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# Analysis and Applications of Quantile Approach on Residual Extropy

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**Abstract** Extropy is a measure of the uncertainty of random variable. Motivated with the wide applicability of quantile functions in modeling and analyzing statistical data, in this paper, we study quantile version of the extropy from residual lifetime variable, "residual quantile extropy" in short. Unlike the residual extropy function, the residual quantile extropy determines the quantile density function uniquely through a simple relationship. Aging classes, stochastic orders and characterization results are derived, using proposed quantile measure of uncertainty. We also suggest some applications related to $(n-i+1)$-out-of-$n$ systems and distorted random variables. Finally, a nonparametric estimator for residual quantile extropy is provided. In order to evaluate the proposed estimator, we use a simulation study and a real data set.

**Keywords** Distorted distribution, Quantile function, Nonparametric estimator, Reliability measures, Residual extropy, Stochastic orders, Uncertainty measure.

**AMS 2010 subject classifications** 94A17, 62G05, 62N05

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## 1. Introduction

The quantile function (qf) of continuous random variable $X$ can be specified in terms of the cumulative distribution function (cdf) $F_X(x)$ as

$$Q_X(u) = F_X^{-1}(u) = \inf\{x \mid F_X(x) \geq u\}, \quad 0 \leq u \leq 1. \tag{1}$$

Although both distribution and quantile functions convey the same information about the distribution of a random variable (rv), qfs have several properties that are not shared by cdfs. For example, $(i)$ the sum of two qfs is again a qf; $(ii)$ the product of two positive qfs is also a qf; $(iii)$ the nondecreasing function of a qf is again a qf; $(iv)$ in many cases, qf is more convenient as it is less influenced by extreme observations, and thus provides a straightforward analysis with a limited amount of information; $(v)$ there are explicit general distribution forms for the qf of order statistics. It is easier to generate random numbers from the qf; $(vi)$ there are probability models having no closed form cdfs. However, they have closed form qfs; and $(vii)$ the use of qfs in the place of cdf provides new models, alternative methodology, easier algebraic manipulations and methods of analysis in certain cases and some new results that are difficult to derive by using distribution function. For more properties and additional information on quantile function, we refer to Gilchrist [6] and Nair et al. [15].
Shannon [29] by developing information theory, introduced a criterion for measurement of uncertainty and called it entropy. The Shannon entropy of nonnegative continuous rv $X$ with probability density function (pdf) $f_X(x)$ is given by $H(X) = -\int_0^{+\infty} f_X(x)\log f_X(x)dx$. Nowadays, this criterion has found a special place in the sciences, including economics, physics, computer, telecommunications, communication theory, reliability and etc. Lad et

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al. [14] provided a completion to theories of information based on entropy, resolving a longstanding question in its axiomatization as proposed by Shannon [29] and followed by Jaynes [9]. They showed that Shannon's entropy function has a complementary dual function which is called " extropy". The extropy of discrete rv $X$ is given by $J(X) = -\sum_{i=1}^{N}(1-p_i)\log(1-p_i)$, where $p_i = P(X = x_i)$. When the range of possibilities for discrete rv X increases, the extropy measure $J(X)$ can be closely approximated by $1 - \frac{1}{2}\sum_{i=1}^{N} p_i^2$, which led to the definition of differential extropy. The differential extropy of nonnegative continuous rv $X$ is given by

$$J(X) = -\frac{1}{2}\int_0^{+\infty} f_X^2(x)dx. \tag{2}$$

Lad et al. [14] analyzed the differential extropy function for densities, showing that relative extropy constitutes a dual to the Kullback-Leibler divergence, widely recognized as the continuous entropy measure. Extropy has several applications. For example, $(i)$ extropy is used to score the forecasting distributions using the total scoring rule (see Gneiting and Raftery [7]); $(ii)$ extropy is interpreted as a measure of the amount of uncertainty represented by the distribution for rv, that is, if the extropy of $X$ is less than that of another rv $Y$, that is, $J(X) \le J(Y)$, then $X$ is said to have more uncertainty than $Y$ (see Qiu et al. [25]); $(iii)$ extropy is used to compare two mixed systems with same signature but with different components (see Qiu et al. [25]). Most recently, Qiu and Jai [24] proposed residual differential extropy (REX) to measure residual uncertainty of a nonnegative absolutely continuous rv as

$$J(X;t) = -\frac{1}{2\bar{F}_X^2(t)}\int_t^{+\infty} f_X^2(x)dx, \quad t \ge 0. \tag{3}$$

For further studies on differential extropy, we refer to Qiu et al. [25], Raqab and Qiu [27], Yang et al. [39], Noughabi and Jarrahiferiz [19], Jose and Sathar [11] and Jahanshahi et al. [10].
Recently, the quantile based methods have been employed effectively to investigate the information properties of such models.

Accordingly, Sunoj and Sankaran [34] introduced quantile versions of the differential Shannon's entropy and dynamic version of it. The quantile based residual entropy is defined by

$$H(X;u) = \ln(1-u) + (1-u)^{-1}\int_u^1 \ln q_X(p)dp, \quad \text{for all } u \in (0,1), \tag{4}$$

where $q_X(u) = Q^{'}(u)$ is the quantile density function (qdf). The measure (4) gives the expected uncertainty contained in the conditional density about the predictability of an outcome of $X$ until $100(1-u)\%$ point of distribution. For the usefulness of information measures based on qf, we refer to Nanda et al. [17], Kang and Yan [12], Baratpour and Khammar [1], Sunoj et al. [35], Khammar and Jahanshahi [13], and the references therein. Motivated with the usefulness of REX, in this paper, we introduce a quantile version based on it, namely residual quantile extropy (RQEX) and prove some of its properties. The RQEX has several advantages. For example, $(i)$ unlike the REX, the RQEX uniquely determines the quantile density function; $(ii)$ we derive RQEX functions for certain qfs which do not have an explicit form for cdfs; $(iii)$ based on the RQEX function, we define a new quantile based stochastic order, to compare the uncertainties of residual lives of two random lives $X$ and $Y$ at the age points $Q_X(u)$ and $Q_Y(u)$ at which $X$ and $Y$ possess equally survival probabilities; and $(iv)$ we provide the new characterizations for some well known lifetime distributions through simple relationships.
The paper is organized as follows. In Section 2, we propose residual differential extropy (REX) in terms of the quantile function called residual quantile extropy (RQEX). Effect of monotone transformations on the RQEX is discussed. We present characterization results for certain distributions in terms of RQEX. Examples are provided to illustrate the results. Some aging classes properties and stochastic comparisons are given in Section 3. In Section 4, we disuss an application of the RQEX based on the distroted rv, which can be useful for reliability analysis of series systems. Finally in Section 5, we provide a nonparametric estimator for RQEX. The proposed estimator is illustrated for simulated and real data sets.

## 2. Residual quantile extropy

In this section, we propose a quantile version of $J(X;x)$ of nonnegative absolutely continuous rv X. First, we recall some notations and preliminary concepts of qf (see Nair et al. [15]).

