0.0053
	δ_2	0.0594	0.0230	0.5471	0.0716	0.0724	0.0359	0.0131	0.4266	0.0596	0.0595
	θ	-0.0063	0.0171	0.5132	0.0679	0.0684	-0.0279	0.0997	1.2340	0.1204	0.1221
100	α_1	-0.0588	0.0401	0.7510	0.0858	0.0853	-0.0347	0.0366	0.7387	0.0787	0.0786
	β_1	0.0014	0.0001	0.0389	0.0037	0.0037	0.0223	0.0006	0.0417	0.0041	0.0041
	δ_1	0.0425	0.0056	0.2416	0.0259	0.0261	0.0281	0.0040	0.2224	0.0201	0.0200
	α_2	-0.0420	0.0602	0.9487	0.1023	0.1029	0.0025	0.0509	0.8854	0.0891	0.0894
	β_2	0.0002	0.0001	0.0303	0.0028	0.0028	0.0224	0.0006	0.0324	0.0034	0.0034
	δ_2	0.0474	0.0128	0.4025	0.0404	0.0405	0.0212	0.0089	0.3599	0.0355	0.0357
	θ	-0.0100	0.0062	0.3074	0.0280	0.0279	-0.0408	0.0381	0.7494	0.0706	0.0706
150	α_1	-0.0370	0.0193	0.5261	0.0417	0.0417	-0.0120	0.0227	0.5892	0.0446	0.0442
	β_1	0.0041	0.0001	0.0341	0.0030	0.0030	0.0210	0.0005	0.0396	0.0029	0.0029
	δ_1	0.0234	0.0023	0.1640	0.0123	0.0124	0.0109	0.0025	0.1919	0.0162	0.0161
	α_2	-0.0213	0.0296	0.6697	0.0473	0.0474	0.0001	0.0305	0.6855	0.0547	0.0553
	β_2	0.0003	0.0001	0.0320	0.0024	0.0024	0.0266	0.0008	0.0357	0.0029	0.0029
	δ_2	0.0220	0.0058	0.2865	0.0239	0.0242	0.0052	0.0048	0.2715	0.0216	0.0215
	θ	0.0026	0.0019	0.1690	0.0143	0.0142	-0.0103	0.0175	0.5173	0.0444	0.0450
200	α_1	-0.0143	0.0093	0.3741	0.0223	0.0223	-0.0173	0.0146	0.4696	0.0316	0.0318
	β_1	-0.0031	0.0001	0.0300	0.0021	0.0021	0.0141	0.0003	0.0379	0.0027	0.0027
	δ_1	0.0150	0.0013	0.1288	0.0094	0.0093	0.0123	0.0016	0.1495	0.0109	0.0108
	α_2	-0.0188	0.0129	0.4394	0.0313	0.0311	0.0049	0.0184	0.5313	0.0408	0.0408
	β_2	0.0013	0.0000	0.0271	0.0019	0.0019	0.0169	0.0004	0.0369	0.0028	0.0027
	δ_2	0.0139	0.0027	0.1974	0.0144	0.0143	0.0034	0.0028	0.2087	0.0152	0.0152
	θ	0.0000	0.0008	0.1086	0.0063	0.0063	-0.0075	0.0092	0.3743	0.0263	0.0264

test, which have been extended to the multivariate case in Zimmerman [29], Justel et al. [8], Genest et al. [37], and Langren¨¦ and Warin [31].

9.1. Goodness of Fit Test for Copulas

The simplest goodness-of-fit test for copulas lies in comparing the distance between a non-parametric estimate $\hat{C_n}$ of C and a parametric estimate C_θ derived from an estimator θ which is consistent when the null hypothesis H_0 holds. Let (F_{1n}, F_{2n}) be the empirical distribution functions of (F_1, F_2), respectively. A natural substitute for the un-observable $U_i = (F_{1i}, F_{2i})$ where $i = 1, \ldots, n$, is given by

$$C(\hat{U}_{1i}, \hat{U}_{2i}) = (\frac{n}{n+1}F_1(X_i), \frac{n}{n+1}F_2(Y_i))$$

