The Topp-Leone Odd Burr III-G Family of Distributions: Model, Properties and Applications

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Abstract   We propose and investigate a new generalized family of distributions called the Topp-Leone Odd Burr III-G (TL-OBIII-G) family of distributions. We present the sub-families of this new family of distributions. Properties of the new family of distributions includs sub-models, quantile function, moments, incomplete and probability weighted moments, distribution of the order statistics, and Rényi entropy are derived. The Maximum likelihood estimation technique is used to estimate the model parameters, and a Monte Carlo simulation study is employed to examine the performance of the model. Two real data sets are used to prove the importance of the TL-OBIII-G family of distributions.

1. Introduction

The limitations of the well-known standard distribution in data modeling have motivated researchers to extend distributions by adding one or more shape parameters to gain flexibility in the new distribution. Some of the generated families in the literature including the odd Burr III family of distributions by Jamal et al. [19], the Nadarajah Haghighi Topp-Leone-G family of distributions by Reyad et al. [33], the Marshall-Olkin Odd Burr III-G family of distributions by Afify et al. [2], the generalized transmuted-G family proposed by Nofal et al. [28], Marshall-Olkin alpha power-G by Nassar et al. [27], beta-G by Eugene et al. [17], Topp Leone odd Lindley-G by Reyad et al. [32], odd Lomax-G by Cordeiro et al. [9], exponentiated Weibull-H by Cordeiro et al. [10], the Burr III Marshal Olkin family of distributions by Bhatti et al. [7]; among others.

The Burr III (BIII) distribution was one of the twelve cumulative distribution functions introduced by Burr [8]. In the actuarial literature it is referred to as the inverse Burr distribution (Klugman et al. [21]) and as the kappa distribution in the meteorological area [25]. The BIII distribution has been studied and also used in various fields of sciences. It has been used in finance, environmental studies, survival analysis, and reliability theory, see Lindsay et al. [24] and Gove et al. [18]. A random variable X has the odd Burr III-G distribution if its cumulative distribution function (cdf) and probability density function (pdf) are given by

$$F(x; \alpha, \beta, \xi) = \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta},$$

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and

$$f(x; \alpha, \beta, \xi) = \alpha\beta \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha}\right)^{-\beta-1} \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha-1} \frac{g(x; \xi)}{\left(\overline{G}(x; \xi)\right)^2},$$

for $\alpha, \beta > 0$, and parameter vector ξ, respectively.

The Topp-Leone transformation was applied to several distributions, for example; Topp-Leone Inverse Weibull distribution by Abbas et al. [1], Type II Topp-Leone Inverse Rayleigh distribution introduced by Mohammed and Yahia [26], Topp-Leone Weibull-Lomax distribution presented by Jamal et al. [20], the Topp-Leone generalized exponential power series distribution by Kunjiratanachot et al. [22] and Aryal et al. [5] developed the Topp-Leone Generated Weibull distribution. Al-Shomrani et al. [3] proposed the Topp-Leone family of distributions with the cdf and pdf given by

$$F_{TL-G}(x; b, \xi) = \left[1 - \overline{G}^2(x; \xi)\right]^b \tag{1}$$

and

$$f_{TL-G}(x; b, \xi) = 2\alpha b \left[1 - \overline{G}^2(x; \xi)\right]^{b-1} \overline{G}(x; \xi) g(x; \xi), \tag{2}$$

respectively, for $b > 0$ and parameter vector ξ.
The basic motivations for developing the TL-OBIII-G family of distributions are:

- to construct and generate distributions with symmetric, left-skewed, right-skewed, reversed-J shapes;
- to provide better fits than other generated distributions under the same transformation;
- to construct heavy-tailed distributions for modeling different real data sets.

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The rest of the results are outlined as follows: Section 2 contain the new TL-OBIII-G family of distributions and its properties such as sub-families, hazard rate function (hrf), quantile function to mention a few. In Section 3 estimation of the parameters of the TL-OBIII-G family of distributions via the method of maximum likelihood is presented, followed by a Monte Carlo simulation study to examine the performance of the maximum likelihood estimates in section 4. Two applications to real data sets are given in section 5, followed by some concluding remarks in section 6

2. The Model and Statistical Properties

In this section, we derive the Topp-Leone Odd Burr III-G (TL-OBIII-G) family of distributions and its statistical properties.

2.1. Topp-Leone Odd Burr III-G Family of Distributions

Inserting the cdf of the OBIII-G distribution into equation (1), we obtain the new family of distribution herein referred to as Topp-Leone Odd Burr III-G family of distributions. The cdf and pdf of the proposed Topp-Leone Odd Burr III-G (TL-OBIII-G) family of distributions are given by

$$F(x; \alpha, \beta, b, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^b \tag{3}$$

and

$$
\begin{aligned}
f(x; \alpha, \beta, b, \xi) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \\
&\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^{2}},
\end{aligned}
\tag{4}
$$

respectively, for $\alpha, \beta, b > 0$ and parameter vector ξ.

The hazard rate function (hrf) and quantile functions of the TL-OBIII-G family of distributions are respectively given below. The hrf is given by

$$
\begin{aligned}
h_F(x; \alpha, \beta, b, \xi) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \\
&\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^{2}} \\
&\times \left(1 - \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]^{-b} \right)^{-1},
\end{aligned}
\tag{5}
$$

for $\alpha, \beta, \delta > 0$ and parameter vector ξ.

The quantile function of the TL-OBIII-G family of distributions is obtained by solving the non-linear equation:

$$
F(x; \alpha, \beta, b, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]^{b} = u,
\tag{6}
$$

for $0 \leq u \leq 1$, that is,

$$
\frac{G(x;\xi)}{\overline{G}(x;\xi)} = \left(\left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}},
$$

so that

$$
G(x;\xi) = \left(\left(\left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}} + 1 \right)^{-1}.
\tag{7}
$$

Consequently, the quantile function for the TL-OBIII-G family of distributions is given by

$$
Q_G(u; \alpha, \beta, b, \xi) = X = G^{-1} \left[\left(\left(\left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right].
\tag{8}
$$

It follows therefore that random numbers can be generated from the TL-OBIII-G family of distributions based on equation (8).

2.2. Sub-Families of TL-OBIII-G Family of Distributions

In this subsection, some sub-families of the TL-OBIII-G family of distributions are presented.

- When $\alpha = 1$, the Topp-Leone Odd Burr III-G (TL-OBIII-G) family of distributions reduces to a new family of distributions with the cdf

$$F(x; \beta, b, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-1} \right)^{-\beta} \right)^2 \right]^b,$$

for $\beta, b > 0$, and parameter vector ξ.

- If $\beta = 1$, we obtain the new family of distributions with the cdf

$$F(x; \alpha, b, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-1} \right)^2 \right]^b,$$

for $\alpha, b > 0$, and parameter vector ξ.

- If $b = 1$ we obtain the new family of distributions with the cdf

$$F(x; \alpha, \beta, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right],$$

for $\alpha, \beta > 0$ and parameter vector ξ.

- If $\beta = \alpha = 1$ we obtain the new family of distributions with the cdf

$$F(x; b, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-1} \right)^{-1} \right)^2 \right]^b,$$

for $b > 0$ and parameter vector ξ.

- If $\beta = b = 1$ we obtain the new family of distributions with the cdf

$$F(x; \alpha, \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-1} \right)^2 \right],$$

for $\alpha > 0$ and parameter vector ξ.

- If $\beta = \alpha = b = 1$ we obtain the new family of distributions with the cdf

$$F(x; \xi) = \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-1} \right)^{-1} \right)^2 \right],$$

for parameter vector ξ.

