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Cumulative Residual Extropy for Pareto Distribution in the Presence of Outliers: Bayesian and Non-Bayesian Methods

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Abstract The extropy is considered to be a complementary dual of the well-known Shannon's entropy and has wide applications in many fields. This article discusses estimating the extropy and cumulative residual extropy of the Pareto distribution using the maximum likelihood and Bayesian methods. We obtain the maximum likelihood of extropies measures in presence of outliers. These estimators are specialized to homogenous case (no-outliers). The Bayesian estimators of both extropy measures are derived based on symmetric and asymmetric loss functions. The Markov chain Monte Carlo methods are used to accomplish some complex calculations. The precision of the Bayesian and the maximum likelihood estimates for both extropy are examined through simulations. Regarding results of simulation study, we conclude that the performances of both estimation methods improve with sample sizes. Also, Bayesian estimates of the extropy and cumulative residual extropy under linear exponential loss function are superior to the Bayesian estimates under the other loss functions in most of cases. As the exact values of extropy and cumulative residual extropy decreased, the mean squared errors and the absolute biases of maximum likelihood and Bayesian estimates of extropy and cumulative residual extropy decreased with number of outliers. The performance for the extropy and cumulative residual extropy estimates increase with number of outliers in almost cases. Generally, there is a great agreement between the theoretical and empirical results. Further performance comparison is conducted by the experiments with real data.

Keywords Pareto distribution; outliers; extropy; cumulative residual extropy; and Metropolis-Hastings algorithm.

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1. Introduction

Pareto distribution is a well-known distribution used to model heavy tailed phenomena. It has many applications in actuarial science, insurance risk, business failures, life testing, hydrology, finance, telecommunication, reliability analysis, physics and engineering. The probability density function (PDF) and survival function (SF) of the Pareto distribution with scale parameter λ and shape parameter γ are given, respectively, by

$$f(x;\gamma,\lambda) = \gamma\,\lambda^{\gamma}\,x^{-(\gamma+1)}\ ;\ \lambda < x < \infty\,,\ \gamma,\lambda > 0\,, \tag{1}$$

and

$$\bar{F}(x;\gamma,\lambda) = \left(\frac{\lambda}{x}\right)^{\lambda}. \tag{2}$$

The estimation of the parameters of the Pareto distribution have been discussed in [1] . Comparisons of methods of estimation for a Pareto distribution of the first kind have been considered in [2]. [3] considered the Bayesian

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survival estimation of the Pareto distribution of the second kind on failure censored data. [4] introduced Bayesian inference for the Pareto lifetime model under progressive censoring with binomial removals. [5] discussed the efficient estimation of the parameters of the Pareto distribution in the presence of outliers. Bayesian inference for the Pareto lifetime model in the presence of outliers under progressive censoring with binomial removals have been discussed in [6].
According to [5], let a set of random variables $X_1, X_2, \ldots, X_n$ exemplify the claim amounts of a motor insurance company. It is considered that claims of some of vehicles (expensive/severely damaged vehicle) are β times higher than normal vehicles. Hence, they assumed that the random variables $(X_1, X_2, \ldots, X_n)$ are such that any k of them (the number of outliers) are distributed as Pareto distribution with the following PDF

$$g(x; \Theta) = \gamma\, (\beta\lambda)^{\gamma}\, x^{-(\gamma+1)}, \;\; \beta\lambda < x < \infty, \;\; \gamma > 0, \;\; \beta > 1, \;\; \lambda > 0, \tag{3}$$

where, λ, β are the scale parameters, γ is the shape parameter, $\Theta \equiv (\gamma, \beta, \lambda)$ is the set of parameters and the remaining $(n-k)$ random variables are distributed as Pareto distribution with parameters γ and λ having PDF (1). The corresponding SF is given by

$$\bar{G}(x; \Theta) = \left(\frac{\beta\lambda}{x}\right)^{\gamma}. \tag{4}$$

The joint distribution of $X_1, X_2, \ldots, X_n$ in the presence of k outliers can be obtained as

$$f(\underline{x}; \Theta) = \frac{\gamma^n \lambda^{n\gamma} \beta^{k\gamma}}{C(n,k)} \prod_{i=1}^{n} x_i^{-(\gamma+1)} \sum_{\underline{A}} \prod_{j=1}^{k} I(x_{A_j} - \beta\lambda), \tag{5}$$

where, $C(n,k) = \frac{n!}{k!(n-k)!}$, $\sum_{\underline{A}} = \Sigma_{A_1=1}^{n-k+1} \Sigma_{A_2=A_1+1}^{n-k+2} \ldots \Sigma_{A_k=A_{k-1}+1}^{n}$, and $I(.)$ represents the indicator function defined as:

$$I(x) = \begin{cases} 1, & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the marginal distribution of the Pareto distribution in the presence of k outliers is expressed as:

$$\begin{aligned} f(x; \Theta) &= \frac{k}{n} g(x; \Theta) + \frac{n-k}{n} f(x; \gamma, \lambda) \\ &= \gamma \lambda^{\gamma} \left(b\beta^{\gamma} + \bar{b}\right) x^{-(\gamma+1)} \end{aligned} \tag{6}$$

The SF is given by:

$$\begin{aligned} \bar{F}(x; \Theta) &= \frac{k}{n} \bar{G}(x; \Theta) + \frac{n-k}{n} \bar{F}(x; \gamma, \lambda) \\ &= \left(\frac{\lambda}{x}\right)^{\gamma} \left(b\beta^{\gamma} + \bar{b}\right), \end{aligned} \tag{7}$$

where, $b = k/n$, $\bar{b} = (1-b) = (n-k)/n$.

Measurement of uncertainty associated with a random variable has been of great interest at all times. [7] proposed a measure of uncertainty associated with a probability distribution called Shannon entropy. Shannon entropy plays a significant role in numerous fields such as financial analysis, data compression, molecular biology, hydrology, meteorology, computer science and information theory. Despite the massive success of Shannon's entropy, this measure has some defects and may not be adequate in every situation. To overcome these defects an alternate measure of uncertainty called extropy has been proposed in the literature that expands Shannon's entropy. As is entropy, the extropy is explicated as a measure of the amount of uncertainty exemplified by X. However, the two measures are various and essentially intertwined with each other. The detection of extropy was stimulated by a problem that manifests in the application of the theory of proper scoring rules for alternate forecast distributions. [8] was introduced the complementary dual of entropy as an alternative measure of uncertainty, called extropy. Entropy and extropy measures relate as the positive and negative images of a photographic film relate to each other. Let X be a non-negative random variable with PDF $f(x)$, then the extropy of X is defined as:

$$J(X) = \frac{-1}{2} \int_0^\infty f^2(x)\ dx\,. \tag{8}$$

Several properties of this new information measure such as the maximum extropy distribution and its statistical applications were studied in [8]. One statistical application of extropy is to score the forecasting distributions. For example, under the total log scoring rule, the expected score of a forecasting distribution equals the negative sum of the entropy and extropy of this distribution provided in [9]. In commercial or scientific areas such as astronomical measurements of heat distributions in galaxies, the extropy has been universally investigated in [10] and [11].

Few works of estimating the extropy have studied and discussed. For example, [12] handled with the estimation of extropy using two estimators and developed a test using extropy. Two estimators for extropy were introduced and a goodness of test for standard uniform distribution was developed in [13]. Some estimators of extropy of a continuous random variable were introduced in [14].