Table 4. MLE of the Parameters of BFGMWGE Distributions: Case 2

θ		0.3					0.8				
n		Bias	MSE	L.ACI	L.BT	L.BP	Bias	MSE	L.ACI	L.BT	L.BP
35	α_1	0.1068	1.5054	4.7937	0.6021	0.5950	0.0561	1.5379	4.8588	0.8162	0.8172
	β_1	0.0715	0.1399	1.4397	0.2925	0.2904	0.0754	0.1438	1.4575	0.2767	0.2713
	δ_1	0.0985	0.1696	1.5683	0.2555	0.2533	0.1223	0.1984	1.6798	0.3514	0.3473
	α_2	0.1931	1.4091	4.5936	0.8444	0.8446	0.2355	1.5892	4.8571	0.6910	0.6903
	β_2	0.0207	0.0705	1.0384	0.1371	0.1376	0.0303	0.0709	1.0377	0.1566	0.1575
	δ_2	0.2790	0.7407	3.1932	0.7487	0.7558	0.2577	0.7616	3.2699	0.6106	0.6106
	θ	0.0728	0.5734	2.9561	0.4056	0.4062	0.2805	1.9210	5.3233	0.5404	0.5358
60	α_1	0.1046	1.1238	4.1373	0.5229	0.5280	0.1433	1.0973	4.0697	0.5033	0.5049
	β_1	0.0418	0.0697	1.0226	0.1301	0.1304	0.0691	0.0748	1.0382	0.1254	0.1260
	δ_1	0.0638	0.1166	1.3158	0.1706	0.1701	0.0452	0.0971	1.2094	0.1556	0.1561
	α_2	0.2260	1.1490	4.1095	0.4125	0.4134	0.2637	1.1930	4.1571	0.4140	0.4120
	β_2	0.0019	0.0418	0.8014	0.1134	0.1143	0.0259	0.0397	0.7746	0.1200	0.1190
	δ_2	0.1729	0.5075	2.7103	0.3224	0.3236	0.1228	0.4263	2.5151	0.3583	0.3564
	θ	0.0195	0.1778	1.6519	0.1940	0.1959	0.0451	0.2239	1.8472	0.2123	0.2117
100	α_1	0.1666	1.0467	3.9588	0.3892	0.3905	0.1563	0.9565	3.7864	0.3646	0.3661
	β_1	0.0294	0.0438	0.8127	0.0851	0.0852	0.0465	0.0448	0.8095	0.0788	0.0788
	δ_1	0.0227	0.0722	1.0504	0.0919	0.0915	0.0252	0.0755	1.0734	0.1017	0.1021
	α_2	0.1785	0.8267	3.4965	0.3258	0.3261	0.2257	0.8638	3.5361	0.3792	0.3766
	β_2	0.0008	0.0248	0.6175	0.0601	0.0598	0.0235	0.0239	0.5999	0.0587	0.0591
	δ_2	0.1106	0.3148	2.1572	0.2177	0.2167	0.0720	0.2884	2.0872	0.2315	0.2317
	θ	-0.0021	0.0947	1.2069	0.1392	0.1397	-0.0056	0.0928	1.1949	0.1356	0.1369
150	α_1	0.1367	0.8798	3.6394	0.2703	0.2708	0.1506	0.8117	3.4837	0.2614	0.2601
	β_1	0.0202	0.0272	0.6421	0.0553	0.0556	0.0418	0.0283	0.6385	0.0513	0.0509
	δ_1	0.0256	0.0691	1.0258	0.0935	0.0934	0.0169	0.0602	0.9600	0.0786	0.0785
	α_2	0.1668	0.7934	3.4317	0.2630	0.2638	0.2810	0.8666	3.4806	0.2995	0.2982
	β_2	-0.0006	0.0192	0.5436	0.0436	0.0437	0.0325	0.0184	0.5164	0.0403	0.0404
	δ_2	0.0973	0.2677	1.9928	0.1766	0.1768	0.0111	0.2047	1.7738	0.1395	0.1390
	θ	0.0040	0.0609	0.9680	0.0752	0.0741	-0.0077	0.0563	0.9298	0.0712	0.0707
200	α_1	0.0884	0.6602	3.1679	0.2161	0.2163	0.0958	0.6026	3.0218	0.2191	0.2197
	β_1	0.0128	0.0212	0.5682	0.0398	0.0398	0.0385	0.0212	0.5509	0.0365	0.0361
	δ_1	0.0250	0.0523	0.8915	0.0645	0.0640	0.0183	0.0472	0.8489	0.0611	0.0608
	α_2	0.1265	0.6112	3.0257	0.2341	0.2337	0.1993	0.6335	3.0228	0.2004	0.2032
	β_2	-0.0062	0.0142	0.4665	0.0332	0.0334	0.0237	0.0137	0.4501	0.0313	0.0318
	δ_2	0.0969	0.2319	1.8502	0.1183	0.1179	0.0296	0.1774	1.6482	0.1199	0.1202
	θ	0.0073	0.0458	0.8389	0.0661	0.0664	-0.0082	0.0372	0.7555	0.0525	0.0519

The scaling factor $\frac{n}{n+1}$ used in defining $\hat{U}_i$ avoids numerical issues that sometimes occur, when a parametric copula density is evaluated at pseudo-observations. A natural estimator $\hat{C}_n$ of C, called the empirical copula, is then defined, for all (u_1, u_2);$u \in [0,1]$, by

$$\hat{C}_n(u_1, u_2) = \frac{1}{n} \sum_{i=1}^{n} I(\hat{U}_{1i} \leq u_{1i}, \hat{U}_{2i} \leq u_{2i}) \tag{41}$$