2.3. Some Specialized Sub Models

In this subsection, we present sub-models of TL-OBIII-G family of distributions by specifying $G(x; \xi)$ and $g(x; \xi)$ in equations (3) and (4).

2.3.1. TL-OBIII-LLoG Distribution If the baseline cdf and pdf are given by $G(x;\lambda) = 1 - \left(1 + x^\lambda\right)^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}\left(1 + x^\lambda\right)^{-2}$, for $\lambda > 0$, and $x > 0$, the cdf and pdf of Topp-Leone Odd Burr III-log-logistic (TL-OBIII-LLoG) distribution are given by

$$F(x;\alpha,\beta,b,\lambda) = \left[1 - \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^b$$

and

$$\begin{aligned}
f(x;\alpha,\beta,b,\lambda) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right) \frac{g(x;\xi)}{\left(\left(1 + x^\lambda\right)^{-1}\right)^2} \\
&\times \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta-1} \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha-1},
\end{aligned}$$

respectively, for $\alpha,\beta,b,\lambda > 0$. The hrf is given by

$$\begin{aligned}
h_F(x;\alpha,\beta,b,\lambda) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right) \frac{g(x;\xi)}{\left(\left(1 + x^\lambda\right)^{-1}\right)^2} \\
&\times \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta-1} \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha-1} \\
&\times \left(1 - \left[1 - \left(1 - \left(1 + \left(\frac{1 - \left(1 + x^\lambda\right)^{-1}}{\left(1 + x^\lambda\right)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^b\right)^{-1}.
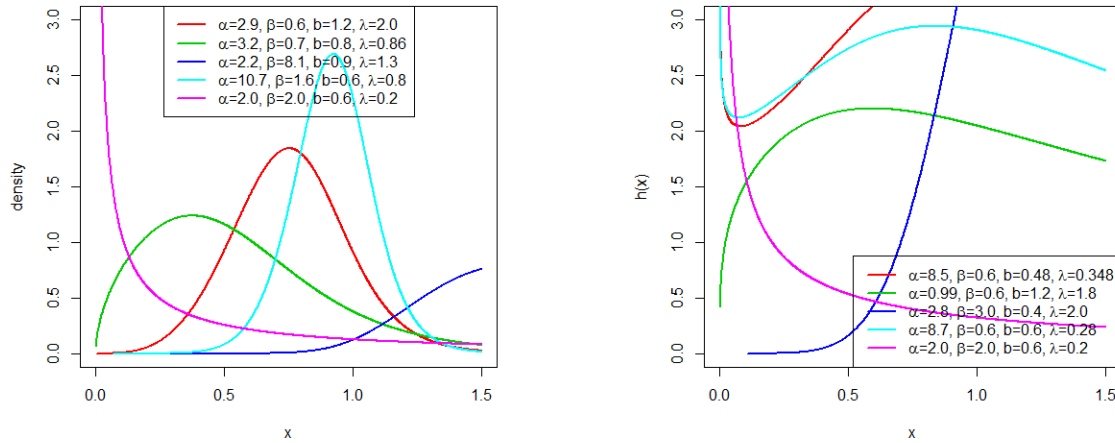\end{aligned}$$

Figure 1. Density and Hazard Rate Function Plots for TL-OBIII-LLoG Distribution

Figure 1 shows the plots of pdf and hrf of the TL-OBIII-LLoG distribution, respectively. The pdf can take several shapes including increasing, right-skewed, left-skewed, unimodal and reverse-J shapes. The TL-OBIII-LLoG hrf displays increasing, decreasing, bathtub, bathtub followed by upside-down bathtub, and upside-down bathtub shapes. These non-monotonic shapes are likely to be encountered in real life situations.

Table 1. **Table of Quantile for TL-OBIII-LLoG Distribution**

	$(\alpha, \beta, b, \lambda)$				
u	$(1.4, 1.0, 0.4, 1.8)$	$(0.6, 1.3, 2.2, 1.0)$	$(0.8, 1.0, 2.6, 1.7)$	$(2.0, 1.0, 1.0, 2.0)$	$(1.0, 1.8, 0.6, 2.5)$
0.1	0.0773	0.2157	0.4168	0.4799	0.3786
0.2	0.1555	0.4281	0.5756	0.5885	0.5050
0.3	0.2330	0.7007	0.7224	0.6643	0.6133
0.4	0.3154	1.0671	0.8773	0.7354	0.7139
0.5	0.4046	1.5836	1.0497	0.8022	0.8174
0.6	0.5027	2.3724	1.2586	0.8735	0.9319
0.7	0.6194	3.6896	1.5376	0.9544	1.0676
0.8	0.7768	6.3088	1.9540	1.0557	1.2502
0.9	1.0327	13.9007	2.7804	1.2141	1.5600

2.3.2. TL-OBIII-Power Distribution The cdf and pdf of power distribution are given by $G(x; \theta, k) = (\theta x)^k$ and $g(x; \theta, k) = k\theta^k x^{k-1}$, for $\theta, k > 0$, and $x \in (0, \frac{1}{\theta})$. The Topp-Leone Odd Burr III-Power (TL-OBIII-P) distribution has cdf and pdf given by

$$F(x; \alpha, \beta, b, \theta, k) = \left[1 - \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1 - (\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^b$$

and

$$
\begin{aligned}
f(x;\alpha,\beta,b,\theta,k) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)\left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta-1} \\
&\times \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha-1}\frac{k\theta^k x^{k-1}}{\left(1-(\theta x)^k\right)^2},
\end{aligned}
$$

respectively, for $\alpha,\beta,b,\theta,k > 0$. The hrf is given by

$$
\begin{aligned}
h_F(x;\alpha,\beta,b,\theta,k) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)\left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta-1} \\
&\times \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha-1}\frac{k\theta^k x^{k-1}}{\left(1-(\theta x)^k\right)^2} \\
&\times \left(1 - \left[1 - \left(1 - \left(1 + \left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^b\right)^{-1}.
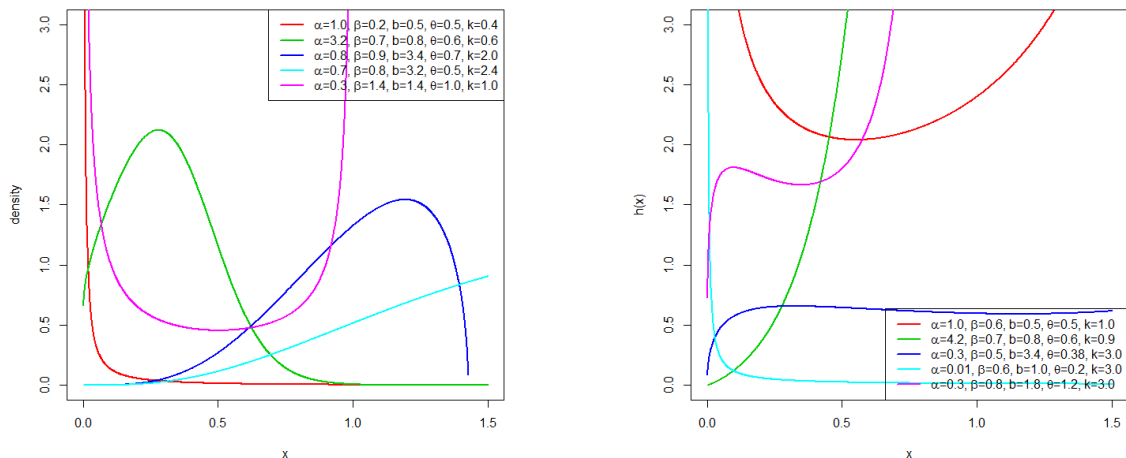\end{aligned}
$$

Figure 2. Density and Hazard Rate Function Plots for TL-OBIII-Power Distribution

Figure 2 shows the plots of pdf and hrf of the TL-OBIII-P distribution, respectively. The pdf can take several shapes including increasing, right-skewed, U-shape and reverse-J shapes. The TL-OBIII-P hrf displays increasing, reverse-J, bathtub, upside-down bathtub followed by bathtub, and upside-down bathtub shapes.