Lately, a cumulative residual extropy (CREx) which is analogous to cumulative residual entropy was proposed in [15], as follows:

$$\Xi(X) = \frac{-1}{2} \int_0^\infty \bar{F}^2(x)\ dx\,. \tag{9}$$

Also, CREx have successfully applied in risk measurement using the Pareto distribution as an example in [15]. From the previous literatures, the Pareto distribution arrest attention from theoretical and statisticians essentially due to its applicability in numerous areas. In the literature, there is no work that has been done about the estimation problem of extropy of Pareto distribution in the presence of outliers. So, our motivation here is to consider the maximum likelihood (ML) and Bayesian estimation methods of extropy and CREx for Pareto distribution in the presence of outliers. The considered loss functions are squared error loss function (SELF), linear exponential loss function (LLF), minimum expected loss function (MLF), and Degroot loss function (DLF). Markov Chain Monte Carlo (MCMC) technique using the Metropolis-Hastings (M-H) algorithm is used due to the complicated forms of extropy and CREx Bayesian estimators. Further, application to real data is employed.

The form of the article is as follows. Expressions of the extropy and CREx for Pareto distribution in the presence of outliers and the homogenous case are derived in Section 2. Section 3 gives the ML and Bayesian estimators of extropy and CREx in the presence of outliers and the homogenous case. The accuracy and precision of both extropy estimates are examined with simulation results and data analysis in Section 4. The paper ends with discussion and summary.

2. Expressions of the Extropy and Cumulative Residual Extropy

In this section, outright expressions for the extropy and CREx for the Pareto distribution in the presence of outliers are derived. Assume X be a random variable following the Pareto distribution, hence the extropy of X is given by substituting (6) in (8) as follows:

$$J(X) = \frac{-1}{2} \int_{D_x} (\gamma\lambda^\gamma)^2 \left(b\beta^\gamma + \bar{b}\right)^2 x^{-2(\gamma+1)}\ dx = \frac{-1}{2} Q. \tag{10}$$

For the purpose of computing extropy, it must be to obtain Q, as follows:

$$Q = (\gamma\lambda^{\gamma})^2(b\beta^{\gamma}+\bar{b})\left[b\beta^{\gamma}\int_{\beta\lambda}^{\infty} x^{-2(\gamma+1)}\ dx + \bar{b}\int_{\lambda}^{\infty} x^{-2(\gamma+1)}\ dx\right]$$
$$= \frac{\gamma^2\lambda^{-1}(b\beta^{\gamma}+\bar{b})}{2\gamma+1}\left[b\beta^{-(\gamma+1)}+\bar{b}\right]. \quad (11)$$

Hence, the extropy of Pareto distribution in the presence of outliers is given by substituting (11) in (10) as follows:

$$J(x) = \frac{-\gamma^2\lambda^{-1}}{2(2\gamma+1)}\left(\frac{b^2}{\beta}+\bar{b}^2+b\bar{b}\beta^{\gamma}+b\bar{b}\beta^{-(\gamma+1)}\right). \quad (12)$$

By the same way, the CREx is obtained by substituting (7) in (9) as follows:

$$\Xi(x) = \frac{-\lambda}{2(2\gamma-1)}\left(b^2\beta+\bar{b}^2+b\bar{b}\beta^{-\gamma+1}+b\bar{b}\beta^{\gamma}\right),\ \gamma > 0.5. \quad (13)$$

Expressions (12) and (13) of extropy and CREx are functions of parameters. For $\beta = 1$ or $k = 0$ in (12) and (13), the required expressions of extropy and CREx of the Pareto distribution in homogenous case (no-outliers) are provided.

3. Estimation of Extropies in the Presence of Outliers

This section handled with the ML and Bayesian estimators of extropy and CREx measures from Pareto distribution in the presence of outliers. Further, the theoretical results are specialized to the homogenous case (i.e., $k = 0$ or $\beta = 1$).

3.1. Estimation of Extropies in the Presence of Outliers

The ML method is widely used due to its desirable properties including consistency, asymptotic efficiency and invariance property. Here, we derive the ML estimators of the extropy and CREx extropy for the Pareto distribution in the presence of k outliers.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the Pareto distribution in the presence of k outliers with PDF (5). The log-likelihood function, denoted by $\ln l$, from a sample of n observations $X_1, X_2, \ldots, X_n$ is given by

$$\ln l = n\ln\gamma + n\gamma\ln\lambda + k\gamma\ln\beta - \ln\left[C(n,k)\right] - (\gamma+1)\Sigma_{i=1}^{n}\ln x_i + \ln\left[\sum_{\underline{A}}\prod_{j=1}^{k} I(x_{A_j}-\beta\lambda)\right].$$

Assuming that the parameter λ is known, also, it is known that the estimate of $\beta\lambda$ is the sample minimum, i.e $\beta\lambda$ is $X_{(1)} = min(X_1, X_2, ..., X_n)$, so,

$$\bar{\beta} = X_{(1)}/\lambda. \quad (14)$$

The partial derivative of the log-likelihood function with respect to γ can be obtained as

$$\frac{\partial \ln l}{\partial\gamma} = \frac{n}{\gamma} + n\ln(\lambda) + k\ln(\beta) - \Sigma_{i=1}^{n}\ln x_i. \quad (15)$$

Substituting (14) in (15) and equating by zero, then, the ML estimator of γ, denoted by $\bar{\gamma}$ is given by

$$\bar{\gamma} = \frac{n}{\Sigma_{i=1}^{n}\ln x_i - k\ln(X_{(1)}) - (n-k)\ln(\lambda)}. \quad (16)$$

Based on invariance property of ML method, the ML estimators of $J(X)$ and $\Xi(X)$, denoted by $\bar{J}(X)$ and $\bar{\Xi}(X)$ are obtained by directly substituting (14) and (16) in (12) and (13), respectively. Furthermore, the ML estimators of $J(X)$ and $\Xi(X)$ are obtained in case of homogenous case by taking $\beta = 1$ or $k = 0$.

3.2. *Extropy and CREx Bayesian estimators*

Here, Bayesian estimators of $J(X)$ and $\Xi(X)$ for the Pareto distribution in the presence of outliers are derived based on SELF, LLF, MLF and DLF. Also, the theoretical results of Bayesian estimators of $J(X)$ and $\Xi(X)$ in the presence of outliers are specialized to the homogenous case (i.e., $k = 0$ or $\beta = 1$).
To compute the Bayesian estimators of $J(X)$ and $\Xi(X)$, we must obtain firstly the Bayesian estimators of λ, β and γ. The Bayesian estimators cannot be obtained in explicit forms. So, the MCMC technique based on M-H algorithm is used to generate samples from the posterior distributions and consequently computing the Bayesian estimators. Consider a random sample of size n from the Pareto distribution with set of parameters Θ, where their likelihood function is defined in (5). Following [6], the joint conjugate prior for (γ, λ) has the following power gamma prior

$$T_1(\gamma,\lambda) = \frac{\delta}{\Gamma(\nu)}(\ln(\mu) - \delta\ln(\rho))^{\nu}\ \lambda^{\delta\gamma-1}\ \gamma^{\nu}\ \mu^{-\gamma}\ ; \gamma > 0\ ,\ 0 < \lambda < \rho\ ,\ \delta, \mu, \rho > 0\ ,\ 0 < \rho^{\delta} < \mu.$$

Also, they assumed the following prior density function for parameter β is given as

$$T_2(\beta) = \frac{1}{\beta\ln(d)}\ ;\ 1 < \beta < d\ ,\ d > 1.$$

Hence, the independence of parameters are considered, then the joint prior distribution for γ, λ and β is given by

$$\pi(\Theta) = \frac{\delta}{\Gamma(\nu)\ln(d)}(\ln(\mu) - \delta\ln(\rho))^{\nu}\ \lambda^{\delta\gamma-1}\ \gamma^{\nu}\ \mu^{-\gamma}\ \beta^{-1}\ ; \gamma > 0\ , 0 < \lambda < \rho\ , 1 < \beta < d\ , d > 1\ \delta, \mu, \rho > 0\ ,\ 0 < \rho^{\delta} < \mu. \tag{17}$$

