STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **10**, March 2022, pp 589–605.
Published online in International Academic Press (www.IAPress.org)

A New Two-parameter Modified Half-logistic Distribution: Properties and Applications

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Abstract This article aims to present and analyse a modified two-parameter version of the Half-Logistic lifetime model. The hazard function, quantile function, asymptotic, linear combination, extreme value, moments, incomplete moments, residual entropies, moment generating function and order statistics, all theoretical properties of this model that are derived and discussed in depth. By performing a simulation analysis, the various techniques of estimation are compared to the estimates of the maximum likelihood of parameters.Finally, two actual data sets have been applied to illustrate the goals of this article.

Keywords Moments; likelihood estimation; Entropy ; Half-Logistic distribution

AMS 2010 subject classifications 60Exx, 60E05

DOI:10.19139/soic-2310-5070-1210

1. Introduction

Analysing lifetimes and Modelling are important in engineering, medicine, economics, etc. In several utilized areas such as lifetime analysis, insurance and finance, we aspire to extend structures of distributions. So, In the literature, many procedures for producing new G families of probabilistic distributions have been suggested and studied. Several attempts have been allocated for identifying many new G families of probabilistic distributions that expand the famous G classes and have a wide of pliability in modelling data in practice. Among them, the generalized G-classes of distributions say G are used in which at least one parameter are added to a baseline distribution. The well-known standard half-logistic (SHL) lifetime distribution have been suggested as a lifetime increasing hazard rate function (see Balakrishnan (1985)). The cumulative function (CDF) of SHL can be written as

$$\mathbf{\Pi}(z) = \left(1 - e^{-z}\right) / \left(e^{-z} + 1\right) |_{z\in\mathbf{R}^+} .$$

The related density function (PDF) of SHL distribution can then be derived as

$$\pi(z) = 2e^{-z} \left(e^{-z} + 1\right)^{-2} |_{z\in\mathbf{R}^+} .$$

In this article, we introduce a novel two-parameter lifetime half-logistic distribution called the modified half-logistic (MHL). The MHL is derived based on modifying the the CDF of the SHL model with adding a new two shape parameters to the standard case. We are motivated to introduce the modified Half-Logistic lifetime model since its PDF can be ”monotonically and right skewed with no peak” and ”monotonically and right skewed with

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

one peak". Also its hazard rate function (HRF) can be "monotonically-increasing", "monotonically-decreasing", "bathtub ", "upside down" and "upside down-bathtub". Additionally, the the modified Half-Logistic lifetime model will be useful in modelling reliability analysis of the right skewed failure time data. The new CDF can then be expressed as

$$F_{\underline{\varsigma}}(z) = \frac{1}{1+e^{-\beta z}}(1-e^{-z})^{\alpha}|_{z\in\mathbf{R}^+}\,, \tag{1}$$

where $\underline{\varsigma} = (\alpha, \beta)$ and $\alpha, \beta \in \mathbf{R}^+$. The PDF and HRF corresponding to (1) are given by

$$f_{\underline{\varsigma}}(z) = e^{-z}(1-e^{-z})^{-1+\alpha}\frac{\beta\, e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta z}}{[1+e^{-\beta z}]^2}, \tag{2}$$

and

$$h_{\underline{\varsigma}}(z) = \frac{e^{-z}(1-e^{-z})^{-1+\alpha}\left[\beta\, e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta z}\right]}{[1+e^{-\beta z}]\,[1+e^{-\beta z} - (1-e^{-z})^{\alpha}]}. \tag{3}$$

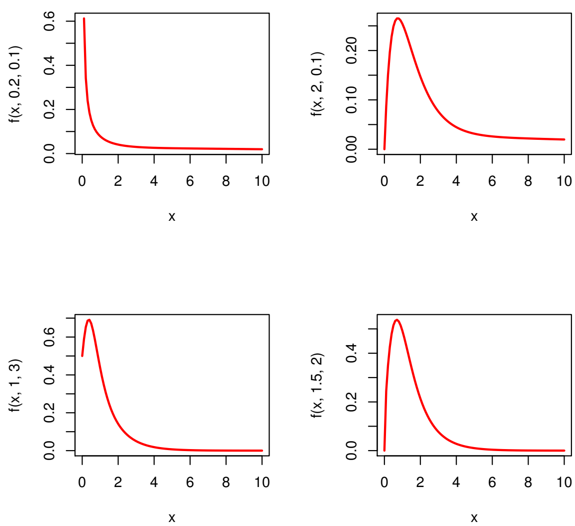

Figure 1. Density function of MHL(α, β) for some selected value of parameters.

For selected parameter values, shapes (1) and (2) show the PDF and HRF of MHL$(\underline{\varsigma})$. The PDF of the modified Half-Logistic lifetime model can be "monotonically and right skewed with no peak" and "monotonically and right skewed with one peak" (see figure 1). The HRF of MHL$(\underline{\varsigma})$ can be "monotonically-increasing", "monotonically-decreasing", "bathtub (**U**-HRF)", "upside down" and "upside down-bathtub (upside down-**U**)" (see figure 2). For $\alpha = \beta = 1$, MHL reduce to standard Half-Logistic distribution.

The rest of this work is arranged in the following manner: In the above, new family of distributions was suggested part 2 explores the proposed distribution's various properties. Asymptotic, quantile function, the mixture for PDF, moments, residual entropy, moment generating function, order statistics, extreme value and some of these properties. In part 3, the method of the maximum likelihood is compared with several estimation methodologies via comprehensive graphical simulation analysis. Data of real sets are analysed to display the specifics of the new family in part 4. In part 5, some concluding remarks are considered.

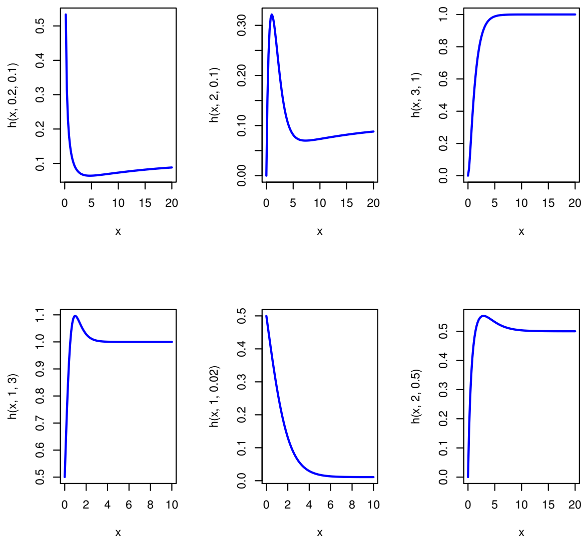

Figure 2. Hazard rate function of MHL(α, β) for some selected value of parameters.

2. Structural Properties

We will apply some structural properties of MHL($\underline{\varsigma}$) distributions in this part.

2.1. Function of quantiles

The function of quantiles (FQ) of the MHL($\underline{\varsigma}$) have an unclosed form. If $U \sim U(0,1)$, then the FQ of MHL($\underline{\varsigma}$) can be obtained by solving non-linear equation $F(z) = u$.