Table 2. **Table of Quantile for TL-OBIII-P Distribution**

u	$(1.4, 1, 0.4, 1.8, 0.3)$	$(0.6, 1.3, 2.2, 1, 1.2)$	$(0.8, 1, 2.6, 1.7, 0.5)$	$(2, 1.0, 1.0, 2, 1.0)$	$(1, 1.8, 0.6, 2.5, 0.5)$
			$(\alpha, \beta, b, a, \theta)$		
0.1	0.7849	0.7447	0.1180	0.2946	0.7221
0.2	1.2038	1.0361	0.1813	0.4059	0.9034
0.3	1.5896	1.2875	0.2397	0.4897	1.0497
0.4	1.9807	1.5323	0.2989	0.5605	1.1866
0.5	2.4011	1.7882	0.3628	0.6240	1.3261
0.6	2.8779	2.0726	0.4353	0.6837	1.4782
0.7	3.4540	2.4114	0.5231	0.7420	1.6568
0.8	4.2202	2.8583	0.6399	0.8019	1.8902
0.9	5.4586	3.5807	0.8293	0.8694	2.2648

2.3.3. TL-OBIII-Lomax Distribution Consider Lomax distribution as the baseline distribution with cdf and pdf are given by $G(x; a, \theta) = 1 - (1 + \theta x)^{-a}$ and $g(x; a, \theta) = a\theta (1 + \theta x)^{-a-1}$, for $a, \theta > 0$, and $x > 0$. The TL-OBIII-Lomax (TL-OBIII-L) distribution has cdf and pdf given by

$$F(x; \alpha, \beta, b, a, \theta) = \left[1 - \left(1 - \left(1 + \left(\frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^b$$

and

$$
\begin{aligned}
f(x; \alpha, \beta, b, a, \theta) = {} & 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\
& \times \left(1 - \left(1 + \left(\frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right) \frac{a\theta (1 + \theta x)^{-a-1}}{\left((1 + \theta x)^{-a} \right)^2} \\
& \times \left(1 + \left(\frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}} \right)^{-\alpha-1},
\end{aligned}
$$

respectively, for $\alpha, \beta, b, a, \theta > 0$. The hrf is given by

$$
\begin{aligned}
h_F(x; \alpha, \beta, b, a, \theta) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{1 - (1+\theta x)^{-a}}{(1+\theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\
&\times \left(1 - \left(1 + \left(\frac{1 - (1+\theta x)^{-a}}{(1+\theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right) \frac{a\theta (1+\theta x)^{-a-1}}{\left((1+\theta x)^{-a} \right)^2} \\
&\times \left(1 + \left(\frac{1 - (1+\theta x)^{-a}}{(1+\theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{1 - (1+\theta x)^{-a}}{(1+\theta x)^{-a}} \right)^{-\alpha-1} \\
&\times \left(1 - \left[1 - \left(1 - \left(1 + \left(\frac{1 - (1+\theta x)^{-a}}{(1+\theta x)^{-a}} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^b \right)^{-1}.
\end{aligned}
$$

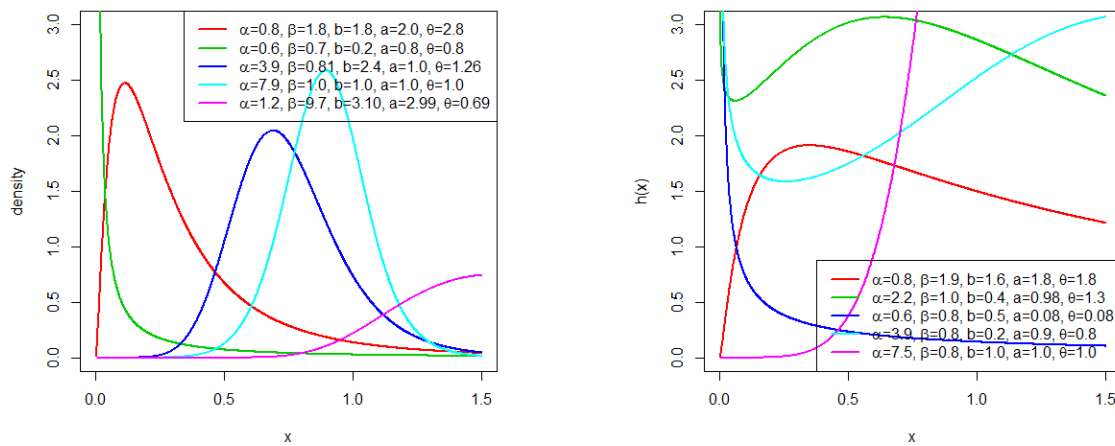Figure 3. Density and Hazard Rate Function Plots for TL-OBIII-L Distribution

Figure 3 shows the plots of pdf and hrf functions of TL-OBIII-L distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, almost symmetric, and reverse-J shapes. The TL-OBIII-L hrf displays increasing, decreasing, bathtub, bathtub followed by upside-down bathtub and upside-down bathtub shapes.

Table 3. **Table of Quantile for TL-OBIII-L Distribution**

	$(\alpha, \beta, b, a, \theta)$				
u	$(1.8, 0.9, 0.8, 0.8, 1.0)$	$(2.0, 0.8, 0.5, 0.4, 2.0)$	$(2.5, 0.6, 0.7, 1.5, 1.8)$	$(1.4, 1.3, 1.9, 2.0, 2.0)$	$(0.8, 1.7, 0.8, 0.4, 1.9)$
0.1	0.1414	0.0448	0.0249	0.1008	0.1205
0.2	0.2545	0.1167	0.0501	0.1340	0.2808
0.3	0.3619	0.2034	0.0749	0.1657	0.5182
0.4	0.4836	0.3169	0.0995	0.1934	0.8791
0.5	0.6176	0.4601	0.1250	0.2282	1.4738
0.6	0.7764	0.6440	0.1597	0.2629	2.5450
0.7	0.9849	0.9101	0.1918	0.3094	4.8028
0.8	1.2809	1.3367	0.2352	0.3733	10.9256
0.9	1.8431	2.2512	0.3060	0.4878	40.3184

2.4. Series Expansion of Density Function

In this section, we write the pdf of TL-OBIII-G family of distribution as a linear combination of exponentiated-G (E-G) distribution, which will be used for further computations. Applying the following expansions

$$(1 - z)^k = \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} z^j \quad \text{and} \quad (1 + z)^{-\beta - 1} = \sum_{l=0}^{\infty} \frac{\Gamma((\beta + 1) + l)}{\Gamma(\beta + 1)l!} (-1)^l z^l,$$

for $|z| < 1$, the pdf of TL-OBIII-G family of distributions can be written as

$$
\begin{aligned}
f(x; \alpha, \beta, b, \xi) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\
&\quad \times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \\
&\quad \times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2j+1} \\
&\quad \times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j,p=0}^{\infty} \binom{b-1}{j} \binom{2j+1}{p} (-1)^{j+p} \\
&\quad \times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta(p+1)-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
= \ & 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!}(-1)^{j+p+l} \\
\times \ & \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha(l+1)-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
= \ & 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!}(-1)^{j+p+l} \\
\times \ & (G(x;\xi))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi)\right)^{\alpha(l+1)-1} g(x;\xi) \\
= \ & 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!}(-1)^{j+p+l} \\
\times \ & (1-(1-G(x;\xi)))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi)\right)^{\alpha(l+1)-1} g(x;\xi) \\
= \ & 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
\times \ & (-1)^{j+p+l} \left(\overline{G}(x;\xi)\right)^{k+\alpha(l+1)-1} g(x;\xi) \\
= \ & 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
\times \ & (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} (G(x;\xi))^v g(x;\xi) \\
= \ & 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
\times \ & (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{v+1}{v+1} (G(x;\xi))^v g(x;\xi) \\
= \ & \sum_{v=0}^{\infty} C_{v+1} g_{v+1}(x;\xi), \hspace{3cm} (9)
\end{aligned}
$$

where $g_{v+1}(x;\xi) = (v+1)[G(x;\xi)]^v g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $v+1 > 0$ and parameter vector ξ, and

$$
\begin{aligned}
C_{v+1} = \ & 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
\times \ & (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{1}{v+1}. \hspace{2cm} (10)
\end{aligned}
$$