Combining (5) and (17), the joint posterior distribution can be written as:

$$\pi^*(\Theta|\,\underline{x}) \propto \lambda^{\gamma(\delta+n)-1} \exp\left[-\gamma\left(\Sigma_{i=1}^{n}\ln x_i + \ln\mu - k\ln\beta\right) - \ln\beta - \Sigma_{i=1}^{n}\ln x_i + (\nu+n)\ln\gamma\right]. \tag{18}$$

Hence, the marginal posterior distributions of γ, β and λ take the following forms, respectively,

$$\pi_1^{**}(\gamma|\,\underline{x}) = \triangle^{-1} e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i + \ln\mu) - \Sigma_{i=1}^{n}\ln x_i + (\nu+n)\ln\gamma]} \int_0^{\rho}\int_1^{d} \lambda^{\gamma(\delta+n)-1}\beta^{k\gamma-1}\ d\beta\, d\lambda\,, \tag{19}$$

$$\pi_2^{**}(\beta|\,\underline{x}) = \triangle^{-1} e^{[-\ln\beta - \Sigma_{i=1}^{n}\ln x_i]} \int_0^{\rho}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1} e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i + \ln\mu - k\ln\beta) + (\nu+n)\ln\gamma]}\ d\gamma\, d\lambda\,, \tag{20}$$

and

$$\pi_3^{**}(\lambda|\,\underline{x}) = \triangle^{-1}\lambda^{-1} e^{[-\Sigma_{i=1}^{n}\ln x_i]} \int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)} e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i + \ln\mu - k\ln\beta) - \ln\beta + (\nu+n)\ln\gamma]}\ d\gamma\, d\beta\,. \tag{21}$$

where $\triangle = \int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1}\ e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i + \ln\mu - k\ln\beta) - \ln\beta - \Sigma_{i=1}^{n}\ln x_i + (\nu+n)\ln\gamma]}\ d\gamma d\beta d\lambda$, is the normalizing constant.

(i) Bayesian Estimator under SELF

A quadratic or SELF is one of the useful symmetric loss functions in nature; i.e. it gives equal importance to both over and under estimation. The SELF is defined as

$$l_{SELF}\left(\widehat{\vartheta}, \vartheta\right) = \left(\widehat{\vartheta} - \vartheta\right)^2.$$

Therefore, the Bayesian estimators of γ, β and λ for Pareto distribution in presence of outliers under SELF, say $\widehat{\gamma}_{SELF}, \widehat{\beta}_{SELF}$ and $\widehat{\lambda}_{SELF}$ are obtained as a posterior mean as follows

$$\widehat{\gamma}_{SELF} = E(\gamma|\underline{x}) = \triangle^{-1} \int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1} e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\ln\beta-\Sigma_{i=1}^{n}\ln x_i+(\nu+n+1)\ln\gamma]}\, d\gamma\, d\beta\, d\lambda, \tag{22}$$

$$\widehat{\beta}_{SELF} = E(\beta|\underline{x}) = \triangle^{-1} \int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1} e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\Sigma_{i=1}^{n}\ln x_i+(\nu+n)\ln\gamma]}\, d\gamma\, d\beta\, d\lambda, \tag{23}$$

$$\widehat{\lambda}_{SELF} = E(\lambda|\underline{x}) = \triangle^{-1} \int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)}\, e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\ln\beta-\Sigma_{i=1}^{n}\ln x_i+(\nu+n)\ln\gamma]}\, d\gamma\, d\beta\, d\lambda. \tag{24}$$

(ii) Bayesian Estimator under LLF

The LLF is asymmetric loss function proposed in [16] . The LLF with parameters τ and h is given by

$$l_{LLF}\left(\widehat{\vartheta},\vartheta\right) = \lambda\left[e^{\tau\left(\widehat{\vartheta}-\vartheta\right)} - \tau\left(\widehat{\vartheta}-\vartheta\right) - 1\right],$$

where τ and h are constants. The sign and magnitude of τ represent the direction and degree of symmetry, respectively $\tau > 0$ means overestimation is more serious than underestimation, and $\tau < 0$ means the opposite). The Bayesian estimators of γ, β and λ for Pareto distribution in presence of outliers under LLF, say $\widehat{\gamma}_{LLF}, \widehat{\beta}_{LLF}$ and $\widehat{\lambda}_{LLF}$ are obtained as follows:

$$\begin{aligned}\widehat{\gamma}_{LLF} &= \frac{-1}{\tau}E[e^{-\tau\gamma}] = \frac{-1}{\tau}\ln\left[\int_0^{\infty} e^{-\tau\gamma}\ \pi_1^{**}(\gamma|\underline{x})\ d\gamma\right]\\ &= \frac{-1}{\tau}\ln\left[\triangle^{-1}\int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1} e^{[-\gamma(\tau+\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\ln\beta-\Sigma_{i=1}^{n}\ln x_i+(\nu+n)\ln\gamma]} d\gamma d\beta d\lambda\right]\end{aligned} \tag{25}$$

$$\widehat{\beta}_{LLF} = \frac{-1}{\tau}\ln\left[\triangle^{-1}\int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1}\ e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\tau\beta-\ln\beta-\Sigma_{i=1}^{n}\ln x_i+(\nu+n)\ln\gamma]}\ d\gamma\, d\beta\, d\lambda\right], \tag{26}$$

$$\widehat{\lambda}_{LLF} = \frac{-1}{\tau}\ln\left[\triangle^{-1}\int_0^{\rho}\int_1^{d}\int_0^{\infty} \lambda^{\gamma(\delta+n)-1}\ e^{[-\gamma(\Sigma_{i=1}^{n}\ln x_i+\ln\mu-k\ln\beta)-\ln\beta-\Sigma_{i=1}^{n}\ln x_i+(\nu+n)\ln\gamma-\tau\lambda]}\ d\gamma\, d\beta\, d\lambda\right]. \tag{27}$$

where, τ is real number.

(iii) Bayesian Estimator under MLF

Minimum expected loss function was developed in [17] which defined by:

$$l_{MLF}\left(\widehat{\vartheta},\vartheta\right) = \frac{\left(\widehat{\vartheta}-\vartheta\right)^2}{\vartheta^2},$$

where, $\widehat{\vartheta}_{MLF}$ is an estimator of ϑ. Hence, the Bayesian estimators of γ, β and λ for Pareto distribution in presence of outliers under MLF, say $\widehat{\gamma}_{MLF}, \widehat{\beta}_{MLF}$ and $\widehat{\lambda}_{MLF}$, are derived as follows:

$$\begin{aligned}\widehat{\gamma}_{MLF} &= \frac{E(\gamma^{-1}|\,\underline{x})}{E(\gamma^{-2}|\,\underline{x})} = \left[\frac{\int_0^\infty \ \gamma^{-1} \ \pi_1^{**}(\gamma|\,\underline{x}) \ d\gamma}{\int_0^\infty \ \gamma^{-2} \ \pi_1^{**}(\gamma|\,\underline{x}) \ d\gamma}\right] \\ &= \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n-1)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n-2)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right], \end{aligned} \tag{28}$$

$$\widehat{\beta}_{MLF} = \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) - 2\ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - 3\ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right], \tag{29}$$

$$\widehat{\lambda}_{MLF} = \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-2} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-3} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right]. \tag{30}$$

