STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **10**, March 2022, pp 519–547.
Published online in International Academic Press (www.IAPress.org)

A New Lifetime Distribution with Properties, Characterizations, Validation Testing and Different Estimation Methods

Haitham M. Yousof[1,*], M. Masoom Ali[2], Khaoula Aidi[3], G.G. Hamedani [4],and Mohamed Ibrahim [5]

[1] *Department of Statistics, Mathematics and Insurance, Benha University, Egypt*
[2] *Department of Mathematical Sciences Ball State University, Muncie, Indiana 47306 USA*
[3]*Laboratory of probability and statistics LaPS,University Badji Mokhtar, Annaba, Algeria*
[4] *Department of Mathematical and Statistical Sciences, Marquette University, USA*
[5]*Department of Applied, Mathematical and Actuarial Statistics, Faculty of Commerce, Damietta University, Damietta, Egypt*

Abstract A new lifetime distribution is proposed and its properties are studied. The new density function has a heavy right skew tails with different shapes. The new failure rate function can be “constant”, “bathtub (**U**)”, “increasing-constant”, “decreasing-constant”, “upside down” and “upside down-**U**”. Complexity of the most integrals related to statistical properties is solved and numerically analyzed. Simple type copula is presented. Numerical calculations for analyzing the skewness and kurtosis are presented. Different estimation methods such as the maximum likelihood estimation method, Cramér-von-Mises estimation method, ordinary least square estimation method, weighted least square estimation method, Anderson Darling estimation method, right tail Anderson Darling estimation method and left tail-Anderson Darling estimation method are considered. Numerical simulation studies are performed. An example of environmental real data set is employed to compare the estimation methods. Another example is presented to measure importance and flexibility of the new model. Using the validation approach proposed by Bagdonavicius and Nikulin (2011) for censored data, we propose the construction of modified chi-square goodness-of-fit tests for the new model. Based on the maximum likelihood estimators on initial data, the modified statistics recover the information lost while grouping data and follow chi-square models. All elements of the modified criteria tests are given explicitly. Numerical example from simulated samples and four real data sets have been analyzed to illustrate the feasibility of the modified test.

Keywords Validation; Morgenstern Family; Modeling; Simulation; Right Tail Anderson Darling; Nadarajah Haghighi; Left Tail-Anderson Darling; Censorship.

AMS 2010 subject classifications: 62N01; 62N02; 62E10.

DOI: 10.19139/soic-2310-5070-1222

1. Introduction and motivation

Among the parametric distributions, the exponential (Exp) model is perhaps the most widely applied model in several fields. The Exp model has “constant” hazard rate function (HRF) which is a useful property in reliability analysis. A new useful generalization of the standard Exp model as an alternative to the gamma (Gam), Weibull (W) and exponentiated exponential (ExpExp) distributions was recently proposed and studied by Nadarajah and Haghighi (2011). The cumulative distribution function (CDF) of the Nadarajah and Haghighi (NH) model is given by

$$G_{a,c}(\mathbf{y}) = 1 - \mathrm{e}^{1-(1+c\mathbf{y})^{a}}|_{\mathbf{y}>0}, \tag{1}$$

*Correspondence to: Haitham M. Yousof (Email: haitham.yousof@fcom.bu.edu.eg).Department of Statistics, Mathematics and Insurance, Benha University, Egypt.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

and the corresponding probability density function (PDF) is

$$g_{a,c}(\mathbf{y}) = ac\,(1+c\mathbf{y})^{a-1}\mathrm{e}^{1-(1+c\mathbf{y})^{a}}|_{\mathbf{y}>0}, \tag{2}$$

where the parameter $a > 0$ controls the shape of the distribution and $c > 0$ is the scale parameter. For $a = 1$, the NH distribution reduces to standard Exp the model. For $c = 1$, the NH distribution reduces to the one parameter NH model. Lemonte (2013) proposed a new exponential-type model called the generalized NH (GNH). Ortega et al. (2015) investigated and studied the Gamma NH (GamNH). Lemonte et al. (2016) proposed and studied the Marshall-Olkin-NH (MONH). Yousof et al. (2017) derived and studied the odd Lindley NH (OLNH) distribution. Yousof and Korkmaz (2017) studied the Topp-Leone NH (TLNH) model. Dias et al. (2018) introduced and studied the beta NH (BNH). Recently, Ibrahim (2020) studied the odd log-logistic NH (OLLNH) and Proportional reversed hazard rate (PRHRNH) models with its corresponding statistical properties and different methods of estimation and Shehata and Yousof (2021) presented and studied a novel two-parameter Nadarajah-Haghighi extension called the Nadarajah-Haghighi distribution.

In this paper, we propose and study a new lifetime model called the Topp Leone Generated Nadarajah Haghighi model based on Rezaei et al. (2017). After studying its properties, different classical estimation methods such as the maximum likelihood estimation method, Cramér-von-Mises estimation method, ordinary least square estimation method, weighted least square estimation method, Anderson Darling estimation method, right tail Anderson Darling estimation method and left tail-Anderson Darling estimation method are considered and the a numerical simulation studies are performed. A real data set related to remission times (in months) of a random sample of 128 bladder cancer patients is employed to compare the estimation methods. Another example is presented to measure importance and flexibility of the new model. Using the validation approach proposed by Bagdonavicius and Nikulin (2011) for censored data, we propose the construction of modified chi-square goodness-of-fit tests for the new model. Based on the maximum likelihood estimators on initial data, the modified statistics recover the information lost while grouping data and follow chi-square models. All elements of the modified criteria tests are given explicitly. Numerical example from simulated samples and real data (survival data on 26 psychiatric inpatients admitted to the university of Iowa hospitals) have been presented to illustrate the feasibility of the modified test.

The PDF and CDF of the Topp Leone Generated G (TLGG) family of distributions are given by

$$f_{\alpha,\theta,\underline{\varphi}}(\mathbf{y}) = 2\alpha\theta g_{\underline{\varphi}}(\mathbf{y})\, G_{\underline{\varphi}}(\mathbf{y})^{\theta\alpha-1}\left[1-G_{\underline{\varphi}}(\mathbf{y})^{\theta}\right]\left[2-G_{\underline{\varphi}}(\mathbf{y})^{\theta}\right]^{\alpha-1}|_{\mathbf{y}\in\mathbb{R}}, \tag{3}$$

and

$$F_{\alpha,\theta,\underline{\varphi}}(\mathbf{y}) = \left\{G_{\underline{\varphi}}(\mathbf{y})^{\theta}\left[2-G_{\underline{\varphi}}(\mathbf{y})^{\theta}\right]\right\}^{\alpha}|_{\mathbf{y}\in\mathbb{R}}, \tag{4}$$

respectively. For $\theta = 1$, we get the TLG family. By inserting (1) and (2) into (3), we can write the PDF of the Topp Leone Generated Nadarajah Haghighi (TLGNH) model as

$$\begin{aligned} f(\mathbf{y}) &= 2\alpha\theta ac\,(1+c\mathbf{y})^{a-1}\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\left[1-\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\right]^{\theta\alpha-1} \\ &\times\left\{1-\left[1-\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\right]^{\theta}\right\}\left\{2-\left[1-\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\right]^{\theta}\right\}^{\alpha-1}|_{\mathbf{y}>0}. \end{aligned} \tag{5}$$

and the corresponding CDF is given by

$$F(\mathbf{y}) = \left(\left[1-\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\right]^{\theta}\left\{2-\left[1-\mathrm{e}^{1-(1+c\mathbf{y})^{a}}\right]^{\theta}\right\}\right)^{\alpha}|_{\mathbf{y}>0}. \tag{6}$$

For $\theta = 1$, TLGNH reduces TLNH (see Yousof and Korkmaz (2017)). Let $Y_{i,j}$ denote a variable of interest for the jth sub-system associated with the ith system. Suppose $Y_{i,j}$ are independent and identical random variables with CDF (1). The variable $Y_{i,j}$ of interest for the system will be

$$Y = \max_{(1<i<\alpha)} \left(\min_{(j=1,2)} Y_{i,j} \right).$$

It is not difficult to show that the CDF of Y is (6). We provide some plots of the PDF and hazard rate function (HRF) of the TLGNH model to show its flexibility. Figure 1(a) displays some plots of the TLGNH PDF for some parameter values of α, θ, a and c. The Plots of hazard rate function (HRF) of the TLGNH model for some parameter values of α, θ, a and c are shown in Figure 1(b). The CDF in (6) can be expressed as

$$F(\mathbf{y}) = \sum_{\kappa=0}^{\infty} \Omega_{(\kappa)} \; H_{(\theta^{\bullet},a,c)}(\mathbf{y}), \tag{7}$$

where $\theta^{\bullet} = (\alpha + \kappa)\,\theta$, $\Omega_{(\kappa)} = (-1)^{\kappa} \left(\frac{1}{2}\right)^{-\alpha+\kappa} \binom{\alpha}{\kappa}$ and $H_{(\theta^{\bullet},a,c)}(\mathbf{y}) = [G(\mathbf{y};a,c)]^{\theta^{\bullet}}$ is the CDF of the ExpNH distribution with power parameter $\theta^{\bullet}$. The corresponding TLGNH density function is obtained by differentiating (7) as

$$f(\mathbf{y}) = \sum_{\kappa=0}^{\infty} \Omega_{(\kappa)} \; \mathbf{h}_{(\theta^{\bullet},a,c)} \left(\mathbf{y}\right), \tag{8}$$

where $\mathbf{h}_{(\theta^{\bullet},a,c)} \left(\mathbf{y}\right) = \theta^{\bullet} g_{a,c}(\mathbf{y}) G_{a,c}(\mathbf{y})^{\theta^{\bullet}-1}$ is the ExpNH density with power parameter $\theta^{\bullet}$.Thus, several of its structural properties can be obtained from Equation (8) and those properties of the ExpNH distribution. Figure 1(a) shows that the PDF of the new version has a right skew tail with different shapes. Figure 1(b) shows that the HRF of the new version has many important failure rates such as the constant ($\alpha = 1$, $\theta = 1$, $a = 1$ and $c = 1$), increasing-constant ($\alpha = 1$, $\theta = 2$, $a = 1$ and $c = 1$), decreasing-constant ($\alpha = 0.01$, $\theta = 0.05$, $a = 1$ and $c = 1$), **U** shape ($\alpha = 0.02$, $\theta = 0.75$, $a = 1.42$ and $c = 1$), upside-down-constant shape ($\alpha = 3$, $\theta = 1$, $a = 1$ and $c = 1$), upside down-**U** shape ($\alpha = 3.25$, $\theta = 1$, $a = 1.4$ and $c = 1$).

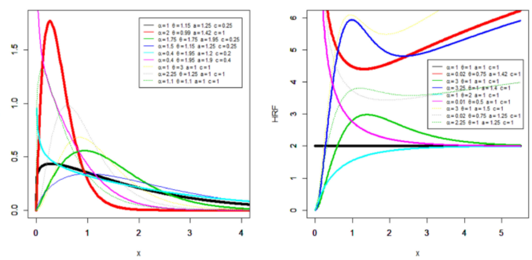

Figure 1. Plots of the TLGNH PDF and HRF for some parameter values.

2. Mathematical properties

2.1. Moments

The $\mathbf{r}^{th}$ ordinary moment of Y is given by $\mu'_{\mathbf{r}} = \mathbf{E}(Y^{\mathbf{r}}) = \sum\limits_{\kappa=0}^{\infty} \Omega_{(\kappa)} \int_{-\infty}^{\infty} \mathbf{y}^{\mathbf{r}} \, \mathbf{h}_{(\theta^{\bullet},a,c)} \left(\mathbf{y}\right) d\mathbf{y}$. Then we obtain

$$\mu'_{\mathbf{r}} = \sum_{\kappa,\mathbf{v}=0}^{\infty} \sum_{\tau=0}^{\mathbf{r}} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})} \Gamma \left(\frac{\tau}{a} + 1, 1 + \mathbf{v} \right), \tag{9}$$

where $\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})} = \Omega_{(\kappa)} \varsigma_{\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})}$, $\varsigma_{\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})} = \frac{\theta^{\bullet}}{c^{\mathbf{r}}} (-1)^{\mathbf{r}+\mathbf{v}-\tau} \, \mathrm{e}^{1+\mathbf{v}} \left(1+\mathbf{v}\right)^{-\left(\frac{\tau}{a}+1\right)} \binom{\theta^{\bullet}-1}{\mathbf{v}} \binom{\mathbf{r}}{\tau}$ and $\Gamma\left(a, \mathbf{y}\right) = \int_{\mathbf{y}}^{\infty} z^{a-1} \mathrm{e}^{-z} dz$ denotes the complementary incomplete gamma function, which can be evaluated in R, MATHEMATICA, etc. The moments in (9) reduce to ($\forall \, \theta^{\bullet} > 0$ integer)

$$\mu'_{\mathbf{r}} = \sum_{\kappa=0}^{\infty} \sum_{\mathbf{v}=0}^{\theta^{\bullet}-1} \sum_{\tau=0}^{\mathbf{r}} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet};\mathbf{r})} \Gamma\left(\frac{\tau}{a}+1, 1+\mathbf{v}\right), \tag{10}$$

When $\mathbf{r}=1$ in (9) or (10), we have the mean of Y. The $\mathbf{r}^{th}$ central moment of Y, say $\mu_{\mathbf{r}}$, is $\mu_{\mathbf{r}} = \mathbf{E}(Y-\mu)^{\mathbf{r}} = \sum_{h=0}^{\mathbf{r}} (-1)^h \binom{\mathbf{r}}{h} (\mu'_1)^{\mathbf{r}} \mu'_{\mathbf{r}-h}$.

Table 1: $E(Y)$, $\mathbf{v}(Y)$, $\mathbf{s}(Y)$ and $\mathbf{k}(Y)$ of the TLGNH distribution.