Table 5. Bayesian Estimation of the Parameters of BFGMWGE Distributions: Case 2

θ		0.3					0.8				
n		Bias	MSE	HDP	L.BT	L.BP	Bias	MSE	HDP	L.BT	L.BP
35	α_1	0.0079	0.2576	1.9912	0.3950	0.3949	-0.0583	0.3152	2.1912	0.3274	0.3221
	β_1	0.0654	0.0075	0.2214	0.0386	0.0385	0.0719	0.0083	0.2198	0.0441	0.0446
	δ_1	0.0685	0.0121	0.3368	0.0562	0.0564	0.0793	0.0169	0.4042	0.0755	0.0765
	α_2	0.0836	0.3831	2.4064	0.3324	0.3352	0.0796	0.4024	2.4695	0.3842	0.3828
	β_2	0.0196	0.0009	0.0910	0.0182	0.0180	0.0275	0.0014	0.1006	0.0135	0.0135
	δ_2	0.1149	0.1387	1.3900	0.2087	0.2088	0.1177	0.1637	1.5189	0.2448	0.2431
	θ	-0.0054	0.0533	0.9057	0.1465	0.1457	-0.0644	0.1986	1.7304	0.2561	0.2535
60	α_1	-0.0385	0.1429	1.4755	0.1853	0.1836	-0.0185	0.1231	1.3748	0.1926	0.1935
	β_1	0.0403	0.0022	0.0904	0.0114	0.0114	0.0663	0.0049	0.0914	0.0115	0.0116
	δ_1	0.0522	0.0067	0.2468	0.0246	0.0246	0.0323	0.0035	0.1948	0.0225	0.0226
	α_2	-0.0242	0.1688	1.6091	0.2051	0.2050	0.0246	0.1586	1.5597	0.2077	0.2086
	β_2	0.0006	0.0002	0.0530	0.0078	0.0076	0.0237	0.0007	0.0501	0.0061	0.0060
	δ_2	0.0923	0.0778	1.0331	0.1295	0.1305	0.0515	0.0549	0.8972	0.1000	0.1006
	θ	0.0057	0.0208	0.5659	0.0771	0.0772	-0.0338	0.0852	1.1374	0.1408	0.1412
100	α_1	-0.0092	0.0792	1.1034	0.1100	0.1095	0.0065	0.0871	1.1578	0.1166	0.1160
	β_1	0.0236	0.0008	0.0652	0.0065	0.0065	0.0426	0.0021	0.0705	0.0065	0.0066
	δ_1	0.0174	0.0023	0.1761	0.0190	0.0191	0.0151	0.0026	0.1899	0.0159	0.0158
	α_2	-0.0020	0.0703	1.0403	0.1002	0.0994	0.0212	0.0662	1.0062	0.0931	0.0929
	β_2	0.0005	0.0001	0.0365	0.0034	0.0034	0.0207	0.0005	0.0396	0.0039	0.0038
	δ_2	0.0346	0.0263	0.6217	0.0614	0.0612	0.0152	0.0226	0.5865	0.0525	0.0527
	θ	0.0086	0.0061	0.3057	0.0313	0.0311	-0.0080	0.0304	0.6832	0.0675	0.0665
150	α_1	-0.0350	0.0465	0.8347	0.0688	0.0691	0.0097	0.0600	0.9602	0.0760	0.0765
	β_1	0.0143	0.0004	0.0532	0.0043	0.0042	0.0380	0.0017	0.0660	0.0054	0.0054
	δ_1	0.0162	0.0021	0.1674	0.0126	0.0127	0.0078	0.0021	0.1766	0.0155	0.0154
	α_2	-0.0137	0.0305	0.6835	0.0502	0.0503	0.0132	0.0335	0.7167	0.0576	0.0581
	β_2	-0.0014	0.0001	0.0379	0.0029	0.0029	0.0275	0.0009	0.0438	0.0036	0.0036
	δ_2	0.0283	0.0132	0.4371	0.0347	0.0346	-0.0018	0.0115	0.4210	0.0379	0.0382
	θ	-0.0014	0.0024	0.1904	0.0155	0.0155	-0.0067	0.0142	0.4663	0.0399	0.0403
200	α_1	-0.0086	0.0198	0.5509	0.0392	0.0386	0.0083	0.0352	0.7357	0.0531	0.0530
	β_1	0.0064	0.0002	0.0495	0.0037	0.0037	0.0292	0.0011	0.0654	0.0044	0.0044
	δ_1	0.0108	0.0010	0.1194	0.0083	0.0083	0.0067	0.0012	0.1354	0.0096	0.0096
	α_2	-0.0041	0.0105	0.4019	0.0280	0.0282	0.0038	0.0198	0.5524	0.0388	0.0384
	β_2	-0.0027	0.0001	0.0293	0.0020	0.0020	0.0173	0.0004	0.0420	0.0030	0.0029
	δ_2	0.0130	0.0052	0.2770	0.0209	0.0209	0.0032	0.0074	0.3379	0.0243	0.0242
	θ	-0.0021	0.0008	0.1091	0.0069	0.0069	-0.0056	0.0078	0.3457	0.0242	0.0244

Genest et al. [30] introduced Multiplier bootstrap-based goodness-of-fit test, this consideration leads naturally to Anderson Darling-type statistics such as

$$R_n = n\int_0^1 \left(\frac{\hat{C}_n(u_1,u_2)-\hat{C}_{\theta n}(u_1,u_2)}{(C_{\theta n}(u_1,u_2)(1-C_{\theta n}(u_1,u_2))+\delta_m)^m}\right)^2 d\hat{C}_n(u_1,u_2) \tag{42}$$

Involving a consistent, rank-based estimator θ_n of θ, and tuning parameters $m \geq \theta$ and $\delta_m \geq 0$. We use the conclusions of Genest to fit FGM copulas by R package. But the main assumption needed on the used data is the correlation coefficient, which varies for each copula function. For example, the range of

Spearman correlation coefficient for FGM copula is $[-0.333, 0.333]$ and the range of Kendall correlation coefficient for FGM copula is $[-0.222, 0.222]$.

9.2. Kolmogorov-Smirnov statistic

In a one-dimensional sample, the empirical distribution changes only in the observed points, and the univariate Kolmogorov-Smirnov statistic is obtained by evaluating the distance between the empirical and theoretical distribution functions in these points. In a multi-dimensional sample, the empirical distribution function jumps on an infinite number of points.
Let X and Y be a random variable with joint density $f(X,Y) = f(x)f(y|x)$ and define the transformation by $W_1 = F(x),\ \ W_2 = F(y|x)$ then W_1, W_2 are distributed as uniform $(0-1)$. We may assume that $u_1 = (x_1,\ y_1)$, ..., $u_n = (x_n,\ y_n)$ is a random sample from two independent uniform $(0-1)$ distributions. For $u = (x,\ y)$, we define the superior distance $D_n^+ = F_n(u) - F(u)$ and the inferior distance $D_n^- = F(u) - F_n(u)$ where F is the distribution function of two independent uniform random variables on (0,1) and F_n is the empirical distribution function. The theorem leads to the following procedure to compute the Kolmogorov-Smirnov statistic for Bivariate distribution is