(iv) Bayesian Estimator under DLF

The DLF was provided in [18] as follows:

$$l_{DLF}\left(\widehat{\vartheta}, \vartheta\right) = \frac{\left(\widehat{\vartheta} - \vartheta\right)^2}{\vartheta^2},$$

where, $\widehat{\vartheta}_{DLF}$ is an estimator of ϑ. Hence, the Bayesian estimators of γ, β and λ for Pareto distribution in presence of outliers under DLF, say $\widehat{\gamma}_{DLF}, \widehat{\beta}_{DLF}$ and $\widehat{\lambda}_{DLF}$, are derived as follows:

$$\begin{aligned}\widehat{\gamma}_{DLF} &= \frac{E(\gamma^{2}|\,\underline{x})}{E(\gamma|\,\underline{x})} = \left[\frac{\int_0^\infty \ \gamma^{2} \ \pi_1^{**}(\gamma|\,\underline{x}) \ d\gamma}{\int_0^\infty \ \gamma \ \pi_1^{**}(\gamma|\,\underline{x}) \ d\gamma}\right] \\ &= \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n+2)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n+1)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right], \end{aligned} \tag{31}$$

$$\widehat{\beta}_{DLF} = \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) + \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)-1} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right], \tag{32}$$

$$\widehat{\lambda}_{DLF} = \left[\frac{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)+1} \ e^{\left[-\gamma(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}{\int_0^\rho \int_1^d \int_0^\infty \lambda^{\gamma(\delta+n)} \ e^{\left[-\gamma\left(\Sigma_{i=1}^n \ln x_i + \ln\mu - k\ln\beta\right) - \ln\beta - \Sigma_{i=1}^n \ln x_i + (\nu+n)\ln\gamma\right]} \ d\gamma\, d\beta\, d\lambda}\right]. \tag{33}$$

Integrals (22) to (33) are difficult to be obtained; therefore the M-H algorithm is employed to generate MCMC samples from posterior density functions (18). After acquiring MCMC samples from the posterior distribution, we can get the Bayes estimate of γ, β and λ. Hence, the Bayesian estimators of $\bar{J}(X)$ and $\bar{\Xi}(X)$ denoted by $\widehat{J}(X)$ and $\widehat{\Xi}(X)$ are obtained after inserting the Bayes estimates of parameters in (12) and (13). Furthermore, for $\beta = 1$ or $k = 0$ the $\widehat{J}(X)$ and $\widehat{\Xi}(X)$ are resulted in homogenous case.

4. Simulation and Application

This section discussed the performance of $J(X)$ and $\Xi(X)$ estimates and provided a real data example to illustrate the theoretical results. The performances of the two methods are investigated through Monte Carlo simulations using R software.

4.1. Simulation experiment

Here, we carried out a simulation study in order to check the behavior of the ML estimates (MLEs) and Bayes estimates (BEs) of $J(X)$ and $\Xi(X)$ for the Pareto distribution in the presence of outliers. The study was executed for number of outliers $k = 0, 1$ and 2 and different sample sizes $n = 10, 20, 30, 40, 50$. The parameter values were selected as set $1 \equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$ and set $2 \equiv (\gamma = 3.5, \beta = 2, \lambda = 0.5)$. The hyper-parameters were selected as $\nu = 0.5, \mu = 5, \delta = 0.2, \rho = 6$ and $d = 6$. Also, the value of τ was selected as $\tau = 2$ and -2. The MLEs of $\bar{J}(X)$ and $\bar{\Xi}(X)$ were computed. Also, BEs were obtained under SELF, LLF, MLF and DLF. Then, the absolute biases (ABs) and mean squared errors (MSEs) for different sample sizes and number of outliers were calculated. Due to the complicated form of the posterior distribution, the MCMC technique is used to generate samples from the posterior distributions. Here, M-H algorithm will be used via R 4.0.3 program. All the results were based on the number of replications $N = 1000$. Finally, we have done the same M–H algorithm presented in [19].

4.2. Numerical outcomes

Here, we summarized the simulation outputs of the theoretical study provided in Section 3. The observed numerical values were recorded in Tables 1–4 and and represented in Figures 1–7.

• As anticipated, it is noticed that the performance of $J(X)$ and $\Xi(X)$ estimates improved with sample sizes.

• The MSEs of the MLEs of $J(X)$ and $\Xi(X)$ in homogenous and outliers cases decrease with sample sizes. Also, the BEs of $J(X)$ and $\Xi(X)$ under different loss functions improved with sample sizes for $k = 0, 1$ and 2 (see for example Figures 1–4 and Tables 1–4).

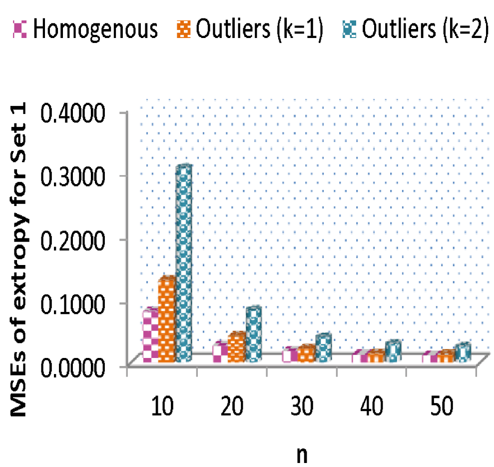


Figure 1: MSEs of MLEs of extropy at $k = 0, 1$ and 2 for Set 1.

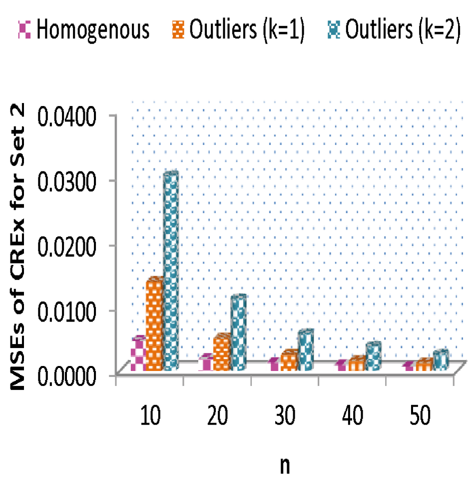


Figure 2: MSEs of MLEs of CREx at $k = 0, 1$ and 2 for Set 2.

• The MSEs and the ABs of the $\widehat{J}(X)$ and $\widehat{\Xi}(X)$ were increasing with number of outliers in majority of situations (see Tables 1–4).

• The MSEs of the $\widehat{J}(X)$ and $\widehat{\Xi}(X)$ under SELF, MLF, DLF and LLF at $\tau = -2$ had the smallest values compared to $\widehat{J}(X)$ and $\widehat{\Xi}(X)$ under SELF, MLF, DLF and LLF at $\tau = -2$ in majority of situations (see for example Figures

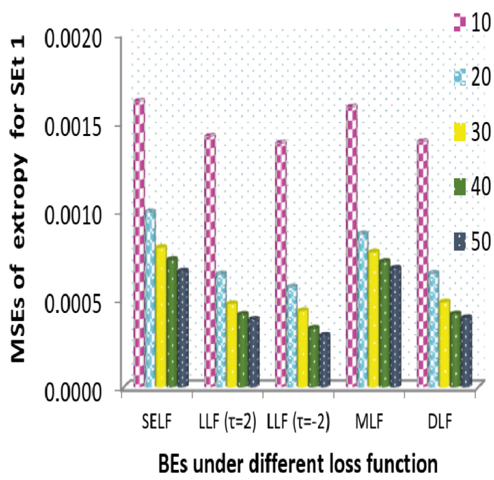


Figure 3: MSEs of extropy BEs at $k = 1$ for Set 1.

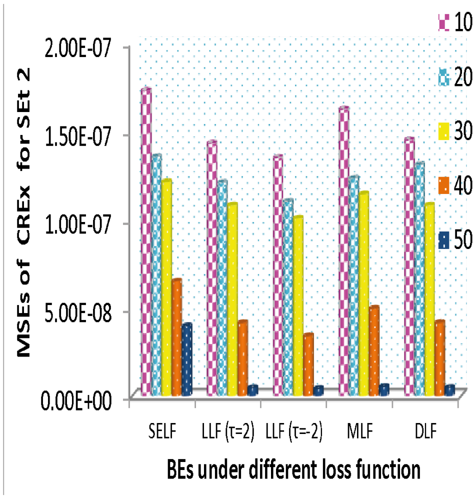


Figure 4: MSEs of CREx BEs at $k = 0$ for Set 2.