Therefore, the pdf of TL-OBIII-G family of distributions can be written as an infinite linear combination of exponentiated-G (E-G) densities. The structural properties of the TL-OBIII-G family of distributions follows from those of E-G distribution.

2.5. *Moments and Generating Function*

From now onwards, let $Y_{v+1} \sim E - G(v+1)$. The rth raw moment, μ'_r of the TL-OBIII-G family of distributions can be obtained from equation (9) as

$$
\begin{aligned}
\mu'_r = E(X^r) &= \int_{-\infty}^{\infty} x^r f(x) dx \\
&= \sum_{v=0}^{\infty} C_{v+1} E(Y_{v+1}^r).
\end{aligned}
$$

The moment generating function (MGF) $M_X(t) = E(e^{tX})$ is given by:

$$
M_X(t) = \sum_{v=0}^{\infty} C_{v+1} M_{v+1}(t),
$$

where $M_{v+1}(t)$ is the mgf of Y_{v+1} and C_{v+1} is given by equation (10).
Tables 4, 5 and 6 presents the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected parameter values of some special cases of the TL-OBIII-G family of distributions.

Table 4. Table of Moments for Selected Parameters for TL-OBIII-LLoG Distribution

	$(\alpha, \beta, b, \lambda)$				
	(0.4, 0.5, 0.2, 0.5)	(0.9, 1.2, 1.0, 0.6)	(0.3, 2.2, 1.0, 0.2)	(3.0, 2.0, 0.5, 0.6)	(0.8, 2.0 , 1.2, 0.5)
$E(X)$	0.0053	0.1599	0.0220	0.3490	0.1208
$E(X^2)$	0.0024	0.0830	0.0111	0.2261	0.0689
$E(X^3)$	0.0015	0.0552	0.0074	0.1637	0.0478
$E(X^4)$	0.0011	0.0412	0.0055	0.1270	0.0365
$E(X^5)$	0.0009	0.0328	0.0044	0.1032	0.0295
$E(X^6)$	0.0007	0.0272	0.0037	0.0866	0.0247
SD	0.0006	0.0232	0.0032	0.0745	0.0213
CV	0.0005	0.0203	0.0028	0.0653	0.0187
CS	0.0005	0.0180	0.0024	0.0581	0.0166
CK	0.0004	0.0162	0.0022	0.0522	0.0150

Table 5. Table of Moments for Selected Parameters for TL-OBIII-Lomax Distribution

	$(\alpha, \beta, b, a, \theta)$				
	(0.4, 0.5, 0.2, 0.5)	(0.9, 1.2, 1.0, 0.6)	(0.3, 2.2, 1.0, 0.2)	(3.0, 2.0, 0.5, 0.6)	(0.8, 2.0 , 1.2, 0.5)
$E(X)$	0.2291	0.0822	0.07376	0.0170	0.1920
$E(X^2)$	0.1075	0.0396	0.0344	0.0036	0.0480
$E(X^3)$	0.0662	0.0257	0.0222	0.0015	0.0155
$E(X^4)$	0.0469	0.0189	0.0163	0.0009	0.0064
$E(X^5)$	0.0360	0.0150	0.0129	0.0006	0.0032
$E(X^6)$	0.0291	0.0124	0.0107	0.0004	0.0019
SD	0.0244	0.0105	0.0091	0.0003	0.0013
CV	0.0210	0.0092	0.0079	0.0003	0.0010
CS	0.0184	0.0081	0.0070	0.0002	0.0007
CK	0.0164	0.0073	0.0063	0.0002	0.0006

Table 6. Table of Moments for Selected Parameters for TL-OBIII-Power Distribution

	$(\alpha, \beta, b, a, \theta)$				
	(0.4, 0.5, 0.2, 0.5, 0.2)	(0.9, 1.2, 1.0, 0.6, 2.0)	(0.3, 2.2, 1.0, 0.2, 3.0)	(0.5, 2.0, 0.5, 0.6, 0.8)	(0.5, 2.0, 1.2, 0.5, 0.2)
$E(X)$	0.0046	0.3398	0.0323	0.1908	0.1570
$E(X^2)$	0.0025	0.2507	0.0233	0.1119	0.0931
$E(X^3)$	0.0017	0.1979	0.0182	0.0799	0.0667
$E(X^4)$	0.0013	0.1632	0.0149	0.0624	0.0522
$E(X^5)$	0.0011	0.1387	0.0127	0.0513	0.0429
$E(X^6)$	0.0009	0.1205	0.0109	0.0436	0.0364
SD	0.0008	0.1065	0.0096	0.0379	0.0316
CV	0.0007	0.0954	0.0087	0.0335	0.0279
CS	0.0006	0.0863	0.0078	0.0301	0.0251
CK	0.0005	0.0788	0.0072	0.0273	0.0228

2.6. *Incomplete Moments*

When it comes to computing Bonferroni and Lorenz curves, incomplete moments are useful. The sth incomplete moment of X denoted by $\eta_s(t)$ is given by

$$\eta_s(t) = \int_{-\infty}^{t} x^s f(x) dx.$$

Using equation (9), we obtain

$$\eta_s(t) = \sum_{v=0}^{\infty} C_{v+1} \int_{-\infty}^{t} x^s g_{v+1}(x; \xi) dx. \tag{11}$$

By setting $s = 1$ in equation (11), we obtain the first incomplete moment of the TL-OBIII-G family of distributions.

2.7. *Rényi Entropy*

Rényi entropy (Rényi [31]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; \alpha, \beta, b, \xi)]^v dx \right), v \neq 1, v > 0. \tag{12}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that

$$
\begin{aligned}
[f(x; \alpha, \beta, b, \xi)]^v &= (2\alpha\beta b)^v \left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{v(b-1)} \\
&\times \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-\beta} \right)^v \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{v(-\beta-1)} \\
&\times \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{v(-\alpha-1)} \frac{g(x; \xi)^v}{\left(\overline{G}(x; \xi) \right)^{2v}}.
\end{aligned}
$$

Using the fact that

$$
\left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{v(b-1)} = \sum_{j=0}^{\infty} \binom{v(b-1)}{j} (-1)^j
$$
$$
\times \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2j},
$$

we write

$$
\begin{aligned}
[f(x;\alpha,\beta,b,\xi)]^v &= (2\alpha\beta b)^v \sum_{j=0}^{\infty} \binom{v(b-1)}{j}(-1)^j \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{v(-\beta-1)} \\
&\times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{-\beta}\right)^{2j+v} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{v(-\alpha-1)} \\
&\times \frac{g(x;\xi)^v}{\left(\overline{G}(x;\xi)\right)^{2v}}.
\end{aligned}
$$