Let $X$ be a nonnegative absolutely continuous rv with cdf $F_X(x)$, pdf $f_X(x)$ and qf $Q_X(u)$ given by (1). If $F_X(x)$ is right continuous and strictly increasing we have $F_X(Q_X(u)) = u$, so that $F_X(x) = u$ implies $x = Q_X(u)$ and $q_X(u)f_X(Q_X(u)) = 1$ for all $u \in [0,1]$. In general, the choice of an appropriate lifetime model is dictated by its ability to capture the failure patterns exhibited in the data. A primary concept used to represent the physical properties of the failure patterns is the hazard rate $h_X(x)$ or equivalently the hazard quantile function (hqf), defined as, for all $u \in (0,1)$

$$H_X(u) = h_X(Q_X(u)) = \frac{f_X(Q_X(u))}{\bar{F}_X(Q_X(u))} = [(1-u)q_X(u)]^{-1}, \tag{5}$$

where $\bar{F}_X(x) = 1 - F_X(x)$ is the survival function (sf) of rv X. We can interpret the hqf since it explains the conditional probability of failure in the next small interval of time given survival until $100(1-u)\%$ point of distribution. Like $h_X(x)$ that determines the cdf or sf uniquely, $H_X(u)$ also uniquely determines the qf.

If we replace $t$ by $Q_X(u)$ in (3) and since $q_X(u)f_X(Q_X(u)) = 1$ for all $u \in [0,1]$, the residual differential extropy of nonnegative absolutely continuous rv X can be rewritten into the quantile form in following definition. Let X be an nonnegative absolutely continuous rv with pdf $f_X(x)$ and qdf $q_X(x)$. The residual quantile extropy (RQEX) of X is defined as

$$\begin{aligned} J(X;u) = J(X;Q_X(u)) &= -\frac{1}{2(1-u)^2}\int_{Q_X(u)}^{+\infty} f_X^2(x)dx \\ &= -\frac{1}{2(1-u)^2}\int_u^1 q_X^{-1}(p)dp, \quad u \in (0,1). \end{aligned} \tag{6}$$

From (6) it easily follows that $J(X;u)$ takes values in $(-\infty, 0]$. Based on the presented definition of the RQEX, we can inference the following cases:

1. The RQEX measures spectrum of the extropy's uncertainty contained in the conditional density about the predictability of an outcome of $X$ until $100(1-u)\%$ point of distribution.
2. RQEX measures the uncertainty of residual life $X_{Q_X(u)}$, that is, RQEX measures the uncertainty of X at age point $Q_X(u)$.
3. When $u \to 0$, RQEX reduces to the quantile extropy, which is the quantile version of (2).

From (5), we can write RQEX in terms of hqf as follows:

$$J(X;u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)H_X(p)dp, \quad u \in (0,1). \tag{7}$$

It is to be noted that by knowing qf, qdf or hqf, the expression for RQEX is quite simple to compute. To study the $J(X;u)$ value for some distributions we provide the following example.

*Example 1*

(i) If $X$ is distributed uniformly on $(a,b)$ with qf $Q_X(u) = a + (b-a)u$, then it can be easily shown that $J(X;u) = -\frac{1}{2(1-u)(b-a)}$.

(ii) If $X$ follows exponential distribution with qf $Q_X(u) = -\frac{1}{\lambda}\ln(1-u)$, $\lambda > 0$, then $J(X;u) = -\frac{\lambda}{4}$ (which does not depend on $u$).

(iii) When $X$ follows Pareto type I distribution with qf $Q_X(u) = a(1-u)^{-\frac{1}{b}}$, $a, b > 0$, then $J(X;u) = -\frac{b^2(1-u)^{\frac{1}{b}}}{2a(2b+1)}$.

(iv) When $X$ is Weibull distribution with qf $Q_X(u) = \left(-\frac{\ln(1-u)}{\lambda}\right)^{\frac{1}{a}}$, $a, \lambda > 0$, then $J(X;u) = -\frac{a\bar{\Gamma}_{-\ln(1-u)}(2-\frac{1}{a},2)}{2\lambda^{\frac{1}{a}}(1-u)^2}$, where $\bar{\Gamma}_x(\alpha,\beta)$ is known as the incomplete gamma function and defined as

$$\bar{\Gamma}_x(\alpha,\beta) = \int_x^\infty y^{\alpha-1}e^{-\beta y}dy, \quad x > 0, \quad \alpha, \ \beta > 0.$$

(v) If $X$ follows loglogistic distribution with qf $Q_X(u) = \frac{1}{a}\left(\frac{u}{1-u}\right)^{\frac{1}{b}}$, $a, b > 0$, then $J(X;u) = -\frac{ab\bar{B}_u\left(2-\frac{1}{b}, 2+\frac{1}{b}\right)}{2(1-u)^2}$, where $\bar{B}_x(\alpha, \beta)$ is known as the incomplete beta function and defined as

$$\bar{B}_x(\alpha, \beta) = \int_x^1 y^{\alpha-1}(1-y)^{\beta-1}dy, \;\; 0 < x < 1, \;\; \alpha, \; \beta > 0.$$

To learn more about the characteristics of the $J(X;u)$, we plot it in some considered distributions. Figure 1 gives the graphs of $J(X;u)$ for uniform distribution with $a = 1, b = 3$ and exponential distribution with $\lambda = 2$. It shown that the $J(X;u)$ for exponential distribution is a constant function of $u$. Figure 1 indicates that the $J(X;u)$ for uniform distribution is a decreasing function of $u$. Hence as $u$ gets larger the uncertainty of rv $X$ gets larger.
Figure 2 provides the graphs of $J(X;u)$ for Pareto type I distribution with $a = 2, b = 4$ and loglogistic distribution with $a = 1, b = 1.5$. It shown that the $J(X;u)$ for Pareto type I (loglogistic) distribution is an increasing (a decreasing) function of $u$. Hence as $u$ gets larger the uncertainty of rv $X$ gets smaller (larger). We see from Figure 1 and Figure 2 that $J(X;u)$ is not monotonic in $u$.

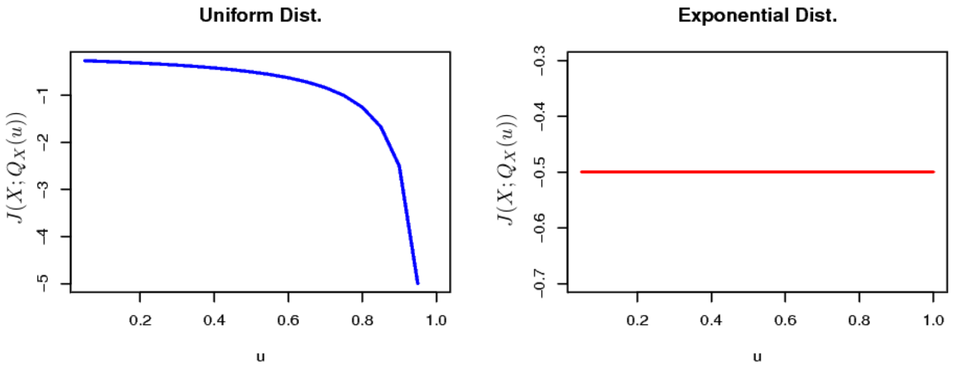


Figure 1. Graph of $J(X;u)$ for $a = 1, b = 3$ (left panel) and $\lambda = 2$ (right panel) in Example 1.

