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The Balakrishnan-Alpha-Beta-Skew-Laplace Distribution: Properties and Applications

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Abstract In this paper, a new form of alpha-beta-skew-Laplace distribution is proposed under Balakrishnan [3] mechanism and investigated some of its related distributions. The moments, distributional properties and some extensions of the proposed distribution have also studied. Finally, the suitability and the appropriateness of the proposed distribution has tested by conducting data fitting experiment and comparing the values of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) with the values of some other related distributions. Likelihood Ratio test is used for discriminating between the nested models.

Keywords Skew Distribution, Balakrishnan Alpha Skew Laplace Distribution, Bimodal Distribution, Likelihood Ratio Test, AIC, BIC

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1. Introduction

The study of skewed distributions was first introduced with the skew-normal distribution by Azzalini [2] by adding an additional parameter in the normal distribution to introduce asymmetry. Thereafter, a lot of research work has so far been carried out (for details see [4], [5, 6, 7],[9], [16, 17]). [1] derived the skew-Laplace distribution and the probability density function (pdf) is given by

$$f_Z(z;\alpha) = 2\Phi(\lambda z)\left(\frac{1}{2}e^{-|z|}\right), z \in R \tag{1}$$

where $\Phi(.)$ is the cumulative distribution function (cdf) of standard Laplace distribution.
For the construction of skew-symmetric distributions [13] proposed a general methodology by implementing the concept of skew function $G(.)$ instead of cdf in [2] where $G(.)$ is a Lebesgue measurable function satisfying, $0 \le G(z) \le 1$ and $z \in R$, almost everywhere. The pdf of the same is given by

$$f_Z(z) = 2\varphi(z)G(z), z \in R \tag{2}$$

where $\varphi(.)$ is the density function. Obviously, by selecting different skew functions in (2), one can develop many numbers of skewed distributions (see for example, [5, 6, 7],[8],[11], [15], [16, 17] among others).
[10] introduced the alpha-skew-Laplace $ASLa(\alpha)$ distribution and the pdf is given by

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$$f_Z(z;\alpha) = \frac{(1-\alpha z)^2+1}{C(\alpha)}\left(\frac{1}{2}e^{-|z|}\right), z \in R \tag{3}$$

where $C(\alpha) = 2\left(\alpha^2+1\right)$.
[19] derived a new form of skew distribution known as the alpha-beta-skew-Laplace *ABSLa*(α, β) distribution and the pdf is given by

$$f_Z(z;\alpha,\beta) = \frac{\left(1-\alpha z-\beta z^3\right)^2+1}{C(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right), z \in R \tag{4}$$

where $C(\alpha,\beta) = 2\left(1+\alpha^2+360\beta^2+24\alpha\beta\right)$.
[18] introduced the Balakrishnan-alpha-skew-Laplace distribution with pdf given by

$$f_Z(z;\alpha) = \frac{\left[(1-\alpha z)^2+1\right]^2}{C_2(\alpha)}\left(\frac{1}{2}e^{-|z|}\right), z \in R \tag{5}$$

where $C_2(\alpha) = 4\left(1+4\alpha^2+6\alpha^4\right)$.
In the present article, [18] and [19] methodology has been implemented to propose a new alpha-beta-skew-Laplace distribution, known the Balakrishnan-alpha-beta-skew-Laplace distribution which is flexible enough to support both unimodal, bimodal as well as multimodal behavior and investigate some of its distributional properties. The propose distribution is useful to model stochastic variables those exhibit skewness with multiple modes in arbitrary locations. To exhibit the applicability of the proposed distribution, the two real life datasets are consider which give better fitting when compared to some other known distributions viz., the normal distribution, the skew normal distribution, the Laplace distribution, the skew-Laplace distribution of [1], the alpha-skew-Laplace distribution of [10], the alpha beta skew Laplace distribution of [19] and the Balakrishnan alpha skew Laplace distribution of [18]. The article is summarized as follows. In Section 2, the Balakrishnan-alpha-beta-skew-Laplace distribution is defined and discusses some of its important distributional properties. The random number generation of the proposed distribution is defined in Section 3. The location-scale extension and maximum likelihood estimation are given in Section 4. In Section 5, some numerical examples based on real life data and Likelihood ratio test are provided. Finally, the article is ended with conclusions in Section 6.