2.2. Asymptotic for CDF, PDF and HRF

Note that $1 - e^{-z} \sim z$, $1 + e^{-\beta z} \sim 2$ as $z \to 0^+$ and $1 - (1 - e^{-z})^{\alpha} \sim \alpha\, e^{-z}$ $1 + e^{-\beta z} \sim 1$ as $z \to +\infty$.

The explicate asymptotic of of CDF, PDF and HRF of MHL($\underline{\varsigma}$) when $z \to 0^+$ are produced by

$$\begin{aligned} F_{\underline{\varsigma}}(z)|_{z\to 0^+} &\sim 0.5\, z^{\alpha}, \\ f_{\underline{\varsigma}}(z)|_{z\to 0^+} &\sim 0.5\, \alpha z^{-1+\alpha}, \\ h_{\underline{\varsigma}}(z)|_{z\to 0^+} &\sim \frac{1}{2 - z^{\alpha}} \alpha\, z^{-1+\alpha}. \end{aligned} \tag{4}$$

The explicate asymptotic of of CDF, PDF and HRF of MHL($\underline{\varsigma}$) when $z \to +\infty$ are produced by

$$\begin{aligned} \left[1 - F_{\underline{\varsigma}}(z)\right]|_{z\to +\infty} &\sim \frac{1}{e^{z}}\alpha\,, \\ f_{\underline{\varsigma}}(z)|_{z\to +\infty} &\sim \frac{1}{e^{z}}\alpha, \\ h_{\underline{\varsigma}}(z)|_{z\to +\infty} &\sim \alpha. \end{aligned} \tag{5}$$

These equations illustrate how parameters affect the MHL distribution's tails.

2.3. Linear Combination for PDF

Using generalized binomial expansion for any $|\frac{a_1}{a_1}| < 1$,

$$\frac{1}{\left(1+\frac{a_1}{a_2}\right)^2} = \sum_{\mathbf{j}_1=0}^{+\infty} \binom{-2}{\mathbf{j}_1} \frac{a_1^{\mathbf{j}_1}}{a_2^{\mathbf{j}_1}} = \sum_{\mathbf{j}_1=0}^{+\infty} (1+\mathbf{j}_1)(-1)^{\mathbf{j}_1} \frac{a_1^{\mathbf{j}_1}}{a_2^{\mathbf{j}_1}}. \tag{6}$$

then

$$\begin{aligned} f_{\underline{S}}(z) &= \frac{e^{-z}(1-e^{-z})^{-1+\alpha}\left[\beta\, e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta\, z}\right]}{\left[1+e^{-\beta\, z}\right]^2} \\ &= \sum_{\mathbf{j}_1=0}^{+\infty} (-1)^{\mathbf{j}_1}(1+\mathbf{j}_1)e^{-\mathbf{j}_1\beta z}e^{-z}(1-e^{-z})^{-1+\alpha}\left[\beta\, e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta\, z}\right] \\ &= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} (-1)^{\mathbf{j}_1+\mathbf{j}_2}(1+\mathbf{j}_1)\binom{-1+\alpha}{\mathbf{j}_2} e^{-\beta\,\mathbf{j}_1\, z}e^{-z}\left[\beta\, e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta\, z}\right] \\ &= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} (-1)^{\mathbf{j}_1+\mathbf{j}_2}(1+\mathbf{j}_1)\binom{-1+\alpha}{\mathbf{j}_2} \left[\begin{array}{c} \alpha\, e^{-(1+\beta\,\mathbf{j}_1+\mathbf{j}_2)\, z} \\ +\beta\, e^{-(\beta(1+\mathbf{j}_1)+\mathbf{j}_2)z} \\ +(\alpha-\beta)e^{-(1+\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2)z} \end{array}\right] \\ &= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \left[\begin{array}{c} \frac{\alpha}{\beta\,\mathbf{j}_1+\mathbf{j}_2+1} f_{\beta\,\mathbf{j}_1+\mathbf{j}_2+1}(z) \\ +\frac{\beta}{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2} f_{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2}(z) \\ +\frac{(\alpha-\beta)}{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2+1} f_{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2+1}(z) \end{array}\right] \end{aligned} \tag{7}$$

where

$$\varpi_{\mathbf{j}_1,\mathbf{j}_2} = (-1)^{\mathbf{j}_1+\mathbf{j}_2}(1+\mathbf{j}_1)\binom{-1+\alpha}{\mathbf{j}_2}$$

and the function $f_{\mathbf{\Omega}}(z) = \mathbf{\Omega}\, e^{-\mathbf{\Omega}\, z}$ refers to the well-known exponential model with parameter $\mathbf{\Omega} \in \mathbf{R}^+$. So, We can point to some mathematical properties using (7).

2.4. Raw moments

Using equation (7), the $\hbar^{th}$ moment of X is given by

$$\begin{aligned} \mathrm{E}(z^\hbar) &= \int_0^{+\infty} z^\hbar\, f(z)dz \\ &= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} (-1)^{\mathbf{j}_1+\mathbf{j}_2}(1+\mathbf{j}_1)\int_0^{+\infty} z^\hbar \binom{-1+\alpha}{\mathbf{j}_2} \\ &\quad \times\left[\alpha\, e^{-(\beta\,\mathbf{j}_1+\mathbf{j}_2+1)\, z} + \beta\, e^{-(\beta(1+\mathbf{j}_1)+\mathbf{j}_2)z} + (\alpha-\beta)e^{-(\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2+1)z}\right] \end{aligned} \tag{8}$$

$$= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \left[\begin{array}{c} \frac{1}{(1+\beta\,\mathbf{j}_1+\mathbf{j}_2)^{\hbar+1}}\alpha\Gamma(\hbar+1) \\ +\frac{1}{(\beta(1+\mathbf{j}_1)+\mathbf{j}_2)^{\hbar+1}}\beta\Gamma(\hbar+1) \\ +\frac{1}{(1+\beta(1+\mathbf{j}_1)+\mathbf{j}_2)^{\hbar+1}}(\alpha-\beta)\,\Gamma(\hbar+1) \end{array}\right] \tag{9}$$

where $\Gamma(\hbar+1) = \int_0^{+\infty} t^\hbar e^{-t}dt$ denote the Gamma function.