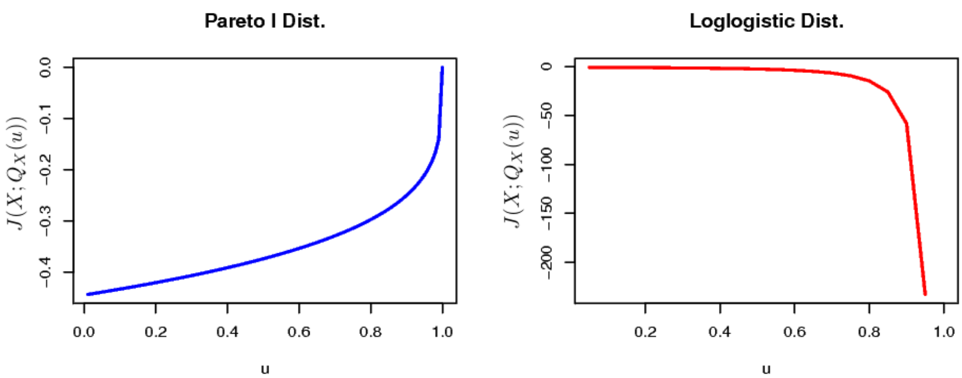


Figure 2. Graph of $J(X;u)$ for $a = 2, b = 4$ (left panel) and $a = 1, b = 1.5$ (right panel) in Example 1.

Unlike the rvs in above example, in following example, we consider two rvs that do not have explicitly known cdf's, though it has closed form qf's.

*Example 2*
Let $X_1$ and $X_2$ be two rvs with respective qf's $q_{X_1}(x)$ and $q_{X_2}(x)$ as follow.

$$q_{X_1}(u) = Ku^{\alpha}(1-u)^{-(A+\alpha)}. \tag{8}$$

$$q_{X_2}(u) = K(1-u)^{-A}\big(-\ln(1-u)\big)^{-M}, \tag{9}$$

where $K$, $\alpha$, $A$ and $M$ are real constants. Then, using (6), RQEX of two rvs $X_1$ and $X_2$ can be computed as

$$\begin{aligned} J(X_1;u) &= -\frac{1}{2K(1-u)^2}\int_u^1 p^{-\alpha}(1-p)^{(A+\alpha)}dp \\ &= -\frac{1}{2K(1-u)^2}\bar{B}_u(1-\alpha, A+\alpha+1), \\ J(X_2;u) &= -\frac{1}{2K(1-u)^2}\int_u^1 (1-p)^{A}\big(-\ln(1-p)\big)^{M}dp \\ &= -\frac{1}{2K(1-u)^2}\int_{-\ln(1-u)}^{\infty} z^{M}e^{-z(A+1)}dz \\ &= -\frac{1}{2K(1-u)^2}\bar{\Gamma}_{-\ln(1-u)}(M+1, A+1). \end{aligned}$$

It is obvious that if X and Y have the same distribution then $J(X;u) = J(Y;u)$, the question that arises is: “What about the converse?”. Differentiating (6) with respect to $u$, we obtain

$$J^{'}(X;u) = \frac{q_X(u)}{2(1-u)^2} + \frac{2J(X;Q_X(u))}{(1-u)}. \tag{10}$$

If $q_X(u) = q_Y(u)$, then $J(X;u) = J(Y;u)$. This implies that underlying quantile density function can be characterized uniquely by RQEX $J(X;u)$. Thus, there is a unique characteristic of $J(X;u)$ unlike the REX $J(X;t)$ in (3), where no such explicit relationship exists between $J(X;x)$ and $f(x)$. In the next theorem we give characterizations of some well known distributions in terms of RQEX through simple relationships.

*Theorem 1*
For a nonnegative random variable $X$, the relationship

$$J(X;u) = -\frac{A(1-u)^B}{2}, \quad A > 0, \tag{11}$$

holds for all $u > 0$, if and only if $X$ is distributed as
(a) uniform distribution, if $B = -1$.
(b) exponential distribution, if $B = 0$.
(c) Pareto type II distribution, if $B = \frac{1}{b}$.

*Proof*
The first part of the proof is straightforward. For the converse part, assume that the relationship (11) holds. Hence, from (6), we have

$$\int_u^1 q_X^{-1}(p)dp = A(1-u)^{B+2}.$$

Differentiating from above equation with respect to u, we can obtain

$$q_X(u) = (A(B+2))^{-1}(1-u)^{-(B+1)}.$$

Therefore, if $B = -1$ and $A = (b-a)^{-1}$; $b > a$, which implies that $Q_X(u) = a + (b-a)u$. Then, we have the uniform distribution $U(a,b)$. If $B = 0$ and $A = \frac{\theta}{2}$; $\theta \geq 0$, which implies that $Q_X(u) = -\theta^{-1}\ln(1-u)$. Then, we have the exponential distribution with parametre $\theta$. Finally, If $B = \frac{1}{b}$ and $A = \frac{1}{b}\big(\frac{(b-a)^2}{a} + a - 1\big)$, that $a$ and $b$ are positive constants, we have $Q_X(u) = a\big((1-u)^{\frac{1}{b}} - 1\big)$. This means, we have the Pareto type II distribution. □

In the following theorem, we characterize the lifetime distributions in Theorem 1 when RQEX is expressed in terms of hqf. The following results have been obtained by using a method similar to Theorem 1, the proof is hence omitted.

*Theorem 2*
For a nonnegative random variable X with the hqf $H_X(x)$, the relationship

$$J(X;u) = -\frac{AH_X(u)}{2}, \tag{12}$$

holds for all $u > 0$, if and only if $X$ is distributed as
(a) uniform distribution, if $A = 1$.
(b) exponential distribution, if $A = 0$.
(c) Pareto type II distribution, if $A = -b$.

Next, we study the effect of a differentiable and invertible transformation on $J(X;u)$. For a nondecreasing, differentible and invertible function $\phi(.)$, we have

$$J(\phi(X);u) = J(\phi(X);Q_{\phi(X)}(u)) = -\frac{\int_u^1 \left(\frac{d}{dp}(Q_X^{-1}(\psi(Q_X(p))))\right)^2 (q_X(p))^{-1}dp}{2\left(1-Q_X^{-1}(\psi(Q_X(u)))\right)^2}, \tag{13}$$

where $\psi(.) = \phi^{-1}(.)$.