α	θ	a	c	$E(Y)$	$\mathbf{v}(Y)$	$\mathbf{s}(Y)$	$\mathbf{k}(Y)$
0.005	1	1	1	0.0012865	0.0008056	43.03816	2722.165
0.01				0.0125722	0.0078321	13.73513	279.9879
0.05				0.0573306	0.0349897	6.377075	62.74027
0.1				0.1040020	0.0621733	4.697541	35.42657
0.5				0.3265344	0.1790888	2.55775	12.62371
1				0.5	0.25	2	9
1.5				0.665002	0.2803381	1.784458	7.855008
2				0.854444	0.2552210	2.136105	8.477581
0.1	0.1	2	0.5	0.001755	0.0006831	29.09206	1191.531
	0.5			0.031439	0.0135368	6.684257	63.90605
	1			0.091814	0.0403234	3.767063	21.77376
	2			0.232250	0.0932072	2.214706	9.137817
	5			0.625637	0.1266310	2.187207	5.712456
	7.5			0.891217	0.0418087	22.99588	−39.4803
1.15	1	0.5	1.25	1.444262	3.5095720	3.856576	31.50422
		1		0.438694	0.1686134	1.904516	8.488894
		2		0.184030	0.0226402	1.346256	5.232966
		5		0.066827	0.0025772	1.081866	4.117977
		10		0.032396	0.0005781	1.002535	3.83987
		30		0.010582	5.982×10^{-5}	0.396782	6.822024
		50		0.006324	2.171×10^{-5}	2.725248	−3.634929
		75		0.004207	4.568×10^{-5}	11.64966	17.63234
1.1	2	2	0.01	37.96826	469.27550	0.816713	3.813323
			0.1	3.796826	4.6927550	0.816713	3.813323
			0.5	0.759365	0.1877102	0.816713	3.813323
			1	0.379683	0.0469276	0.816713	3.813323
			10	0.037968	0.0004693	0.816714	3.815184
1	2.5	2.5	5	0.064986	0.0010394	0.6880018	3.465930
1.5	1.5	2.5	5	0.055744	0.0009465	0.8364045	3.774259
1.5	1.5	1.5	0.5	1.036926	0.3895478	1.080701	4.620780
0.05	1.5	1.5	0.05	1.263646	8.6120670	4.050257	25.12157
0.05	0.5	0.5	0.05	0.978414	46.636750	20.88281	862.6479
0.05	0.05	0.5	0.05	0.013230	0.5271864	184.4357	69559.24
0.01	0.01	0.5	0.05	0.000114	0.0043332	2000.385	8271313

Table 2: $E(Y)$, $\mathbf{v}(Y)$, $\mathbf{s}(Y)$ and $\mathbf{k}(Y)$ of the NH distribution.

a	c	$E(Y)$	$\mathbf{v}(Y)$	$\mathbf{s}(Y)$	$\mathbf{k}(Y)$
0.5	0.5	4	160	4.869908	48.96
0.75		3.286205	14.30818	2.686034	14.90616
1		2	4	2	9
2		0.7578722	0.394141200	1.2539130	4.772774
5		0.2614665	0.038373660	0.9138792	3.524559
10		0.1247885	0.008206707	0.8134962	3.230566
20		0.0609891	0.001900823	0.7653901	3.135202
30		0.0403548	0.000823778	0.7496473	3.325968
40		0.0301529	0.000457592	0.7415721	2.889868
50		0.0240682	0.000290664	0.7515864	3.762883
75		0.0159976	0.000127897	1.5130750	0.554984
100		0.0119803	7.15836×10^{-5}	0.3326859	6.986337
10	0.01	6.2394260	20.516770	0.8134962	3.230566
	0.1	0.6239426	0.2051677	0.8134962	3.230566
	0.5	0.1247885	0.0082067	0.8134962	3.230566
	1	0.0623943	0.0020517	0.8134962	3.230566
	2	0.0311971	0.0005129	0.8131856	2.861325
	5	0.0124789	8.207×10^{-5}	1.2185510	4.845482
	10	0.0062394	1.996×10^{-5}	4.0543040	−6.663687
	12.5	0.0049915	1.896×10^{-5}	1.5014750	2.38749
	15	0.0041596	1.734×10^{-6}	49.367260	73.36594

2.2. Numerical analysis for some measures

Numerical calculations for analyzing the $E(Y)$, variance $(\mathbf{v}(Y))$, skewness $(\mathbf{s}(Y))$ and kurtosis $(\mathbf{k}(Y))$ are calculated in Table 1 for some selected values of parameter. Based on Table 1 we note that $\mathbf{s}_{\text{TLGNH}}(Y)$ is always positive, $\mathbf{k}_{\text{TLGNH}}(Y)$ can be more than 3 or less than 3. Based on Tables 1 and 2 we note that, the $\mathbf{s}_{\text{TLGNH}}(Y) \in (0.397, 2000.4)$, whereas the $\mathbf{s}_{\text{NH}}(Y) \in (0.741, 49.367)$. Further, the spread of $\mathbf{k}_{\text{TLGNH}}(Y)$ is ranging from -39.480 to 8271313, whereas $\mathbf{k}_{\text{NH}}(Y)$ only varies only from -6.664 to 73.365.

2.3. Moment generating function

The moment generating function (M·G·F) $M_Y(\tau) = \mathbf{E}\left(\mathrm{e}^{\tau Y}\right)$ of Y can be derived from equation (9) or (10) as

$$M_Y(\tau) = \sum_{\kappa,\mathbf{v},\mathbf{r}=0}^{\infty} \sum_{\tau=0}^{\mathbf{r}} \frac{\tau^{\mathbf{r}}}{\mathbf{r}!} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})} \Gamma\left(\frac{\tau}{a}+1, 1+\mathbf{v}\right),$$

or $\forall\, \theta^{\bullet} > 0$ integer we have

$$M_Y(\tau) = \sum_{\kappa,\mathbf{r}=0}^{\infty} \sum_{\mathbf{v}=0}^{\theta^{\bullet}-1} \sum_{\tau=0}^{\mathbf{r}} \frac{\tau^{\mathbf{r}}}{\mathbf{r}!} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})} \Gamma\left(\frac{\tau}{a}+1, 1+\mathbf{v}\right).$$

2.4. Incomplete moments

The s^{th} incomplete moment, say $\mathbf{I}_{s,Y}(\tau)$, of Y can be expressed from (8) as

$$\mathbf{I}_s(\tau) = \sum_{\kappa=0}^{\infty} \Omega_{(\kappa)} \int_{-\infty}^{\tau} \mathbf{y}^s\, \mathbf{h}_{(\theta^{\bullet},a,c)}(\mathbf{y})\, d\mathbf{y}$$

and then

$$\mathbf{I}_{s,Y}(\tau)=\sum_{\kappa,\mathbf{v}=0}^{\infty}\sum_{\tau=0}^{s}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},s)}\left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right)-\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})(1+c\,\tau)^{a}\right)\right],$$

or $\forall\,\theta^{\bullet}>0$ integer we have

$$\mathbf{I}_{s,Y}(\tau)=\sum_{\kappa=0}^{\infty}\sum_{\mathbf{v}=0}^{\theta^{\bullet}-1}\sum_{\tau=0}^{s}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},s)}\left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right)-\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})(1+c\,\tau)^{a}\right)\right].$$

2.5. The moment of the residual life

The m^{th} moment of the residual life can be written as $K_{m,Y}(\tau)|_{(Y>\tau \text{ and } m=1,2,\ldots)}=\mathbf{E}[(Y-\tau)^{m}]$ which can be expressed as

$$K_{m,Y}(\tau)|_{(Y>\tau \text{ and } m=1,2,\ldots)}=\frac{1}{1-F(\tau)}\int_{\tau}^{\infty}(Y-\tau)^{m}dF(\mathbf{y}).$$

Then we have

$$K_{m,Y}(\tau)=\frac{1}{1-F(\tau)}\sum_{\kappa,\mathbf{v}=0}^{\infty}\sum_{\tau=0}^{m}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},m)^{\bullet}}\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right),$$

where $\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},m)^{\bullet}}=\Omega_{(\kappa)}^{\bullet}\zeta_{\mathbf{v},\tau}^{(\theta^{\bullet},m)}$ and $\Omega_{(\kappa)}^{\bullet}=\Omega_{(\kappa)}\sum_{\mathbf{r}=0}^{m}\binom{m}{\mathbf{r}}(-\tau)^{m-\mathbf{r}}$. Or $\forall\,\theta^{\bullet}>0$ integer we have

$$K_{m,Y}(\tau)=\frac{1}{1-F(\tau)}\sum_{\kappa=0}^{\infty}\sum_{\mathbf{v}=0}^{\theta^{\bullet}-1}\sum_{\tau=0}^{m}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},m)^{\bullet}}\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right).$$

2.6. The life expectation (LE)

The **LE** at age τ can be defined by $\mathbf{LE}_{Y}(\tau)|_{(Y>\tau)}=\mathbf{E}\left[(Y-\tau)\right]=K_{1,Y}(\tau)$, which represents the additional life length for a certain item(s) which is (are) alive at age τ, where

$$\mathbf{LE}_{Y}(\tau)=\frac{1}{1-F(\tau)}\sum_{\kappa=0}^{\infty}\sum_{\mathbf{v}=0}^{\theta^{\bullet}-1}\sum_{\tau=0}^{1}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},1)^{\bullet}}\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right)$$

2.7. The moment of the reversed residual life

The m^{th} moment of the reversed residual life $M_{m.Y}(\tau)|_{(Y\leq\tau,\ \tau>0 \text{ and } m=1,2,\ldots)}=\mathbf{E}\left[(\tau-Y)^{m}\right]$ which can be expressed as

$$M_{m,Y}(\tau)|_{(Y\leq\tau,\ \tau>0 \text{ and } m=1,2,\ldots)}=\frac{1}{F(\tau)}\int_{0}^{\tau}(\tau-Y)^{m}dF(\mathbf{y}).$$

Then, we have

$$M_{m,Y}(\tau)=\frac{1}{F(\tau)}\sum_{\kappa,\mathbf{v}=0}^{\infty}\sum_{\tau=0}^{m}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},m)^{\bullet\bullet}}\left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right)-\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})(1+c\,\tau)^{a}\right)\right],$$

where $\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})^{\bullet\bullet}}=\Omega_{(\kappa)}^{\bullet\bullet}\zeta_{\mathbf{v},\tau}^{(\theta^{\bullet},\mathbf{r})}$ and $\Omega_{(\kappa)}^{\bullet\bullet}=\Omega_{(\kappa)}\sum_{\mathbf{r}=0}^{m}(-1)^{\mathbf{r}}\binom{m}{\mathbf{r}}\tau^{m-\mathbf{r}}$. Or $\forall\,\theta^{\bullet}>0$ integer we have

$$M_{m.Y}(\tau)=\frac{1}{F(\tau)}\sum_{\kappa=0}^{\infty}\sum_{\mathbf{v}=0}^{\theta^{\bullet}-1}\sum_{\tau=0}^{m}\zeta_{\kappa,\mathbf{v},\tau}^{(\theta^{\bullet},m)^{\bullet\bullet}}\left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right)-\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})(1+c\,\tau)^{a}\right)\right].$$

2.8. *The mean inactivity time (MIT)*

The (**MIT**) is given by $\mathbf{MIT}_Y(\tau)|_{(Y\leq\tau)} = \mathbf{E}[(\tau - Y)] = M_{1,Y}(\tau)$,which presents the elapsed waiting time since the failure of an item occurred in $(0, \tau)$. The MIT of the TLGNH model can be obtained as

$$\mathbf{MIT}_Y(\tau) = \frac{1}{F(\tau)} \sum_{\kappa,\mathbf{v}=0}^{\infty} \sum_{\tau=0}^{1} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^\bullet,1)^{\bullet\bullet}} \left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right) - \Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\,(1+c\,\tau)^{a}\right)\right],$$

or $\forall\, \theta^\bullet > 0$ integer we get

$$\mathbf{MIT}_Y(\tau) = \frac{1}{F(\tau)} \sum_{\kappa=0}^{\infty} \sum_{\mathbf{v}=0}^{\theta^\bullet-1} \sum_{\tau=0}^{1} \zeta_{\kappa,\mathbf{v},\tau}^{(\theta^\bullet,1)^{\bullet\bullet}} \left[\Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\right) - \Gamma\left(\frac{\tau}{a}+1,(1+\mathbf{v})\,(1+c\,\tau)^{a}\right)\right]$$

2.9. *Reliability estimation*

The reliability $\mathbf{R}\left(Y_1, Y_2|_{(Y_1>Y_2)}\right)$ of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let Y_1 and Y_2 be two independent random variables with TLGNH$(\alpha_1, \theta_1, a, c)$ and TLGNH$(\alpha_2, \theta_2, a, c)$ distributions respectively. Thus $\mathbf{R}\left(Y_1, Y_2|_{(Y_1>Y_2)}\right)$ can be expressed as

$$\mathbf{R}\left(Y_1, Y_2|_{(\mathbf{y}_1>\mathbf{y}_2)}\right) = \Pr(Y_1 > Y_2) = \sum_{\kappa,\mathbf{v}=0}^{\infty} \nu_{\kappa,\mathbf{v}}\,.$$

where

$$\nu_{\kappa,\mathbf{v}} = \sum_{\kappa,\mathbf{v}=0}^{\infty} \frac{(-1)^{\kappa+\mathbf{v}}\, 2^{\alpha_1+\alpha_2-\kappa-\mathbf{v}}}{[(\alpha_2+\mathbf{v})\,\theta_2]\,\{[(\alpha_1+\kappa)\,\theta_1] + [(\alpha_2+\mathbf{v})\,\theta_2]\}} \binom{\alpha_1}{\kappa}\binom{\alpha_2}{\mathbf{v}}.$$

3. Characterizations

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated moments. This type of characterization initiated by Galambos and Kotz (1978) and followed by other authors such as Kotz and Shanbhag (1980), Glänzel et al. (1984), Glänzel (1987), Glänzel and Hamedani (2001) and Kim and Jeon (2013), to name a few. For example, Kim and Jeon (2013) proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present three characterizations of the TLGNH distribution based on: (i) conditional expectation (truncated moment) of certain function of a random variable; (ii) the reversed hazard function and (iii) in terms of the conditional expectation of a function of a random variable.

3.1. *Characterizations based on two truncated moments*

This subsection deals with the characterizations of TLGNH distribution in terms of a simple relationship between two truncated moments. We will employ Theorem 1 of Glänzel (1987) given in the Appendix A. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 3.1.1. Let Y be a continuous random variable and let

$$q_1(y) = \frac{\left\{2 - \left[1 - e^{1-(1+cy)^a}\right]^\theta\right\}^{1-\alpha}\left[1 - e^{1-(1+cy)^a}\right]^{1-\theta\alpha}}{\left\{1 - \left[1 - e^{1-(1+cy)^a}\right]^\theta\right\}}$$

and

$$q_2(x) = q_1(x)\, e^{1-(1+cy)^a}|_{y\in\mathbb{R}}.$$

Then Y has pdf (5) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(y) = \frac{1}{2}e^{1-(1+cy)^a}|_{y\in\mathbb{R}}.$$

Proof. If Y has pdf (5), then

$$(1 - F(y))\, E\left[q_1(Y) \mid Y \geq y\right] = 2\theta\alpha e^{1-(1+cy)^a}|_{y\in\mathbb{R}},$$

and

$$(1 - F(y))\, E\left[q_2(Y) \mid Y \geq y\right] = \theta\alpha e^{2-2(1+cy)^a}|_{y\in\mathbb{R}},$$

and hence

$$\xi(y) = \frac{1}{2}e^{1-(1+cy)^a}|_{y\in\mathbb{R}},$$

we also have

$$\xi(y)\, q_1(y) - q_2(y) = -\frac{1}{2}q_1(y)\, e^{1-(1+cy)^a} < 0|_{y\in\mathbb{R}}.$$

Conversely, if ξ is of the above form, then

$$s'(y) = \frac{\xi'(y)\, q_1(y)}{\xi(y)\, q_1(y) - q_2(y)} = ac\,(1 + cy)^{a-1}|_{y\in\mathbb{R}},$$

and $s(y) = (1 + cy)^a$. Now, according to Theorem 1, Y has density (5).

Corollary 3.1.1. Suppose Y is a continuous random variable. Let $q_1(y)$ be as in Proposition 3.1.1. Then Y has density (5) if and only if there exist functions q_2 and ξ defined in Theorem 1 for which the following first order differential equation holds

$$\frac{\xi'(y)\, q_1(y)}{\xi(y)\, q_1(y) - q_2(y)} = ac\,(1 + cy)^{a-1}|_{y\in\mathbb{R}}.$$

Corollary 3.1.2. The differential equation in Corollary 3.1.1 has the following general solution

$$\xi(x) = e^{-1+(1+cy)^a}\left[-\int ac\,(1 + cy)^{a-1}\, e^{1-(1+cy)^a}\,(q_1(x))^{-1}\, q_2(x) + D\right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 3.1.1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem 1.

3.2. Characterization based on reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(y) = \frac{f(y)}{F(y)}, \;\; x \in \text{support of } F.$$

In this subsection we present a characterizations of the TLGNH which is not of the above trivial form.