The moment generating function (MGF) of X by using equation (7) is given by the n-th moment of X is given by

$$m_Z(t) = \mathrm{E}(e^{t\,z}) = \int_0^{+\infty} e^{t\,Z}\, f_{\varsigma}(z)dz$$
$$= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \int_0^{+\infty} e^{t\,z}\left[\alpha\, e^{-(\beta\,\mathbf{j}_1+\mathbf{j}_2+1)\,z} + \beta\, e^{-(\beta(1+\mathbf{j}_1)+\mathbf{j}_2)z} + (\alpha-\beta)e^{-(\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2+1)z}\right]$$
$$= \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \left[\begin{array}{c} \frac{1}{1+\beta\,\mathbf{j}_1+\mathbf{j}_2-t}\alpha \\ +\frac{1}{\mathbf{j}_2+\beta(1+\mathbf{j}_1)-t}\beta \\ +\frac{1}{1+\beta(1+\mathbf{j}_1)+\mathbf{j}_2-t}\,(\alpha-\beta) \end{array}\right] \tag{10}$$

Theorem 1
All moments of MHL exist.
Proof: First note that $1 \le e^{-\beta\,z}+1 \le 2$ and

$$\alpha + \beta\, e^{-(\beta-1)z} + (\alpha-\beta)e^{-\beta\,z} < \alpha+\beta+(\alpha-\beta) \text{ for any } z \in \mathbf{R}^{+},$$

then

$$\mathrm{E}(z^{\hbar}) = \int_0^{+\infty} z^{\hbar}\, f(z)dz$$
$$= \int_0^{+\infty} z^{\hbar}\, \frac{e^{-z}(1-e^{-z})^{-1+\alpha}\left[\beta\, e^{-(\beta-1)z}+\alpha+(\alpha-\beta)e^{-\beta\,z}\right]}{\left[1+e^{-\beta\,z}\right]^2} \tag{11}$$
$$< 2\alpha \int_0^{+\infty} z^{\hbar}\, e^{-z}(1-e^{-z})^{-1+\alpha} < +\infty. \tag{12}$$

It concludes all moments of MHL exist.

Now, we obtain first four moments for some value of parameters, and then we can compute mean, variance, kurtosis and skewness. Figure 3 shows that measures.

2.5. Conditional moments

Here, we obtain the conditional moments of the MHL(ς) distribution.
Therefore

$$E(z^{\hbar}|Z \le z) = \frac{1}{F_{\varsigma}(z)} \int_0^{z} t^{\hbar}\, f_{\varsigma}(t)\, dt$$

Using equation (7)

$$E(z^{\hbar}|Z \le z) =$$
$$\frac{1}{F_{\varsigma}(z)} \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \left[\begin{array}{c} \frac{1}{1+\beta\,\mathbf{j}_1+\mathbf{j}_2}\alpha\,\gamma\left(\hbar+1, \frac{z}{1+\beta\,\mathbf{j}_1+\mathbf{j}_2}\right) \\ +\frac{1}{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2}\beta\,\gamma\left(\hbar+1, \frac{z}{\beta(1+\mathbf{j}_1)+\mathbf{j}_2}\right) \\ +\frac{1}{1+\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2}(\alpha-\beta)\,\gamma\left(\hbar+1, \frac{z}{1+\beta(1+\mathbf{j}_1)+\mathbf{j}_2}\right) \end{array}\right],$$

where $\gamma(\vartheta_1,\vartheta_2) = \int_0^{\vartheta_1} u^{\vartheta_2-1}e^{-u}du$ denote the lower incomplete gamma function.
Also

$$E(z^{\hbar}|z \ge z) = \frac{1}{1-F_{\varsigma}(z)} \int_z^{+\infty} f(t)t^{\hbar}\, dt.$$

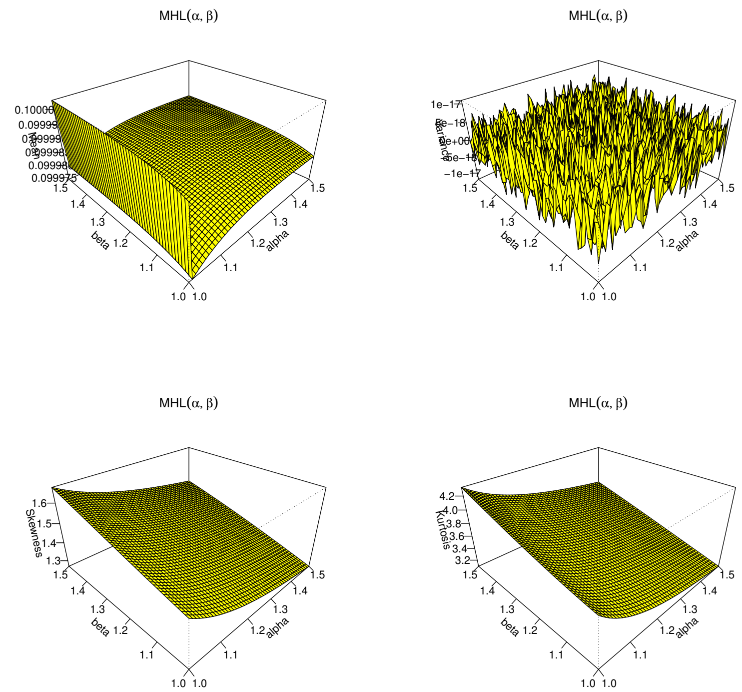

Figure 3. 3D plots of mean, variance, skewness and kurtosis of MHL distribution as a function of (α, β)

With the same way one can obtain

$$E(z^{\hbar}|Z \geq z) = \frac{1}{1-F_{\underline{\varsigma}}(z)} \sum_{\mathbf{j}_1,\mathbf{j}_2=0}^{+\infty} \varpi_{\mathbf{j}_1,\mathbf{j}_2} \left[\begin{array}{c} \frac{1}{1+\beta\,\mathbf{j}_1+\mathbf{j}_2} \alpha\,\Gamma\left(\hbar+1, \frac{z}{1+\beta\,\mathbf{j}_1+\mathbf{j}_2}\right) \\ +\frac{1}{\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2} \beta\Gamma\left(\hbar+1, \frac{z}{\beta(1+\mathbf{j}_1)+\mathbf{j}_2}\right) \\ +\frac{1}{1+\beta\,(1+\mathbf{j}_1)+\mathbf{j}_2} (\alpha-\beta)\Gamma\left(\hbar+1, \frac{z}{1+\beta(1+\mathbf{j}_1)+\mathbf{j}_2}\right) \end{array} \right], \tag{13}$$

where $\Gamma(\vartheta_1, \vartheta_2) = \int_{\vartheta_1}^{+\infty} u^{\vartheta_2-1} e^{-u} du$ denote the upper incomplete gamma function.

2.6. Extreme Value

If $z = (z_1 + z_2 ... + z_\hbar)/\hbar$ indicates to the mean of the sample. Then, using the central limit theorem, $\hbar^{0.5}(z - E(z))/\sqrt{\mathrm{Var}(z)}$ is the standard normal model as $\hbar \to +\infty$. Then, the converges of the relevant extreme values $M_{1,\hbar}^{[Z]} = \max(z_1, z_2, ..., z_\hbar)$ and $m_{1,\hbar}^{[Z]} = \min(z_1, z_2, ..., z_\hbar)$ could be useful. Then for the (1), we get

$$z^{\alpha} = \lim_{t\to 0} \frac{1}{F_{\underline{\varsigma}}(t)} F_{\underline{\varsigma}}(t\,z),$$

and

$$\mathrm{e}^{-z} = \lim_{t\to+\infty} \frac{1}{1-F_{\underline{\varsigma}}(t)} \left[1 - F_{\underline{\varsigma}}(t + z\,\tau(t))\right].$$

Thus, according to Theorem 1.6.2 of Leadbetter et al. (2012), there must be constants $a_\hbar, b_\hbar, c_\hbar \in \mathbf{R}^+$ and $d_\hbar \in \mathbf{R}^+$ in order for

$$\Pr\left(\frac{M_{1,\hbar}^{[Z]} - b_\hbar}{a_\hbar^{-1}} \leq z\right) \to e^{-e^{-z}},$$

and

$$\Pr\left(\frac{m_{1,\hbar}^{[Z]} - d_\hbar}{c_\hbar^{-1}} \leq z\right) \to 1 - e^{-z},$$

as $\hbar \to +\infty$. It is possible also to determine the shape of the norming constants.