2. A New Alpha-Beta-Skew-Laplace Distribution

In this section, we define a new family of distribution known as Balakrishnan-alpha-beta-skew-Laplace (*BABSLa*) distribution and discuss some of its distributional properties.
Definition 1: If a random variable Z has a density function

$$f_Z(z;\alpha,\beta) = \frac{\left[\left(1-\alpha z-\beta z^3\right)^2+1\right]^2}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right), z \in R \tag{6}$$

then it is said to be Balakrishnan-alpha-beta-skew-Laplace distribution with skewness parameters $\alpha \in R$ and $\beta \in R$. We denote it as *BABSLa*$_2(\alpha,\beta)$. Where,

$C_2(\alpha,\beta) = C_2(\alpha) + 192\beta\left[15\alpha^3+1260\alpha^2\beta+\alpha\left(2+75600\beta^2\right)+130\left(\beta+83160\beta^3\right)\right]$ and $C_2(\alpha)$ is define above.

Some particular cases are: When

- $\beta = 0$, then we get the $BASLa_2(\alpha)$ distribution of [18] and is given by

$$f_Z(z;\alpha) = \frac{\left[(1-\alpha z)^2+1\right]^2}{C_2(\alpha)}\left(\frac{1}{2}e^{-|z|}\right), z \in R$$

where $C_2(\alpha)$ is define above.
- $\alpha = 0$, then we get

$$f_Z(z;\beta) = \frac{\left[\left(1-\beta z^3\right)^2+1\right]^2}{4\left(119750400\beta^4+1440\beta^2+1\right)}\left(\frac{1}{2}e^{-|z|}\right), z \in R$$

.
This equation is known as Balakrishnan-beta-skew-Laplace $BBSLa_2(\beta)$ distribution.
- $\alpha = \beta = 0$, then we get the standard Laplace $La(0,1)$ distribution and is given by

$$f_Z(z) = \left(\frac{1}{2}e^{-|z|}\right), z \in R$$

.
- $\alpha \to \pm\infty$, then we get the bimodal-Laplace $BLa(4)$ distribution (see [12]) given by

$$f_Z(z) = \frac{z^4}{24}\left(\frac{1}{2}e^{-|z|}\right), z \in R$$

.
- $\beta \to \pm\infty$, then we get the bimodal-Laplace $BLa(12)$ distribution (see [12]) given by

$$f_Z(z) = \frac{z^{12}}{479001600}\left(\frac{1}{2}e^{-|z|}\right), z \in R$$

.
- $Z \sim BABSLa_2(\alpha,\beta)$, then $-Z \sim BABSLa_2(-\alpha,-\beta)$.

The pdf of $BABSLa_2(\alpha,\beta)$ distribution for different choices of the parameters α and β are plotted in Figure 1. It is observed from Figure 1 that the proposed distribution is very flexible to support unimodal, bimodal and multimodal behaviors and has at most four modes and is also proved in Theorem 1. The parameters α and β of the proposed distribution have significant effects on the skewness and the probable number of modes. Also, the plot of the pdf of $BABSLa_2(\alpha,\beta)$ distribution approaches to Laplace as α and β tends to zero.

Theorem 1
The pdf of $BABSLa_2(\alpha,\beta)$ distribution has at most four modes.

Proof
Differentiating (5) with respect to z, we have

$$f'_Z(z;\alpha,\beta) = \frac{\partial f_Z(z;\alpha,\beta)}{\partial z} = \frac{\partial}{\partial z}\frac{\left[\left(1-\alpha z-\beta z^3\right)^2+1\right]^2}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right)$$

$$= \frac{1}{C_2(\alpha,\beta)}\frac{\partial}{\partial z}\left[\left(1-\alpha z-\beta z^3\right)^2+1\right]^2\left(\frac{1}{2}e^{-|z|}\right)$$