Using the expansion

$$
\begin{aligned}
\left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{-\beta}\right)^{2j+v} &= \sum_{p=0}^{\infty} \binom{2j+v}{p}(-1)^p \\
&\times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{-\beta p},
\end{aligned}
$$

we write

$$
\begin{aligned}
[f(x;\alpha,\beta,b,\xi)]^v &= (2\alpha\beta b)^v \sum_{j,p=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{v(-\alpha-1)} \\
&\times (-1)^{j+p}\left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{-\beta(v+p)-v} \frac{g(x;\xi)^v}{\left(\overline{G}(x;\xi)\right)^{2v}}.
\end{aligned}
$$

Considering

$$
\left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right)^{-\beta(v+p)-v} = \sum_{l=0}^{\infty}(-1)^l \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha l},
$$

we have

$$
\begin{aligned}
[f(x;\alpha,\beta,b,\xi)]^v &= (2\alpha\beta b)^v \sum_{j,p,l=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}(-1)^{j+p+l}g(x;\xi)^v \\
&\times \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}(G(x;\xi))^{-\alpha(v+l)-v}\left(\overline{G}(x;\xi)\right)^{\alpha(v+l)-v} \\
&= (2\alpha\beta b)^v \sum_{j,p,l=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}(-1)^{j+p+l}g(x;\xi)^v \\
&\times \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}(1-(1-G(x;\xi)))^{-\alpha(v+l)-v}\left(\overline{G}(x;\xi)\right)^{\alpha(v+l)-v} \\
&= (2\alpha\beta b)^v \sum_{j,p,l,k=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}(-1)^{j+p+l}g(x;\xi)^v \\
&\times \frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!}\frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}\left(\overline{G}(x;\xi)\right)^{k+\alpha(v+l)-v} \\
&= (2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}\binom{k+\alpha(v+l)-v}{q} \\
&\times (-1)^{j+p+l}\frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!}\frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}(G(x;\xi))^q g(x;\xi)^v.
\end{aligned}
$$

Consequently, Rényi entropy for the TL-OBIII-G family of distributions is given by

$$
\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\Bigg[(2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}\binom{k+\alpha(v+l)-v}{q} \\
&\times (-1)^{j+p+l}\frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!}\frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}\frac{1}{\left[\frac{q}{v}+1\right]^v} \\
&\times \int_0^{\infty}\left(\left[\frac{q}{v}+1\right](G(x;\xi))^{\frac{q}{v}}g(x;\xi)\right)^v dx\Bigg] \\
&= \frac{1}{1-v}\log\left[\sum_{j,p,l,k,q=0}^{\infty} w_{j,p,l,k,q}\exp((1-v)I_{REG})\right],
\end{aligned}
\tag{13}
$$

for $v>0$, $v\neq 1$, where

$I_{REG}=\frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[\frac{q}{v}+1\right](G(x;\xi))^{\frac{q}{v}}g(x;\xi)\right)^v dx\right]$ is the Rényi entropy of E-G distribution with power parameter $\frac{q}{v}+1$, and

$$
\begin{aligned}
w_{j,p,l,k,q} &= (2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j}\binom{2j+v}{p}\binom{k+\alpha(v+l)-v}{q} \\
&\times (-1)^{j+p+l}\frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!}\frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!}\frac{1}{\left[\frac{q}{v}+1\right]^v}.
\end{aligned}
$$

2.8. Order Statistics

Let $X_1, X_2, ..., X_n$ be a simple random sample from the TL-OBIII-G family of distributions. Using the binomial expansion

$(1 - F(x))^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F(x)]^m$, the pdf of the ith order statistic can be expressed as

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n! f(x)}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} \\
&= \frac{n! f(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} [F(x)]^{m+i-1}.
\end{aligned}
\tag{14}
$$

Based on equations (3) and (4), we can write

$$
\begin{aligned}
f(x) F(x)^{m+i-1} &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b(m+i)-1} \\
&\times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \\
&\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^2} \\
&= \sum_{j=0}^{\infty} \binom{b(m+i)-1}{j} (-1)^j \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2j+1} \\
&\times 2\alpha\beta b \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^2} \\
&= 2\alpha\beta b \sum_{j,p=0}^{\infty} \binom{b(m+i)-1}{j} \binom{2j+1}{p} (-1)^{j+p} \\
&\times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta(p+1)-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^2} \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(m+i)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \\
&\times (-1)^{j+p+l} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha(l+1)-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi) \right)^2} \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(m+i)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \\
&\times (-1)^{j+p+l} (G(x;\xi))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi) \right)^{\alpha(l+1)-1} g(x;\xi) \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(m+i)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} (-1)^{j+p+l} \\
&\times (1 - (1 - G(x;\xi)))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi) \right)^{\alpha(l+1)-1} g(x;\xi)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b(m+i)-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times \quad (-1)^{j+p+l} \left(\overline{G}(x;\xi)\right)^{k+\alpha(l+1)-1} g(x;\xi) \\
&= 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b(m+i)-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times \quad (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{v+1}{v+1} \left(G(x;\xi)\right)^{v} g(x;\xi) \\
&= \sum_{v=0}^{\infty} b_{v+1}^{*} g_{v+1}(x;\xi),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
b_{v+1}^{*} &= 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b(m+i)-1}{j}\binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times \quad (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{1}{v+1}.
\end{aligned}
$$

Substituting equation (15) into equation (14), we obtain

$$
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{v=0}^{\infty} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} b_{v+1}^{*} g_{v+1}(x;\xi), \tag{16}
$$

where $g_{v+1}(x;\xi) = (v+1)[G(x;\xi)]^{v} g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $v+1 > 0$ and parameter vector ξ. Thus, the density function of the TL-OBIII-G order statistics is a linear combination of E-G densities.

2.9. Probability Weighted Moments (PWMs)

The $(p,r)^{th}$ PWMs of X with TL-OBIII-G distribution denoted $M_{p,r}$ is given by

$$
M_{p,r} = E(X^p (F(X))^r) = \int_{-\infty}^{\infty} x^p (F(x))^r f(x)dx.
$$

Using equations (3) and (4), we can write

$$
\begin{aligned}
f(x)(F(x))^r &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b(r+1)-1} \\
&\times \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \\
&\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j=0}^{\infty} \binom{b(r+1)-1}{j} (-1)^j \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2j+1} \\
&\times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j,p=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} (-1)^{j+p} \\
&\times \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta(p+1)-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} (-1)^{j+p+l} \\
&\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha(l+1)-1} \frac{g(x;\xi)}{\left(\overline{G}(x;\xi)\right)^2} \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} (-1)^{j+p+l} \\
&\times (G(x;\xi))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi)\right)^{\alpha(l+1)-1} g(x;\xi) \\
&= 2\alpha\beta b \sum_{j,p,l=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} (-1)^{j+p+l} \\
&\times (1-(1-G(x;\xi)))^{-\alpha(l+1)-1} \left(\overline{G}(x;\xi)\right)^{\alpha(l+1)-1} g(x;\xi) \\
&= 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times (-1)^{j+p+l} \left(\overline{G}(x;\xi)\right)^{k+\alpha(l+1)-1} g(x;\xi) \\
&= 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{v+1}{v+1} (G(x;\xi))^v g(x;\xi) \\
&= \sum_{v=0}^{\infty} C_{v+1}^* g_{v+1}(x;\xi),
\end{aligned}
\tag{17}
$$

where $g_{v+1}(x; \xi) = (v+1)[G(x; \xi)]^v g(x; \xi)$ is the exponentiated-G (E-G) pdf with the power parameter $v + 1 > 0$ and parameter vector ξ, and

$$
\begin{aligned}
C^*_{v+1} &= 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b(r+1)-1}{j}\binom{2j+1}{p}\frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!}\frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\
&\times \quad (-1)^{j+p+l+k}\binom{k+\alpha(l+1)-1}{v}\frac{1}{v+1}.
\end{aligned}
$$

Consequently, the PWMs of the TL-OBIII-G family of distributions can be written as

$$
M_{p,r} = \int_{-\infty}^{\infty} x^p \sum_{v=0}^{\infty} C^*_{v+1} g_{v+1}(x; \xi)dx = \sum_{v=0}^{\infty} C^*_{v+1}\int_{-\infty}^{\infty} x^p g_{v+1}(x; \xi)dx
$$

which shows that the $(p, r)^{th}$ PWMs of TL-OBIII-G family of distributions can be obtained from the moments of the E-G distribution.

3. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) technique is the commonly used method of parameter estimation among others in the literature. Here, we employ the MLE to estimate model parameters of TL-OBIII-G family of distributions. Let $x_1, x_2, ..., x_n$ be a random sample from TL-OBIII-G family of distributions with the parameter vector $\Delta = (\alpha, \beta, b, \xi)^T$, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ is given by

$$
\begin{aligned}
\ell_n(\Delta) &= n\ln(2\alpha\beta b) + (b-1)\sum_{i=1}^{n}\ln\left[1 - \left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha}\right)^{-\beta}\right)^2\right] \\
&+ \sum_{i=1}^{n}\ln\left(1 - \left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha}\right)^{-\beta}\right) + (-\alpha - 1)\sum_{i=1}^{n}\ln\left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right) \\
&+ (-\beta - 1)\sum_{i=1}^{n}\ln\left(1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^{-\alpha}\right) + \sum_{i=1}^{n}\ln g(x; \xi) - 2\sum_{i=1}^{n}\ln\left(\overline{G}(x; \xi)\right).
\end{aligned}
$$

Elements of the score vector $U(\mathbf{\Delta}) = \left(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \xi_k} \right)$ can be readily obtained. Taking derivative of the log-likelihood function with respect to each component of the parameter vector α, β, b, and ξ, we obtain

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n} \ln \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right) + (-\beta - 1) \sum_{i=1}^{n} \frac{\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \ln \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)}{\left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)} \\
&\quad - \left((b-1) \sum_{i=1}^{n} \frac{2\beta \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1}}{\left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]} \right. \\
&\quad \times \left. \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \ln \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right) \right) \\
&\quad + \sum_{i=1}^{n} \frac{\beta \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \ln \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)}{\left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)}, \\
\frac{\partial \ell_n}{\partial \beta} &= \frac{n}{\beta} + (b-1) \sum_{i=1}^{n} \frac{2 \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \ln \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)}{\left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right]} \\
&\quad - \sum_{i=1}^{n} \ln \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right), \\
\frac{\partial \ell_n}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n} \ln \left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right],
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \xi_k} &= \left((b-1) \sum_{i=1}^{n} \frac{2\alpha\beta \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1}}{\left[1 - \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right)^{2} \right] \left[\overline{G}(x;\xi) \right]^{2}} \right. \\
&\quad \times \left. \left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \frac{\partial g(x_i;\xi)}{\partial \xi_k} \right) \\
&\quad - \sum_{i=1}^{n} \frac{\alpha\beta \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta-1} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha-1} \frac{\partial g(x_i;\xi)}{\partial \xi_k}}{\left(1 - \left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^{-\alpha} \right)^{-\beta} \right) \left[\overline{G}(x;\xi) \right]^{2}}
\end{aligned}
$$

$$+ \quad (-\alpha - 1) \sum_{i=1}^{n} \frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right) \left[\overline{G}(x;\xi)\right]^2}$$

$$+ \quad (\beta + 1) \sum_{i=1}^{n} \frac{\alpha \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha-1} \frac{\partial g(x_i;\xi)}{\partial \xi_k}}{\left(1 + \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{-\alpha}\right) \left[\overline{G}(x;\xi)\right]^2} + \sum_{i=1}^{n} \frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{g(x;\xi)} - 2 \sum_{i=1}^{n} \frac{\frac{\partial \overline{G}(x;\xi)}{\partial \xi_k}}{(\overline{G}(x;\xi))}.$$

$$(19)$$

The maximum likelihood estimates (MLEs), say $\hat{\mathbf{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\xi})$ can be obtained by equating the system of non linear equations $\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \xi_k}$ to zero and solving simultaneously. These equations cannot be solved analytically as they are not in closed form. Thus, numerical methods such as Newton-Raphson procedure can be used to solve them numerically. We maximize the likelihood function using the function nlm in R ([30]).The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. We also obtain the goodness-of-fit statistics: Cramér-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by Chen and Balakrishnan [13], as well as Kolmogorov-Smirnov (KS) statistic and its p-value. Note that for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and KS, smaller values are preferred.

4. Simulation Study

The performance of the TL-OBIII-LLoG distribution when $\beta = 1$ is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400) via the R package. Simulation results for other parameter values are available on request from the authors. We simulate $N = 1000$ samples for the true parameters values given in Table 7. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 7. **Monte Carlo Simulation Results: Mean, Average Bias and RMSE**

Parameter	n	I (1.2 ,1.0 ,2.1.4)			II (1.0, 2.0, 2.4)			III (1.0, 3.0, 2.0)		
		Mean	Average Bias	RMSE	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	1.2151	0.0151	0.1254	1.0577	0.0577	0.1700	1.0383	0.0383	0.1646
	50	1.2105	0.0105	0.0930	1.0347	0.0347	0.1065	1.0377	0.0377	0.1095
	100	1.2056	0.0056	0.0649	1.0216	0.0216	0.0745	1.01595	0.0159	0.0690
	200	1.1973	-0.0027	0.0469	1.0096	0.0096	0.0506	1.0121	0.0121	0.0470
	400	1.2017	0.0017	0.0357	1.0037	0.0037	0.0339	1.0085	0.0085	0.0380
b	25	1.0465	0.0465	0.1831	2.0189	0.0189	0.3054	3.1034	0.1034	0.7515
	50	1.0232	0.0232	0.1410	2.0202	0.0202	0.2077	3.0286	0.0286	0.4635
	100	1.0163	0.0163	0.1008	2.0030	0.0030	0.1401	2.9788	-0.0212	0.2998
	200	1.0151	0.0152	0.0668	1.9979	-0.0021	0.0978	3.0087	0.0087	0.2200
	400	1.0035	0.0035	0.0489	2.0033	0.0033	0.0757	3.0031	0.0031	0.1555
λ	25	1.4122	0.0122	0.1120	2.4087	0.0086	0.0617	2.0148	0.0148	0.0920
	50	1.4079	0.0079	0.0818	2.4052	0.0052	0.0421	2.0190	0.0190	0.0739
	100	1.4035	0.0035	0.0576	2.4023	0.0024	0.0322	1.9979	-0.0020	0.0533
	200	1.3969	-0.0031	0.0405	2.3994	-0.0006	0.0234	1.9853	-0.0147	0.0423
	400	1.4009	0.0009	0.0306	2.3991	-0.0009	0.0159	1.9891	-0.0109	0.0359

From the results, we can clearly verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