2.7. Order Statistics

Let $z_1, z_2, \cdots, z_\hbar$ is a random sample (RS) from (1). Suppose $z_{1:\hbar} \leq z_{2:\hbar} \leq \cdots \leq z_{\hbar:\hbar}$ indicates to the corresponding order statistics. Then, the PDF corresponding to the i^{th} order statistic can be derived from

$$F_{i:\hbar}(z) = \sum_{\kappa=i}^{\hbar}\binom{\hbar}{\kappa}F_{\underline{\varsigma}}(z)^{\kappa}\left[1 - F_{\underline{\varsigma}}(z)\right]^{\hbar-\kappa} = \sum_{\kappa=i}^{\hbar}\sum_{\zeta=0}^{\hbar-\kappa}(-1)^{\zeta}\binom{\hbar}{\kappa}\binom{-\kappa+\hbar}{\zeta}F_{\underline{\varsigma}}(z)^{\kappa+\zeta} \tag{14}$$

By differentiating from equation 14 with respect to x, the PDF of the i-th order statistic of a MHL($\underline{\varsigma}$) distribution can be written as

$$\begin{aligned}
f_{i:\hbar}(z) &= \sum_{\kappa=i}^{\hbar}\sum_{\zeta=0}^{\hbar-\kappa}(-1)^{\zeta}\binom{\hbar}{\kappa}\binom{\hbar-\kappa}{\zeta}(\kappa+\zeta)f_{\underline{\varsigma}}(z)F_{\underline{\varsigma}}(z)^{\kappa+\zeta-1} \\
&= \sum_{\kappa=i}^{\hbar}\sum_{\zeta=0}^{\hbar-\kappa}(-1)^{\zeta}\binom{\hbar}{\kappa}\binom{\hbar-\kappa}{\zeta}\frac{(\kappa+\zeta)e^{-z}}{(1+e^{-\beta z})^{\kappa+\zeta+1}}\left\{\begin{array}{c}(1-e^{-z})^{\alpha(\kappa+\zeta)-1} \\ \times\left[\beta e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta z}\right]\end{array}\right\} \\
&= \sum_{l,\Im=0}^{+\infty}\sum_{\kappa=i}^{\hbar}\sum_{\zeta=0}^{\hbar-\kappa}c_{\zeta,\kappa,l,\Im}e^{-z}\left[\beta e^{-(\beta-1)z} + \alpha + (\alpha-\beta)e^{-\beta z}\right]
\end{aligned} \tag{15}$$

where

$$c_{\zeta,\kappa,l,\Im} = (-1)^{\zeta+\Im}\binom{\hbar}{\kappa}\binom{\hbar-\kappa}{\zeta}\binom{-\kappa-\zeta-1}{l}\binom{\alpha(\kappa+\zeta)-1}{\Im}.$$

Then, the τ^{th} moment of the i^{th} order statistic can be derived from

$$E\left(Z_{i:\hbar}\right) = \sum_{l,\Im=0}^{+\infty}\sum_{\kappa=i}^{\hbar}\sum_{\zeta=0}^{\hbar-\kappa}c_{\zeta,\kappa,l,\Im}\left[\begin{array}{c}\frac{1}{(1+\Im+\beta l)^{1+\tau}}\alpha\Gamma\left(\tau+1\right) \\ +\frac{1}{[\Im+\beta(1+l)]^{1+\tau}}\beta\Gamma\left(\tau+1\right) \\ +\frac{1}{[1+\Im+\beta(1+l)]^{1+\tau}}(\alpha-\beta)\Gamma\left(\tau+1\right)\end{array}\right]$$

2.8. Residual Entropy

Residual Entropy is important measure of information. The Residual entropy of z is given by

$$\mathcal{E}(Z) = -\int_0^{+\infty}\log\left[F_{\underline{\varsigma}}(z)\right]F_{\underline{\varsigma}}(z)dz$$

After some simple algebra using geometric expansion and generalized binomial expansion, for MHL($\underline{\varsigma}$) we can obtain,

$$-\log\left[F_{\underline{\varsigma}}(z)\right]F_{\underline{\varsigma}}(z)=\sum_{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3=0}^{+\infty}\varrho_{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3}(\alpha)\left[\begin{array}{c}\frac{1}{e^{z(1+\beta\,\mathbf{j}_2+\mathbf{j}_1+\mathbf{j}_3)}}\alpha\\+\frac{1}{e^{z(\beta(1+\mathbf{j}_1+\mathbf{j}_2)+\mathbf{j}_3)}}(-1)^{1+\mathbf{j}_1}\end{array}\right]$$

where

$$\varrho_{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3}(\alpha)=\frac{1}{1+\mathbf{j}_1}(-1)^{\mathbf{j}_2+\mathbf{j}_3}\binom{\alpha}{\mathbf{j}_3}.$$

Then we obtain $*(z)$ as follow

$$\mathcal{E}(Z)=\sum_{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3=0}^{+\infty}\varrho_{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3}(\alpha)\left[\frac{1}{1+\beta\,\mathbf{j}_2+\mathbf{j}_1+\mathbf{j}_3}\alpha+\frac{1}{\beta(1+\mathbf{j}_1+\mathbf{j}_2)+\mathbf{j}_3}(-1)^{1+\mathbf{j}_1}\right],$$