1. Compute the maximum distance in the observed points $D_n^1 = max(D_n^+(u_i)); i = 1, ..., n$
2. Compute the maximum and minimum distances in the intersection points, $D_n^2 = max\{D_n^+(x_j, y_i)|x_j > x_i, \quad y_j < y_i\}; j, i = 1, ..., n$ and $D_n^3 = \frac{2}{n} - min\{D_n^+(x_j, y_i)|x_j > x_i, \quad y_j < y_i\}; j, i = 1, ..., n$.
3. Compute the maximum distance among the projections of the observed points on the right unit square border, $D_n^4 = \frac{1}{n} - min\{D_n^+(1, y_i)\}; i = 1, ..., n$.
4. Compute the maximum distance among the projections of the observed points on the top unit square border, $D_n^5 = \frac{1}{n} - min\{D_n^+(x_i, 1)\}; i = 1, ..., n$.
5. Compute the maximum $D_n = max\{D_n^1,\ D_n^2,\ D_n^3,\ D_n^4,\ D_n^5\}$

For more information and proof of this, see Justel et al. [8].

9.3. Anderson-Darling

If $X,\ Y$, is a random sample from the Bivariate distribution and $F(X,Y)$ is a cumulative function with uniform $(0-1)$. The Anderson¨CDarling test is based on the distance

$$\begin{aligned} A^2 &= n\int_0^\infty \frac{[F_n(x,\ y) - F(x,\ y)]^2}{F(x,\ y)\,[1 - F(x,\ y)]} dF(x,\ y) \\ &= -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\ln(F_i(x,\ y)) + \ln(1 - F_{n+1-i}(x,\ y))\right] - n;\ \ i = 1, ..., n \end{aligned} \tag{43}$$

9.4. Cramer-von Mises

The Cramer-von Mises statistic w^2 for testing two dimensional gaussianity is defined as (Koziol, 1982)

$$\begin{aligned} W^2 &= n\int_0^\infty \left[F_n(x,\ y) - F(x,\ y)\right]^2 dF(x,\ y) \\ &= \frac{1}{12} + \sum_{j=1}^{n} \left[F_j(x,\ y) - \frac{2j-1}{2n}\right]^2;\ \ j = 1, ..., n \end{aligned} \tag{44}$$

10. Application of Real Data

In this section, we analyze medical data of kidney patients, and diabetic nephropathy data. We study the parameter estimation of the appropriate distribution of each data set, where the correlation between the two variables (bivariate data) is low. And through this, we access a fit model specialized in the study of weak relations and the extent of their impact and effectiveness. In the goodness-of-fit test for copulas, we use a parametric bootstrap N=10000 time and the empirical copula estimate. We compared the proposed Bivariate distributions by different Bivariate distributions as Bivariate FGM Gamma (BFGMG), which it is discussed by [5], Bivariate FGM Weibull (BFGMW), which it is discussed by [7] and Bivariate FGM generalized exponential (BFGMGE), which it is discussed by [25].

10.1. The Medical Data of kidney Patients

The data set for 30 patients from McGilchrist [26]. This data represents the recurrence time of infection for kidney patients, where X: refers to first recurrence time, and Y: to second recurrence time. Elaal and Jarwan [25] discussed the estimation of the parameters of FGM bivariate generalized exponential distribution for this data. The MLE, SE, KSS, and its p-value for the marginals distributions are listed in Table 10. Figures 10 show the fitted pdf and estimated cdf, which support our results (KSS-test) in Table 10. By using Genest of goodness-of-fit test for copulas to fit of FGM by R package then the statistic is $R_n = 0.29031$ and $\hat{\theta} = 0.46704$ with p-value= 0.3974. Note that the data corresponds to an FGM copula and the data fit for marginals distribution, where P-value > 0.05. Now, we fit the BFGMWGE, BFGMWGR and BFGMWGITL model on this data. In the enclosed Table 12, we provide the MLE, SE, KSS, CVM and AD values for the competitive models. The proposed distributions are the best distribution comparing to BFGMW, BFGMG, and BFGMGE distribution. The BFGMWGP distribution is the best model to fit this data because it has the least measure of KSS, CVM, and AD. In the enclosed Table 6, we provide the Bayesian estimation and SE for the BFGMWGE, BFGMWGR and BFGMWGITL models. History plots, MCMC convergence of $\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2$ and θ are represented for BFGMWGE, BFGMWGR and BFGMWGITL models respectively in Figures 11, 12 and 13.

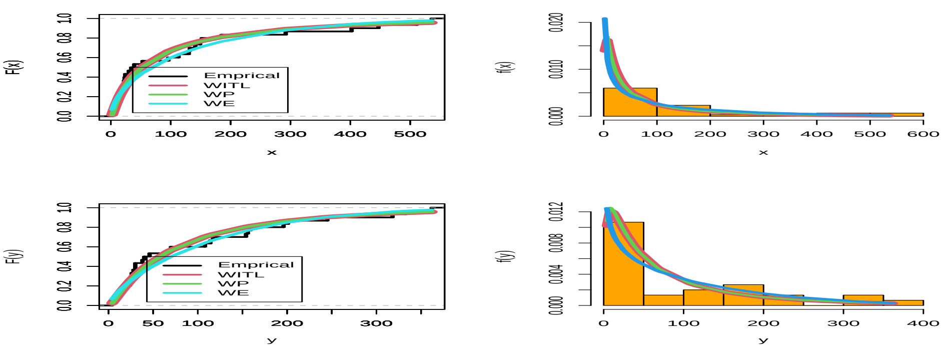

Figure 10. CDF and PDF for marginal distributions of X and Y:kidney patients data