3–6 and Tables 3 and 4).

• As the exact values of $J(X)$ and $\Xi(X)$ decreased, the MSEs and the ABs of MLEs and BEs of extropy and

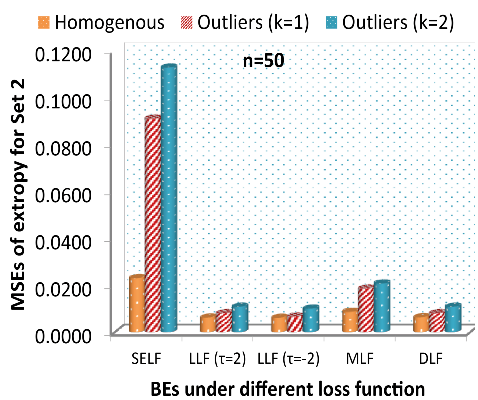


Figure 5: MSEs of Extropy BEs for Set 2 at $k = 0, 1, 2$ and $n = 50$.

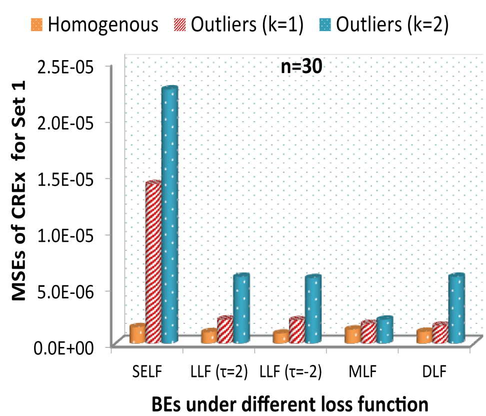
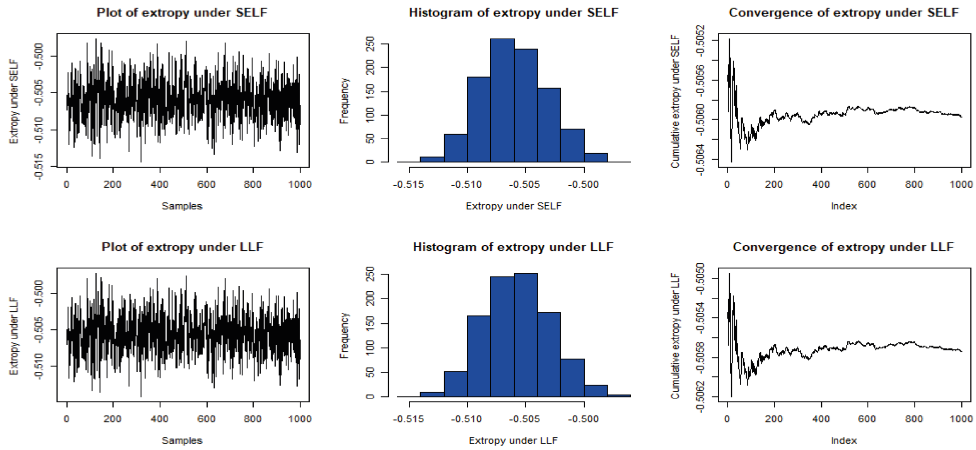


Figure 6: MSEs of CREx BEs for Set 1 at $k = 0, 1, 2$ and $n = 30$.

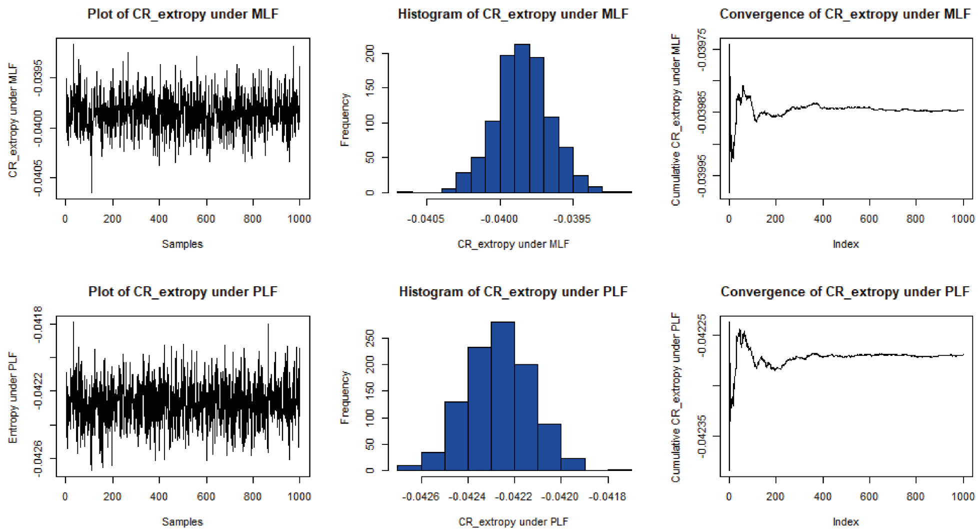
CREx decreased with k.

• The MSEs and ABs of $\overline{J}(X)$ and $\widehat{J}(X)$ increased with the value of γ in the presence of outliers and in the homogenous case. While the MSEs and ABs of $\overline{\Xi}(X)$ and $\widehat{\Xi}(X)$ decreased with the value of γ in outliers and homogenous cases.

• History plots for different estimates of $J(X)$ and $\Xi(X)$ under the symmetric and asymmetric loss functions look like a horizontal band with no long upward or downward trends which are indicators to convergence (see for example Figure 7 (a) for outlier case and Figures 7 (b) for homogenous case).



(a) Extropy under SELF and LLF at $\tau = 2, k = 1$ and $n = 50$ for Set 1.



(b) CREx under MLF and DLF at $k = 0$ and $n = 50$ for Set 2.

Figure 7: Bayesian estimators for extropy and CREx in outliers and homogenous cases for Set 1and Set 2

Table 1. ABs and MSEs of MLEs for Extropy in Homogenous and Outliers Cases.

			Set 1 $\equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$							Set 2 $\equiv (\gamma = 3.5, \beta = 2, \lambda = 0.5)$			
			$k = 0$							$k = 0$			
n		10	20	30	40	50	n		10	20	30	40	50
Exact value				-0.56250			Exact value				-0.56125		
MLE	AB	0.0784	0.0290	0.0140	0.0109	0.0147	MLE	AB	0.1455	0.1019	0.0372	0.0119	0.0101
	MSE	0.0800	0.0269	0.0181	0.0124	0.0111	MLE	MSE	0.2952	0.1116	0.0866	0.0674	0.0598
			$k = 1$							$k = 1$			
Exact value		-0.4246	-0.4905	-0.5138	-0.5258	-0.5330	Exact value		-1.1841	-1.3502	-1.4089	-1.4388	-1.4570
MLE	AB	0.1606	0.0848	0.0482	0.0319	0.0292	MLE	AB	0.1864	0.2317	0.1510	0.01106	0.0956
	MSE	0.1294	0.0419	0.0219	0.0137	0.0125	MLE	MSE	0.4380	0.1481	0.0947	0.0746	0.0634
			$k = 2$							$k = 2$			
Exact value		-0.3112	-0.4246	-0.4679	-0.4905	-0.5044	Exact value		-0.8971	-1.1841	-1.2931	-1.3502	-1.3852
MLE	AB	0.2802	0.1371	0.0892	0.0633	0.0527	MLE	AB	0.8052	0.4723	0.3184	0.2594	0.2062
	MSE	0.4057	0.0623	0.0296	0.0193	0.0149	MLE	MSE	0.6783	0.2683	0.1453	0.1079	0.0894

Table 2. ABs and MSEs of MLEs for CREx in Homogenous and Outliers Cases.