3. Estimation

3.1. Maximum-likelihood estimation (MLE)

Suppose $z_1, z_2, \dots, z_\hbar$ be a any RS of volume $\hbar$ from the MHL($\underline{\varsigma}$) model. The log-likelihood function for a vector of parameters $\underline{\varsigma}=(\alpha,\beta)^T$ is:

$$\begin{aligned} l(\underline{\varsigma}) &= -\sum_{\zeta=1}^{\hbar}z_\zeta+(-1+\alpha)\sum_{\zeta=1}^{\hbar}\log(1-e^{-z_\zeta})+\sum_{\zeta=1}^{\hbar}\log(\alpha+\beta\,e^{-(\beta-1)\,z_\zeta}+(\alpha-\beta)\,e^{-\beta\,z_\zeta})\\ &- 2\sum_{\zeta=1}^{\hbar}\log(1+e^{-\beta\,z_\zeta}) \end{aligned} \quad (16)$$

the function $l(\underline{\varsigma})$ can then be maximized directly or by resolving the non-linear equations By differentiating (16), The components of the common score vector $U(\underline{\varsigma})$ can be derived as

$$U_\alpha(\underline{\varsigma})=\sum_{\zeta=1}^{\hbar}\log(1-e^{-z_\zeta})+\sum_{\zeta=1}^{\hbar}\frac{1+e^{-\beta\,z_\zeta}}{\beta\,e^{-(\beta-1)\,z_\zeta}+\alpha+(\alpha-\beta)\,e^{-\beta\,z_\zeta}}$$

and

$$U_\beta(\underline{\varsigma})=\sum_{\zeta=1}^{\hbar}\frac{(1-z_\zeta\,\beta)e^{-(\beta-1)\,z_\zeta}-[(\alpha-\beta)z_\zeta+1]\,e^{-\beta\,z_\zeta}}{\beta\,e^{-(\beta-1)\,z_\zeta}+\alpha+(\alpha-\beta)\,e^{-\beta\,z_\zeta}}-2\sum_{\zeta=1}^{\hbar}\frac{z_\zeta\,e^{-\beta\,z_\zeta}}{1+e^{-\beta\,z_\zeta}},$$

3.2. Different methods

•Methods of Least and weighted least squares

Swain et al. (1988) first developed ”the Least Squares (LSE) and weighted Least Squares Estimators (WLSE)” by minimizing the following two functions , respectively,

$$S_{\text{LSE}}(\underline{\varsigma})=\sum_{\zeta=1}^{\hbar}\left(F\left(t_{\zeta:\hbar};(\underline{\varsigma})\right)-\frac{1}{\hbar+1}\zeta\right)^2|_{\zeta=1,2,\dots,\hbar}$$

and

$$S_{\text{WLSE}}(\underline{\varsigma})=\sum_{\zeta=1}^{\hbar}\xi_\zeta(\hbar)\left(F\left(t_{\zeta:\hbar};\underline{\varsigma}\right)-\frac{1}{\hbar+1}\zeta\right)^2,$$

where

$$\xi_{\zeta}\left(\hbar\right)=\frac{(\hbar+1)^{2}(\hbar+2)}{\zeta(\hbar-\zeta+1)}$$

•Method of Cramér–von–Mises

The Cramér–von–Mises Estimator (CME) (see Choi and Bulgren (1968)) are determined by minimizing

$$S_{\text{CME}}\left(\underline{\varsigma}\right)=\frac{1}{12\hbar}+\sum_{\zeta=1}^{\hbar}\left(F\left(t_{\zeta:\hbar};\underline{\varsigma}\right)-\frac{1}{2\hbar}\left(2\,\zeta-1\right)\right)^{2}.$$

•Methods of Anderson–Darling and right-tailed Anderson–Darling

The Anderson Darling (ADE) and Right-Tailed estimators (RTADE) (see Anderson and Darling (1952)) can be obtained by minimizing

$$S_{\text{ADE}}\left(\underline{\varsigma}\right)=-\hbar-\frac{1}{\hbar}\sum_{\zeta=1}^{\hbar}(2\zeta-1)\left[\log F\left(t_{\zeta};\underline{\varsigma}\right)+\log\overline{F}\left(t_{\hbar+1-\zeta};\underline{\varsigma}\right)\right]$$

and

$$S_{\text{RTADE}}\left(\underline{\varsigma}\right)=\frac{\hbar}{2}-2\sum_{\zeta=1}^{\hbar}F\left(t_{\zeta};\underline{\varsigma}\right)-\frac{1}{\hbar}\sum_{\zeta=1}^{\hbar}(2\zeta-1)\log\overline{F}\left(t_{\hbar+1-\zeta};\underline{\varsigma}\right),$$

where $\overline{F}\left(\cdot\right)=1-F\left(\cdot\right)$.

•Method of maximum product of spacings

The MPSs (see Cheng , Amin (1979, 1983)) are obtained by maximizing

$$G\left(\underline{\varsigma}\right)=\left[\prod_{\zeta=1}^{\hbar+1}D_{\zeta}(\underline{\varsigma})\right]^{\frac{1}{\hbar+1}},$$

where

$$D_{\zeta}(\underline{\varsigma})=F\left(t_{\zeta:\hbar};\underline{\varsigma}\right)-F\left(t_{\zeta-1:\hbar};\underline{\varsigma}\right)|_{\zeta=1,2,\ldots,\hbar},$$

and $F\left(t_{0:\hbar};\underline{\varsigma}\right)=0$ and $F\left(t_{\hbar+1:\hbar};\underline{\varsigma}\right)=1$.

3.3. Simulation study

To examine the above-mentioned estimators, we look at the one model that was applied in this part and look at the MSE of those estimators for various samples. as example, given what has been described before, for $(\underline{\varsigma})=(0.9,0.6),(2,1),(3.1,0.4)$.

The review of each way of parameter estimations for the MHL distribution with regards to sample of volume of n is supposed. To do this, a simulation analysis is made based on the steps that follow:

1. **Step 1.**Create 10,000 samples of volume n from (1). This operation is carried out using the quantile function and generated data of uniform distribution.
2. **Step 2.** Calculate the estimates for the 10,000 samples, say $\left(\hat{\alpha}_{\zeta},\hat{\beta}_{\zeta}\right)$ for $\zeta=1,2,...,10^{4}$.
3. **Step 3.** Calculate the biases and mean squared errors as follows

$$Bias_{\varepsilon}(\hbar)=\frac{1}{10000}\sum_{\zeta=1}^{10000}\left(\hat{\varepsilon}_{\zeta}-\varepsilon\right)$$

and

$$MSE_{\varepsilon}(\hbar) = \frac{1}{10000}\sum_{\zeta=1}^{10000}\left(\hat{\varepsilon}_{\zeta} - \varepsilon\right)^2$$

We computed the $bias_{\varepsilon}(\hbar)$ and the $MSE_{\varepsilon}(\hbar)$ for $\varepsilon = \varsigma$ and $\hbar = 30, 70, \cdots 500$ using the optim function and the well-known Nelder-Mead method by statistical package R. Figures 4-6 display the results.