*Proof*
Let $Z = \phi(X)$ with pdf $f_Z(x)$ and cdf $F_Z(x)$ be nonnegative rv. We know that $F_Z(x) = P(\phi(X) \le x) = P(X \le \phi^{-1}(x)) = F_X(\phi^{-1}(x)) = F_X(\psi(x))$ and hence $f_Z(x) = f_X(\psi(x))\psi^{'}(x)$. Thus, equation (3) can be expressed as

$$J(Z;t) = -\frac{1}{2\bar{F}_X^2(\psi(t))}\int_t^\infty \left(f_X(\psi(x))\psi^{'}(x)\right)^2 dx. \tag{14}$$

Letting $x = Q_X(u)$ in the above equation, we see that (14) is equivalent to

$$J(Z;u) = -\frac{1}{2\left(1-F_X(\psi(Q_X(u)))\right)^2}\int_u^1 \left(f_X(\psi(Q_X(p)))\psi^{'}(Q_X(p))\right)^2 dQ_X(p). \tag{15}$$

On the other hand, relation (1) gives $F_X(\psi(Q_X(u))) = Q_X^{-1}(\psi(Q_X(u)))$ that implies

$$f_X(\psi(Q_Z(u)))\psi^{'}(Q_X(u))q_X(u) = \frac{d}{du}(Q_X^{-1}(\psi(Q_X(u)))).$$

Substituting these expressions in (15), we get the desired result. □

*Example 3*
Let $X$ have the exponential distribution with failure rate $\lambda$. Then $Y = X^{\frac{1}{\alpha}}$ has the Weibull distribution with $Q_Y(u) = \left(-\frac{\ln(1-u)}{\lambda}\right)^{\frac{1}{\alpha}}$. By taking $\phi(X) = X^{\frac{1}{\alpha}}$, we get $\psi(x) = \phi^{-1}(x) = x^\alpha$ and this implies that

$$Q_X^{-1}\left(h(Q_X(u)\right) = 1 - e^{-\lambda(Q_X(u))^\alpha} \quad \text{and} \quad \psi^{'}(Q_X(u))q_X(u) = \alpha(Q_X(u))^{\alpha-1}q_X(u).$$

So, $\frac{d}{du}\left(Q_X^{-1}(\psi(Q_X(u))\right) = \alpha\lambda(Q_X(u))^{\alpha-1}e^{-\lambda(Q_X(u))^\alpha}q_X(u)$. Substitating these expressions in (13), we obtain

$$\begin{aligned} J(\phi(X);u) &= -\frac{(\alpha\lambda)^2\int_u^1 Q_X(p)^{2(\alpha-1)}e^{-2\lambda(Q_X(p))^\alpha}q_X(p)dp}{2e^{-2\lambda(Q_X(u))^\alpha}} \\ &= -\frac{\alpha\bar{\Gamma}_{-\ln(1-u)}(2-\frac{1}{\alpha},2)}{2\lambda^{\frac{1}{\alpha}}(1-u)^2}. \end{aligned}$$

## 3. Ageing classes and stochastic comparisons

The applications of ageing classes (nonparametric classes of distributions) and stochastic orders can be seen in reliability, engineering, survival analysis, biological science, maintenance and biometrics. For example, reliability analysts are interested in comparing stochastic systems using stochastic orders. Also, they are interested in modeling survival data and using classifications of life distributions based on some aspects of aging. In this section, we study the ageing and ordering properties based on RQEX. First, we define the following aging classes, using the monotonicity of the RQEX function. We say that $X$ has an increasing (decreasing) RQEX, shortly written as IRQEX (DRQEX), if $J(X;u)$ is nondecreasing (nonincreasing) in $u$; $u \geq 0$. Now we derive upper and lower bounds for RQEX depending on the hqf. From the relationship (10) it holds that if $X$ is IRQEX (DRQEX), then $J(X;u) \geq (\leq) - \frac{1}{4(1-u)q(u)}$. Thus, it follows that if X is IRQEX (DRQEX), then from (5) we have

$$J(X;u) \geq (\leq) - \frac{H_X(u)}{4}. \tag{16}$$

*Example 4*
Consider Example 1 and Figures 1 and 2. For exponential distribution, $J(X;u)$ is constant and hence it is the boundary of IRQEX and DRQEX classes and from (16), the RQEX for exponential distributions is equal to $-\frac{\lambda}{4}$. For uniform distribution, $J^{'}(X;u) = -\frac{1}{2(b-a)(1-u)^2} < 0$ and therefore uniform rv belongs to DRQEX class. It is clear from (16) that the upper bound of the RQEX for uniform distributions is $-\frac{1}{4(1-u)(b-a)}$. For Pareto type I distribution, $J^{'}(X;u) = \frac{b(1-u)^{\frac{1}{b}-1}}{2a(2b+1)} > 0$ and this gives Pareto I rv belongs to IRQEX class. This shows that for Pareto I distributions, the RQEX is at least $-\frac{a(1-u)^{\frac{1}{b}}}{4b}$.

*Remark 1*
We say that X is increasing (decreasing) hazard rate [IHR (DHR)] if $H_X(u)$ is increasing (decreasing) in $u$. The uniform distribution has increasing hazard rate (IHR) while its RQEX is decreasing. But, the Pareto type I distribution has decreasing hazard rate (DHR) while its RQEX is increasing. Thus, we can say that IHR (DHR) property does not imply IRQEX (DRQEX) property. Note that, the monotonicity of $h_X(x)$ and $H_X(u)$ are identical.

Qiu et al. [25], compared the uncertainties of two rvs X and Y by comparing their REX functions at the same time points $t$, without defining explicitly a stochastic order there. In the following definition, we proposed a stochastic order based on the REX functions. Also, based on the RQEX functions, we introduce a stochastic order so as to compare the uncertainties of X and Y at the age points $Q_X(p)$ and $Q_Y(p)$ at which X and Y possess equally survival probabilities. The random variable $X$ is said to be smaller than $Y$ in the

- RXE ordering denoted by $X \overset{REX}{\leqslant} Y$, if $J(X;t) \leq J(Y;t)$ for all $t \geq 0$.
- RQXE ordering denoted by $X \overset{RQEX}{\leqslant} Y$, if $J(X;u) \leq J(Y;u)$ for all $u \in [0,1]$. Base on the following example, we show that RXE and RQEX orders do not seem to have been discussed in literature.

*Example 5*
Let $X$ and $Y$ have two Pareto type I distribution with sfs $\bar{F}_X(x) = \left(\frac{1}{x}\right)^{\frac{1}{2}}$ and $\bar{G}_X(x) = \left(\frac{1}{x}\right)^{\frac{1}{3}}$, respectively. From definition of residual extropy, we obtain

$$J(X;t) = -\frac{t^{\frac{1}{2}}}{2} \geq -\frac{t^{\frac{2}{3}}}{2} = J(Y;t), \quad t \geq 1.$$

On the other hand, from part (iii) of Example 1, we have

$$J(X;u) = -\frac{(1-u)^2}{16} \leq -\frac{(1-u)^3}{30} = J(Y;u), \quad u \in (0,1).$$

Thus, $X \overset{RQEX}{\leqslant} Y \nRightarrow X \overset{REX}{\leqslant} Y$. Also, interchanging the roles of $X$ and $Y$ implies that $X \overset{REX}{\leqslant} Y \nRightarrow X \overset{RQEX}{\leqslant} Y$.

In the next theorem we show that the RQEX order is closed under increasing convex transformation. We use the following lemma in the proof of the next theorem.

*Lemma 1*
(Barlow and Proschan [2]). Let $f(u,x)$ be a function on the interval $(a,b)$, not necessarily nonnegative, where $-\infty \leq a < b \leq \infty$. Let $g(x)$ be any nonnegative increasing function in $x$. If $\int_u^b f(u,x)dx \geq 0$ for all $u \in (a,b)$, then $\int_a^b f(u,x)g(x)dx \geq 0$.

*Theorem 3*
If $X \overset{RQEX}{\leqslant} Y$, and if $\phi(.)$ is a nonnegative, increasing and convex function for all $x \geq 0$ such that $\phi(0) = 0$, then $\phi(X) \overset{RQEX}{\leqslant} \phi(Y)$.