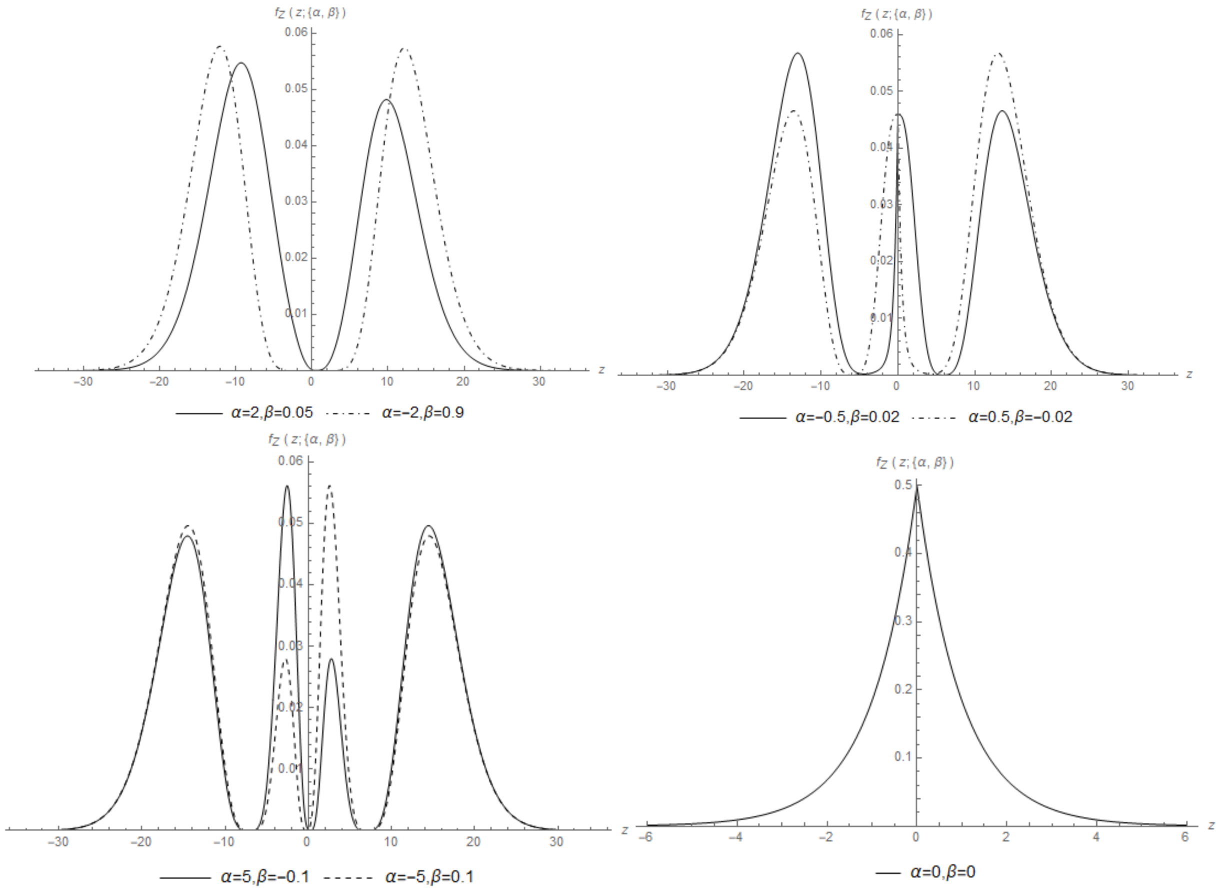

Figure 1. Plots of the pdf of *BABSLa*$_2(\alpha\,,\beta\,)$

$$= \frac{1}{C_2(\alpha,\beta)}\frac{\partial}{\partial z}[4 - 8\alpha z + 8\alpha^2 z^2 - 4z^3\left(\alpha^3+2\beta\right) + \alpha z^4\left(\alpha^3+16\beta\right) - 12\alpha^2\beta z^5 + 4\beta z^6\left(\alpha^3+2\beta\right) - 12\alpha\beta^2 z^7 + 6\alpha^2\beta^2 z^8 - 4\beta^3 z^9 + 4\alpha\beta^3 z^{10} + z^{12}\beta^4]\left(\frac{1}{2}e^{-|z|}\right)$$