10.2. Diabetic Nephropathy Data

In this section, we have considered the duration of diabetes and serum creatinine (SrCr). As it was already known that the patients are diabetic and we are estimating the complication arising out of it (using the values of SrCr the data has been classified into two categories namely diabetic nephropathy (DN) (SrCr $\geq$ 1.4mg/dl) and nondiabetic nephropathy (SrCr $<$ 1.4mg/dl) groups). From the available reports of 200 patients, reports of SrCr were available for each patient. The pathological reports of these patients were collected from the database of Dr. Lal's path lab from January 2012 to August 2013. Grover et al. [23] discussed this data, which consists of the mean duration of diabetes of 132 types 2 diabetic nephropathy patients for different time intervals. These data are obtained in Table 7. The MLE, SE, KS distance, and its p-value for the marginals distributions are listed in Table 8. Figure 14 shows the fitted pdf and estimated

Table 6. Bayesian estimation : kidney patients data

	BFGMWGE		BFGMWGITL		BFGMWGP	
	estimate	SE	estimate	SE	estimate	SE
α_1	2.2511	0.6506	2.9303	1.3517	0.7776	0.6354
β_1	0.6331	0.0899	1.9730	0.3894	1.8156	0.4275
δ_1	0.0024	0.0006	0.1364	0.0517	0.2120	0.0769
α_2	2.4644	1.0205	0.3613	0.3232	0.4333	0.4035
β_2	0.7699	0.1174	1.9105	0.5077	2.2096	0.5501
δ_2	0.0031	0.0014	0.3402	0.1427	0.2439	0.0782
θ	0.4340	0.4603	0.3234	0.3699	0.8745	0.3565

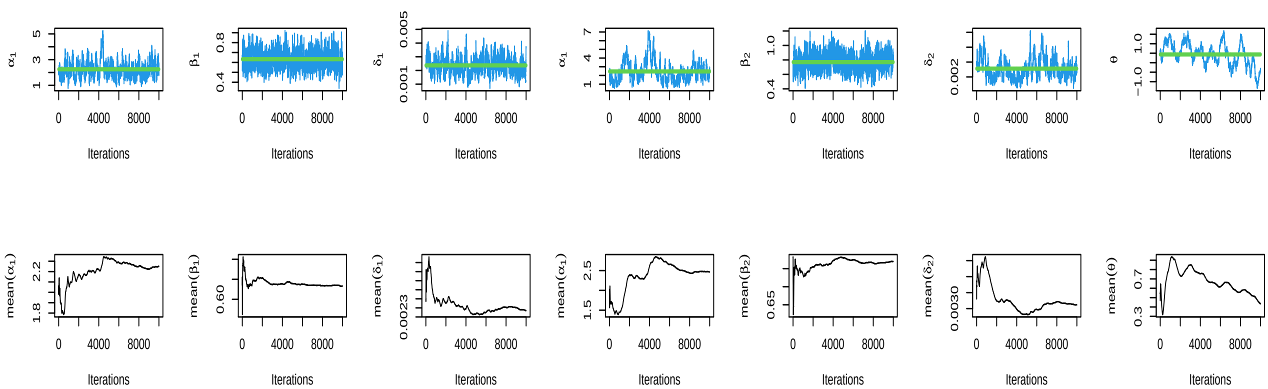

Figure 11. The MCMC plots for parameters of BFGMWGE based on kidney patients data

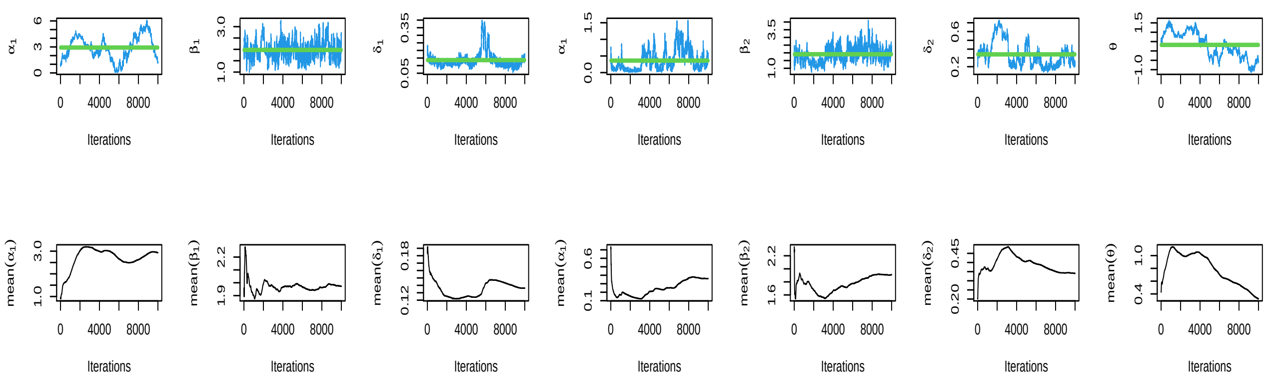

Figure 12. The MCMC plots for parameters of BFGMWGITL based on kidney patients data

cdf, which support our results (KS-test) in Table 8. By using Genest of goodness-of-fit test for copulas to fit of FGM by R package then the statistic is $R_n = 0.4807$ and $\hat{\theta} = 0.0659$ with p-value= 0.14. Note that the data corresponds to an FGM copula and the data fit for marginals distribution, where P-value > 0.05. Now, we fit the BFGMWGE, BFGMWGR, and BFGMWGITL model on this data. The proposed