Set 1 $\equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$							Set 2 $\equiv (\gamma = 3.5, \beta = 2, \lambda = 0.5)$						
$k = 0$							$k = 0$						
n		10	20	30	40	50	n		10	20	30	40	50
Exact value				-0.12500			Exact value				-0.04167		
MLE	Bias	0.0478	0.0237	0.0208	0.0132	0.0006	MLE	Bias	0.0056	0.0034	0.0022	0.0008	0.0006
	MSE	0.1028	0.0335	0.0208	0.0157	0.0115	MLE	MSE	0.0046	0.0019	0.0013	0.0009	0.0007
$k = 1$							$k = 1$						
Exact value		-0.1215	-0.1224	-0.1236	-0.1239	-0.1248	Exact value		-0.0375	-0.0395	-0.0409	-0.0410	-0.0412
MLE	Bias	0.0555	0.0340	0.0235	0.0148	0.0080	MLE	Bias	0.0949	0.0530	0.0359	0.0254	0.0206
	MSE	0.1502	0.0419	0.0276	0.0166	0.0123	MLE	MSE	0.0137	0.0050	0.0025	0.0016	0.0013
$k = 2$							$k = 2$						
Exact value		-0.1195	-0.1219	-0.1228	-0.1237	-0.1242	Exact value		-0.0357	-0.393	-0.0398	-0.0399	0.0410
MLE	Bias	0.1035	0.0374	0.0293	0.0114	0.0080	MLE	Bias	0.1559	0.0953	0.0644	0.0523	0.0421
	MSE	0.2297	0.0479	0.0287	0.0185	0.0133	MLE	MSE	0.0301	0.0111	0.0057	0.0038	0.0026

Table 3. ABs and MSEs of BEs for Extropy in Homogenous and Outliers Cases.

Set 1 $\equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$						
$k = 0$						
n		10	20	30	40	50
Exact value				-0.56250		
SELF	AB	1.1797E-03	1.7250E-03	1.0282E-03	9.7250E-04	8.1797E-04
	MSE	2.4492E-02	7.0504E-04	7.1419E-04	6.9169E-04	5.4091E-04
LLF ($\nu = 2$)	AB	2.9640E-02	9.8057E-04	7.9870E-04	6.2235E-04	2.7558E-04
	MSE	2.4496E-03	4.3383E-04	3.6506E-04	3.4246E-04	3.6061E-04
LLF ($\nu = -2$)	AB	2.9640E-02	8.8057E-04	6.3841E-04	5.8858E-04	6.9100E-04
	MSE	1.2496E-03	3.9383E-04	3.5963E-04	2.4689E-04	2.4528E-04
MLF	AB	2.4074E-02	5.3633E-04	3.6634E-04	1.6916E-04	2.9733E-04
	MSE	5.7160E-03	8.4604E-04	7.1596E-04	6.3465E-04	7.3879E-04
DLF	AB	2.9802E-02	3.8569E-04	6.1788E-04	7.2220E-04	6.7032E-04
	MSE	1.2654E-03	4.3401E-04	3.9959E-04	3.5258E-04	3.4521E-04
$k = 1$						
Exact value		-0.42463	-0.49052	-0.51383	-0.52575	-0.53298
SELF	AB	3.4674E-02	1.9975E-02	1.3531E-02	9.8712E-03	9.0786E-03
	MSE	1.6190E-03	9.9065E-04	7.8938E-04	7.2129E-04	6.5619E-04
LLF ($\nu = 2$)	AB	3.4447E-02	1.8347E-02	1.2383E-02	9.2582E-03	7.0125E-03
	MSE	1.4206E-03	6.3910E-04	4.7105E-04	4.1086E-04	3.8231E-04
LLF ($\nu = -2$)	AB	3.4294E-02	1.7578E-02	1.1955E-02	7.8834E-03	6.9902E-03
	MSE	1.3815E-03	5.6528E-04	4.3105E-04	3.3216E-04	2.9191E-04
MLF	AB	3.3960E-02	1.7032E-02	1.2480E-02	8.9628E-03	6.9879E-03
	MSE	1.5858E-03	8.6638E-04	7.6356E-04	7.1021E-04	6.7194E-04
DLF	AB	3.4277E-02	1.8443E-02	1.2935E-02	9.3572E-03	7.9698E-03
	MSE	1.3904E-03	6.4264E-04	4.8041E-04	4.1287E-04	3.9154E-04
$k = 2$						
Exact value		-0.31115	-0.424623	-0.46788	-0.49052	-0.50443
SELF	AB	6.1378E-02	3.4701E-02	2.5571E-02	2.0747E-02	1.5592E-02
	MSE	4.0448E-03	1.6961E-03	1.2003E-03	9.4894E-04	8.8997E-04
LLF ($\nu = 2$)	AB	6.0448E-02	3.4746E-02	2.4033E-02	2.8329E-02	1.4954E-02
	MSE	3.7863E-03	1.4563E-03	8.3837E-04	6.8507E-04	5.7615E-04
LLF ($\nu = -2$)	AB	6.0094E-02	3.3738E-02	2.3033E-02	1.9329E-02	1.4854E-02
	MSE	3.6644E-03	1.3669E-03	8.2837E-04	6.4907E-04	5.2615E-04
MLF	AB	5.9379E-02	3.4891E-02	2.2808E-02	1.7970E-02	1.4253E-02
	MSE	3.8472E-03	1.7018E-03	1.0147E-03	8.9154E-04	8.0907E-04
DLF	AB	6.0337E-02	3.8729E-02	2.4125E-02	2.9310E-02	1.5684E-02
	MSE	3.7946E-03	1.5551E-03	8.4265E-04	6.9836E-04	5.8556E-04

Note: E-a: stands for $10^{-}a$.

Continued Table 3.

Set 1 $\equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$						
$k = 0$						
n		10	20	30	40	50
Exact value				-1.53125		
SELF	AB	4.9271E-01	4.2498E-01	3.8485E-01	3.3586E-01	1.2515E-01
	MSE	2.2071E-01	1.8089E-01	1.4835E-01	1.1303E-01	2.2912E-02
LLF ($\nu = 2$)	AB	1.8333E-01	1.6260E-01	1.3453E-01	1.0228E-01	6.4583E-02
	MSE	3.3962E-02	2.7427E-02	1.8375E-02	1.0721E-02	6.0756E-03
LLF ($\nu = -2$)	AB	1.8126E-01	1.6140E-01	1.2348E-01	1.0154E-01	6.3457E-02
	MSE	3.1239E-02	2.6389E-02	1.7654E-02	1.0347E-02	5.9874E-03
MLF	AB	1.4261E-01	1.5001E-01	1.6171E-01	1.7408E-01	2.8643E-01
	MSE	2.1461E-02	2.3529E-02	2.1069E-02	1.1116E-02	8.5603E-03
DLF	AB	1.8327E-01	1.6344E-01	1.3479E-01	1.0245E-01	6.5643E-02
	MSE	3.3940E-02	2.6701E-02	1.8569E-02	1.0754E-02	6.1898E-03
$k = 1$						
Exact value		-1.18408	-1.35015	-1.40885	-1.43882	-1.45701
SELF	AB	4.9122E-01	4.9209E-01	4.9077E-01	4.9003E-01	1.9876E-01
	MSE	2.4249E-01	2.4165E-01	2.4118E-01	2.4046E-01	9.0836E-02
LLF ($\nu = 2$)	AB	2.0663E-01	2.0577E-01	2.0494E-01	2.0445E-01	3.0954E-02
	MSE	4.3112E-02	4.2722E-02	4.2366E-02	4.2186E-02	7.9107E-03
LLF ($\nu = -2$)	AB	2.0213E-01	2.0433E-01	2.0348E-01	2.0342E-01	2.9864E-02
	MSE	4.0865E-02	4.0346E-02	4.0023E-02	3.9874E-02	6.5432E-03
MLF	AB	1.3429E-01	1.3208E-01	1.3179E-01	1.3090E-01	1.2576E-01
	MSE	1.9680E-02	1.9286E-02	1.8640E-02	1.8516E-02	1.8464E-02
DLF	AB	2.0657E-01	2.0588E-01	2.0570E-01	2.0439E-01	3.0894E-02
	MSE	4.3458E-02	4.3091E-02	4.2898E-02	4.2159E-02	7.9342E-03
$k = 2$						
Exact value		-0.89705	-1.18408	-1.29312	-1.35015	-1.38517
SELF	AB	4.9901E-01	4.4839E-01	4.3426E-01	4.1079E-01	3.3514E-01
	MSE	2.6758E-01	2.0135E-01	1.8887E-01	1.6904E-01	1.1255E-01
LLF ($\nu = 2$)	AB	2.9436E-01	1.7702E-01	1.6747E-01	1.5185E-01	1.0252E-01
	MSE	4.2998E-02	3.1690E-02	2.8369E-02	2.3381E-02	1.0787E-02
LLF ($\nu = -2$)	AB	1.9324E-01	1.7654E-01	1.5642E-01	1.4653E-01	9.2518E-03
	MSE	4.1346E-02	3.0764E-02	1.9864E-02	2.1347E-02	9.8779E-03
MLF	AB	2.2851E-01	1.4382E-01	1.4790E-01	1.5454E-01	1.5252E-01
	MSE	5.5014E-02	2.1792E-02	2.2978E-02	2.4913E-02	2.0637E-02
DLF	AB	1.9551E-01	1.7694E-01	1.6740E-01	1.5179E-01	1.0246E-01
	MSE	4.9977E-02	3.1662E-02	2.8347E-02	2.3363E-02	1.0775E-02