$$= \frac{1}{C_2(\alpha,\beta)}[4\left\{\frac{\partial}{\partial z}\left(\frac{1}{2}e^{-|z|}\right)\right\} - 8\alpha\left\{\frac{\partial}{\partial z}z\left(\frac{1}{2}e^{-|z|}\right)\right\} + 8\alpha^2\left\{\frac{\partial}{\partial z}z^2\left(\frac{1}{2}e^{-|z|}\right)\right\} - 4(\alpha^3+2\beta)\left\{\frac{\partial}{\partial z}z^3\left(\frac{1}{2}e^{-|z|}\right)\right\} + \alpha(\alpha^3+16\beta)\left\{\frac{\partial}{\partial z}z^4\left(\frac{1}{2}e^{-|z|}\right)\right\} - 12\alpha^2\beta\left\{\frac{\partial}{\partial z}z^5\left(\frac{1}{2}e^{-|z|}\right)\right\} + 4\beta(\alpha^3+2\beta)\left\{\frac{\partial}{\partial z}z^6\left(\frac{1}{2}e^{-|z|}\right)\right\} - 12\alpha\beta^2\left\{\frac{\partial}{\partial z}z^7\left(\frac{1}{2}e^{-|z|}\right)\right\} + 6\alpha^2\beta^2\left\{\frac{\partial}{\partial z}z^8\left(\frac{1}{2}e^{-|z|}\right)\right\} - 4\beta^3\left\{\frac{\partial}{\partial z}z^9\left(\frac{1}{2}e^{-|z|}\right)\right\} + 4\alpha\beta^3\left\{\frac{\partial}{\partial z}z^{10}\left(\frac{1}{2}e^{-|z|}\right)\right\} + \beta^4\left\{\frac{\partial}{\partial z}z^{12}\left(\frac{1}{2}e^{-|z|}\right)\right\}] \quad (7)$$

Now, we have

$\frac{\partial}{\partial z}\left(\frac{1}{2}e^{-|z|}\right)=-|z|'\left(\frac{1}{2}e^{|z|}\right)$, $\frac{\partial}{\partial z}\left\{z\left(\frac{1}{2}e^{-|z|}\right)\right\}=-\left(|z|'z-1\right)\left(\frac{1}{2}e^{-|z|}\right)$,

$\frac{\partial}{\partial z}\left\{z^2\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z\left(|z|'z-2\right)\left(\frac{1}{2}e^{-|z|}\right)$, $\frac{\partial}{\partial z}\left\{z^3\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^2\left(|z|'z-3\right)\left(\frac{1}{2}e^{-|z|}\right)$,

$\frac{\partial}{\partial z}\left\{z^4\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^3\left(|z|'z-4\right)\left(\frac{1}{2}e^{-|z|}\right)$, $\frac{\partial}{\partial z}\left\{z^5\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^4\left(|z|'z-5\right)\left(\frac{1}{2}e^{-|z|}\right)$,

$\frac{\partial}{\partial z}\left\{z^6\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^5\left(|z|'z-6\right)\left(\frac{1}{2}e^{-|z|}\right)$, $\frac{\partial}{\partial z}\left\{z^7\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^6\left(|z|'z-7\right)\left(\frac{1}{2}e^{-|z|}\right)$,

$\frac{\partial}{\partial z}\left\{z^8\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^7\left(|z|'z-8\right)\left(\frac{1}{2}e^{-|z|}\right)$, $\frac{\partial}{\partial z}\left\{z^9\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^8\left(|z|'z-9\right)\left(\frac{1}{2}e^{-|z|}\right)$,

$\frac{\partial}{\partial z}\left\{z^{10}\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^9\left(|z|'z-10\right)\left(\frac{1}{2}e^{-|z|}\right)$ and $\frac{\partial}{\partial z}\left\{z^{12}\left(\frac{1}{2}e^{-|z|}\right)\right\}=-z^{11}\left(|z|'z-12\right)\left(\frac{1}{2}e^{-|z|}\right)$.

Putting these values in (7), we get

$$f'_Z(z;\alpha,\beta)=\frac{1}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right)[(\beta^2z^6+2\alpha\beta z^4-2\beta z^3+\alpha^2z^2-2\alpha z+2)$$
$$\{\left(\beta^2z^6+2\alpha\beta z^4-2\beta z^3+\alpha^2z^2-2\alpha z+2\right)-4\left(\alpha\left(4\beta z^3-1\right)+3\beta z^2\left(\beta z^3-1\right)+\alpha^2z\right)|z|'\}] \quad (8)$$

Since (8) has at most seven zeros, the function $f_Z(z;\alpha,\beta)$ can have at most four modes. □