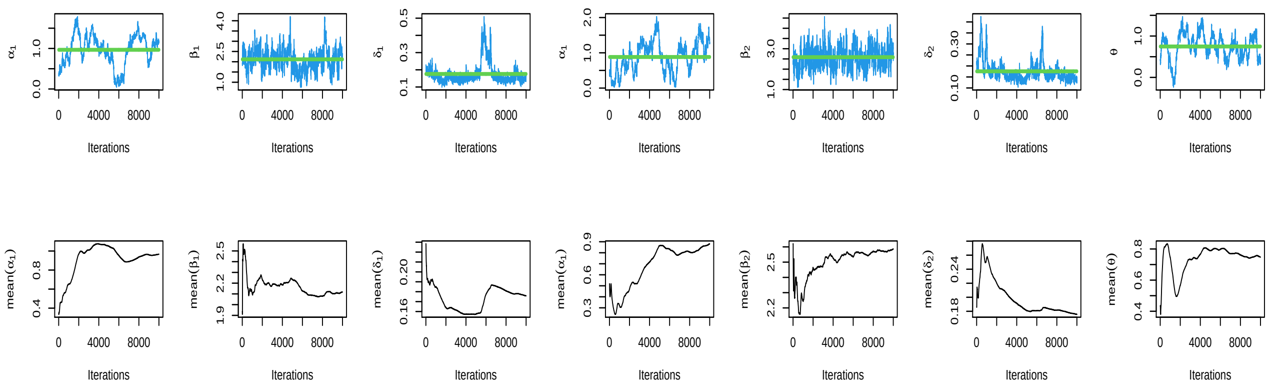

Figure 13. The MCMC plots for parameters of BFGMWGP based on kidney patients data

distributions are the best distribution comparing to BFGMW, BFGMG, and BFGMGE distribution. The BFGMWGR distribution is the best model to fit this data because it has the least measure of KSS, CVM, and AD. In the enclosed Table 12, we provide the MLE, standard error (SE), KSS, CVM, and AD values for the competitive models. In the enclosed Table 9, we provide the Bayesian estimation and SE for the BFGMWGE, BFGMWGR, and BFGMWGITL models. History plots, MCMC convergence of $\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2$ and θ are represented for BFGMWGE, BFGMWGR and BFGMWGITL models respectively in Figures 15, 16 and 17.

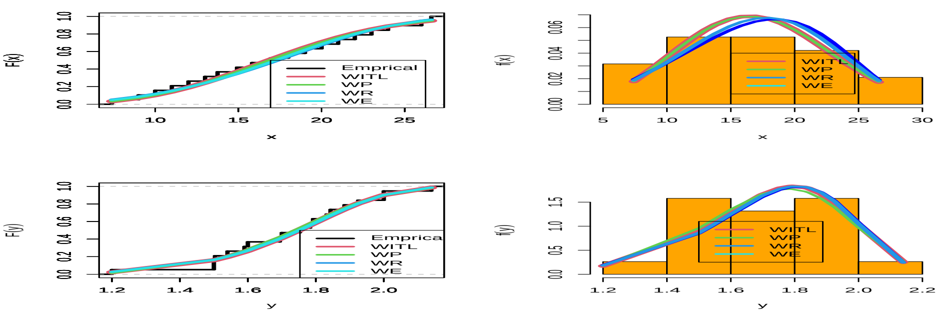

Figure 14. CDF and PDF for marginal distributions with x and y:diabetic nephropathy data

Table 7. Mean duration of diabetes of 132 type 2 diabetic nephropathy patients for different time intervals

No.	Interval	Mean of duration of diabetes (t) from data	Mean SrCr from data	No.	Interval	Mean of duration of diabetes (t) from data	Mean SrCr from data
1	$t \leq 8$	7.4	1.925	11	$17 < t \leq 18$	18	1.832
2	$8 < t \leq 9$	9	1.5	12	$18 < t \leq 19$	19	1.59
3	$9 < t \leq 10$	10	2	13	$19 < t \leq 20$	20	1.7833
4	$10 < t \leq 11$	11	1.6	14	$20 < t \leq 21$	21	1.2
5	$11 < t \leq 12$	12	1.7	15	$21 < t \leq 22$	22	1.792
6	$12 < t \leq 13$	13	1.7533	16	$22 < t \leq 23$	23	1.5
7	$13 < t \leq 14$	13.75	1.54	17	$23 < t \leq 24$	24	1.5033
8	$14 < t \leq 15$	14.92	1.694	18	$25 < t \leq 26$	26	2
9	$15 < t \leq 16$	15.8286	1.8843	19	$t > 26$	26.6	2.14
10	$16 < t \leq 17$	16.9333	1.8433				

Table 8. MLE estimates, SE, KSS and p-values for marginal distributions with x and y:diabetic nephropathy data