Note: E-a: stands for 10^-a.

4.3. Application to real data

Here, the real data set was used to illustrate the method proposed in previous section. The real data set was studied in [5]. The data represent 20 claim amounts. For insurance company one of its services is motor insurance. A claim of at least 500,000 Rials (Iranian Rials) as compensation for the motor insurance can be made. The vehicles involved are of different costs, of which some of them may have a very high cost. Claim amounts vary according to the damage to the vehicles. The company had assumed that claims of expensive/severely damaged vehiclesare1.5 times higher than the normal vehicles. The validity of the fitted model was checked in [5].
Regarding the methods of estimation presented in section 3, the outputs have shown in Table 5 and Figures 8 and 9.

The observed results in Table 5 and Figures 8 and 9 showed that the MSEs of $\overline{J}(X)$, $\overline{\Xi}(X)$, $\widehat{J}(X)$ and $\widehat{\xi}(X)$ in the presence of outliers were greater than the corresponding in homogenous case. The MSEs of $\widehat{J}(X)$ and $\widehat{\xi}(X)$ in the presence of outliers and in the homogenous case under LLF at $\tau = -2$ had the smallest values compared to MSEs of other BEs of both extropies. Also, we concluded that the extropy and CREx estimates increased with number of outliers.

Table 4. ABs and MSEs of BEs for CREx in Homogenous and Outliers Cases.

Set 2 $\equiv (\gamma = 1.5, \beta = 2, \lambda = 0.5)$						
$k = 0$						
n		10	20	30	40	50
Exact value				-0.12500		
SELF	AB	1.0146E-04	3.5476E-05	2.3422E-06	2.2194E-06	1.9683E-06
	MSE	2.3456E-05	1.9705E-06	1.4368E-06	3.4268E-07	1.0560E-07
LLF ($\nu = 2$)	AB	3.6784E-05	4.4217E-05	1.0433E-06	1.0236E-06	1.9855E-07
	MSE	1.7454E-05	1.1892E-06	9.8754E-07	3.9867E-08	3.9875E-08
LLF ($\nu = -2$)	AB	1.6578E-04	7.6655E-05	1.0232E-06	1.0045E-06	1.7864E-07
	MSE	1.5564E-05	1.1068E-06	8.7653E-07	3.2305E-07	3.5643E-08
MLF	AB	3.7643E-05	1.7130E-04	1.5432E-05	1.0226E-05	1.2670E-06
	MSE	3.6542E-05	1.5433E-06	1.2316E-06	1.8759E-07	1.9883E-08
DLF	AB	1.5433E-04	1.2458E-05	1.2940E-05	1.0236E-06	1.9985E-07
	MSE	1.5672E-05	1.2001E-05	1.0232E-06	3.9867E-08	3.9870E-08
$k = 1$						
Exact value		-0.12152	-0.12243	-0.12357	-0.12386	-0.12481
SELF	AB	1.3589E-04	8.3706E-05	2.4777E-06	8.2343E-05	4.8309E-05
	MSE	3.7482E-06	2.2130E-06	1.4158E-05	4.8797E-07	1.5001E-07
LLF ($\nu = 2$)	AB	1.5700E-04	6.1418E-05	1.0602E-05	2.5191E-05	2.2338E-05
	MSE	1.9804E-06	1.1205E-06	1.1093E-06	1.1009E-07	7.1583E-08
LLF ($\nu = -2$)	AB	1.7505E-04	8.5757E-05	1.0602E-05	2.0630E-05	1.8386E-05
	MSE	1.6423E-06	1.1176E-06	1.0593E-06	1.0875E-07	7.0445E-08
MLF	AB	2.0413E-04	3.0097E-05	3.0860E-05	1.3541E-05	1.2670E-05
	MSE	4.3943E-06	1.7219E-06	1.4266E-06	6.3238E-07	1.4683E-07
DLF	AB	1.8457E-04	6.2726E-05	1.0777E-05	2.4766E-06	1.8364E-05
	MSE	1.6521E-06	1.2213E-06	1.0702E-06	3.0899E-07	7.1783E-08
$k = 2$						
Exact value		-0.11954	-0.12189	-0.122782	-0.12368	-0.12423
SELF	AB	7.9160E-04	2.1308E-04	5.1231E-06	6.1442E-04	1.2462E-04
	MSE	5.1460E-05	4.1015E-06	2.2567E-06	7.5636E-07	2.5683E-07
LLF ($\nu = 2$)	AB	6.7858E-04	1.1736E-04	9.1384E-05	1.1037E-05	1.9298E-06
	MSE	2.3415E-05	1.8343E-06	1.5035E-06	1.5395E-05	9.7012E-06
LLF ($\nu = -2$)	AB	7.0097E-04	1.0672E-04	9.0884E-05	1.1166E-06	1.1283E-07
	MSE	2.2320E-05	1.7381E-06	1.4055E-06	5.6428E-07	1.8556E-07
MLF	AB	3.3780E-04	2.2720E-05	1.9312E-04	4.7520E-05	8.9813E-06
	MSE	5.0942E-05	3.1797E-06	1.6068E-06	9.0754E-07	3.5680E-07
DLF	AB	6.9975E-04	1.0709E-04	9.1005E-05	1.0973E-04	1.2836E-05
	MSE	2.5315E-05	1.8384E-06	1.6060E-06	4.7658E-07	1.8856E-07

Note: E-a: stands for $10^{-}a$.

5. Discussion and summery

In this study, we considered the Bayesian and non-Bayesian estimators of extropy and cumulative residual extropy for the Pareto distribution in the presence of k outliers and in the homogenous case. We obtained the ML estimators as well as Bayesian estimators under symmetric and asymmetric loss functions for the considered extropies measures. The MCMC techniques were employed to compute the Bayes estimates based on M-H algorithm. The performance of extropies estimates for Pareto distribution was investigated in terms of their absolute biases and mean squared errors. One application to real data and simulation issue were considered. Regarding the outcomes of simulation results, we conclude that the precision of extropy and CREx estimates from both methods of estimation was improved with increasing sample sizes. Generally, as the exact values of extropy and CREx decreased, the MSEs and the ABs of both MLEs and BEs of extropy and CREx decreased with increasing number of outliers. The Bayesian estimates of extropy and CREx under LLF at $\tau = -2$ were superior to the observed estimates under other selected loss functions in majority of situations. Finally, the real data analysis confirmed the theoretical and simulated studies.