Proposition 3.2.1. Suppose Y is a continuous random variable. Then, Y has density (5) if and only if its hazard function $r_F(y)$ satisfies the following first order differential equation

$$r'_F(y) + ac(1+cy)^{a-1} r_F(y)$$
$$= 2\alpha\theta ace^{1-(1+cy)^a} \frac{d}{dy}\left\{ \frac{\left\{1-\left[1-e^{1-(1+cy)^a}\right]^\theta\right\}}{\left\{2-\left[1-e^{1-(1+cy)^a}\right]^\theta\right\}\left[1-e^{1-(1+cy)^a}\right]} \right\}|_{y\in\mathbb{R}}.$$

Proof. Is straightforward and hence omitted.

3.3. Characterizations based on the Conditional Expectation of a Function of the Random Variable

Hamedani (2013) established the following proposition which can be used to characterize the TLGNH distribution.

Proposition 3.3.1. Suppose $Y:\Omega\to(a,b)$ is a continuous random variable with cdf F. If $\psi(y)$ is a differentiable function on (a,b) with $\lim_{x\to b^-}\psi(x)=1$. Then, for $\delta\neq 1$,

$$E[\psi(Y) \mid Y\le y] = \delta\psi(y)\,|_{y\in(a,b)},$$

implies that

$$\psi(y) = (G(y))^{\frac{1}{\delta}-1}\,|_{y\in(a,b)}.$$

Remark 3.3.1. Let $(a,b)=\mathbb{R}$, $\psi(x) = \left(\left\{2-\left[1-e^{1-(1+cy)^a}\right]^\theta\right\}\left[1-e^{1-(1+cy)^a}\right]^\theta\right)$ and $\delta=\frac{\alpha}{\alpha+1}$, then Proposition 3.3.1 presents a characterization of TLGHNH distribution. Clearly, there are other suitable functions than the one we employed for simplicity.

4. Classical estimation methods for uncensored data

Consider the following estimation methods: maximum likelihood estimation (MLE) method; Cramér-von-Mises estimation (CVME) method, ordinary least square estimation (OLSE) method; weighted least square estimation (WLSE) method; Anderson Darling estimation (A-D-E) method; right tail-Anderson Darling estimation (ADE-RT) method and left tail-Anderson Darling estimation (ADE-LT) method. All these methods are discussed in the statistical literature with more details. In this work, we may ignore many of its derivation details for avoiding the replication.

4.1. Maximum likelihood estimation (MLE)

Let $Y_1, Y_2, ..., Y_\hbar$ be any random sample (R·S) of size $\hbar$ from the TLGNH distribution with $\underline{\mathbf{\Psi}} = (\alpha,\theta,a,c)^{\mathbf{T}}$ as the vector of the model parameters. The log likelihood function $\left(\ell^{(\hbar)}_{[\underline{\mathbf{\Psi}}]}\right)$ for $\underline{\mathbf{\Psi}}$ may be expressed as

$$\begin{array}{l}\ell_{[\underline{\Psi}]}^{(\hbar)}=\hbar\log 2+\hbar\log\alpha+\hbar\log\theta+\hbar\log a+\hbar\log c\\ +(a-1)\sum_{\kappa=1}^{\hbar}\log(1+c\mathbf{y}_\kappa)+\sum_{\kappa=1}^{\hbar}\left[1-(1+c\mathbf{y})^a\right]+(\theta\alpha-1)\sum_{\kappa=1}^{\hbar}\log\mathbf{p}_\kappa\\ +\sum_{\kappa=1}^{\hbar}\log\left\{1-\mathbf{p}_\kappa^\theta\right\}+(\alpha-1)\sum_{\kappa=1}^{\hbar}\log\left\{2-\mathbf{p}_\kappa^\theta\right\},\end{array} \tag{11}$$

where $\mathbf{p}_\kappa=1-\mathrm{e}^{1-(1+c\mathbf{y}_\kappa)^a}$. The score vector elements are

$$\mathbf{U}_{(\alpha)}=\frac{\hbar}{\alpha}+\theta\sum_{\kappa=1}^{\hbar}\log\mathbf{p}_\kappa+\sum_{\kappa=1}^{\hbar}\log\left(2-\mathbf{p}_\kappa^\theta\right)$$

$$\mathbf{U}_{(\theta)}=\frac{\hbar}{\theta}+\alpha\sum_{\kappa=1}^{\hbar}\log\mathbf{p}_\kappa-\sum_{\kappa=1}^{\hbar}\frac{\mathbf{p}_\kappa^\theta\log\mathbf{p}_\kappa}{1-\mathbf{p}_\kappa^\theta}-(\alpha-1)\sum_{\kappa=1}^{\hbar}\frac{\mathbf{p}_\kappa^\theta\log\mathbf{p}_\kappa}{2-\mathbf{p}_\kappa^\theta},$$

$$\begin{aligned}\mathbf{U}_{(a)} \quad = \quad & \frac{\hbar}{a}+\sum_{\kappa=1}^{\hbar}\log(1+c\mathbf{y}_\kappa)-\sum_{\kappa=1}^{\hbar}(1+c\mathbf{y}_\kappa)^a\log(1+c\mathbf{y}_\kappa)\\ & +(\theta\alpha-1)\sum_{\kappa=1}^{\hbar}\frac{\mathbf{v}_\kappa}{\mathbf{p}_\kappa}-\sum_{\kappa=1}^{\hbar}\frac{\theta\mathbf{v}_\kappa\mathbf{p}_\kappa^{\theta-1}}{1-\mathbf{p}_\kappa^\theta}-(\alpha-1)\sum_{\kappa=1}^{\hbar}\frac{\theta\mathbf{v}_\kappa\mathbf{p}_\kappa^{\theta-1}}{2-\mathbf{p}_\kappa^\theta}\end{aligned}$$

and

$$\begin{aligned}\mathbf{U}_{(c)} \quad = \quad & \frac{\hbar}{c}+(a-1)\sum_{\kappa=1}^{\hbar}\frac{\mathbf{y}_\kappa}{1+c\mathbf{y}_\kappa}-a\sum_{\kappa=1}^{\hbar}\mathbf{y}_\kappa(1+c\mathbf{y}_\kappa)^{a-1}\\ & +(\theta\alpha-1)\sum_{\kappa=1}^{\hbar}\frac{u_\kappa}{\mathbf{p}_\kappa}-\theta\sum_{\kappa=1}^{\hbar}\frac{u_\kappa\mathbf{p}_\kappa^{\theta-1}}{1-\mathbf{p}_\kappa^\theta}-\theta\,(\alpha-1)\sum_{\kappa=1}^{\hbar}\frac{u_\kappa\mathbf{p}_\kappa^{\theta-1}}{2-\mathbf{p}_\kappa^\theta},\end{aligned}$$

where $\mathbf{v}_\kappa=(1+c\mathbf{y}_\kappa)^a\mathrm{e}^{1-(1+c\mathbf{y}_\kappa)^a}\log(1+c\mathbf{y}_\kappa)$ and $u_\kappa=\mathbf{y}_\kappa(1+c\mathbf{y}_\kappa)^a\mathrm{e}^{1-(1+c\mathbf{y}_\kappa)^a}$. Setting $\mathbf{U}_{(\alpha)}=\mathbf{U}_{(\theta)}=\mathbf{U}_{(a)}=\mathbf{U}_{(c)}=0$ and solving them simultaneously yields the MLE $\widehat{\underline{\Psi}}=(\widehat{\alpha},\widehat{\theta},\widehat{a},\widehat{c})^{\mathbf{T}}$ of $\underline{\Psi}=(\alpha,\theta,a,c)^{\mathbf{T}}$.

4.2. CVME method

The CVME of the parameters α,θ,a and c are obtained via minimizing the following expression with respect to (w.r.t.) to the parameters α,θ,a and c respectively, where

$$\mathbf{CVME}_{(\underline{\Psi})}=\frac{1}{12}\hbar^{-1}+\sum_{\kappa=1}^{\hbar}\left[F_{\underline{\Psi}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)-\varrho_{(\kappa,\hbar)}\right]^2,$$

where $\varrho_{(\kappa,\hbar)}=\left[(2\kappa-1)/2\hbar\right]$ and

$$\mathbf{CVME}_{(\underline{\Psi})}=\sum_{\kappa=1}^{\hbar}\left(\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\varrho_{(\kappa,\hbar)}\right)^2.$$

The, CVME of the parameters α,θ,a and c are obtained by solving the two following non-linear equations

$$\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\varrho_{(\kappa,\hbar)}\right]\mathbf{\Delta}_{(\alpha)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\Psi}\right)=0,$$

$$\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\varrho_{(\kappa,\hbar)}\right]\mathbf{\Delta}_{(\theta)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\Psi}\right)=0,$$

$$\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\varrho_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(a)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)=0,$$

and

$$\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\varrho_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(c)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)=0,$$

where $\boldsymbol{\Delta}_{(\alpha)}(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}})=\partial F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)/\partial a$, $\boldsymbol{\Delta}_{(\theta)}(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}})=\partial F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)/\partial b$, $\boldsymbol{\Delta}_{(a)}(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}})=\partial F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)/\partial b$ and $\boldsymbol{\Delta}_{(c)}(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}})=\partial F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)/\partial c$.

4.3. OLSE method

Let $F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)$ denote the C·D·F of TLGNH model and let $\mathbf{y}_1<\mathbf{y}_2<\cdots<\mathbf{y}_\hbar$ be the $\hbar$ ordered RS. The OLSEs are obtained upon minimizing

$$\mathbf{OLSE}(\underline{\boldsymbol{\Psi}})=\sum_{\kappa=1}^{\hbar}\left[F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)-\Upsilon_{(\kappa,\hbar)}\right]^2,$$

then, we have

$$\mathbf{OLSE}(\underline{\boldsymbol{\Psi}})=\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]^2,$$

where $\Upsilon_{(\kappa,\hbar)}=\frac{\kappa}{\hbar+1}$.The LSEs are obtained via solving the following non-linear equations

$$0=\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(\epsilon)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right),$$

$$0=\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(\theta)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right),$$

$$0=\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(a)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right),$$

and

$$0=\sum_{\kappa=1}^{\hbar}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]\boldsymbol{\Delta}_{(c)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right),$$

where $\boldsymbol{\Delta}_{(\alpha)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)$, $\boldsymbol{\Delta}_{(\theta)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)$, $\boldsymbol{\Delta}_{(a)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)$ and $\boldsymbol{\Delta}_{(c)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)$ are defined above.

4.4. WLSE method

The WLSE is obtained by minimizing the function $\mathbf{WLSE}(\underline{\boldsymbol{\Psi}})$ w.r.t. α,θ,a and c

$$\mathbf{WLSE}\left(\underline{\boldsymbol{\Psi}}\right)=\sum_{\kappa=1}^{\hbar}\vartheta_{(\kappa,\hbar)}\left[F_{\underline{\boldsymbol{\Psi}}}\left(\mathbf{y}_{[\kappa:\hbar]}\right)-\Upsilon_{(\kappa,\hbar)}\right]^2,$$

where $\vartheta_{(\kappa,\hbar)}=\left[(1+\hbar)^2(2+\hbar)\right]/\left[\kappa(1+\hbar-\kappa)\right]$. The WLSEs are obtained by solving

$$0=\sum_{\kappa=1}^{\hbar}\left\{\begin{array}{c}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^c}\right]^b\right\}\right)^a-\Upsilon_{(\kappa,\hbar)}\right]\\ \times\vartheta_{(\kappa,\hbar)}\boldsymbol{\Delta}_{(\alpha)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\boldsymbol{\Psi}}\right)\end{array}\right\},$$

$$0=\sum_{\kappa=1}^{\hbar}\left\{\begin{array}{c}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\right\}\right)^{a}-\Upsilon_{(\kappa,\hbar)}\right]\\ \times\vartheta_{(\kappa,\hbar)}\mathbf{\Delta}_{(\theta)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\mathbf{\Psi}}\right)\end{array}\right\},$$

$$0=\sum_{\kappa=1}^{\hbar}\left\{\begin{array}{c}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\right\}\right)^{a}-\Upsilon_{(\kappa,\hbar)}\right]\\ \times\vartheta_{(\kappa,\hbar)}\mathbf{\Delta}_{(a)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\mathbf{\Psi}}\right)\end{array}\right\},$$

and

$$0=\sum_{\kappa=1}^{\hbar}\left\{\begin{array}{c}\left[\left(\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\left\{2-\left[1-\mathrm{e}^{1-(\mathbf{y}_{[\kappa:\hbar]}+1)^{c}}\right]^{b}\right\}\right)^{a}-\Upsilon_{(\kappa,\hbar)}\right]\\ \times\vartheta_{(\kappa,\hbar)}\mathbf{\Delta}_{(c)}\left(\mathbf{y}_{[\kappa:\hbar]};\underline{\mathbf{\Psi}}\right)\end{array}\right\}.$$

4.5. The ADE method

The ADE $\widehat{\alpha}_{(\text{ADE})}$, $\widehat{\theta}_{(\text{ADE})}$, $\widehat{a}_{(\text{ADE})}$ and $\widehat{c}_{(\text{ADE})}$ are obtained by minimizing the function

$$\mathbf{ADE}_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)=-\hbar-\hbar^{-1}\sum_{\kappa=1}^{\hbar}\left[\begin{array}{c}(2\kappa-1)\\ \times\left\{\begin{array}{c}\log F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[\kappa:\hbar]})\\ +\log\left[1-F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[-\kappa+1+\hbar:\hbar]})\right]\end{array}\right\}\end{array}\right].$$

The parameter estimates $\widehat{\alpha}_{(\text{ADE})}$, $\widehat{\theta}_{(\text{ADE})}$, $\widehat{a}_{(\text{ADE})}$ and $\widehat{c}_{(\text{ADE})}$ are derived by solving the nonlinear equations

$$0=\partial\left[\mathbf{ADE}_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\alpha,0=\partial\left[\mathbf{ADE}_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\theta,$$

$$0=\partial\left[\mathbf{ADE}_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial a,$$

and

$$0=\partial\left[\mathbf{ADE}_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial c.$$

4.6. The ADE-RT method

The ADE(R-T) $\widehat{\alpha}_{(\text{ADE-RT})}$, $\widehat{\theta}_{(\text{ADE-RT})}$, $\widehat{a}_{(\text{ADE-RT})}$ and $\widehat{c}_{(\text{ADE-RT})}$ are obtained by minimizing

$$ADE-RT_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)=\frac{1}{2}\hbar-2\sum_{\kappa=1}^{\hbar}F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[\kappa:\hbar]})-\frac{1}{\hbar}\sum_{\kappa=1}^{\hbar}\left(\begin{array}{c}(2\kappa-1)\\ \times\left\{\log\left[1-F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[-\kappa+1+\hbar:\hbar]})\right]\right\}\end{array}\right).$$

The estimates $\widehat{\alpha}_{(\text{ADE-RT})}$, $\widehat{\theta}_{(\text{ADE-RT})}$, $\widehat{a}_{(\text{ADE-RT})}$ and $\widehat{c}_{(\text{ADE-RT})}$ follow by solving the nonlinear equations

$$0=\partial\left[ADE-RT_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\alpha,0=\partial\left[ADE-RT_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\theta,$$

$$0=\partial\left[ADE-RT_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial a\text{ and }0=\partial\left[ADE-RT_{\left(\mathbf{y}_{[\kappa:\hbar]},\mathbf{y}_{[-\kappa+1+\hbar:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial c.$$