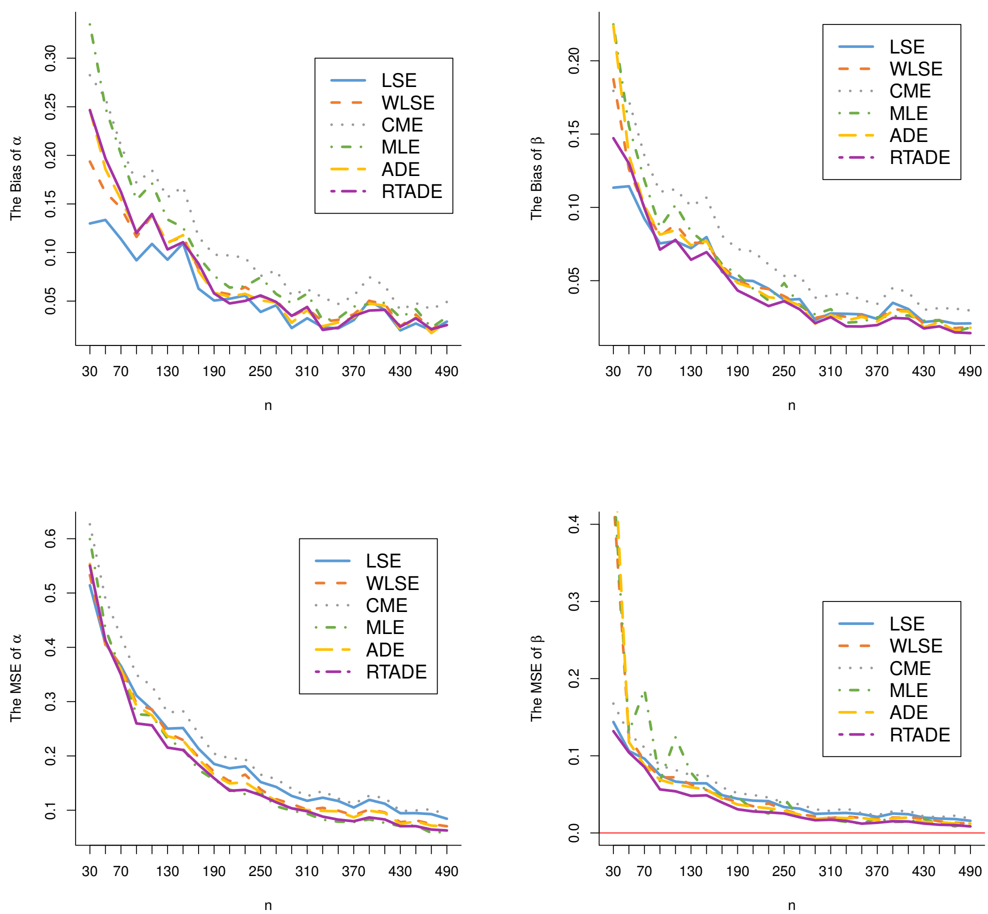

Figure 4. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (0.9, 0.6)$

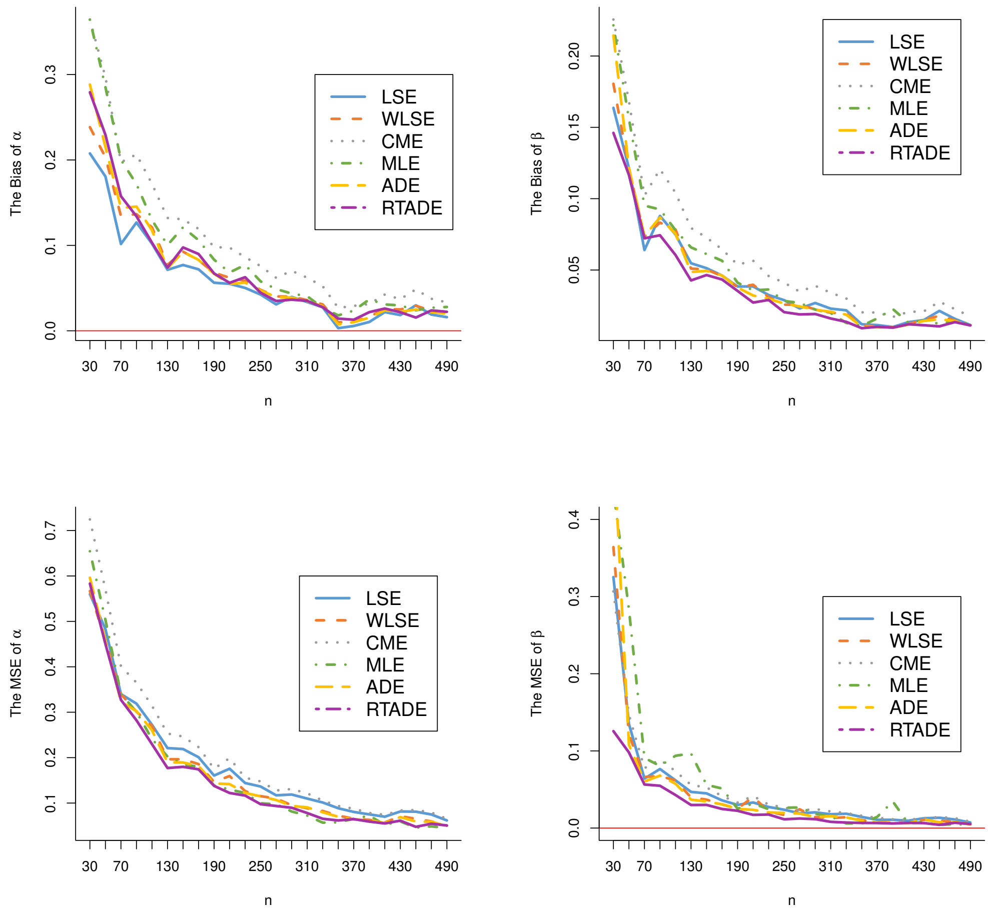

Figure 5. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (2, 1)$

As can be seen in the MSE plots for two parameters, as the sample volume grows, both methods approach zero, proving the truth of these numerical computations and estimation methods for the MHL distribution parameters. Moreover,

- for estimating α, the LSE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating β, the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating α,the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating β, the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.

3.4. Applications

In this part, two applications are presented by MHL model and a number of famous models. The well-known statistical tests ”Anderson-Darling (A^*) and Cramér–von Mises (W^*)”, are selected for comparison. The