5. Applications

The TL-OBIII-LLoG distribution is fitted to two real data sets and compared to fits of several non-nested four parameter distributions. The TL-OBIII-LLoG distribution is compared to the new modified Weibull (NMW) distribution introduced by Doostmoradi et al. [15], beta generalized exponential (BGE) distribution introduced by Barreto-Souza et al. [6], beta generalized Lindley (BGL) distribution by Oluyede and Yang [29], exponentiated modified Weibull distribution by Elbatal [16], Weibull Lomax (WL) distribution by Tahir et al. [34], generalized Weibull log-logistic (GWLLoG) distribution by Cordeiro et al. [11] and Burr III Marshall Olkin-Lindley (BIIIMO-L) distribution by Bhatti et al. [7]. The pdf's of the four parameter NMW, BGE, BGL, EMW, WL, GWLLoG, BIIIMO-L distributions are given in equations (20), (21), (22), (23), (24), (25) and (26) respectively, that is,

$$g_{NMW}(x) = \left(\alpha\gamma x^{\gamma-1}e^{\alpha x^\gamma} + \lambda\beta x^{\lambda-1}e^{-\beta x^\lambda}\right)e^{-e^{\alpha x^\gamma}+e^{-\beta x^\lambda}}, \quad x > 0, \tag{20}$$

$$g_{BGE}(x) = \frac{\alpha\lambda}{B(a,b)}e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha a-1}\left(1 - \left(1 - e^{-\lambda x}\right)^\alpha\right)^{b-1}, \quad x > 0, \tag{21}$$

$$\begin{aligned}
g_{BGL}(x) &= \frac{\alpha\lambda^2}{B(a,b)(1+\lambda)}(1+x)e^{-\lambda x}\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{a\alpha-1} \\
&\times \left[1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right)^a\right]^{b-1}, \quad x > 0,
\end{aligned} \tag{22}$$

$$g_{EMW}(x) = \gamma\left[\delta + \lambda\theta^\alpha x^{\lambda-1}\right]e^{-(\delta x+(\theta x)^\lambda)}\left[1 - e^{-(\delta x+(\theta x)^\lambda)}\right]^{\delta-1}, \quad x > 0, \tag{23}$$

$$g_{WL}(x) = \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\alpha-1}\left(1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right)^{b-1}$$

$$\times \quad \exp\left(-a\left(\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right)^{b}\right), \quad x>0, \tag{24}$$

$$g_{GWLLoG}(x) = \frac{\alpha\beta\theta x^{\theta-1}}{a^{\theta}}\exp\left(-\alpha\left[\log\left(1+\left(\frac{x}{a}\right)^{\theta}\right)\right]^{\beta}\right)$$

$$\times \quad \left(1+\left(\frac{x}{a}\right)^{\theta}\right)^{-1}\left[\log\left(1+\left(\frac{x}{a}\right)^{\theta}\right)\right]^{\beta-1}, \quad x>0, \tag{25}$$

and

$$g_{BIIIMO-L}(x) = \left(-\log\left[\frac{\lambda\left(1+\frac{bx}{1+b}\right)\exp(-bx)}{1-\bar{\lambda}\left(1+\frac{bx}{1+b}\right)\exp(-bx)}\right]\right)^{-\beta-1}$$

$$\times \quad \left(1+\left(-\log\left[\frac{\lambda\left(1+\frac{bx}{1+b}\right)\exp(-bx)}{1-\bar{\lambda}\left(1+\frac{bx}{1+b}\right)\exp(-bx)}\right]\right)^{-\beta}\right)^{-\alpha-1}, \quad x>0.$$

$$\times \quad \frac{\frac{\alpha\beta b^2(1+x)\exp(-bx)}{1+b}}{\left(1+\frac{bx}{1+b}\right)\exp(-bx)}, \quad x>0. \tag{26}$$

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [12]) are given in Figure 4 and Figure 5 for the two data sets considered in this section. For the probability plot, we plotted $F_{TL-OBIII-LLoG}(x_{(j)};\hat{\alpha},\hat{\beta},\hat{b},\hat{\lambda})$ against $\frac{j-0.375}{n+0.25}$, $j=1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares (SS) as

$$SS = \sum_{j=1}^{n}\left[F_{TL-OBIII-LLoG}(x_{(j)};\hat{\alpha},\hat{\beta},\hat{b},\hat{\lambda})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [13] as well as Kolmogorov-Smirnov (KS) statistic, its P-value and SS are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^* and K-S, the better the fit.

5.1. Growth Hormone Data

The first data consists of the estimated time since growth hormone medication until the children reached the target age. The data was used by Alizadeh et al. [4] to show the superiority of the exponentiated power Lindley power series (EPLPS) class of distributions distributions. The data are:
2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.
Estimates of the parameters of TL-OBIII-LLoG distribution and the non-nested models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W*, A*, KS and its p-value as well as SS are given in Table 8. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 4.

Table 8. Estimates of Models for Growth Hormone Data

Model		Estimates						Statistics					
	α	β	b	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
TL-OBIII-LLoG	0.0710	0.5180	168.7000	19.0390	155.4418	163.4418	164.7751	169.6632	0.0389	0.2685	0.0873	0.9522	0.0389
	(8.4675×10^{-03})	(9.9284×10^{-02})	(1.6615×10^{-04})	(3.3992×10^{-05})									
	α	λ	a	b									
BGE	0.7487	0.0323	6.3824	19.9198	159.7534	167.7534	169.0867	173.9748	0.0944	0.6166	0.1184	0.7097	0.0907
	(0.5541)	(0.0111)	(7.1232)	(0.5568)									
	α	γ	λ	β									
NMW	0.0006	2.8548	4.0927	0.0018	168.0796	176.0796	177.413	182.301	0.0551	0.3873	0.1618	0.3187	0.1439
	(0.0014)	(0.8086)	(0.7964)	(0.0021)									
	α	θ	a	b									
BGL	0.1179	1.1290×10^{-07}	3.0140×10^{-01}	1.0100×10^{01}	342.7928	350.8082	352.1415	357.0296	0.0685	0.4632	0.4833	1.579×10^{-07}	2.6551
	(9.1449×10^{-03})	(2.4083×10^{-06})	(8.1743×10^{-04})	(2.6267×10^{-05})									
	γ	δ	λ	θ									
EMW	6.5130	0.4820	17.0010	1.0000×10^{-04}	158.1951	166.1951	167.5284	172.4165	0.0703	0.4747	0.1012	0.8656	0.0754
	(2.2693)	(0.07981)	(1.7949×10^{-16})	(3.0495×10^{-11})									
	a	b	α	β									
WL	0.0002	4.3897	0.3987	0.0380	164.004	172.0040	173.3373	178.2254	0.1508	0.9496	0.1504	0.4069	0.1533
	(0.0003)	(0.0111)	(0.0653)	(0.0507)									
	α	β	a	θ									
GWLLoG	0.2962	2.7285	2.3479	1.4831	159.4803	167.4803	168.8137	173.7017	0.0863	0.5686	0.1158	0.7352	0.0829
	(0.5048)	(1.3142)	(1.6100)	(0.7834)									
	α	β	λ	b									
BIIIMO-L	0.1177	0.1109	0.1097	5.8707×10^{-06}	556.1171	564.1172	565.4505	570.3386	0.0723	0.4860	0.8985	2.2000×10^{-16}	8.8178
	(0.0205)	(0.0168)	(0.0185)	(3.3596×10^{-05})									