4.7. The ADE-LT method

The ADE(L-T) $\widehat{\alpha}_{(ADE-LT)}$, $\widehat{\theta}_{(\text{ADE-LT})}$, $\widehat{a}_{(\text{ADE-LT})}$ and $\widehat{c}_{(\text{ADE-LT})}$ are obtained by minimizing

$$ADE-LT_{\left(\mathbf{y}_{[\kappa:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)=-\frac{3}{2}\hbar+2\sum_{\kappa=1}^{\hbar}F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[\kappa:\hbar]})-\frac{1}{\hbar}\sum_{\kappa=1}^{\hbar}(2\kappa-1)\log F_{(\underline{\mathbf{\Psi}})}(\mathbf{y}_{[\kappa:\hbar]}).$$

The estimates $\widehat{\alpha}_{(ADE-LT)}$, $\widehat{\theta}_{(\text{ADE-LT})}$, $\widehat{a}_{(\text{ADE-LT})}$ and $\widehat{c}_{(\text{ADE-LT})}$ follow by solving the nonlinear equations

$$0=\partial\left[ADE-LT_{\left(\mathbf{y}_{[\kappa:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\alpha,0=\partial\left[ADE-LT_{\left(\mathbf{y}_{[\kappa:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\theta,$$

$$0=\partial\left[ADE-LT_{\left(\mathbf{y}_{[\kappa:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial a\text{ and }0=\partial\left[ADE-LT_{\left(\mathbf{y}_{[\kappa:\hbar]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial c.$$

5. Comparing classical methods

5.1. Comparing classical methods under uncensored schemes via simulation experiments

A numerical simulation is performed in to compare the classical estimation methods. The simulation study is based on $N = 1000$ generated data sets from the TLGNH version where $\hbar = 60, 100, 150$ and 500 and **I**= $(\alpha = 2, \theta = 0.5, a = 0.6, c = 1.2)$, **II**= $(\alpha = 3, \theta = 0.9, a = 0.2, c = 2)$ and **III**= $(\alpha = \theta = a = c = 0.9)$. The estimation methods are compared in terms of their average of estimates $\left(AVs_{(\underline{\Psi})}\right)$ and mean-standard errors $\left(MSEs_{(\underline{\Psi})}\right)$. The confidence intervals $\left(\text{C·Is}_{(\underline{\Psi})}\right)$ have been also calculated, where

$$MSE_{(\alpha)} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\alpha}_i - \alpha\right)^2, MSE_{(\theta)} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\theta}_i - \theta\right)^2,$$

$$MSE_{(a)} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{a}_i - a\right)^2 \text{ and } MSE_{(c)} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{c}_i - c\right)^2.$$

Table 3 gives the simulation results for parameters $\alpha = 2, \theta = 0.5, a = 0.6$ and $c = 1.2$. Table 4 gives the simulation results for parameters $\alpha = 3, \theta = 0.9, a = 0.2$ and $c = 2$. Table 5 gives the simulation results for parameters $\alpha = \theta = a = 0.9$ and $c = 0.9$. From Tables 3-5 we note that the $\text{MSEs}_{(\underline{\Psi})}$ tend to 0 when $\hbar$ increases which means incidence of consistency property and $\text{AVs}_{(\underline{\Psi})}$ tend to actual values when $\hbar$ increases.

Table 3: Simulation results for parameters $\alpha = 2, \theta = 0.5$.

	n	$\widehat{\alpha}$			$\widehat{\theta}$		
		$\text{AVs}_{(\widehat{\alpha})}$	$MSEs_{(\widehat{\alpha})}$	$\text{CIs}_{(\widehat{\alpha})}$	$\text{AVs}_{(\widehat{\theta})}$	$MSEs_{(\widehat{\theta})}$	$\text{CIs}_{(\widehat{\theta})}$
MLE	60	2.03746	0.07659	(1.57657,2.64027)	0.50465	0.00215	(0.42140,0.60261)
CVME		2.04123	0.10680	(1.50907,2.77609)	0.50473	0.00261	(0.41687,0.61468)
OLSE		2.10173	0.12292	(1.54948,2.85836)	0.51428	0.00291	(0.42381,0.62646)
WLSE		2.11091	0.11657	(1.56604,2.87465)	0.51596	0.00279	(0.42624,0.63025)
ADE		2.02823	0.08798	(1.53312,2.71042)	0.50346	0.00232	(0.41767,0.60975)
ADE-RT		2.03860	0.11318	(1.48132,2.78802)	0.50422	0.00273	(0.41035,0.61509)
ADE-LT		2.01984	0.08154	(1.55769,2.66591)	0.50214	0.00226	(0.42114,0.60598)
MLE	100	2.02883	0.04325	(1.67611,2.47490)	0.50371	0.00123	(0.44309,0.57974)
CVME		2.02653	0.05867	(1.62214,2.57332)	0.50323	0.00146	(0.43616,0.58805)
OLSE		2.06213	0.06430	(1.64827,2.62221)	0.50888	0.00157	(0.44069,0.59456)
WLSE		2.06789	0.06125	(1.67545,2.61109)	0.50995	0.00152	(0.44504,0.59419)
ADE		2.02027	0.04930	(1.64163,2.51999)	0.50260	0.00132	(0.43820,0.58428)
ADE−RT		2.02819	0.06402	(1.58576,2.57437)	0.50337	0.00158	(0.42950,0.58785)
ADE−LT		2.01762	0.04505	(1.65456,2.48992)	0.50226	0.00126	(0.43896,0.57917)
MLE	150	2.01078	0.02880	(1.70780,2.37678)	0.50105	0.00082	(0.44893,0.56026)
CVME		2.00796	0.03655	(1.66670,2.42084)	0.50061	0.00093	(0.44406,0.56484)
OLSE		2.03136	0.03848	(1.68552,2.45013)	0.50435	0.00096	(0.44731,0.56878)
WLSE		2.03654	0.03601	(1.70352,2.44484)	0.50526	0.00091	(0.45042,0.56809)
ADE		2.00514	0.03142	(1.69414,2.39271)	0.50037	0.00085	(0.44739,0.56209)
ADE−RT		2.01421	0.04105	(1.65046,2.41696)	0.50153	0.00102	(0.44166,0.56240)
ADE−LT		2.00369	0.02972	(1.69550,2.37517)	0.50012	0.00084	(0.44711,0.56205)
MLE	200	2.01223	0.02127	(1.74218,2.30970)	0.50150	0.00061	(0.45360,0.5505)
CVME		2.01052	0.02808	(1.70976,2.35588)	0.50119	0.00072	(0.45199,0.55522)
OLSE		2.02811	0.02936	(1.72424,2.37727)	0.50399	0.00074	(0.45444,0.55845)
WLSE		2.03244	0.02738	(1.73099,2.36632)	0.50476	0.00070	(0.45541,0.55747)
ADE		2.00707	0.02375	(1.71675,2.31975)	0.50084	0.00065	(0.45229,0.55231)
ADE−RT		2.01562	0.03503	(1.68269,2.41563)	0.50190	0.00087	(0.44793,0.56366)
ADE−LT		2.00451	0.02196	(1.72451,2.29794)	0.50041	0.00063	(0.45203,0.54900)

(continued): simulation results for parameters $a = 0.6$ and $c = 1.2$.

	n	$\widehat{a}$			$\widehat{c}$		
		AVs$_{(\widehat{a})}$	$MSEs_{(\widehat{a})}$	CIs$_{(\widehat{a})}$	AVs$_{(\widehat{c})}$	$MSEs_{(\widehat{c})}$	CIs$_{(\widehat{c})}$
MLE	60	0.60823	0.00477	(0.49066,0.75835)	1.21896	0.04484	(0.86879,1.72803)
CVME		0.60981	0.00973	(0.45060,0.81909)	1.22214	0.05714	(0.83019,1.73363)
OLSE		0.59223	0.00893	(0.43897,0.79597)	1.17919	0.05307	(0.80121,1.67562)
WLSE		0.58950	0.00794	(0.44502,0.78340)	1.16990	0.04938	(0.80027,1.65507)
ADE		0.60288	0.00638	(0.466103,0.78833)	1.21412	0.04873	(0.84467,1.71727)
ADE-RT		0.60479	0.00537	(0.47847,0.76522)	1.21718	0.04411	(0.86280,1.68120)
ADE-LT		0.61190	0.01088	(0.44216,0.83895)	1.23115	0.06718	(0.80853,1.80250)
MLE	100	0.60645	0.00281	(0.050751,0.71082)	1.21283	0.02480	(0.91740,1.53707)
CVME		0.60436	0.00509	(0.47734,0.75909)	1.20979	0.03060	(0.89505,1.59032)
OLSE		0.59398	0.00488	(0.46999,0.74520)	1.18430	0.02932	(0.87740,1.55117)
WLSE		0.59229	0.00435	(0.47497,0.73280)	1.17852	0.02764	(0.87329,1.52527)
ADE		0.60165	0.00354	(0.48853,0.72462)	1.20702	0.02662	(0.90555,1.54480)
ADE−RT		0.60129	0.00308	(0.50186,0.72743)	1.20609	0.02528	(0.92550,1.55818)
ADE−LT		060491	0.00559	(0.47036,0.76620)	1.21257	0.03434	(0.88524,1.61560)
MLE	150	0.60473	0.00167	(0.52933,0.69040)	1.21160	0.01551	(0.99028,1.48708)
CVME		0.60567	0.00337	(0.50256,0.73151)	1.21326	0.02030	(0.95833,1.52155)
OLSE		0.59870	0.00323	(0.49745,0.72262)	1.19613	0.01949	(0.94598,1.49962)
WLSE		0.59722	0.00288	(0.50425,0.71385)	1.19133	0.01831	(0.95069,1.47975)
ADE		0.60267	0.00228	(0.51743,0.70891)	1.20970	0.01734	(0.97535,1.49998)
ADE−RT		0.60159	0.00196	(0.52436,0.70119)	1.20632	0.01604	(0.99568,1.49125)
ADE−LT		0.60626	0.00377	(0.49787,0.73608)	1.21596	0.02332	(0.94393,1.53663)
MLE	200	0.60170	0.00121	(0.53610,0.67337)	1.20450	0.01158	(1.00999,1.41771)
CVME		0.60299	0.00254	(0.51585,0.70778)	1.20688	0.01535	(0.99024,1.46303)
OLSE		0.59779	0.00248	(0.51162,0.70153)	1.19408	0.01498	(0.98028,1.44696)
WLSE		0.59664	0.00221	(0.51480,0.69870)	1.19019	0.01412	(0.98103,1.44745)
ADE		0.60089	0.00174	(0.52676,0.68698)	1.20438	0.01322	(0.99826,1.44968)
ADE−RT		0.60030	0.00162	(0.52549,0.68120)	1.20253	0.01326	(0.99359,1.44183)
ADE−LT		0.60365	0.00284	(0.51258,0.71872)	1.0963	0.01757	(0.98542,1.50272)

Table 4: Simulation results for parameters $\alpha = 3, \theta = 0.9$.

	n	$\widehat{\alpha}$			$\widehat{\theta}$		
		AVs$_{(\widehat{\alpha})}$	$MSEs_{(\widehat{\alpha})}$	CIs$_{(\widehat{\alpha})}$	AVs$_{(\widehat{\theta})}$	$MSEs_{(\widehat{\theta})}$	CIs$_{(\widehat{\theta})}$
MLE	60	3.04877	0.16191	(2.38287,3.92684)	0.90647	0.00596	(0.76909,1.07377)
CVME		3.05443	0.24078	(2.23088,4.13721)	0.90669	0.00775	(0.74957,1.09280)
OLSE		3.14899	0.27806	(2.29548,4.26333)	0.92370	0.00864	(0.76263,1.11280)
WLSE		3.15697	0.28055	(2.32539,4.26811)	0.92522	0.00869	(0.76837,1.11269)
ADE		3.04235	0.19797	(2.29968,4.06564)	0.90585	0.00695	(0.75746,1.08973)
ADE-RT		3.07138	0.25082	(2.24852,4.18341)	0.90987	0.00807	(0.75020,1.09942)
ADE-LT		3.01680	0.16170	(2.32683,3.88390)	0.90132	0.00598	(0.76279,1.06192)
MLE	100	3.03709	0.09287	(2.50031,3.68293)	0.90557	0.00348	(0.79979,1.02730)
CVME		3.04516	0.14131	(2.39152,3.84826)	0.90631	0.00460	(0.78281,1.04568)
OLSE		3.10120	0.15564	(2.43406,3.91921)	0.91646	0.00496	(0.79132,1.05726)
WLSE		3.10400	0.15219	(2.43989,3.90009)	0.91703	0.00485	(0.79077,1.05524)
ADE		3.03040	0.11091	(2.46244,3.77999)	0.90444	0.00394	(0.79318,1.04514)
ADE−RT		3.04830	0.14403	(2.39748,3.87727)	0.90696	0.00474	(0.78306,1.05109)
ADE−LT		3.02174	0.09935	(2.46429,3.68940)	0.90303	0.00369	(0.79130,1.02945)
MLE	150	3.03592	0.06142	(2.59514,3.56082)	0.90570	0.00234	(0.81630,1.00261)
CVME		3.01376	0.09497	(2.47816,3.68171)	0.90119	0.00315	(0.79978,1.01883)
OLSE		3.07388	0.10175	(2.50964,3.74896)	0.91218	0.00328	(0.80628,1.02948)
WLSE		3.07472	0.10134	(2.50441,3.72445)	0.91236	0.00327	(0.80494,1.02799)
ADE		3.00771	0.07070	(2.54121,3.58907)	0.90059	0.00254	(0.80888,1.00732)
ADE−RT		3.03610	0.10388	(2.50887,3.75080)	0.90524	0.00343	(0.80413,1.03207)
ADE−LT		3.02284	0.06446	(2.55717,3.56992)	0.90362	0.00242	(0.81017,1.00504)
MLE	200	3.02413	0.04677	(2.63168,3.46807)	0.90369	0.00179	(0.82223,0.98688)
CVME		3.02005	0.07110	(2.54841,3.60764)	0.90267	0.00236	(0.81383,1.00669)
OLSE		3.04795	0.07352	(2.53744,3.59998)	0.90778	0.00241	(0.81227,1.00507)
WLSE		3.05032	0.07298	(2.55118,3.59443)	0.90824	0.00240	(0.81442,1.00507)
ADE		3.01060	0.05345	(2.57513,3.47962)	0.90142	0.00194	(0.81731,0.98987)
ADE−RT		3.01841	0.07147	(2.51056,3.59105)	0.90233	0.00240	(0.80592,1.00305)
ADE−LT		3.01762	0.04920	(260092,3.48709)	0.90275	0.00184	(0.82015,0.99192)

(continued): simulation results for parameters $a = 0.2$ and $c = 2$.