*Proof*
Since $X \overset{RQEX}{\leqslant} Y$, we have for all $u \in (0,1)$

$$-\frac{1}{2(1-u)^2}\int_u^1 q_X^{-1}(p)dp \leq -\frac{1}{2(1-u)^2}\int_u^1 q_Y^{-1}(p)dp, \tag{17}$$

where follows that for all $u \in (0,1)$, $q_X^{-1}(u) \geq q_Y^{-1}(u)$ and so $Q_X(u) \leq Q_Y(u)$. Since $\phi(.)$ is nonnegative, increasing and convex, we obtain $[\phi^{'}(Q_X(u))]^{-1} \geq [\phi^{'}(Q_Y(u))]^{-1}$. Thus, from (17) and Lemma 1, we get

$$\begin{aligned}
&J(\phi(Y);u) - J(\phi(X);u)\\
&= \frac{1}{2}\left[\frac{\int_u^1 \left(q_X(p)\phi^{'}(Q_X(p))\right)^{-1}(p)dp}{(1-u)^2} - \frac{\int_u^1 \left(q_Y(p)\phi^{'}(Q_Y(p))\right)^{-1}(p)dp}{(1-u)^2}\right] \geq 0,
\end{aligned}$$

for all $u \in (0,1)$. □

*Corollary 1*
Let $Z_1 = a_1X + b_1$ and $Z_2 = a_2Y + b_2$, $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$. If $X \overset{RQEX}{\leqslant} Y$ and $a_1 \geq a_2$, then $Z_1 \overset{RQEX}{\leqslant} Z_2$.

*Remark 2*
Let X be a nonnegative continuous rv, and $\phi(.)$ be a nonnegative increasing function defined on $[0,\infty)$ with $\phi(0) = 0$. We call $\phi(X)$ as the generalized scale transform of X. If function $\phi(.)$ is increasing convex with $\phi(0) = 0$, then $\phi(.)$ is called a risk preference function, and $\phi(X)$ is called the risk preference transform of X. Therefore by Theorem 3, we can say that, $RQEX$ order has closure property under the convex generalized scale transform and risk preference transform.

(Vineshkumar et al. [36]) The random variable $X$ is said to be smaller than $Y$ in the hazard quantile function ordering denoted by $X \overset{HQ}{\leqslant} Y$, if $H_X(u) \geq H_Y(u)$ for all $u \in (0,1)$. By using definition of the hazard quantile function order and from (7), the following result is obvious, the proof is hence omitted.

*Theorem 4*
If $X \overset{HQ}{\leqslant} Y$, then $X \overset{RQEX}{\leqslant} Y$.

Vineshkumar et al. [36] provided the relationships between the orders based on reliability measures in distribution functions and quantile based reliability measures. We express some of these relationships in the following lemma. For comprehensive discussion on various concepts of stochastic orderings based on reliability measures in distribution functions one can see Shaked and Shanthikumar [28].

*Lemma 2*
For the continuous nonnegative rvs $X$ and $Y$,
(a) $X \overset{disp}{\leqslant} Y \Leftrightarrow X \overset{HQ}{\leqslant} Y$.

(b) If $X$ or $Y$ is DFR (IFR), then $X \overset{hr}{\leqslant} Y \Rightarrow (\Leftarrow)\ X \overset{HQ}{\leqslant} Y$.

(c) If $X$ and $Y$ have the same lower end of the support and if $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u \in (0,1)$, then $X \overset{st}{\leqslant} Y \Rightarrow X \overset{HQ}{\leqslant} Y$.

Next theorem is related to the $RQEX$ ordering and orderings based on reliability measures in distribution functions.

*Theorem 5*
Let $X$ and $Y$ be two nonnegative rvs having continuous qdfs $q_X(u)$ and $q_Y(u)$ and qf's $Q_X(u)$ and $Q_Y(u)$, respectively. Then,

(a) $X \overset{disp}{\leqslant} Y \Leftrightarrow X \overset{RQEX}{\leqslant} Y$.

(b) If $X$ or $Y$ is DFR, then $X \overset{hr}{\leqslant} Y \Rightarrow X \overset{RQEX}{\leqslant} Y$.

(c) If $X$ and $Y$ have the same lower end of the support and if $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u \in (0,1)$, then $X \overset{st}{\leqslant} Y \Rightarrow X \overset{RQEX}{\leqslant} Y$.

*Proof*
Using Lemma 2 and Theorem 4, the proof is completed. □

Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) nonnegative rvs having sfs $\bar{F}_X(x)$. If $X_{i:n}$ denotes the $i$th order statistics in this sample of size $n$, then the lifetime of a series system is determined by $X_{1:n}$ and the lifetime of a parallel system is determined by $X_{n:n}$ with sfs $\bar{F}_{1:n}(x)$ and $\bar{F}_{n:n}(x)$, respectively. The quantile based uncertainty measures of order statistics is useful to compare the uncertainties of lifetimes of $(n-i+1)$-out-of-$n$ systems. The following example can be viewed as some direct applications of part (b) of Theorem 5 in the area of order statistics.

*Example 6*
Let $X_1, X_2, ..., X_n$ be iid nonnegative DFR rvs having continuous qfs $Q_X(u)$. Then,

$(a)$ $X_{i:n} \overset{RQEX}{\leqslant} X_{i+1:n}$. That is $J(X_{i:n}; u)$ is a increasing function of $i$.

$(b)$ $X_{1:n} \overset{RQEX}{\leqslant} X_{1:n-1}$.

$(c)$ $X_{n-1:n-1} \overset{RQEX}{\leqslant} X_{n:n}$.

$(d)$ We know that $X \overset{hr}{\leqslant} X_{1:n}$. Thus, $X \overset{RQEX}{\leqslant} X_{1:n}$. Also, since $X_{n:n} \overset{hr}{\leqslant} X$, we have $X_{n:n} \overset{RQEX}{\leqslant} X$.

## 4. Application of RQEX

In this section, we study the problem of evaluating the RQEX for the distorted rv and the proportional hazard rates model. We suggest our application to reliability analysis of series systems.
Let $\Delta$ be the set of continuous, nondecreasing and piecewise differentiable functions $\varphi : [0,1] \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. These functions are known as distortion functions. Sordo and Suarez-Llorens [30] used these distortion functions to classes of variability measures. Gupta et al. [8] used these function for the analysis of random lifetimes of coherent systems. Using distorted functions, Di Crescenzo et al. [4] defined the distorted rv as follows. Let X be an nonnegative absolutely continuous rv with sf $\bar{F}_X(x)$. For each distortion function $\varphi \in \Delta$, $X_\varphi$ is the distorted rv induced by $\varphi$ with sf

$$\bar{F}_{X_\varphi}(x) = \varphi(\bar{F}_X(x)). \tag{18}$$

Denneberg [3] and Wang [37, 38] introduced distorted distributions in the context of actuarial science for real problems. Yaari [40] and Schmeidler [32] used these distributions in the rank dependent expected utility model. Navarro et al. [18] proposed some ordering preservation results for generalized distorted distributions which can be used to obtain preservation results for coherent systems with non-identically distributed components. With simple calculations, we can obtain the qf and qdf of the distorted rv $X_\varphi$ as

$$Q_{X_\varphi}(u) = Q_X(1-\varphi^{-1}(1-u)) \text{ and } q_{X_\varphi}(u) = \frac{q_X(1-\varphi^{-1}(1-u))}{\varphi'(\varphi^{-1}(1-u))}, \quad u \in (0,1), \tag{19}$$

respectively. Now, recalling (6) and from (19), we can obtain the RQEX of $X_\varphi$ as follows:

$$J(X_\varphi; u) = -\frac{1}{2(1-u)^2}\int_u^1 \frac{\varphi'(\varphi^{-1}(1-p))}{q_X(1-\varphi^{-1}(1-p))}dp, \quad u \in (0,1).$$