The values in the table above showed that the TL-OBIII-LLoG distribution has the smallest values of AIC, AICC, BIC, W*, A*, KS and the largest p-value compared to the corresponding values for the fitted non-nested models. So, we conclude that the TL-OBIII-LLoG distribution could be chosen as the best fittting model for the growth hormone data.
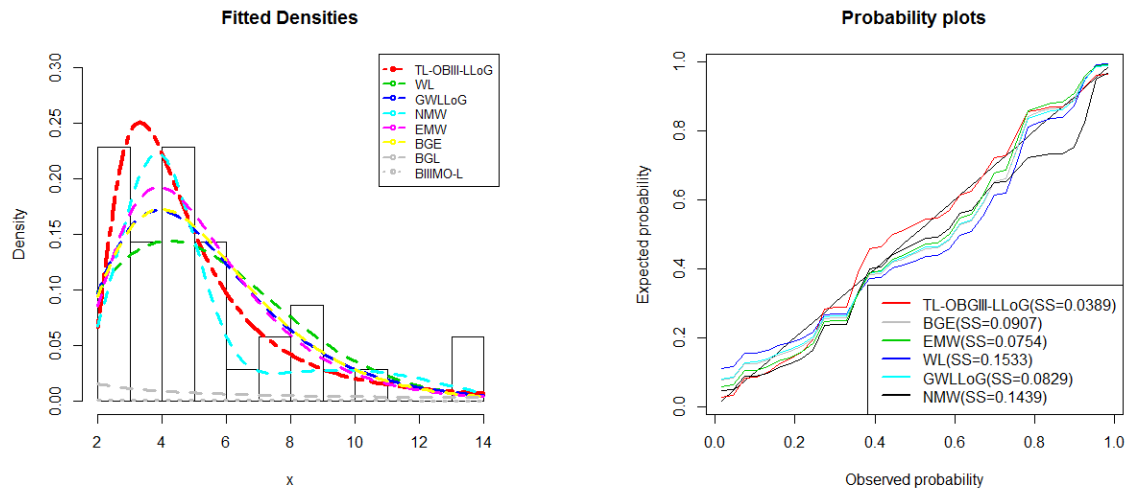

Figure 4. Fitted Densities and Probability Plots of the Growth Hormone Data

5.2. *Repair Lifetimes of an Airborne Transceiver Data*

The second data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver with size n=46 from Leiva et al. [23] and Chhikara and Folks [14]. These data are:
0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.
Estimates of the parameters of TL-OBIII-LLoG distribution and non-nested models (standard error in parentheses),

AIC, BIC, and the goodness-of-fit statistics W*, A*, KS and its p-value as well as SS are given in Table 9. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 5.

Table 9. Estimates of Models for Repair Lifetimes of an Airborne Transceiver Data

Model	Estimates				Statistics								
	α	β	b	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
TL-OBIII-LLoG	0.4601	0.1392	144.0183	1.5403	199.9626	207.9626	208.9382	215.2772	0.0515	0.3127	0.0762	0.9520	0.0508
	(15.4843)	(0.0108)	(0.0043)	(51.8267)									
	α	λ	a	b									
BGE	10.9759	1.2799	0.1848	0.1855	201.8082	209.8083	210.7839	217.1229	0.0563	0.4558	0.1109	0.6228	0.0890
	(20.9289)	(0.7340)	(0.3193)	(0.0620)									
	α	γ	λ	β									
NMW	0.0377	1.0873	1.3624	0.2518	218.5555	226.5555	227.5311	233.8701	0.0681	0.4514	0.1604	0.1870	0.3884
	(0.0333)	(0.2863)	(0.2228)	(0.0803)									
	α	θ	a	b									
BGL	0. 1179	1.3719×10^{-07}	0.3014	10.1000	382.0483	390.1551	391.1307	397.4696	0.0837	0.5519	0.4630	5.4390×10^{-09}	3.2424
	(8.1272×10^{-03})	(2.4697×10^{-06})	(7.7978×10^{-04})	(2.2593×10^{-05})									
	γ	δ	λ	θ									
EMW	0.9582	0.2693	17.0010	1.0000×10^{-04}	209.9658	217.9658	218.9414	225.2804	0.1441	1.0004	0.1519	0.2385	0.1889
	(0.1897)	(0.0543)	(3.4209×10^{-18})	(5.8159×10^{-13})									
	a	b	α	β									
WL	1.6956e+03	2.5654	0.0198	0.2078	200.603	208.603	209.5786	215.9176	0.0563	0.3634	0.0937	0.8132	0.0509
	(7.6319e-05)	(1.3635)	(0.0158)	(0.4281)									
	α	β	a	θ									
GWLLoG	8.2995	0.1697	35.4344	5.3096	208.9809	216.9809	217.9565	224.2954	0.1295	0.8973	0.1199	0.5218	0.1147
	(1.8675)	(0.1866)	(1.2987)	(5.7670)									
	α	β	λ	b									
BIIIMO-L	0.1184	0. 1121	0.1107	1.1385×10^{-05}	659.9823	667.9820	668.9577	675.2966	0.0902	0.6006	0.8711	2.2000×10^{-16}	11.4417
	(0.0187)	(0.0175)	(0.0163)	(1.8930×10^{-05})									

In Table 9 the values of AIC, AICC, BIC, W*, A*, KS for TL-OBIII-LLoG distribution are smaller than those for fitted non-nested distributions, which indicates that the TL-OBIII-LLoG distribution provides a better fit for the repair lifetimes of an airborne transceiver data. Also, we can conclude that the TL-OBIII-LLoG distribution is the better fit from the p-value of the KS statistic as it is large for TL-OBIII-LLoG distribution as compared to those from the other non-nested distributions for the repair lifetimes of an airborne transceiver data.
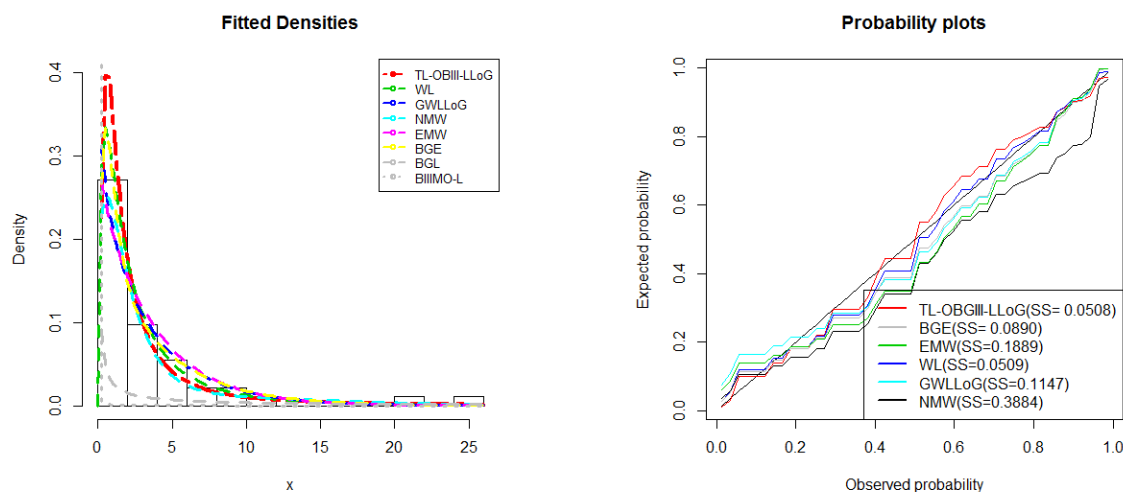

Figure 5. Fitted Densities and Probability Plots of the Repair Lifetimes of an Airborne Transceiver Data

6. Concluding Remarks

We have developed a new family of distributions called the Topp-Leone Odd Burr III-G (TL-OBIII-G) family of distributions. Properties of this new distribution such as sub-models, quantile function, moments, incomplete moments, probability weighted moments, order statistics and Rényi entropy were studied. The maximum likelihood estimates (MLEs) have been computed via the maximum likelihood estimation method. Monte Carlo simulation study was employed to check the perfomance of the MLEs. Lastly, we illustrated that the TL-OBIII-G family of distributions is useful for lifetime applications by fitting its special case of TL-OBIII-LLoG distribution to two real data sets.

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