	n	$\widehat{a}$			$\widehat{c}$		
		AVs$_{(\widehat{a})}$	$MSEs_{(\widehat{a})}$	CIs$_{(\widehat{a})}$	AVs$_{(\widehat{c})}$	$MSEs_{(\widehat{c})}$	CIs$_{(\widehat{c})}$
MLE	60	0.20085	0.00016	(0.17801,0.22617)	2.02950	0.15903	(1.33450,2.90873)
CVME		0.20072	0.00020	(0.17589,0.23092)	2.03667	0.18631	(1.31221,2.99788)
OLSE		0.19808	0.00019	(0.17365,0.22780)	1.95692	0.17368	(1.26386,2.88582)
WLSE		0.19790	0.00018	(0.17448,0.22561)	1.94955	0.16787	(1.26720,2.82527)
ADE		0.20011	0.00018	(0.17542,0.22918)	2.03078	0.17644	(1.30762,2.96539)
ADE-RT		0.19982	0.00015	(0.17833,0.22481)	2.01887	0.17925	(1.31497,2.93339)
ADE-LT		0.20126	0.00023	(0.17455,0.23538)	2.05073	0.17032	(1.36077,2.95996)
MLE	100	0.20042	0.00009	(0.18375,0.22028)	2.00866	0.08935	(1.49545,2.62268)
CVME		0.20006	0.00011	(0.18110,0.22217)	2.01028	0.10483	(1.44805,2.71419)
OLSE		0.19848	0.00011	(0.17968,0.22045)	1.96263	0.10150	(1.41551,2.65097)
WLSE		0.19844	0.00010	(0.18095,0.21968)	1.95931	0.09698	(1.43674,2.65712)
ADE		0.20002	0.00010	(0.18002,0.21992)	2.012686	0.09562	(1.43239,2.65315)
ADE−RT		0.19978	0.00009	(0.18228,0.21916)	2.00647	0.10591	(1.43562,2.68662)
ADE−LT		0.20035	0.00014	(0.17934,0.22478)	2.01948	0.09916	(1.48003,2.69421)
MLE	150	0.20032	0.00006	(0.18617,0.21604)	1.99932	0.05972	(1.56432,2.52531)
CVME		0.20055	0.00008	(0.18435,0.21847)	2.02273	0.07456	(1.53736,2.58587)
OLSE		0.19880	0.00008	(0.18275,0.21692)	1.96944	0.06869	(1.49331,2.53585)
WLSE		0.19881	0.00007	(0.18366,0.21675)	1.96859	0.06642	(1.50865,2.54301)
ADE		0.20032	0.00007	(0.18512,0.21773)	2.01960	0.06392	(1.55450,2.55406)
ADE−RT		0.19985	0.00007	(0.18377,0.21520)	2.00390	0.10591	(1.43562,2.68662)
ADE−LT		0.19994	0.00009	(0.18267,0.22009)	2.00446	0.09916	(1.48003,2.69421)
MLE	200	0.20027	0.00005	(0.18768,0.21381)	2.00155	0.04635	(1.61251,2.47792)
CVME		0.20013	0.00006	(0.18520,0.21562)	2.00833	0.05448	(1.56972,2.49101)
OLSE		0.19931	0.00006	(0.18536,0.21589)	1.98379	0.05362	(1.57504,2.49744)
WLSE		0.19927	0.00005	(0.18589,0.21502)	1.98158	0.05203	(1.57860,2.47539)
ADE		0.20005	0.00005	(0.18686,0.21446)	2.00986	0.04851	(1.61840,2.47845)
ADE−RT		0.20013	0.00005	(0.18728,0.21467)	2.01003	0.07473	(1.50155,2.55262)
ADE−LT		0.200047	0.00007	(0.18438,0.21828)	2.00399	0.04892	(1.5958,2.48203)

Table 5: Simulation results for parameters $\alpha = 0.9, \theta = 0.9$.

	n	$\widehat{\alpha}$			$\widehat{\theta}$		
		AVs$_{(\widehat{\alpha})}$	$MSEs_{(\widehat{\alpha})}$	CIs$_{(\widehat{\alpha})}$	AVs$_{(\widehat{\theta})}$	$MSEs_{(\widehat{\theta})}$	CIs$_{(\widehat{\theta})}$
MLE	60	0.91714	0.01532	(0.70985,1.19068)	0.90877	0.00833	(0.75326,1.10630)
CVME		0.92153	0.02230	(0.68685,1.26358)	0.91126	0.01009	(0.74120,1.13157)
OLSE		0.95136	0.02730	(0.68227,1.29879)	0.93147	0.01203	(0.73783,1.15215)
WLSE		0.95601	0.02666	(0.69845,1.28935)	0.93485	0.01180	(0.74898,1.14647)
ADE		0.90880	0.01607	(0.68886,1.18535)	0.90470	0.00829	(0.73984,1.09421)
ADE-RT		0.92864	0.02484	(0.66982,1.26621)	0.91592	0.01118	(0.73431,1.13590)
ADE-LT		0.91599	0.01672	(0.68838,1.18845)	0.90964	0.00891	(0.73633,1.10839)
MLE	100	0.91380	0.00865	(0.74702,1.11187)	0.90851	0.00483	(0.77718,1.04994)
CVME		0.91110	0.01156	(0.73092,1.14799)	0.90571	0.00542	(0.77619,1.06342)
OLSE		0.91799	0.01224	(0.72911,1.14871)	0.91041	0.00574	(0.77336,1.06384)
WLSE		0.92018	0.01142	(0.74218,1.14167)	0.91202	0.00540	(0.78288,1.06099)
ADE		0.90876	0.00964	(0.73712,1.11525)	0.90526	0.00497	(0.77895,1.04819)
ADE−RT		0.91275	0.01570	(0.69959,1.19326)	0.90637	0.00712	(0.75468,1.08817)
ADE−LT		0.89482	0.00826	(0.73587,1.07943)	0.89522	0.00458	(0.77249,1.02723)
MLE	150	0.90455	0.00541	(0.77382,1.04984)	0.90203	0.00307	(0.80155,1.00656)
CVME		0.90837	0.00781	(0.74653,1.08772)	0.90446	0.00374	(0.78786,1.02544)
OLSE		0.91519	0.00800	(0.76314,1.10094)	0.90921	0.00377	(0.80055,1.03390)
WLSE		0.91742	0.00754	(0.76622,1.09878)	0.91083	0.00360	(0.80287,1.03282)
ADE		0.90808	0.00650	(0.76173,1.07615)	0.90505	0.00337	(0.79639,1.02436)
ADE−RT		0.90583	0.00895	(0.73548,1.09590)	0.90266	0.00422	(0.78325,1.03076)
ADE−LT		0.90113	0.00565	(0.76337,1.05713)	0.90010	0.00311	(0.79635,1.01339)
MLE	200	0.90575	0.00402	(0.78203,1.02947)	0.90328	0.00228	(0.80808,0.99851)
CVME		0.90459	0.00585	(0.77268,1.06518)	0.90215	0.00280	(0.80813,1.01074)
OLSE		0.91583	0.00645	(0.77517,1.08109)	0.90989	0.00306	(0.80953,1.02094)
WLSE		0.91717	0.00597	(0.77919,1.07690)	0.91091	0.00287	(0.81142,1.01833)
ADE		0.90532	0.00487	(0.77070,1.05177)	0.90311	0.00254	(0.80219,1.01001)
ADE−RT		0.90589	0.00691	(0.76011,1.07841)	0.90289	0.00323	(0.79934,1.01700)
ADE−LT		0.90531	0.00447	(0.78238,1.04023)	0.90335	0.00246	(0.808881,1.00237)

(continued): simulation results for parameters $a = 0.9$ and $c = 0.9$.

	n	$\widehat{a}$			$\widehat{c}$		
		AVs$_{(\widehat{a})}$	$MSEs_{(\widehat{a})}$	CIs$_{(\widehat{a})}$	AVs$_{(\widehat{c})}$	$MSEs_{(\widehat{c})}$	CIs$_{(\widehat{c})}$
MLE	60	0.91887	0.01019	(0.75898,1.13752)	0.92232	0.02007	(0.70315,1.21796)
CVME		0.91176	0.01986	(0.67759,1.24197)	0.91309	0.02722	(0.63746,1.29004)
OLSE		0.88624	0.02171	(0.66446,1.24724)	0.88300	0.02953	(0.61954,1.30042)
WLSE		0.88318	0.01905	(0.67960,1.21949)	0.87759	0.02692	(0.62677,1.26996)
ADE		0.90643	0.01371	(0.71388,1.16048)	0.91155	0.02319	(0.66531,1.25146)
ADE-RT		0.90068	0.01176	(0.72641,1.13936)	0.90281	0.02067	(0.67499,1.20885)
ADE-LT		0.91378	0.0217	(0.67124,1.29335)	0.91640	0.03672	(0.62577,136440)
MLE	100	0.90642	0.00598	(0.77114,1.06681)	0.90606	0.01169	(0.71650,1.13245)
CVME		0.90745	0.01111	(0.72476,1.14089)	0.90836	0.01532	(0.69261,1.17805)
OLSE		0.90168	0.01144	(0.72893,1.14563)	0.90150	0.01574	(0.69719,1.18702)
WLSE		0.89964	0.00972	(0.73312,1.12242)	0.89833	0.01402	(0.69835,1.15927)
ADE		0.90105	0.00754	(0.75150,1.09299)	0.90309	0.01263	(0.71558,1.15231)
ADE−RT		0.90360	0.00713	(0.74528,1.09175)	0.90618	0.02067	(0.67499,1.20885)
ADE−LT		0.92092	0.01382	(0.73228,1.17297)	0.92556	0.01968	(0.70728,1.22570)
MLE	150	0.90770	0.00380	(0.79549,1.04031)	0.90931	0.00752	(0.75465,1.09628)
CVME		0.904667	0.00798	(0.75488,1.10097)	0.90521	0.01102	(0.72729,1.13694)
OLSE		0.89759	0.00729	(0.75050,1.07804)	0.89688	0.01007	(0.72305,1.10811)
WLSE		0.89583	0.00627	(0.75814,1.06576)	0.89410	0.00915	(0.72518,1.09560)
ADE		0.90001	0.00515	(0.77254,1.05733)	0.90081	0.00856	(0.73716,1.10868)
ADE−RT		0.90311	0.00448	(0.78322,1.04250)	0.90517	0.00796	(0.74698,1.09139)
ADE−LT		0.90940	0.00851	(0.74668,1.10647)	0.91149	0.01210	(0.72176,1.14630)
MLE	200	0.90307	0.00263	(0.81390,1.01396)	0.90365	0.00533	(0.77691,1.06796)
CVME		0.90511	0.00569	(0.77131,1.05747)	0.90582	0.00789	(0.74854,1.08527)
OLSE		0.89474	0.00591	(0.76202,1.06126)	0.89356	0.00817	(0.73756,1.08847)
WLSE		0.89380	0.00504	(0.77133,1.04942)	0.89180	0.00736	(0.74299,1.07714)
ADE		0.90159	0.00405	(0.78906,1.03711)	0.90261	0.00672	(0.75790,1.08222)
ADE−RT		0.90187	0.00328	(0.79886,1.02458)	0.90320	0.00582	(0.76555,1.06913)
ADE−LT		0.90284	0.00665	(0.75953,1.07475)	0.90338	0.00941	(0.73293,1.11543)

5.2. Comparing classical methods under uncensored schemes via an application

For comparing classical methods, an application to real data set is introduced. We consider the Cramér-Von Mises ($W^{\bullet}$) ans KS (and its corresponding P-value) statistis. The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (see Lee and Wang (2003)). Figure 2 gives the total time test (TTT) plot, nonparametric Kernel density estimation, box plot and Quantile-Quantile (Q-Q) plot for the remission data. Based on the TTT plot, the HRF of this data is ”upside down”. It is clear that the data contains some extreme values (see the box plot and it corresponding Q-Q plot). From Table 6, the CVME method is the best method with

$\widehat{\alpha} = 1.498$, $\widehat{\theta} = 1.303, \widehat{a} = 0.523$, $\widehat{c} = 0.265$, W$^\bullet$**=0.03588,** KS=**0.02733 and** P-value=**0.99998** however all other methods performed well. Figures 3-9 give the ECDF, KMS and PP plot plot for MLE, CVME, OLSE, OLSE, ADE, ADE-RT and ADE-LT respectively.

Table 6: Application results for comparing methods.

	Estimates				Criteria		
Method	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{a}$	$\widehat{c}$	W$^\bullet$	KS	P-value
MLE	1.255	1.297	0.533	0.218	0.03703	0.04295	0.97222
CVME	**1.498**	**1.303**	**0.523**	**0.265**	**0.03588**	**0.02733**	**0.99998**
OLSE	1.899	1.2023	0.461	0.355	0.04301	0.02940	0.99989
WLSE	1.444	1.361	0.462	0.327	0.03894	0.03844	0.99154
ADE	1.322	1.385	0.516	0.262	0.03601	0.03173	0.99951
ADE-RT	1.369	1.389	0.496	0.290	0.03635	0.03118	0.99965
ADE-LT	1.060	1.560	0.607	0.196	0.04472	0.03140	0.99960

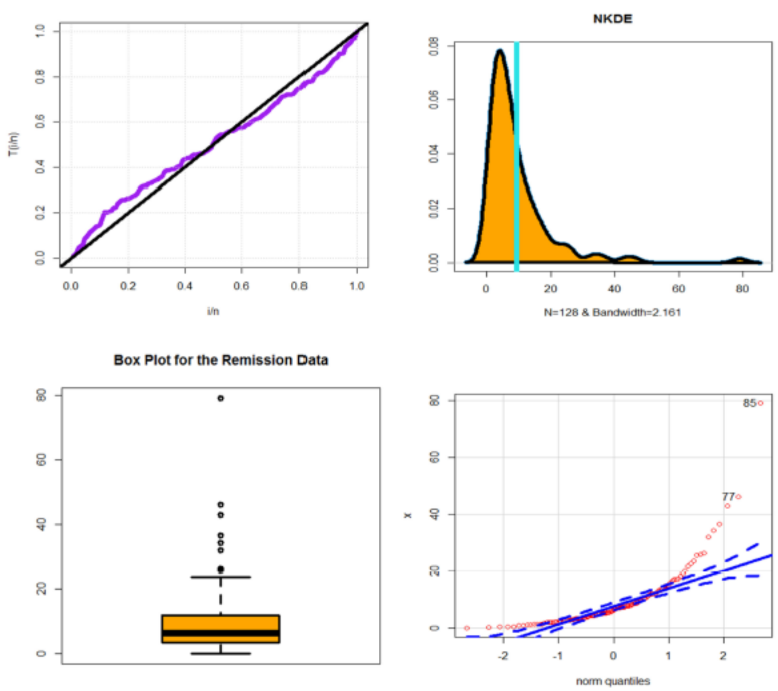

Figure 2. TTT plot, box plot and Q-Q plot for the remission data.

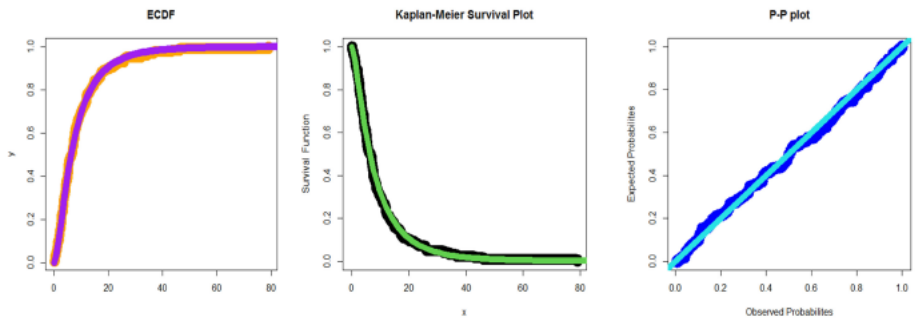

Figure 3. ECDF, KMS and PP plot plot for MLE method.

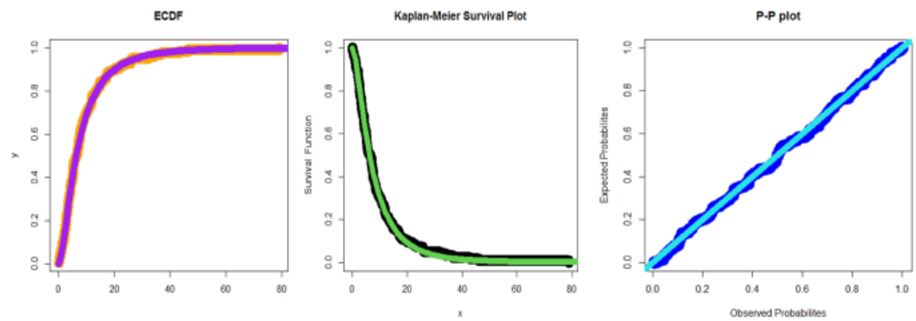

Figure 4. ECDF, KMS and PP plot plot for CVME method.