The proportional hazard rates model is a special case of distortion functions, which is a flexible model to accommodate both monotonic as well as nonmonotonic failure rates even though the baseline failure rate is monotonic. Recently, Parsa et al. [21] proposed a charactrization of this model in terms of the Gini-type index. For $\varphi(x) = x^\delta$, $\delta > 0$, the distorted rv $X_\varphi$ correspond to the proportional hazard rates model with sf, qf and qdf

$$\bar{F}_{X_\varphi}(x) = (\bar{F}_X(x))^\delta, \;\; Q_{X_\varphi}(u) = Q_X(1-(1-u)^{\frac{1}{\delta}}) \text{ and } q_{X_\varphi}(u) = \frac{1}{\delta}(1-u)^{\frac{1}{\delta}-1}q_X(1-(1-u)^{\frac{1}{\delta}}), \tag{20}$$

respectively. Now, recalling (6) and from (20), we have

$$\begin{aligned} J(X_\varphi; u) &= -\frac{\delta}{2(1-u)^2}\int_u^1 (1-p)^{1-\frac{1}{\delta}} q_X^{-1}(1-(1-p)^{\frac{1}{\delta}})dp \\ &= -\frac{\delta^2}{2(1-u)^2}\int_{1-(1-u)^{\frac{1}{\delta}}}^1 (1-z)^{2(\delta-1)} q_X^{-1}(z)dz, \end{aligned} \tag{21}$$

where the last equation is obtained by taking $z = 1-(1-p)^{\frac{1}{\delta}}$.

*Example 7*
Let $X_1, X_2, ..., X_n$ be random lifetimes of $n$ iid components which are connected in a series system having sfs $\bar{F}_X(x)$. We know that, the lifetime of a series system determined by $X_{1:n}$ have sf $\bar{F}_{1:n}(x) = (\bar{F}_X(x))^n$ and satisfies the proportional hazard rates model for $\delta = n \in \mathbb{N}$. Here, we consider two scenarios for the distribution of random lifetimes $X_i$ as follows:
$(i)$ The random lifetimes $X_i$ have exponential distribution with sf $\bar{F}_X(x) = e^{-\theta x}$ and qdf $q_X(x) = \frac{1}{\theta(1-x)}$, $\theta > 0$. Now, from (21), the RQEX of the series system lifetime $X_{1:n}$ can be computed as

$$J(X_{1;n}; u) = -\frac{n\theta}{4},$$

where is independent of $u$. In this case, $J(X_{1:n}; u)$ is a decreasing function of $n \in \mathbb{N}$ (the number of components).
$(ii)$ The random lifetimes $X_i$ have generalized exponential distribution with sf $\bar{F}_X(x) = 1-(1-e^{-\theta x})^\gamma$ and qdf $q_X(x) = \frac{x^{\frac{1}{\gamma}-1}}{\theta\gamma(1-x)^{\frac{1}{\gamma}}}$, $\theta > 0$, $\gamma > 0$. From (21), we have

$$J(X_{1:n}; u) = -\frac{n^2\theta\gamma}{2(1-u)^2}\int_{1-(1-u)^{\frac{1}{n}}}^1 z^{1-\frac{1}{\gamma}}(1-z)^{2(n-1)}(1-z^{\frac{1}{\gamma}})dz.$$

Figure 3 provide some plots of the RQEX of the series system lifetime for various values of $n, \theta, \gamma$, which show that increasing $n$ (the number of units) lead to decreasing RQEX.

## 5. Estimate of RQEX

The purpose of this section is to provide a nonparametric estimator for the RQEX. According to equation (6), for estimating RQEX we need to estimate the function of the $q_X(x)$. Soni et al. [31] proposed the following estimator

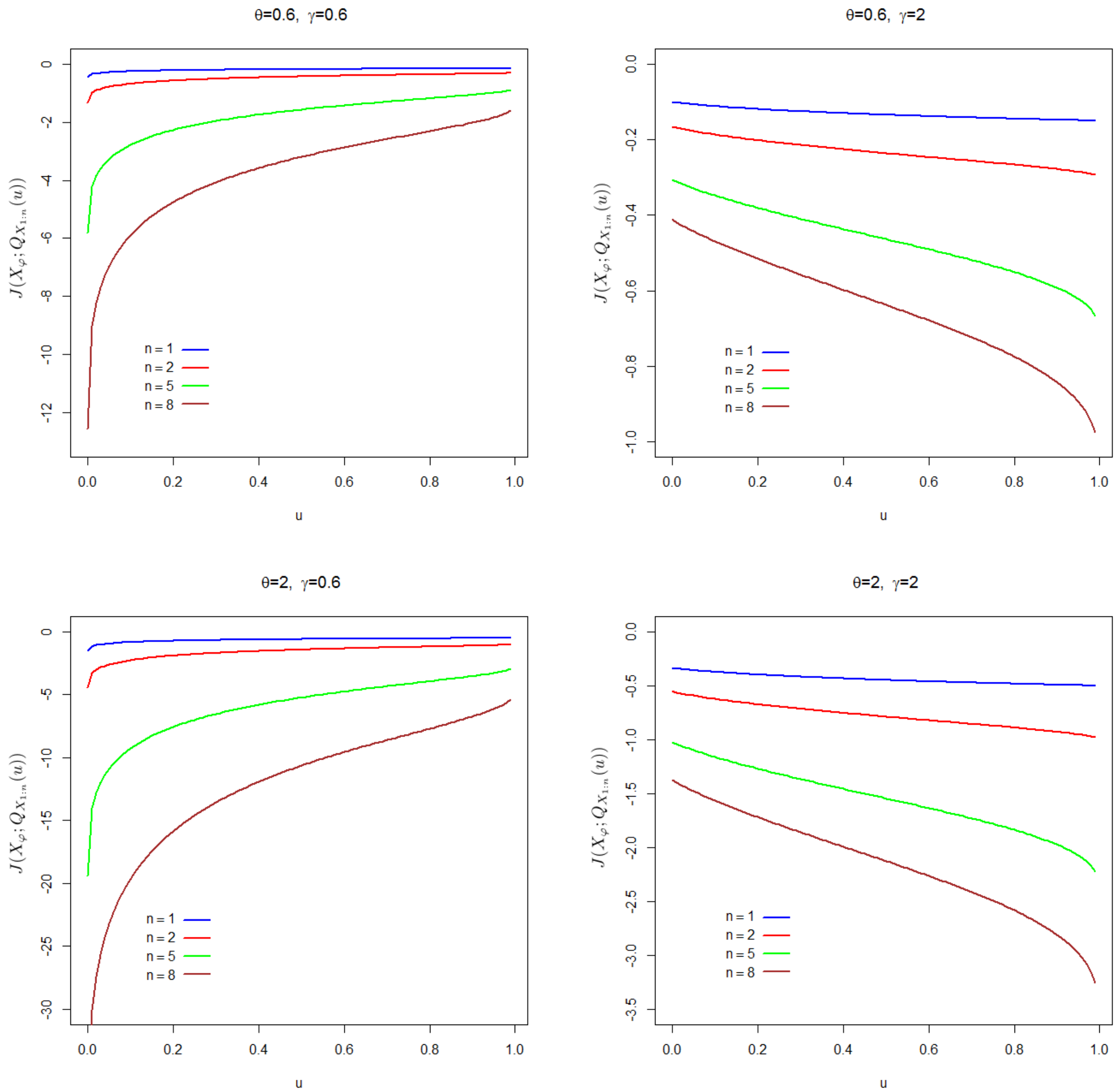


Figure 3. The RQEX plots for series system of Example 7, for some selected values of $n, \theta$ and $\gamma$.