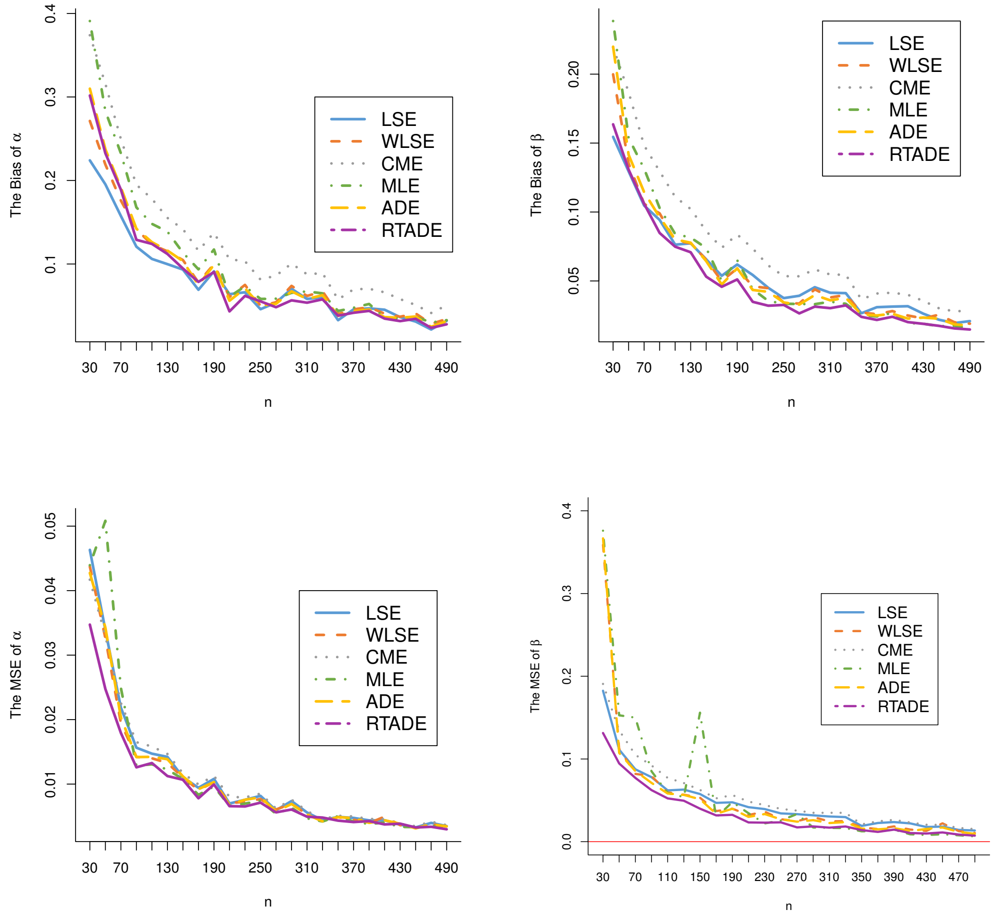

Figure 6. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (3.1, 0.4)$

exponentiated half-logistic (ESHL) distribution (Kang and Seo, 2011), Kumaraswamy standard Half-Logistic distribution (KwSHL) (Cordeiro and de Castro (2011)) , the Beta standard Half-Logistic (BSHL) (Jones (2004)), McDonald standard Half-Logistic (McSHL) model (Oliveria et.al (2016)), new standard odd log-logistic HL (NOLL-SHL) distribution (Alizadeh et al. (2019)), weibull distrbution (W), Generalized Exponential (GE) distribution (Kundu and Gupta (1998)), Log Normal (LN) distribution, Gamma (Ga) distribution, Lindley (Li) distribution (Ghitany et al. (2008)), the Power-Lindley (PL) model (Ghitany et al. (2013)) and the Nadarajah-Haghighi (NH) model (Nadarajah and Haghighi (2011)) have been have been selected for comparison in two examples for comparison in two examples. All competitive CDFs can be found in the in Appendix. Based on his bestselling fame, the MLE method is used for model parameters estimation.

3.5. Data set I

The first data are the failure times of 20 Electric Bulbs were reported by Murthy (2004, p180) . The data are: 1.32, 12.37, 10.56, 21.82, 19.61, 36.63, 0.39, 21.35, 12.62, 5.36, 7.71, 7.22, 6.56, 5.05, 11.58, 3.60, 1.33, 3.53, 12.42, 8.92. Table 1 gives all estination results for this real life data set. Due to Table 1, the MHL distribution is chosen as the best HL version for modeling the failure times data. The histograms of the failure times data set and the shapes of fitted PDF are displayed in figure 7.

Table 1. Results for data set I

model	estimated parameters (se)			W^*	A^*	p-value
MHL (α, β)	3.367 (0.881)	0.227 (0.049)		**0.043**	**0.293**	**0.968**
NOLL-SHL (α, β)	2.852 (0.613)	0.246 (0.053)		10.751	58.781	$< 2.2 \times 10^{-16}$
ESHL (α)	3.454 (0.620)			0.104	0.635	0.007
KwSHL (α, β)	1.453 (0.510)	0.298 (0.059)		0.107	0.650	0.306
BSHL (α, β)	1.313 (0.396)	0.297 (0.060)		0.110	0.670	0.324
McSHL (α, β, c)	0.126 (0.079)	0.220 (0.063)	18.307 (13.132)	0.044	0.303	0.914
Li(α)	0.403 (0.052)			0.135	0.819	0.380
PL(α, β)	0.419 (0.094)	0.975 (0.117)		0.133	0.806	0.457
GE(α, β)	0.309 (0.066)	1.560 (0.404)		0.110	0.669	0.411
NH(α, β)	0.096 (0.109)	1.879 (1.592)		0.141	0.850	0.598
LN(α, β)	1.072 (0.159)	0.888 (0.112)		0.062	0.397	0.771
Ga(α, β)	1.487 (0.343)	0.350 (0.096)		0.114	0.692	0.406
W(α, β)	0.155 (0.057)	1.227 (0.169)		0.127	0.767	0.438

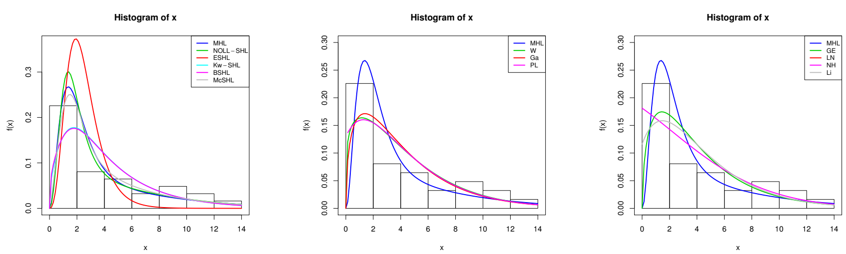

Figure 7. Histogram and fitted pdfs for data set I.

3.6. Data set II

. The second real data set is related to failure time of 50 components (unit: 1000 hours) were reported by Murthy (2004, p100). The data are: 0.036, 0.058, 0.102, 0.103, 0.114, 0.116, 0.061, 0.074, 0.078, 0.086, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 10.94, 11.02, 13.88, 1.600, 0.148, 0.183, 0.192, 0.254, 0.262,

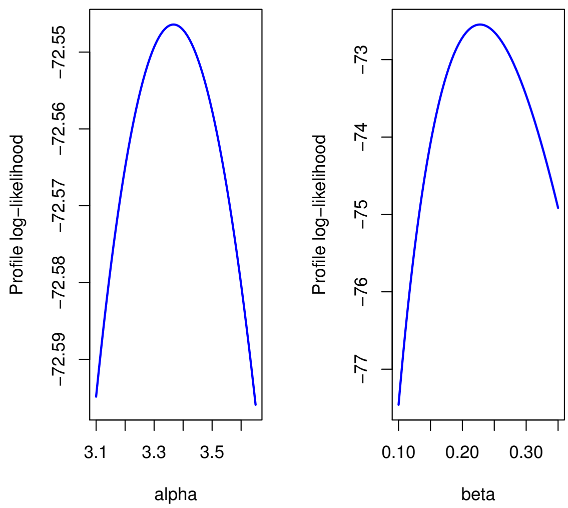

Figure 8. Unimodality of profile likelihood functions of parameters for first data set I.