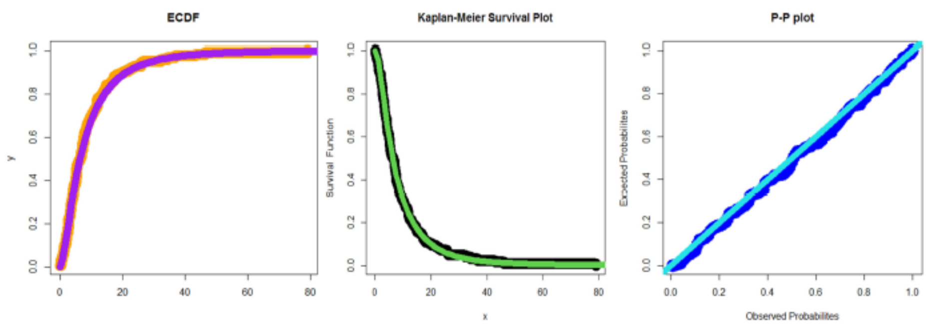

Figure 5. ECDF,KMS and PP plot plot for OLSE method.

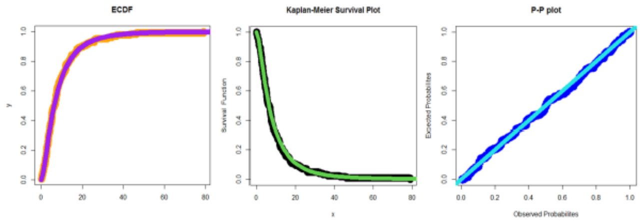

Figure 6. ECDF, KMS and PP plot plot for WLSE method.

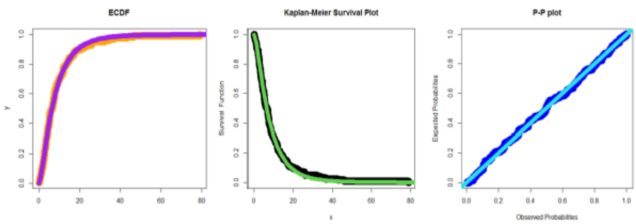

Figure 7. ECDF, KMS and PP plot plot for ADE method.

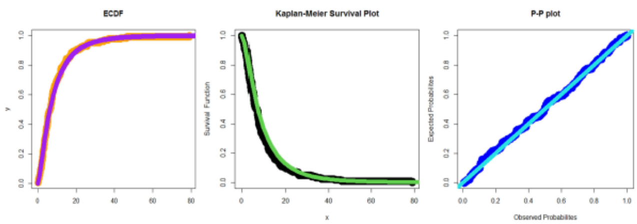

Figure 8. ECDF, KMS and PP plot plot for ADE-RT method.

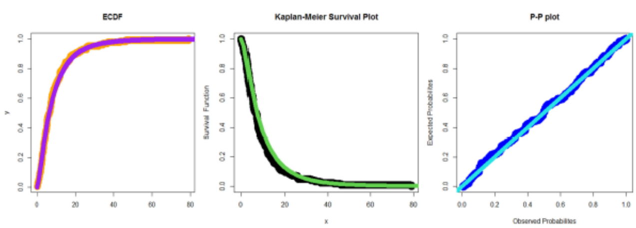

Figure 9. ECDF, KMS and PP plot plot for ADE-LT method.

6. Comparing the competitive models under uncensored schemes

The TLGNH is compared with the generalized exponential (GExp) (Gupta and Kundu (1999)), beta exponential (BExp) (Nadarajah and Kotz (2006)), Kumaraswamy exponential (KuwExp) (Cordeiro and de Castro (2011)) and NH models. Selecting the best model is decided using the estimated −log-likelihood, Akaike Information Criterion ($C_{(1)}$), Consistent Akaike Information Criteria ($C_{(2)}$), Bayesian Information Criterion ($C_{(3)}$) and Hannan-Quinn Information Criterion ($C_{(4)}$). All calculations are obtained by **maxLik routine in R programme**. Figure 10 gives the estimated PDF (EPDF) and estimated CDF (ECDF). Figure 11 gives the estimated survival function (ESF) and estimated HRF (EHRF). The results of this application are listed in Tables 7 and 8. Table 7 gives the MLEs and the corresponding standard errors (SEs). Table 8 gives the results of $C_{(1)}$, $C_{(2)}$, $C_{(3)}$ and $C_{(4)}$. These results show that the TLGNH model has the lowest $C_{(1)}$, $C_{(2)}$, $C_{(3)}$ and $C_{(4)}$. So, it could be chosen as the best model under these four criteria. Clearly, the TLGNH model provides a very close fit. Other useful real data sets for modeling propose can be found in Ibrahim (2020), Elgohari and Yousof (2020) and Ibrahim et al. (2020a).

Table 7: The MLEs and SEs for remission data.

Model	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{a}$	$\widehat{c}$
TLGNH	**1.2555**	**1.2970**	**0.5333**	**0.2182**
	(1.7989)	**(1.7915)**	**(0.142)**	**(0.2092)**
BuXNH		2.040	0.1115267	(24.1076)
		(1.4752)	(0.0359)	(56.038)
GExp	1.2179			0.1211
	(0.1488)			(0.0135)
BExp	1.1726		26.8142	0.0046
	(0.1312)		(0.6327)	(0.0006)
KuwExp	1.4512		0.2816	0.4105
	(0.0240)		(0.0274)	(0.0154)
NH			0.9227	0.1216
			(0.1515)	(0.0344)
Exp				0.1067
				(0.0094)
ExpExp			1.2178	0.1211
			(0.1488)	(0.0136)

Table 8: The $-\hat{\ell}$, $C_{(1)}$, $C_{(2)}$, $C_{(3)}$ and $C_{(4)}$ for remission data

Model	$-\hat{\ell}$	$C_{(1)}$, $C_{(2)}$, $C_{(3)}$, $C_{(4)}$
TLGNH	**410.3825**	**828.7649, 829.0901, 840.173, 833.4001**
G-Exp	413.0776	830.1552, 832.3487, 840.7113, 835.6316
B-Exp	413.3671	832.7342, 835.0594, 846.1423, 839.3693
Kuw-Exp	412.4602	830.9204, 831.1139, 839.4765, 834.3968
NH	414.2255	832.4510, 832.5470, 838.1550, 834.7686
Exp-Exp	413.0776	830.1552, 830.2512, 835.8592, 832.4728
Exp	414.3419	830.6838, 830.7156, 833.5358, 831.8426

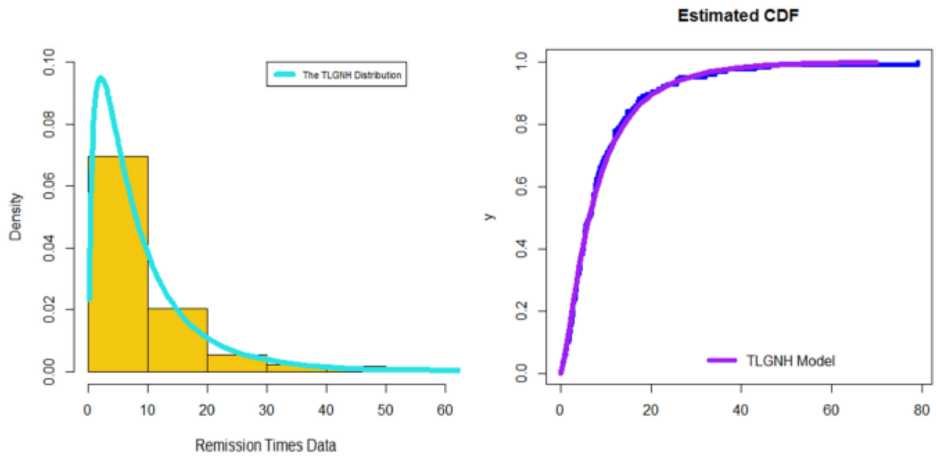

Figure 10. The EPDF and ECDF for the remission data.

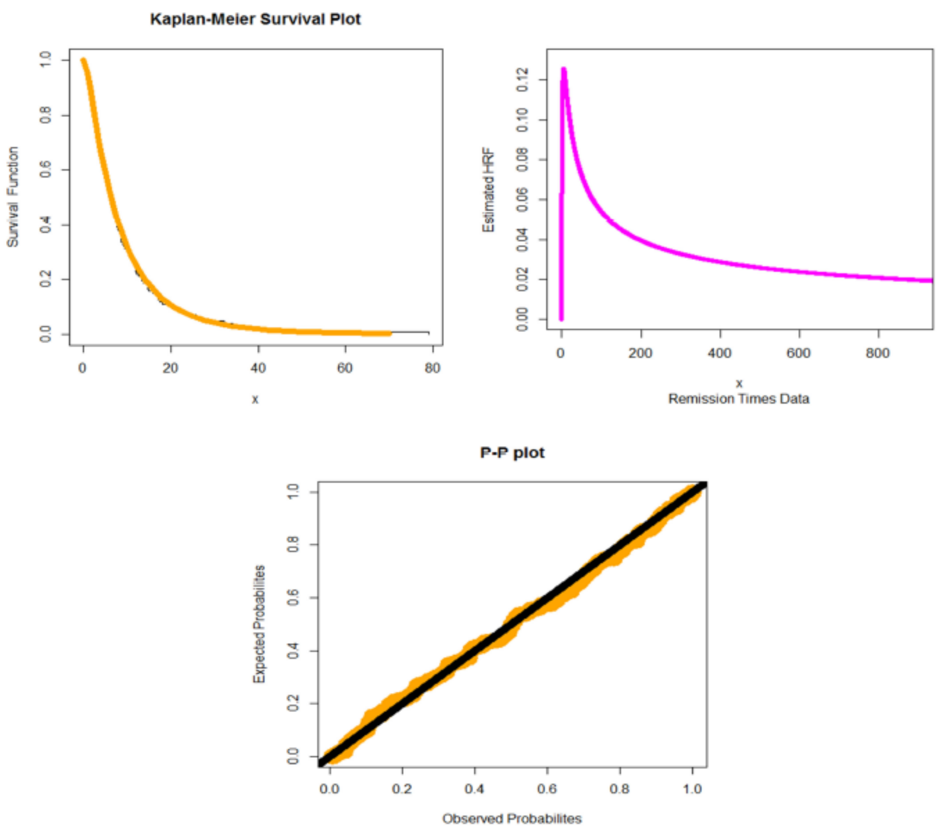

Figure 11. The ESF, EHRF and P-P plot.

7. Validation under censorship

7.1. Maximum likelihood estimation for censored data

In reliability studies and survival analysis, data are often censored. If $Y_1, Y_2, ..., Y_\hbar$ is a censored sample from the TLGNH distribution, each observation can be written as

$$\mathbf{y}_k = \min(Y_k, \mathbf{C}_k)|_{(k=1,..,\hbar)},$$

where Y_k are failure times and $\mathbf{C}_k$ censoring times. The likelihood function is

$$Ł_\hbar(\underline{\Psi}) = \Pi_{k=1}^{\hbar} f(y_k)^{\delta_k} S(y_k)^{1-\delta_k}|_{(\ \delta_k = 1_{X_k < \mathbf{C}_k})},$$

where $S(y_k) = 1 - F(y_k)$ refers to the survival function. The right censoring is assumed to be non informative, so the log-likelihood function can be written as:

$$Ł_\hbar(\underline{\Psi}) = \sum_{h=1}^{\hbar} \delta_k \log f(y_k) + \sum_{k=1}^{k} (1-\delta_k) \log S(y_k).$$

Let $\varpi_k = 1 + c\mathbf{y}_k$, $\varrho_k = 1 - \mathrm{e}^{1-(1+c\mathbf{y}_k)^a}$, $v_k = \varpi^a \mathrm{e}^{1-\varpi^a} \log(\varpi)$ and $u_k = \mathbf{y}_k \varpi^a \mathrm{e}^{1-\varpi^a}$, then

$$Ł_\hbar(\underline{\Psi}) = \sum_{k=1}^{\hbar} \delta_k \left[\begin{array}{c} \log(2) + \log(\alpha\theta c a) \\ +(a-1)\log(\varpi_k) + (1-\varrho_k) \\ +(\alpha\theta - 1)\log(\varrho_k) \\ + \log(1-\varrho_k) + (\alpha-1)\log(2-\varrho_k^\theta) \end{array} \right] + \sum_{k=1}^{\hbar} (1-\delta_k) \log\left(1 - \varrho_{(k,\theta)}^\alpha\right),$$

where $\varrho_{(k,\theta)} = \left[\varrho_k^\theta \left(2 - \varrho_k^\theta\right)\right]$. The maximum likelihood estimators $\widehat{\alpha}$,$\widehat{\theta}$, $\hat{a}$ and $\hat{c}$ of the unknown parameters α, θ, a and c are derived from the nonlinear following score equations:

$$\frac{\partial Ł}{\partial \alpha} = \sum_{k=1}^{\hbar} \delta_i \left[\frac{1}{\alpha} + \theta \log \varrho_k + \log(2 - \varrho_k^\theta)\right] - \sum_{k=1}^{\hbar} (1-\delta_k) \frac{\varrho_{(k,\theta)}^\alpha \log\left[\varrho_k^\theta \left(2 - \varrho_k^\theta\right)\right]}{1 - \varrho_{(k,\theta)}^\alpha}$$

$$\frac{\partial Ł}{\partial \theta} = \sum_{k=1}^{\hbar} \delta_i \left[\frac{1}{\theta} + \alpha \log \varrho_k - \frac{\varrho_k^\theta \log \varrho_k}{1 - \varrho_k^\theta} + \frac{(\alpha-1)\, \varrho_k^\theta \log \varrho_k}{2 - \varrho_k^\theta}\right] - \sum_{k=1}^{\hbar} 2\alpha\,(1-\delta_k) \frac{\varrho_k^\theta \log \varrho_k \left(1 - \varrho_k^\theta\right) \varrho_{(k,\theta)}^{\alpha-1}}{1 - \varrho_{(k,\theta)}^\alpha}$$

$$\frac{\partial Ł}{\partial a} = \sum_{k=1}^{\hbar} \delta_i \left[\begin{array}{c} \frac{1}{a} - \varpi_k^a \log \varpi_k + \frac{(\alpha\theta-1)v_k}{\varrho_k} \\ + \log \varpi_k - \frac{\theta v_k \varrho_k^{\theta-1}}{1-\varrho_k^\theta} - \frac{(\alpha-1)\theta v_k \varrho_k^{\theta-1}}{2-\varrho_k^\theta} \end{array} \right] - \sum_{k=1}^{\hbar} 2\alpha\theta\,(1-\delta_k) \frac{\varpi_k^a \log \varpi_k \mathrm{e}^{1-\varpi_k^a} \varrho_k^{\theta-1} \left(1 - \varrho_k^\theta\right) \varrho_{(k,\theta)}^{\alpha-1}}{1 - \varrho_{(k,\theta)}^\alpha}$$

$$\frac{\partial Ł}{\partial c} = \sum_{k=1}^{\hbar} \delta_i \left[\begin{array}{c} \frac{1}{c} + \frac{(a-1)\mathbf{y}_k}{\varpi_k} - a\mathbf{y}_k \varpi_k^{a-1} + \frac{(\alpha\theta-1)u_k}{\varrho_k} \\ - \frac{\theta u_k \varrho_k^{\theta-1}}{1-\varrho_k^\theta} - \frac{(\alpha-1)\theta u_k \varrho_k^{\theta-1}}{2-\varrho_k^\theta} \end{array} \right] - \sum_{k=1}^{\hbar} 2\alpha\theta a\,(1-\delta_k) \frac{\mathbf{y}_k \varpi_k^{a-1} \mathrm{e}^{1-\varpi_k^a} \varrho_k^{\theta-1} \left(1 - \varrho_k^\theta\right) \varrho_{(k,\theta)}^{\alpha-1}}{1 - \varrho_{(k,\theta)}^\alpha}$$

The explicit form of $\widehat{\alpha}$,$\hat{\theta}$, $\hat{a}$ and $\widehat{c}$ cannot be obtained, so we use numerical methods.