for $q_X(x)$

$$q_n(u) = \frac{1}{nh(n)} \sum_{i=1}^{n} \frac{K(\frac{i/n-u}{h(n)})}{f_n(X_{i:n})}, \tag{22}$$

where $h(n)$ is the bandwidth, $K(\cdot)$ an appropriate kernel function and $f_n(t)$ is a kernel type density estimator of the form

$$f_n(t) = \frac{1}{nh(n)} \sum_{i=1}^{n} K\left(\frac{t - X_i}{h(n)}\right). \tag{23}$$

We use the estimator (22) to estimate $q_X(x)$ in (6). Thus utilizing (22) and (23), an estimator for RQEX is as follows

$$\hat{J}(X;u) = -\frac{S_1(u,n,h(n))}{2(1-u)^2(nh(n))^{-1}}, \tag{24}$$

where

$$S_1(u,n,h(n)) = \int_u^1 \left[\sum_{t=1}^{n}\left(K(\frac{t/n-p}{h(n)})/f_n(X_{t:n})\right)\right]^{-1} dp. \tag{25}$$

For finding the estimator, we consider the Epanechnikov kernel that gives the optimal kernel (Prakasa Rao [22]). By placing $K(u)=\frac{3}{4}(1-u^2)I_{(|u|\le 1)}$ in (25), we have

$$S_1(u,n,h(n)) = \int_u^1 \left[\sum_{t=1}^{n}\left(\frac{\frac{3}{4}(1-(\frac{t/n-p}{h(n)})^2)I_{(|\frac{t/n-p}{h(n)}|\le 1)}}{f_n(X_{t:n})}\right)\right]^{-1} dp. \tag{26}$$

Note that, $\frac{t}{n}-h(n)\le p\le \frac{t}{n}+h(n)$, so, by considering all possible cases and intersection of intervals, the following situations can be considered:
$I$: For $0<u\le \frac{1}{n}+h(n)$, suppose that $t^*=\min\{t|(\frac{t}{n}-h(n))>u\}$.
If $\frac{t^*}{n}-h(n)<\frac{1}{n}+h(n)$, then

$$\begin{aligned}
S_1(u,n,h(n)) &= \int_u^{\frac{t^*}{n}-h(n)} \sum_{t=1}^{t^*-1}\big(S_2(p,n,h(n))\big)^{-1}dp \\
&+ \sum_{j=t^*}^{\min([2nh(n)],n)} \int_{\frac{j}{n}-h(n)}^{\frac{j+1}{n}-h(n)} \sum_{t=1}^{j}\big(S_2(p,n,h(n))\big)^{-1}dp \\
&+ \int_{a_1}^{\frac{1}{n}+h(n)} \sum_{t=1}^{\min([2nh(n)]+1,n)} \big(S_2(p,n,h(n)\big)^{-1}dp \\
&+ \sum_{i=1}^{t_0-1}\int_{\frac{i}{n}+h(n)}^{\min(n,\frac{i+1}{n}+h(n))} \sum_{t=i+1}^{\min([i+2nh(n)],n)} (S_2(p,n,h(n))^{-1}dp,
\end{aligned}$$

where, $S_2(p,n,h(n))=\sum_{t=1}^{n}\{\frac{3}{4}(1-(\frac{t/n-p}{h(n)})^2)I_{(|\frac{t/n-p}{h(n)}|\le 1)}/f_n(X_{t:n})\}$, $[x]$ is the integer part of $x$, $a_1=\frac{\min([2nh(n)]+1,n)}{n}-h(n)$ and $t_0=\min\{t|(\frac{t}{n}+h(n))\ge 1\}$.
Otherwise if, $\frac{t^*}{n}-h(n)\ge \frac{1}{n}+h(n)$, then

$$\begin{aligned}
S_1(u,n,h(n)) &= \int_u^{\frac{1}{n}+h(n)} \sum_{t=1}^{\min([n(u+h(n))],n)} \big(S_2(p,n,h(n)\big)^{-1}dp \\
&+ \sum_{i=1}^{t_0-1}\int_{\frac{i}{n}+h(n)}^{\min(n,\frac{i+1}{n}+h(n))} \sum_{t=i+1}^{\min([i+2nh(n)],n)} \big(S_2(p,n,h(n)\big)^{-1}dp.
\end{aligned}$$

$II$: For $\frac{k}{n}+h(n)<u\le \frac{k+1}{n}+h(n)$, where $k=1,\ldots,n-1$, we obtain

$$\begin{aligned}
S_1(u,n,h(n)) &= \int_u^{\min(1,\frac{k+1}{n}+h(n))} \sum_{t=k+1}^{\min([n(u+h(n))],n)} \big(S_2(p,n,h(n)\big)^{-1}dp \\
&+ \sum_{i=k+1}^{t_0-1}\int_{\frac{i}{n}+h(n)}^{\min(n,\frac{i+1}{n}+h(n))} \sum_{t=i+1}^{\min([i+2nh(n)],n)} \big(S_2(p,n,h(n)\big)^{-1}dp.
\end{aligned}$$

In following example, we employ simulation study to examine the performance of the proposed estimator (24) using Mean Square Error (MSE).

*Example 8*
If X be a rv following the Davies distribution (see Gilchrist [6], page 140), that do not have any closed form expressions for cdf and pdf, then qf and qdf are given, respectively, by

$$Q_X(u) = c\frac{u^{\lambda_1}}{(1-u)^{\lambda_2}}, \quad 0 \le u < 1, \quad \lambda_1, \lambda_2, c > 0,$$

and

$$q_X(u) = c\frac{u^{\lambda_1 - 1}}{(1-u)^{\lambda_2+1}}\left[\lambda_1(1-u) + \lambda_2 u\right],$$

where $c$ is scale parameter, $\lambda_1$ and $\lambda_2$ are shape parameters. When $c = 1$, then RQEX is

$$J(X;u) = \frac{-1}{2(1-u)^2}\int_u^1 \left[\frac{p^{(\lambda_1-1)}(\lambda_1(1-p)+\lambda_2 p)}{(1-p)^{\lambda_2+1}}\right]^{-1} dp. \tag{27}$$

To investigate the importance of the proposed estimator, we calculated MSE of the proposed estimator RQEX (27) for $\lambda_1 = 1$ and $\lambda_2 = \frac{3}{2}$. We repeated the simulation study with 1000 repeat for sample sizes $n = 50$, 100 and 200. The optimum bandwidth is determined based on minimum of MSE and the final results equalled to 0.3, the MSEs are presented in Table 1. Based on Table 1 and Figure 4, we can see that the MSE decreases when the sample size increases and for low quantiles, the proposed estimator has acceptable function and intuitively it can be said that it is also consistent. Figure 5 provides some plots of the RQEX for some selected values of $\lambda_1$ and $\lambda_2$ of the Davies distribution in Example 8. Note that the RQEX increases when $u$ becomes larger.