	WITL				WP			
	x		y		x		y	
	estimate	SE	estimate	SE	estimate	SE	estimate	SE
α	0.6141	4.6597	0.0429	0.1085	0.0171	0.0403	0.0088	0.0106
β	5.8175	2.0004	6.0985	1.3951	5.8144	1.5103	1.7074	1.3160
δ	0.3173	0.2977	1.8251	0.5602	0.3757	0.1184	4.7591	4.1651
KSS	0.0764		0.1071		0.0761		0.1132	
P-Value	0.9994		0.9813		0.9995		0.9681	
	WR				WE			
	x		y		x		y	
	estimate	SE	estimate	SE	estimate	SE	estimate	SE
α	1.4808	1.1778	9.1179	33.7029	2.8505	15.2920	12.2151	83.5543
β	1.3305	0.2938	3.6530	0.7960	2.6852	0.9970	7.0162	1.6569
δ	0.0030	0.0012	0.2615	0.1959	0.0269	0.0390	0.2910	0.2067
KSS	0.0887		0.1130		0.0878		0.1102	
P-Value	0.9950		0.9685		0.9956		0.9750	

Table 9. Bayesian estimation : diabetic nephropathy data

	BFGMWGE		BFGMWGITL		BFGMWGR	
	estimate	SE	estimate	SE	estimate	SE
α_1	2.2511	0.6506	2.9303	1.3517	0.7776	0.6354
β_1	0.6331	0.0899	1.9730	0.3894	1.8156	0.4275
δ_1	0.0024	0.0006	0.1364	0.0517	0.2120	0.0769
α_2	2.4644	1.0205	0.3613	0.3232	0.4333	0.4035
β_2	0.7699	0.1174	1.9105	0.5077	2.2096	0.5501
δ_2	0.0031	0.0014	0.3402	0.1427	0.2439	0.0782
θ	0.4340	0.4603	0.3234	0.3699	0.8745	0.3565

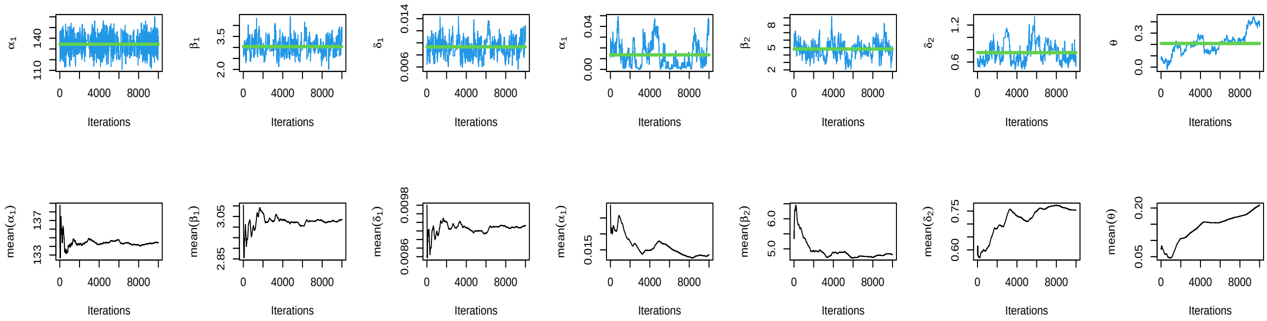

Figure 15. The MCMC plots for parameters of BFGMWGE, based on diabetic nephropathy data

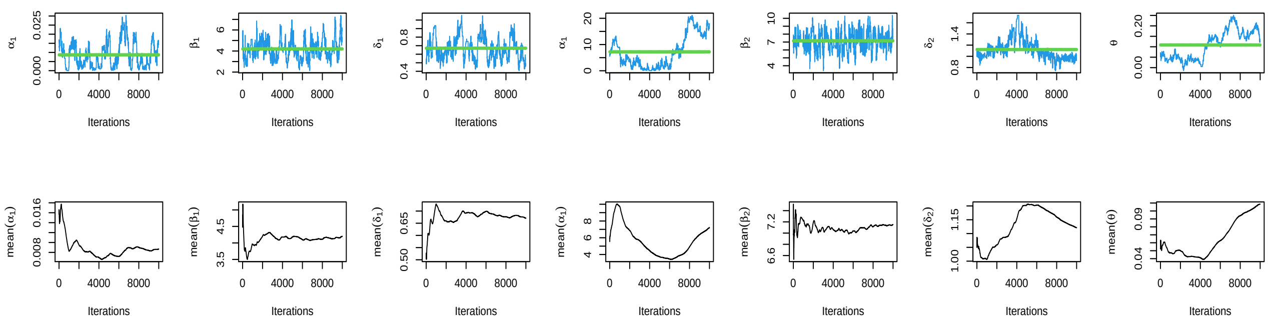

Figure 16. The MCMC plots for parameters of BFGMWGITL based on diabetic nephropathy data

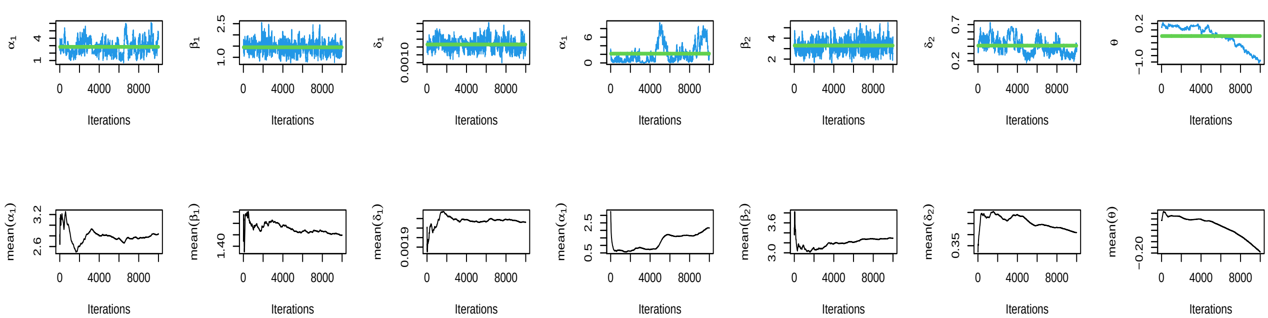

Figure 17. The MCMC plots for parameters of BFGMWGR based on diabetic nephropathy data