7.2. Test statistic for right censored data

Let $Y_1, Y_2, ..., Y_\hbar$ be $\hbar$ i.i.d. random variables grouped into r classes I_j. To assess the adequacy of a parametric model F_0

$$\mathbf{H}_0 : \Pr(Y_k \leq \mathbf{y} \mid \mathbf{H}_0) = F_0(\mathbf{y}; \underline{\mathbf{\Psi}})|_{(\mathbf{y}\geq 0,\ \ \underline{\mathbf{\Psi}}=(\mathbf{\Psi}_1,\ldots,\mathbf{\Psi}_s)^T\in\Theta\subset R^s, F_0(\mathbf{y})=F_0(\mathbf{y};\underline{\mathbf{\Psi}}))}$$

when data are right censored and the parameter vector $\underline{\mathbf{\Psi}}$ is unknown, Bagdonavicius and Nikulin (2011) proposed a statistic test T^2 based on the vector

$$\mathbf{Z}_j = \frac{1}{\sqrt{n}}(\mathbf{U}_j - \mathrm{e}_j)|_{(j=1,2,\ldots,r\ \ ,\ \text{with} r\succ s)}.$$

This one represents the differences between observed and expected numbers of failures ($\mathbf{U}_j$ and e_j) to fall into these grouping intervals $I_j = (\rho_{j-1}, \rho_j]$ with $\rho_0 = 0$, $\rho_r = \tau$, where τ is a finite time. The authors considered ρ_j as random data functions such as the r intervals chosen have equal expected numbers of failures e_j. The statistic test T^2 is defined by

$$T^2 = \mathbf{Z}^T \widehat{\sum}^{-} \mathbf{Z} \ = \sum_{j=1}^{r} \frac{1}{\mathbf{U}_j}(\mathbf{U}_j - \mathrm{e}_j)^2 + \mathbf{Q}$$

where $\mathbf{Z} = (\mathbf{Z}_1, ..., \mathbf{Z}_k)^T$ and $\widehat{\sum}^{-}$ is a generalized inverse of the covariance matrix $\widehat{\Sigma}$ and

$$\begin{aligned}
\mathbf{Q} &= \mathbf{W}^T \widehat{\mathbf{G}}^{-} \mathbf{W},\ \widehat{\mathbf{A}}_j = \mathbf{U}_j/n, \mathbf{U}_j = \sum_{i:Y_k\in I_j} \delta_k, \mathbf{W} = (\mathbf{W}_1, ..., \mathbf{W}_s)^T, \\
\widehat{\mathbf{G}} &= [\widehat{g}_{ll'}]_{s*s},\ \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{\mathbf{C}}_{lj}\widehat{\mathbf{C}}_{l'j}\hat{\mathbf{A}}_j^{-1},\ \widehat{\mathbf{C}}_{lj} = \frac{1}{n}\sum_{k:Y_k\in I_j} \delta_k \frac{\partial}{\partial \underline{\mathbf{\Psi}}} \ln\left[\hbar_{\underline{\mathbf{\Psi}}}(\mathbf{y}_k)\right], \\
\widehat{i}_{ll'} &= \frac{1}{\hbar}\sum_{k=1}^{\hbar} \delta_k \frac{\partial \ln h(\mathbf{y}_k, \widehat{\underline{\mathbf{\Psi}}})}{\partial \underline{\mathbf{\Psi}}_l} \frac{\partial \ln\left[\hbar_{\widehat{\underline{\mathbf{\Psi}}}}(\mathbf{y}_k)\right]}{\partial \underline{\mathbf{\Psi}}_{l'}},
\end{aligned}$$

and

$$\widehat{\mathbf{W}}_l = \sum_{j=1}^{r} \widehat{\mathbf{C}}_{\leq j} \widehat{\mathbf{A}}_j \mathbf{Z}_j|_{(l,l'=1,\ldots.,s)},$$

wher $\widehat{\underline{\mathbf{\Psi}}}$ is the maximum likelihood estimator of $\underline{\mathbf{\Psi}}$ on initial non-grouped data and $\hbar_{\underline{\mathbf{\Psi}}}(\mathbf{y}_k)$ refers to the HRF of the TLGNH. Under the null hypothesis $\mathbf{H}_0$, the limit distribution of the statistic T^2 is a chi-square with $r = rank(\Sigma)$ degrees of freedom. The description and applications of modified chi-square tests are discussed in Voinov et al. (2013). The interval limits ρ_j for grouping data into j classes I_j are considered as data functions and defined by

$$\hat{\rho}_j = \hbar^{-1}\left(\frac{E_j - \sum_{l=1}^{k-1} H_{\underline{\mathbf{\Psi}}}(\mathbf{y}_l)}{\hbar - k + 1}, \underline{\mathbf{\Psi}}\right)|_{\left(\ \hat{\rho}_r = \max\left(Y_{(\hbar)}, \tau\right)\right)},$$

such as the expected failure times e_j to fall into these intervals are $\mathrm{e}_j = \frac{1}{r}E_r$ for any j with $E_r = \sum_{k=1}^{\hbar} H_{\underline{\mathbf{\Psi}}}(\mathbf{y}_k)$, where $H_{\underline{\mathbf{\Psi}}}(\mathbf{y}_l) = -\ln[1 - F(\mathbf{y})]$ refers to the cumulative rate function. The distribution of T_n^2 statistic test is chi-square (see Voinov et al. (2013)). For useful details see Ibrahim et al. (2019, 2020b), Yadav et al. (2020), Abouelmagd et al. (2019a,b), Goual et al. (2019a,b), Goual and Yousof (2020).

7.3. Criteria test for the TLGNH

For testing the null hypothesis $\mathbf{H}_0$ that data belong to the TLGNH model, we construct a modified chi-squared type goodness-of-fit test based on the statistic T^2. Suppose that τ is a finite time, and observed data are grouped into $r > s$ sub-intervals

$$I_j = (\rho_{j-1}, \rho_j] \text{ of } [0, \tau].$$

The limit intervals ρ_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j are obtained as

$$E_j = \frac{-j}{r-1} \sum_{k=1}^{\hbar} \log\left(1 - \varrho_{(k,\theta)}^{\alpha}\right)|_{(j=1,..r-1)}$$

7.4. Estimated matrix $\widehat{\mathbf{W}}$ et $\widehat{\mathbf{C}}$

The components of the estimated matrix $\widehat{\mathbf{W}}$ are derived from the estimated matrix $\widehat{\mathbf{C}}$ which is given by:

$$\widehat{\mathbf{C}}_{1j} = \frac{1}{\hbar} \sum_{k: y_k \in I_j}^{\hbar} \delta_k \left\{ \frac{1}{\alpha} + \theta \log \varrho_k + \log(2 - \varrho_k^{\theta}) + \frac{\varrho_{(k,\theta)}^{\alpha} \log\left[\varrho_k^{\theta}\left(2 - \varrho_k^{\theta}\right)\right]}{1 - \varrho_{(k,\theta)}^{\alpha}} \right\}$$

$$\widehat{\mathbf{C}}_{2j} = \frac{1}{\hbar} \sum_{k: y_k \in I_j}^{\hbar} \delta_k \left[\begin{array}{c} \frac{1}{\theta} + \alpha \log \varrho_k - \frac{\varrho_k^{\theta} \log \varrho_k}{1-\varrho_k^{\theta}} \\ + \frac{(\alpha-1)\varrho_k^{\theta} \log \varrho_k}{2-\varrho_k^{\theta}} + \frac{2\alpha \varrho_k^{\theta} \log \varrho_k \left(1-\varrho_k^{\theta}\right) \varrho_{(k,\theta)}^{\alpha-1}}{1-\varrho_{(k,\theta)}^{\alpha}} \end{array} \right]$$

$$\widehat{\mathbf{C}}_{3j} = \frac{1}{\hbar} \sum_{k: y_k \in I_j}^{\hbar} \delta_k \left[\begin{array}{c} \frac{1}{a} - \varpi_k^{a} \log \varpi_k + \frac{(\alpha\theta-1)v_k}{\varrho_k} + \log \varpi_k - \frac{\theta v_k \varrho_k^{\theta-1}}{1-\varrho_k^{\theta}} - \frac{(\alpha-1)\theta v_k \varrho_k^{\theta-1}}{2-\varrho_k^{\theta}} \\ + \frac{2\alpha\theta \varpi_k^{a} \log \varpi_k \mathrm{e}^{1-\varpi_k^{a}} \varrho_k^{\theta-1}\left(1-\varrho_k^{\theta}\right)\varrho_{(k,\theta)}^{\alpha-1}}{1-\varrho_{(k,\theta)}^{\alpha}} \end{array} \right]$$

$$\widehat{\mathbf{C}}_{4j} = \frac{1}{\hbar} \sum_{k: y_k \in I_j}^{\hbar} \delta_k \left[\begin{array}{c} \frac{1}{c} + \frac{(a-1)\mathbf{y}_k}{\varpi_k} - a\mathbf{y}_k \varpi_k^{a-1} + \frac{(\alpha\theta-1)u_k}{\varrho_k} - \frac{\theta u_k \varrho_k^{\theta-1}}{1-\varrho_k^{\theta}} - \frac{(\alpha-1)\theta u_k \varrho_k^{\theta-1}}{2-\varrho_k^{\theta}} \\ +2\alpha\theta a\left(1-\delta_k\right) \frac{\mathbf{y}_k \varpi_k^{a-1} \mathrm{e}^{1-\varpi_k^{a}} \varrho_k^{\theta-1}\left(1-\varrho_k^{\theta}\right)\varrho_{(k,\theta)}^{\alpha-1}}{1-\varrho_{(k,\theta)}^{\alpha}} \end{array} \right]$$

and

$$\widehat{\mathbf{W}}_l = \sum_{j=1}^{r} \widehat{\mathbf{C}}_{lj} A_j^{-1} \mathbf{Z}_j|_{(l=1,..,m \text{ and } j=1,..,r)}$$

Therefore the quadratic form of the test statistic can be obtained easily:

$$T_n^2\left(\underline{\mathbf{\Psi}}\right) = \sum_{j=1}^{r} \frac{\left(\mathbf{U}_j - \mathrm{e}_j\right)^2}{\mathbf{U}_j} + \widehat{\mathbf{W}}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \widehat{\mathbf{C}}_{lj} \widehat{\mathbf{C}}_{l'j} \hat{\mathbf{A}}_j^{-1} \right]^{-1} \widehat{\mathbf{W}}$$

7.5. Simulations

7.5.1. Maximum likelihood estimation To show the praticability of this test, we carry out an important simulation study by generating $N = 10,000$ right censored samples with different percentage (15% and 30%) of right censoring and different sizes ($n = 25, 50, 130, 350, 500$) from the TLGNH model with parameters $\alpha = 2, \theta = 0.7$, $a = 0.5$ and $c = 1.5$. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Ravi, 2009), we calculate the maximum likelihood estimators of the unknown parameters and their mean squared errors (MSE). The results are given in Table 9.

Table 9: Mean Simulated values of MLEs $\hat{\alpha}, \hat{a}, \hat{c}$ and $\hat{\theta}$
and their corresponding square mean errors

MLEs & MSEs$\downarrow$ $n \rightarrow$	25	50	130	350	500	1000	1500
$\hat{\alpha}$	1.8798	1.8828	1.8983	1.9312	1.9989	1.9995	1.9999
MSE	0.0078	0.0072	0.0061	0.0049	0.0038	0.0022	0.0008
$\hat{\theta}$	0.6818	0.6853	0.6912	0.6963	0.7012	0.7008	0.7004
MSE	0.0072	0.0065	0.0053	0.0042	0.0028	0.0016	0.0009
$\hat{a}$	0.5596	0.5531	0.5412	0.5283	0.5162	0.5098	0.5043
MSE	0.0092	0.0089	0.0073	0.0052	0.0031	0.0019	0.0010
$\hat{c}$	1.4789	1.4823	1.4932	1.4989	1.5022	1.5015	1.5006
MSE	0.0089	0.0073	0.0062	0.0055	0.0048	0.0032	0.0021

The maximum likelihood estimated parameter values, presented in Table 9, agree closely with the true parameter values.

7.5.2. Test statistic T^2 Then, we calculate the test statistic T^2 for each sample with respect to the TLGNH model and we compare the obtained values with the theoretical levels of significance ($\varepsilon = 0.01$, 0.05 , 0.1). The results are summarized in Table 10 and Table 11.

Table 10: Simulated levels of significance for T^2
against their theoretical values (15% of censorship)

$\varepsilon \downarrow n \rightarrow$	25	20	130	350	500
1%	0.0073	0.0082	0.0089	0.0095	0.0098
5%	0.0373	0.0389	0.0420	0.0473	0.0492
10%	0.0872	0.0897	0.00923	0.0982	0.0995

Table 11: Simulated levels of significance for T^2
against their theoretical values (30% of censorship)

$\varepsilon \downarrow n \rightarrow$	25	20	130	350	500
1%	0.0078	0.0085	0.0092	0.0097	0.0102
5%	0.0381	0.0392	0.0415	0.0468	0.0484
10%	0.0884	0.0892	0.0945	0.0978	0.0999

As we can see empirical proportions of rejection of the null hypothesis $\mathbf{H}_0$ for $\varepsilon = 1\%, 5\%$ and 10% levels of significance for all sample sizes and for different percentage of censorship (Table 10 and Table 11) are very close to the theoretical ones. Therefore, the test statistic T^2, proposed in this work, can be applied to fit data to TLGNH

7.6. Four censored data sets for validation

In this section, we introduce four applications of the TLGNH distribution to four real data sets: The first data set (see John et al. (1997)) has reported survival data on 26 psychiatric inpatients admitted to the university of Iowa hospitals during the years 1935-1948. This sample is part of a larger study of psychiatric Inpatients discussed by Tsuang and Woolson (1977). Data for each patient consists of age at first admission to the hospital, sex, number of years of follow-up (years from admission to death or censoring) and patient status at the followup time. The data is given as 1, 1, 2, 11, 14, 22, 22, 24, 25, 26, 28, 30*, 30*, 31*, 31*, 32, 33*, 33*, 34*, 35, 35*, 35*, 36*, 37*, 39*, 40. (* indicates the censorship). The second data set consists sample data from 50 patients with acute myeloid leukemia, reported to the International Register of Bone Marrow Transplants (see John et al. (1997)). These patients had an allogeneic bone marrow transplant where the HLA (Histocompatibility Leukocyte Antigen) homolog marrow was used to rebuild their immune systems. The data required for this study is shown below 0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.5, 2.763, 2.993, 3.224, 3.421, 4.178, 4 .441*, 5.691, 5.855* ,6.941*, 6.941, 7.993*, 8.882, 8.882, 9.145*, 11.480, 11.513, 12.105*, 12.796, 12.993*, 13.849*, 16.612*, 17.138*, 20.066, 20.329*, 22.368*, 26.776*, 28.717*, 28.717*, 32.928*, 33.783* , 34.211*, 34.770*, 39.539*, 41.118*, 45.033*, 46.053*, 46.941*, 48.289*, 57.401*, 58.322*, 60.625*. The third data set, we apply the results obtained from this study to real data established from reliability (Crowder et al., 1991**).** In an experiment to obtain information on the strength of a certain type of braided cord after the weather, the forces of 48 pieces of cord having resisted for a determined time were studied. The right censored force values observed are given as: 26.8*, 29.6*, 33.4*, 35*, 36.3, 40*, 41.7, 41.9*, 42.5*, 43.9, 49.9, 50.1, 50.8, 51.9, 52.1, 52.3, 52.3, 52.4, 52.6, 52.7, 53.1, 53.6, 53.6, 53.9, 53.9, 54.1, 54.6, 54.8, 54.8, 55.1, 55.4, 55.9, 56, 56.1, 56.5, 56.9, 57.1, 57.1, 57.3, 57.7, 57.8, 58.1, 58.9, 59, 59.1, 59.6, 60.4, 60.7. Fourth data set (see Sedmak et al. (1989)). This study is designed to determine if female breast cancer patients, originally classified as lymph node negative by standard light microscopy (SLM), could be more accurately classified by immunohistochemical (IH) examination of their lymph nodes with an anticytokeratin monoclonal antibody cocktail, identical sections of lymph nodes were sequentially examined by SLM and IH. The significance of this study is that 16% of patients with negative axillary lymph nodes, by standard pathological examination, develop recurrent disease within 10 years. Forty-five female breast-cancer patients with negative axillary lymph nodes and a minimum 10-year follow-up were selected from The Ohio State University Hospitals Cancer Registry. Of the 45 patients, 9 were immunoperoxidase positive, and the remaining 36 remained negative. The data required for this study is shown as**:**19, 25, 30, 34, 37, 46, 47, 51, 56, 57, 61, 66, 67, 74, 78, 86, 122* , 123* , 130* , 130* , 133* , 134* , 136* , 141* , 143* ,148* , 151* , 152* ,153* ,154* , 156* , 162* , 164* , 165* , 182* ,189* . Times to death (in months) for breast cancer patients with different immunohistochemical responses. All results of these application are listed in Table 12. Based on Table 12, we can claim that the TLGNH model can be used in modeling these data.