0.379, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 14.73, 15.08. Tables 2 gives all estination results for this real life data set. The histograms of the second failure times data set and the relevant shapes of fitted PDF are displayed in figure 8. Due to Table 2, the MHL distribution is chosen as the best HL version for modeling the failure times data.

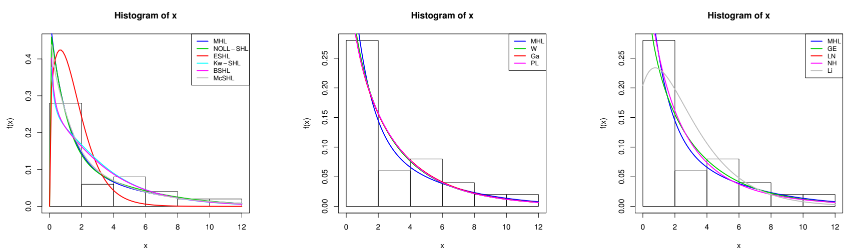

Figure 9. Histogram and fitted pdfs for data set II.

4. Conclusions

In this article, we presented and analysed a novel two-parameter version of the well-known Half-Logistic lifetime model. The new model is called the modified Half-Logistic lifetime model. The hazard function of modified Half-Logistic lifetime model can be "monotonically-increasing","monotonically-decreasing", "bathtub (**U**)", "upside down" and "upside down-bathtub (upside down-**U**)". The PDF of the modified Half-Logistic lifetime model can be "monotonically and right skewed with no peak" and "monotonically and right skewed with one peak". The hazard function, quantile function, asymptotic, linear combination, extreme value, moments, residual entropies, incomplete moments, moment generating function and order statistics, all theoretical properties of this model that are derived and discussed in depth. By performing a simulation analysis, The various techniques of estimation are

Table 2. Results for data set II

model	estimated parameters (se)			W^*	A^*	p-value
MHL (α, β)	0.975 (0.206)	0.298 (0.051)		**0.068**	**0.407**	**0.806**
NOLL-SHL (α, β)	1.001 (0.171)	0.327 (0.057)		18.04	100.03	$< 1.2 \times 10^{-16}$
ESHL (α)	1.229 (0.713)			0.157	0.865	0.0005
KwSHL (α, β)	0.584 (0.151)	0.349 (0.055)		0.157	0.860	0.207
BSHL (α, β)	0.624 (0.124)	0.337 (0.056)		0.147	0.806	0.280
McSHL (α, β, c)	0.151 (0.081)	0.271 (0.061)	5.595 (3.413)	0.082	0.477	0.668
Li(α)	0.565 (0.058)			0.135	0.741	0.012
PL(α, β)	0.770 (0.108)	0.759 (0.079)		0.108	0.606	0.511
GE(α, β)	0.301 (0.058)	0.813 (0.144)		0.103	0.573	0.499
NH(α, β)	0.615 (0.370)	0.717 (0.227)		0.090	0.528	0.497
LN(α, β)	0.344 (0.198)	1.406 (0.140)		0.081	0.654	0.712
Ga(α, β)	0.821 (0.142)	0.283 (0.066)		0.102	0.571	0.498
W(α, β)	0.413 (0.084)	0.881 (0.099)		0.096	0.547	0.534

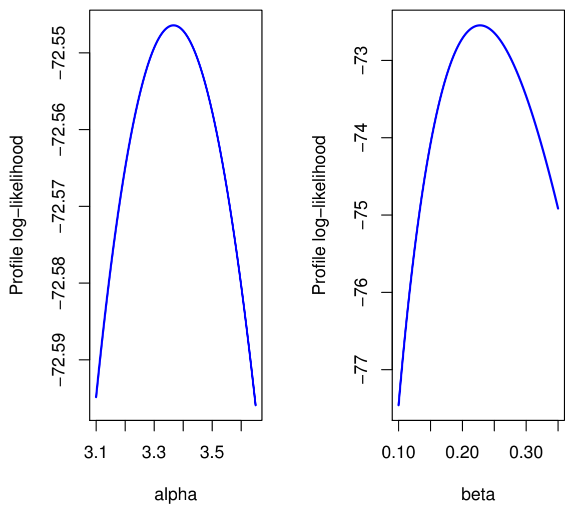

Figure 10. Unimodality of profile likelihood functions of parameters for first data set II.

compared to the estimates of the maximum likelihood of parameters. Finally, to illustrate the goals of this article, two actual data sets have been applied.

Appendix: CDF of competitive models in application section

$$F_{ESHL}(z;\alpha)=\left(\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}\right)^{\alpha}|_{z,\alpha\in\mathbf{R}^+},$$

$$F_{NOLL-SHL}(z;\alpha,\beta)=\frac{\left(\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}\right)^{\alpha}}{\left(\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}\right)^{\alpha}+\left(1-\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}\right)^{\beta}}|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{\kappa wSHL}(z;\alpha,\beta)=1-\left[1-\left(\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}\right)^{\alpha}\right]^{\beta}|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{BSHL}(z;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\int_0^{\frac{1-\mathrm{e}^{-z}}{e^{-z}+1}}t^{-1+\alpha}(1-t)^{\beta-1}dt|_{z,\alpha,\beta\in\mathbf{R}^+},$$

where $B(\vartheta_1,\vartheta_2)=\int_0^1 u^{\vartheta_1-1}(1-u)^{\vartheta_2-1}du$ denote the beta function.

$$F_{McSHL}(z;\alpha,\beta,c)=\frac{1}{B(\alpha,\beta)}\int_0^{\left(\frac{1-e^{-z}}{e^{-z}+1}\right)^{c}}t^{-1+\alpha}(1-t)^{\beta-1}dt|_{z,\alpha,\beta,c\in\mathbf{R}^+},$$

$$F_{Li}(z;\alpha)=1-\left(1+\frac{1}{1+\alpha}\alpha\,z\right)e^{-\alpha\,z}|_{z,\alpha\in\mathbf{R}^+}$$

$$F_{PL}(z;\alpha,\beta)=1-\left(1+\frac{1}{1+\alpha}\alpha\,z^{\beta}\right)e^{-\alpha\,z^{\beta}}|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{GE}(z;\alpha,\beta)=(1-e^{-\alpha\,z})^{\beta}|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{NH}(z;\alpha,\beta)=1-e^{1-(1+\alpha\,z)^{\beta}}|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{LN}(z;\alpha,\beta)=N\left(\frac{1}{\beta}\left(\log(z)-\alpha\right)\right)|_{z,\alpha,\beta\in\mathbf{R}^+},$$

where $N(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt$ denote the CDF of standard Normal random variable.

$$F_{Ga}(z;\alpha,\beta)=\frac{1}{\Gamma(\alpha)}\int_0^{z}t^{-1+\alpha}e^{-\beta\,t}dt|_{z,\alpha,\beta\in\mathbf{R}^+},$$

$$F_{W}(z;\alpha,\beta)=1-e^{-\alpha\,z^{\beta}}|_{z,\alpha,\beta\in\mathbf{R}^+}.$$

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