Continued Table 4.

Set 2 $\equiv (\gamma = 3.5, \beta = 2, \lambda = 0.5)$						
$k = 0$						
n		10	20	30	40	50
Exact value				-0.04167		
SELF	AB	1.3818E-04	1.1480E-04	9.6758E-05	5.8320E-05	1.9643E-05
	MSE	1.7307E-07	1.3558E-07	1.2153E-07	6.5157E-08	3.9613E-08
LLF ($\nu = 2$)	AB	2.4941E-05	2.0807E-05	3.2664E-03	1.9069E-04	6.3864E-05
	MSE	1.4321E-07	1.2108E-07	1.0809E-07	4.1400E-08	4.5139E-09
LLF ($\nu = -2$)	AB	2.4767E-05	1.2207E-05	1.1881E-05	1.6256E-05	6.4863E-06
	MSE	1.3505E-07	1.1011E-07	1.0076E-07	3.4098E-08	4.0138E-09
MLF	AB	8.4442E-05	4.0369E-05	3.5536E-05	1.9297E-05	6.6125E-06
	MSE	1.6270E-07	1.2350E-07	1.1451E-07	4.9600E-08	5.2485E-09
DLF	AB	2.4971E-05	2.2307E-05	3.2660E-03	1.9257E-04	6.4864E-05
	MSE	1.4508E-07	2.4108E-07	1.0806E-07	4.1500E-08	4.6239E-09
$k = 1$						
Exact value		-0.03754	-0.03953	-0.04086	-0.04095	-0.04123
SELF	AB	3.7300E-03	2.0244E-03	1.5682E-03	9.0026E-04	4.7206E-03
	MSE	1.4363E-05	4.2268E-06	2.5320E-06	8.4332E-07	2.2348E-05
LLF ($\nu = 2$)	AB	1.3228E-03	4.8633E-04	4.8212E-04	2.8254E-04	1.5736E-03
	MSE	1.9137E-06	4.6534E-07	3.1120E-07	9.5872E-08	2.5026E-06
LLF ($\nu = -2$)	AB	1.2628E-03	4.4673E-04	3.2675E-04	2.7665E-04	1.5676E-03
	MSE	1.8537E-06	4.0334E-07	3.1015E-07	9.3972E-08	2.4866E-06
MLF	AB	1.1054E-03	6.8966E-04	4.8815E-04	3.3653E-04	1.3465E-03
	MSE	1.5957E-06	5.7747E-07	3.1720E-07	1.4246E-07	1.8652E-06
DLF	AB	1.4328E-03	4.9733E-04	4.8212E-04	2.8654E-04	1.5776E-03
	MSE	1.9737E-06	4.7334E-07	3.1120E-07	9.7872E-08	2.5166E-06
$k = 2$						
Exact value		-0.03567	-0.03934	-0.03978	-0.03989	-0.04097
SELF	AB	4.4749E-02	1.7124E-02	1.3089E-02	1.1458E-02	5.7627E-03
	MSE	2.0091E-03	2.9846E-04	1.7186E-04	1.3838E-04	3.3306E-05
LLF ($\nu = 2$)	AB	1.4305E-02	4.6548E-03	4.2982E-03	3.1460E-03	1.9064E-03
	MSE	2.0759E-04	1.8573E-04	1.8714E-05	1.0019E-05	3.6775E-06
LLF ($\nu = -2$)	AB	1.4265E-02	4.4767E-03	4.9725E-03	2.9725E-03	1.9024E-03
	MSE	2.0655E-04	2.7840E-05	1.7998E-05	1.0998E-05	3.6601E-06
MLF	AB	3.9914E-03	3.7580E-03	2.6680E-03	3.7166E-03	1.5347E-03
	MSE	1.1570E-04	1.5739E-05	7.4594E-06	4.5390E-06	2.1319E-06
DLF	AB	1.4134E-02	5.0846E-03	4.2974E-03	2.9729E-03	1.9063E-03
	MSE	2.0273E-04	2.7852E-05	1.8706E-05	1.1001E-05	9.6770E-06

Note: E-a: stands for $10^{-}a$.

Table 5. Estimates of Extropy and CREx and their MSEs (in brackets) for 20 claim amounts.

		Extropy			CREx		
		$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$
MLEs		-0.05389 (0.31204)	-0.05162 (0.29391)	-0.04198 (0.28161)	-0.32097 (0.02667)	-0.30438 (0.02888)	-0.28829 (0.03149)
BEs	SELF	-5.6247E-01 (2.3954E-06)	-4.9073E-01 (3.2578E-06)	-4.2439E-01 (3.2995E-06)	-1.4359E-01 (2.0023E-07)	-1.3457E-01 (4.6755E-07)	-1.2485E-01 (5.4524E-07)
	LLF ($\tau = 2$)	-5.6207E-01 (9.0337E-07)	-4.9022E-01 (1.3207E-06)	-4.2493E-01 (1.3619E-06)	-1.4367E-01 (1.3745E-07)	-1.3446E-01 (1.50143E-07)	-1.2494E-01 (2.4221E-07)
	LLF ($\tau = -2$)	-5.6224E-01 (6.7159E-07)	-4.9076E-01 (1.0484E-06)	-4.2438E-01 (1.1606E-06)	-1.4356E-01 (9.9551E-08)	-1.3435E-01 (1.3759E-07)	-1.2473E-01 (1.6938E-07)
	MLF	-5.6201E-01 (2.0418E-06)	-4.8971E-01 (3.0202E-06)	-4.2545E-01 (3.2529E-06)	-1.4385E-01 (2.6303E-07)	-1.3434E-01 (3.6546E-07)	-1.2464E-01 (4.2122E-07)
	DLF	-5.6232E-01 (9.0351E-07)	-4.9796E-04 (1.13596E-06)	-4.2493E-01 (1.3630E-06)	-1.4372E-01 (1.3758E-07)	-1.3476E-01 (1.5209E-07)	-1.2498E-01 (2.4238E-07)

Note: E-a: stands for $10^{-}a$.

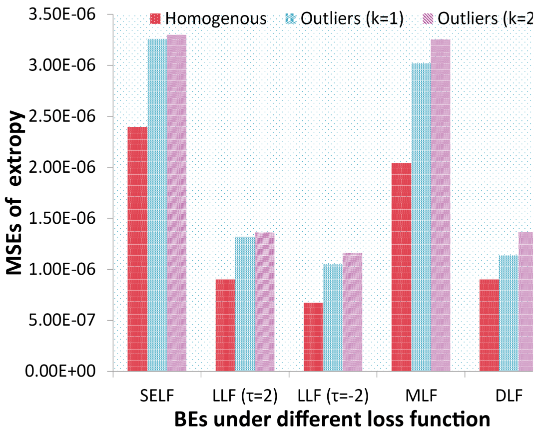


Figure 8: MSEs of Extropy BEs for real data set at $k = 0, 1, 2$.

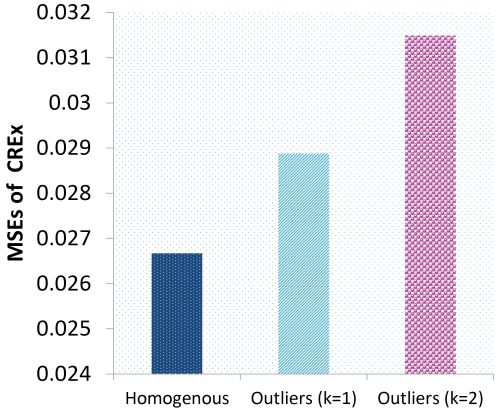


Figure 9: MSEs of CREx MLEs for real data set at $k = 0, 1, 2$.

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