Table 12: The values of Y^2 for each data

Data↓Estimated & Y^2 →	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{c}$	$Y^2\|_{\alpha=0.05}$
Data set **I** (survival data)	1.568	0.243	1.214	2.465	$7.6359 < \chi^2_4 = 9.4877$
Data set **II** (leukemia data)	2.463	0.832	1.861	3.526	$8.8694 < \chi^2_5 = 11.0705$
Data set **III** (reliability data)	0.934	0.139	0.814	1.324	$7.2348 < \chi^2_5 = 11.0705$
Data set **IV** (cancer data)	1.239	0.763	1.086	2.237	$6.9764 < \chi^2_4 = 9.4877$

Below are results of the modified test statistic under the right censored data sets:

- For the data of survival data on 26 psychiatric inpatients admitted to the university of Iowa hospitals :

$$Y^2_{4,0.05} = 7.6359 < \chi^2_{4,0,05} = 9.4877. \text{ Decision} \Rightarrow \text{ accept } \mathbf{H}_0.$$

- For the data of 50 patients with acute myeloid leukemia:

$$Y^2_{5,0.05} = 8.8694 < \chi^2_{5,0,05} = 11.0705. \text{ Decision} \Rightarrow \text{ accept } \mathbf{H}_0.$$

- For the reliability strength of 48 pieces of cord having resisted for a determined time data:

$$Y^2_{5,0.05} = 7.2348 < \chi^2_{5,0,05} = 11.0705. \text{ Decision} \Rightarrow \text{ accept } \mathbf{H}_0.$$

- For the breast cancer patients data (45 patients):

$$Y^2_{4,0.05} = 6.9764 < \chi^2_{4,0,05} = 9.4877. \text{ Decision} \Rightarrow \text{ accept } \mathbf{H}_0.$$

8. Conclusions

In this work, a new Nadarajah-Haghighi model is proposed. Some of its properties are mathematically derived. The new density has a heavy right skew tails with different shapes. The new failure rate function has the "**U** shape", "constant shape", "increasing-constant shape", "decreasing-constant shape", "upside down shape" and "upside down-**U** shape". Numerical calculations for analyzing the skewness and kurtosis are presented and we noted that:

1- The skewness of the new model is always positive.

1- The kurtosis of the new model can be more than 3 or less than 3.

2- The skewness of the new model ranges in the interval (0.397, 2000.4), whereas the skewness of the standard Nadarajah-Haghighi model can be in the interval (0.741, 49.367). Further, the spread of kurtosis for the new model is from -39.480 to 8271313, whereas the spread for the standard Nadarajah-Haghighi kurtosis varies only from -6.664 to 73.365.

We considerd many estimation methods such as the maximum likelihood estimation method; Cramér-von-Mises estimation method; ordinary least square estimation method; weighted least square estimation method; Anderson Darling estimation method; right tail Anderson Darling estimation method; left tail-Anderson Darling estimation method. Numerical simulation studies are executed for testing performance of all themethod. An example of environmental real data set is employed to compare the estimation methods. Another example is presented to measure importance and flexibility of the new model. Using the validation approach proposed by Bagdonavicius and Nikulin (2011) for censored data, we propose the construction of modified chi-square goodness-of-fit tests for the new model. Based on the maximum likelihood estimators on initial data, the modified statistics recover the information lost while grouping data and follow chi-square models. All elements of the modified criteria tests are given explicitly. Comprehensive numerical example from simulated samples and four real data sets have been analyzed to illustrate the feasibility of the modified test.

Appendix

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $Y : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(Y\right) \mid Y \geq y\right] = \mathbf{E}\left[q_1\left(Y\right) \mid Y \geq y\right] \xi\left(y\right), \quad y \in H,$$

is defined with some real function ξ. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\xi \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F\left(y\right) = \int_a^y C \left| \frac{\xi'\left(u\right)}{\xi\left(u\right) q_1\left(u\right) - q_2\left(u\right)} \right| \exp\left(-s\left(u\right)\right)\, du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi'\, q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Note: The goal is to have the function $\xi\left(y\right)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel, 1990), in particular, let us assume that there is a sequence $\{Y_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_1 , q_2 and ξ_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_1 \to q_1$, $q_2 \to q_2$ for some continuously differentiable real functions q_1 and q_2 . Let, finally, Y be a random variable with distribution F . Under the condition that $q_1\ (Y)$ and $q_2\ (Y)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence Y_n converges to Y in distribution if and only if ξ_n converges to ξ , where

$$\xi(y) = \frac{\mathbf{E}[q_2(Y) \mid Y \geq y]}{\mathbf{E}[q_1(Y) \mid Y \geq y]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1 , q_2 and ξ , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$. A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics. In some cases, one can take $q_1(y) \equiv 1$, which reduces the condition of Theorem 1 to $\mathbf{E}[q_2(Y) \mid Y \geq y] = \xi(y)$, $y \in H$. We, however, believe that employing three functions q_1 , q_2 and ξ will enhance the domain of applicability of Theorem 1.

REFERENCES

1. Abouelmagd, T. H. M., Hamed, M. S., Handique, L., Goual, H., Ali, M. M., Yousof, H. M. and Korkmaz, M. C. (2019). A new class of distributions based on the zero truncated Poisson distribution with properties and applications, Journal of Nonlinear Sciences & Applications, 12(3), 152-164.
2. Abouelmagd, T. H. M., Hamed, M. S., Hamedani, G. G., Ali, M. M., Goual, H., Korkmaz, M. C. and Yousof, H. M. (2019). The zero truncated Poisson Burr X family of distributions with properties, characterizations, applications, and validation test. Journal of Nonlinear Sciences and Applications, 12(5), 314-336.
3. Abdul-Moniem, I. B. (2016). Exponentiated Nadarajah and Haghighi exponential distribution. International Journal of Mathematical Analysis and Applications, **2**, 68-73.
4. Ahmed, A., Muhammed, H. Z. and Elbatal, I. (2015). A new class of extension exponential distribution. International Journal of Applied Mathematical Sciences, **8**, 13–30.
5. Bagdonavičius, V. and Nikulin, M. (2011). Chi-squared goodness-of-fit test for right censored data, International Journal of Applied Mathematics and Statistics, vol. 24, 30-50.
6. Bagdonavičius, V., Levuliene, R., J. and Nikulin, M., (2013). Chi-squared goodness-of-fit tests for parametric accelerated failure time models, Communications in Statistics- Theory and Methods, Volume: 42 Issue: 15, 2768-2785.
7. Bourguignon, M., Lima, M. C. S., Leao,J., Nascimento, A. D. C., Pinho, L. G. B. and Cordeiro, G. M. (2015). A new generalized gamma distribution with applications. American Journal of Mathematical and Management Sciences, 34, 309–342.
8. Cordeiro, G. M. and Castro, M. (2011). A new family of generalized distributions. Journal of statistical computation and simulation, **81(7)**, 883-898.
9. Crowder, M. J., Kimber A. C., Smith R. L and Sweeting, T. J. (1991). Statistical analysis of reliability data, CHAPMAN &HALL/CRC
10. Dias, C. R. B., Alizadeh, M. and Cordeiro, G. M. (2018). The beta Nadarajah-Haghighi distribution. Hacettepe Journal of Mathematics and Statistics, 47, 1302 13203.
11. El Damcese, M. A. and Ramadan, D. A. (2015). Studies on properties and estimation problems for modified extension of exponential distribution. International Journal of Computer Applications, 125.
12. Elgohari, H. and Yousof, H. M. (2020). New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. Statistics, Optimization & Information Computing, 8(4), 972-993. https://doi.org/10.19139/soic-2310-5070-1036
13. Elsayed, H. A. H. and Yousof, H. M. (2019). The Burr X Nadarajah Haghighi distribution: statistical properties and application to the exceedances of flood peaks data. Journal of Mathematics and Statistics, 15, 146-157.
14. John P. Klein, Melvin L. Moeschberger (1997), Survival Analysis :Techniques for Censored and Truncated Data , Statistics for Biology and Health
15. Galambos, J. and Kotz, S. Characterizations of probability distributions. A unified approach with emphasis on exponential and related models, Lecture Notes in Mathematics, p.675. Springer, Berlin (1978).
16. Glänzel, W., A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, (1987), 75–84.
17. Glänzel, W., Some consequences of a characterization theorem based on truncated moments, *Statistics: A Journal of Theoretical and Applied Statistics*, 21 (4), (1990), 613–618.
18. Glänzel, W, Telcs, A, Schubert, A. Characterization by truncated moments and its application to Pearson-type distributions, Z. Wahrsch. Verw. Gebiete 66, 173-182 (1984).
19. Glänzel, W. and Hamedani, G.G. Characterizations of the univariate distributions, Studia Scien. Math. hung., 37, 83-118 (2001).
20. Hamedani, G.G., On certain generalized gamma convolution distributions **II**, Technical Report, No. 484, MSCS, Marquette University, 2013.

21. Goual, H. and Yousof, H. M. (2020). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics, 47(3), 393-423.
22. Goual, H., Yousof, H. M. and Ali, M. M. (2019a). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pakistan Journal of Statistics and Operation Research, 15(3), 745-771.
23. Goual, H., Yousof, H. M. and Ali, M. M. (2020b). Lomax inverse Weibull model: properties, applications, and a modified Chi-squared goodness-of-fit test for validation. Journal of Nonlinear Sciences & Applications, 13(6), 330-353.
24. Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. Australian & New Zealand Journal of Statistics, 41(2), 173-188.
25. Ibrahim, M. (2020b). The generalized odd Log-logistic Nadarajah Haghighi distribution: statistical properties and different methods of estimation. Journal of Applied Probability and Statistics, 15(2), 61-84.
26. Ibrahim, M., Altun, E. and Yousof, H. M. (2020a). A new distribution for modeling lifetime data with different methods of estimation and censored regression modeling. Statistics, Optimization & Information Computing, 8(2), 610-630.
27. Ibrahim, M., Altun, E., Goual, H., and Yousof, H. M. (2020b). Modified goodness-of-fit type test for censored validation under a new Burr type XII distribution with different methods of estimation and regression modeling. Eurasian Bulletin of Mathematics, 3(3), 162-182.
28. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H. and Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Communications for Statistical Applications and Methods, 26(5), 473-495.
29. Kim, J.H. and Jeon, Y. Credibility theory based on trimming, Insur. Math. Econ. 53(1), 36-47 (2013).
30. Kotz, S. and Shanbhag, D.N. Some new approach to probability distributions, Adv. Appl. Probab. 12, 903-921 (1980).
31. Lemonte, A. J. (2013). A new exponential type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics & Data Analysis, **62**, 149–170.
32. Lemonte, A. J., Cordeiro, G. M. and Arenas, G. M. (2016). A new useful threeparameter extension of the exponential distribution. Statistics, **50**, 312–337.
33. Lee, E.T. and Wang, J. W. (2003) Statistical Methods for Survival Data Analysis, third ed. Wiley, New York.
34. Lima, S. R. L. (2015). The half normal generalized family and kummaraswamy Nadarajah Haghighi distribution. Master thesis, Universidade Federal de Pernumbuco.
35. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. Reliability engineering & system safety, **91(6)**, 689-697.
36. Ortega, E. M. M., Lemonte, A. J., Silva, G. O. and Cordeiro, G. M. (2015). New flexible models generated by gamma random variables for lifetime modeling. Journal of Applied Statistics, 42, 2159–2179.
37. Vedo, V. T., Nascimento, A. D. C., Miranda, W. R. Lima, M. C. S. Pinho, L. G. B. and Cordeiro, G. M. (2016). Some computational and theoretical aspects of the exponentiated generalized Nadarajah-Haghighi distribution. Chilean Journal of Statistics, forthcoming.
38. Ravi, V. and Gilbert, P. D. (2009). BB: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function, J. Statist. Software, 32(4).
39. Rezaei, S., Sadr, B. B., Alizadeh, M., & Nadarajah, S. (2017). Topp–Leone generated family of distributions: Properties and applications. Communications in Statistics-Theory and Methods, 46(6), 2893-2909.
40. Sedmak, D. D., Meineke, T. A., Knechtges, D. S. and Anderson, J. (1989). Prognostic significance of cytokeratin-positive breast cancer metastases. Modern pathology: an official journal of the United States and Canadian Academy of Pathology, Inc, 2(5), 516-520.
41. Tsuang, M. T. and Woolson, R. F. (1977). Mortality in patients with schizophrenia, mania, depression and surgical conditions. The British Journal of Psychiatry, 130, 162–166.
42. Lemonte, A. J. (2013). A new exponential type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics & Data Analysis, 62, 149-170.
43. Nadarajah, S.and Haghighi, F. (2011). An extension of the exponential distribution. Statistics. 45, 543–558.
44. Shehata, W. A. and Yousof, H. M. (2021). A novel two-parameter Nadarajah-Haghighi extension: properties, copulas, modeling real data and different estimation methods. Statistics, Optimization & Information Computing, forthcoming.
45. Yadav, A. S., Goual, H., Alotaibi, R. M., Ali, M. M. and Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry, 12(1), 57.
46. Yousof, H. M. and Korkmaz, M. C. (2017). Topp-Leone Nadarajah-Haghighi distribution: mathematical properties and applications, International Journal of Applied Mathematics. Journal of Statisticians: Statistics and Actuarial Sciences, 2, 119-128.
47. Yousof, H. M., Korkmaz, M. C. Hamedani G. G. (2017). The odd Lindley Nadarajah-Haghighi distribution. J. Math. Comput. Sci., 7, 864-882.