Table 1. The MSEs of the proposed estimator RQEX for the Davies distribution with $c = 1$, $\lambda_1 = 1$ and $\lambda_2 = \frac{3}{2}$.

| | u | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 50 | 0.0028 | 0.0015 | 0.0018 | 0.0021 | 0.0027 | 0.0050 | 0.0130 | 0.0442 | 0.2494 |
| 100 | 0.0026 | 0.0011 | 0.0007 | 0.0005 | 0.0004 | 0.0006 | 0.0027 | 0.0127 | 0.0832 |
| 200 | 0.0021 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0034 | 0.0284 |

*Example 9*
In this example, we consider a real data set reported in Rai et al. [26], to clarify the performance of the proposed estimator RQEX, $\hat{J}(X;u)$. The data shows the failure time of three systems and is presented in Table 2. The $\hat{J}(X;u)$ for different values of $u$, are given in Table 2. Based on this table, notice that the $\hat{J}(X;u)$ decreases when $u$ becomes larger. Ebrahimi [5] stated, one thing most engineers are agreed upon is that highly uncertain components or systems are inherently not reliable. Based on their idea and Table 3, we can conclude that the first system is more reliable in comparison with the other two systems. This result also coincides with the performance of the first estimator proposed by Subhash et al. [33] for this real data set.

## 6. Conclusion

In this paper, residual quantile extropy which is a quantile version of the extropy based on residual lifetime variable is proposed. Aging classes, stochastic orders and characterization results are derived. The problem of evaluating the RQEX for the distorted rv and the proportional hazard rates model was studied. We suggested our application to reliability analysis of series systems. Finally, the nonparametric estimator for RQEX is provided and based on a simulation study and a real data set the proposed estimator was evaluated.

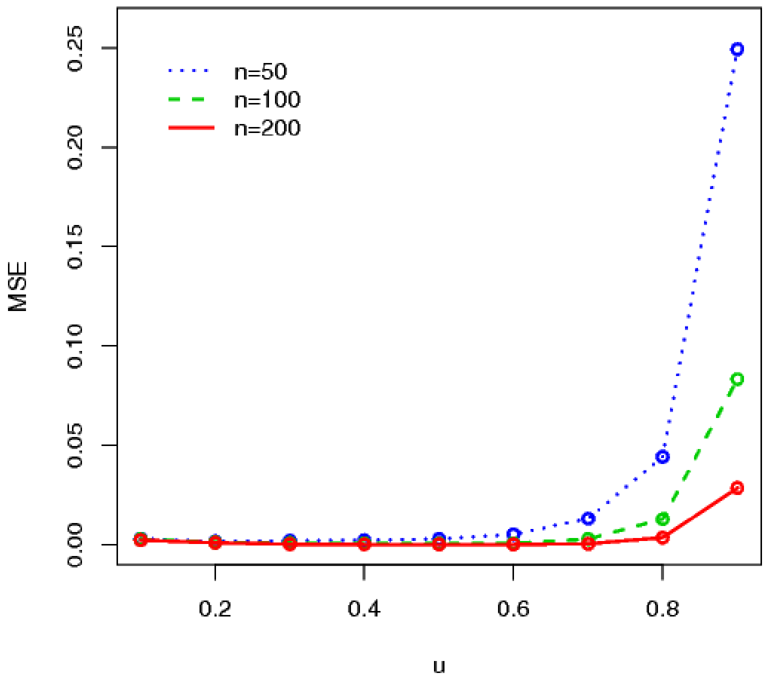


Figure 4. The MSEs of the proposed estimator RQEX for the Davies distribution with $c = 1$, $\lambda_1 = 1$ and $\lambda_2 = \frac{3}{2}$.

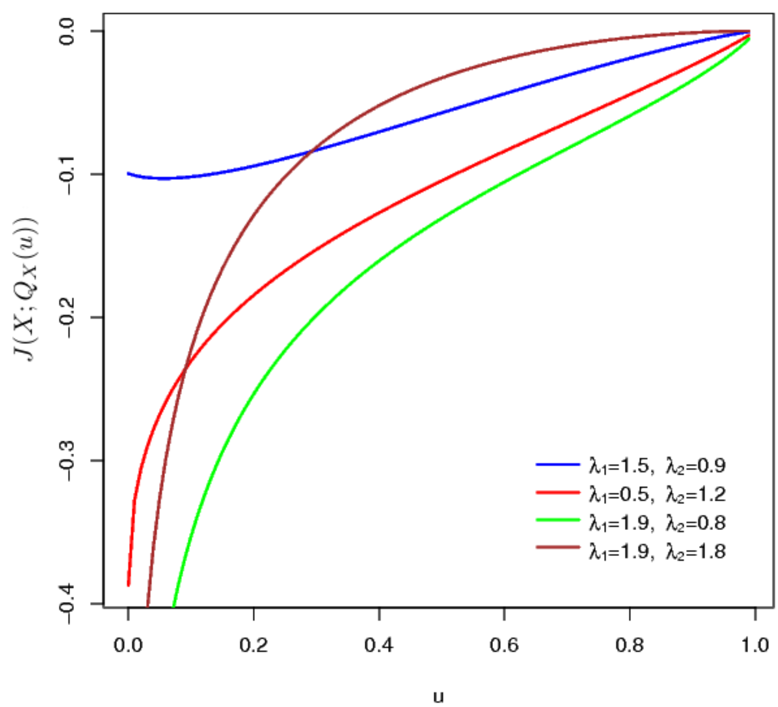


Figure 5. The RQEX plots for the Davies distribution with $c = 1$ and some selected values of $\lambda_1$ and $\lambda_2$

Table 2. Failure data of three systems.

| System 1 | System 2 | System 3 |
|---|---|---|
| 55.6 | 1.4 | 0.3 |
| 72.7 | 35.0 | 32.6 |
| 111.9 | 46.8 | 33.4 |
| 121.9 | 65.9 | 241.7 |
| 303.6 | 181.1 | 396.2 |
| 326.9 | 712.6 | 480.8 |
| 1568.4 | 1005.7 | 588.9 |
| 1913.5 | 1029.9 | 1043.9 |
| | 1675.7 | 1136.1 |
| | 1787.5 | 1288.1 |
| | 1867.0 | 1408.1 |
| | | 1439.4 |
| | | 1604.8 |

Table 3. $\hat{J}(X;u)$ for the real data set.

| | u | | | | |
|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.5 | 0.8 | 0.9 |
| System 1 | -0.260 | -0.359 | -0.519 | -0.923 | -2.044 |
| System 2 | -0.140 | -0.189 | -0.252 | -0.701 | -1.567 |
| System 3 | -0.119 | -0.156 | -0.215 | -0.605 | -1.360 |


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