Table 10. MLE estimates, SE, KSS and p-values for marginal distributions with x and y:kidney patients data

	WITL				WP				WE			
	x		y		x		y		x		y	
	estimate	SE	estimate	SE	estimate	SE	estimate	SE	estimate	SE	estimate	SE
α	0.8689	4.2303	0.5876	2.7828	0.3091	1.2899	0.4002	1.9649	2.0437	0.6949	1.9631	0.9537
β	1.9050	1.0397	2.3019	1.1354	1.9211	1.2073	2.5782	1.2837	0.6352	0.0923	0.7804	0.1209
δ	0.1940	0.3500	0.2156	0.3148	0.2342	0.3623	0.1987	0.2701	0.0025	0.0007	0.0033	0.0013
KSS	0.1240		0.1284		0.1209		0.1270		0.1540		0.1511	
P-Value	0.7457		0.7059		0.7726		0.7184		0.4753		0.5001	

Table 11. The Estimates and the Corresponding SE of Parameters of BFGMW-G family and other distributions for Medical Data

		α_1	β_1	δ_1	α_2	β_2	δ_2	θ	KSS	CVM	AD
BFGMWGP	estimate	0.3346	1.9352	0.2284	0.5182	2.6204	0.1858	0.4348	0.4292	0.6370	3.5598
	SE	1.3044	1.1093	0.3271	4.0053	1.8564	0.3907	0.4917			
BFGMWGITL	estimate	0.9621	1.9203	0.1874	0.7352	2.3371	0.2019	0.4284	0.4508	0.7595	4.1599
	SE	4.5177	0.9791	0.3241	4.8329	1.4615	0.4034	0.4849			
BFGMGE	estimate	0.6634	0.0063		0.9262	0.0096		1.0000	0.7527	3.1098	17.3384
	SE	0.1657	0.0018		0.2342	0.0025		0.3760			
BFGMW	estimate	0.7510	99.7970		0.9287	95.2921		0.3996	0.7545	3.1596	18.0501
	SE	0.1053	25.6070		0.1322	19.8060		0.4990			
BFGMG	estimate	0.6681	0.0055		0.9244	0.0093		0.0100	0.7655	3.4709	20.8518
	SE	0.1467	0.0017		0.2090	0.0028		0.4949			
BFGMWGE	estimate	2.0437	0.6352	0.0025	1.9805	0.7810	0.0033	0.4670	0.8469	4.2151	26.3500
	SE	0.6831	0.0920	0.0007	1.0119	0.1232	0.0014	0.5198			

Table 12. The Estimates and the Corresponding SE of Parameters of BFGMW-G family and other distributions for diabetic nephropathy data

		α_1	β_1	δ_1	α_2	β_2	δ_2	θ	KSS	CVM	AD
BFGMWGITL	par	0.0157	4.8459	0.5208	5.5389	7.2635	1.0806	0.0528	0.0733	0.0269	0.1961
	SE	0.0345	1.3815	0.1874	25.2875	1.4924	0.4932	0.5577			
BFGMWGR	par	2.8252	1.4220	0.0021	3.2501	3.5002	0.3225	0.0873	0.1461	0.0333	0.2151
	SE	1.2043	0.2653	0.0003	8.6865	0.7385	0.1791	0.5758			
BFGMWGE	par	133.8298	3.1032	0.0098	0.0290	5.5988	0.5796	0.0750	0.3480	0.1369	0.6656
	SE	5.8496	0.4250	0.0019	0.1381	1.9469	0.3695	0.5666			
BFGMGE	par	16.5454	0.2005		142.1023	3.0788		-0.0937	0.7154	1.9793	12.3323
	SE	9.4825	0.0410		82.3462	0.3771		0.7963			
BFGMW	par	3.3775	19.0293		9.0052	1.8211		0.0620	0.7492	2.1794	14.1803
	SE	0.6207	1.3722		1.5905	0.0494		0.5619			
BFGMG	par	8.2572	0.4851		58.6296	33.9827		-0.0020	0.7528	2.0719	15.7759
	SE	2.6267	0.1591		18.9709	11.0419		0.5667			

11. Conclusion

In this paper, we have proposed a new bivariate family based on the FGM copula function and WG family, which will be denoted as the BFGMWG family. Moreover, the bivariate exponential WG distribution, bivariate inverted Topp-Leone WG distribution, bivariate Rayleigh WG, and bivariate Pareto WG based on FGM copula function have been obtained. Statistical properties for this family have been discussed, therefore, it can be used quite effectively in life testing data as medical data of kidney patients, and diabetic nephropathy data. Parameters estimation have been discussed by the maximum likelihood method and Bayesian method by using Metropolis-Hastings. Hence, we can argue that Bayesian estimation is the best performing estimator for the BFGMWG family. Additionally, the new BFGMWG family can be used as an alternative to a more traditional bivariate distribution for different applications. The FGMBWG family works better because the marginal functions have the same basic distribution and it has closed forms for moment generating function and product moments. Three types of confidence intervals are considered here namely asymptomatic, bootstrap, and Bayesian credible interval to discuss the interval estimation of the unknown parameters. Some measures of this family have been discussed as Kendall's tau correlation and median regression model, Kolmogorov-Smirnov statistic, Anderson-Darling, Anderson-Darling, and goodness of fit test for copula.

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