Theorem 2
The cdf of *BABSLa*$_2(\alpha,\beta)$ distribution is given by
when $z\leq 0$

$$F_Z(z;\alpha,\beta)=\frac{e^z}{2C_2(\alpha,\beta)}[4+\alpha^4b_1-8b_2\beta+8b_3\beta^2-4b_4\beta^3+\beta^4\{-1320(z-9)z^8-95040(z-7)z^6-$$
$$3991680(z-5)z^4-79833600(z-3)z^2-479001600(z-1)\}+4\alpha^3b_5+2\alpha^2\{4b_7\left(z^2-2z+2\right)-$$
$$6\beta b_6\left(z^2-5z+20\right)z^3+3\beta^2\left(z^2-8z+56\right)z^6\}+4\alpha\{2\beta\left(b_9\left(z^2-4z+12\right)z^2+48\right)-3\beta^2$$
$$\left(b_8z^5\left(z^2-7z+42\right)-5040\right)-2b_{10}z+\beta^3\left(\left(z^2-10z+90\right)z^8+3628800\right)+2\}]$$

when $z>0$

$$F_Z(z;\alpha,\beta)=\frac{e^{-z}}{2C_2(\alpha,\beta)}[-4-\alpha^4b_1+8b_{18}\beta-8b_{19}\beta^2+4b_{20}\beta^3-\beta^4\{z^{12}+12z^{11}+132z^{10}+1320(z+9)$$
$$z^8+95040(z+7)z^6+3991680(z+5)z^4+79833600(z+3)z^2+479001600(z+1)\}-4\alpha^3b_{11}-2\alpha^2$$
$$\{b_{14}\left(4z^2+8z+8\right)+b_{13}\beta\left(6z^5+30z^4+120z^3\right)+\beta^2\left(3z^8+24z^7+168z^6\right)\}+8b_{12}e^z-4\alpha$$
$$\{\beta^2\left(3b_{15}z^7+21b_{15}z^6+126b_{15}z^5-15120\right)+\beta\left(2b_{16}z^4+8b_{16}z^3+24b_{16}z^2+96\right)+$$
$$2b_{17}z+\beta^3\left(z^{10}+10z^9+90z^8+3628800\right)-2\}] \quad (9)$$

where $b_1=z^4-4z^3+12z^2-24z+24$, $b_2=z^3-3z^2+6z-6$,

$b_3=z^6-6z^5+30z^4-120z^3+360z^2-720z+720$,

$b_4=z^9-9z^8+72z^7-504z^6+3024z^5-15120z^4+60480z^3-181440z^2+362880z-362880$,

$b_5 = 720\beta + \beta z^6 - 6\beta z^5 + 30\beta z^4 - (120\beta + 1)z^3 + (360\beta + 3)z^2 - 6(120\beta + 1)z + 6$, $b_6 = 168\beta + 1$,

$b_7 = 15120\beta^2 + 90\beta + 1$, $b_8 = 240\beta + 1$, $b_9 = 75600\beta^2 + 315\beta + 2$,

$b_{10} = 1814400\beta^3 + 7560\beta^2 + 48\beta + 1$,

$b_{11} = 720\beta + \beta z^6 + 6\beta z^5 + 30\beta z^4 + (120\beta - 1)z^3 + (360\beta - 3)z^2 + (720\beta - 6)z - 6$,

$b_{12} = 6\alpha^4 + 720\alpha^3\beta + \alpha^2\left(60480\beta^2 + 4\right) + 96\alpha\left(37800\beta^3 + \beta\right) + 119750400\beta^4 + 1440\beta^2 + 1$,

$b_{13} = 168\beta - 1$, $b_{14} = 15120\beta^2 - 90\beta + 1$, $b_{15} = 240\beta - 1$, $b_{16} = 75600\beta^2 - 315\beta + 2$,

$b_{17} = 1814400\beta^3 - 7560\beta^2 + 48\beta - 1$, $d_{18} = z^3 + 3z^2 + 6z + 6$,

$d_{19} = z^6 + 6z^5 + 30z^4 + 120z^3 + 360z^2 + 720z + 720$ and

$d_{20} = z^9 + 9z^8 + 72z^7 + 504z^6 + 3024z^5 + 15120z^4 + 60480z^3 + 181440z^2 + 362880z + 362880$.

Proof
We have,

$$F_Z(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{\left[\left(1 - \alpha z - \beta z^3\right)^2 + 1\right]^2}{C_2(\alpha,\beta)} \left(\frac{1}{2}e^{-|z|}\right) dz$$

$$= \frac{1}{C_2(\alpha,\beta)} \int_{-\infty}^{z} \left[\left(1 - \alpha z - \beta z^3\right)^2 + 1\right]^2 \left(\frac{1}{2}e^{-|z|}\right) dz$$

$$= \frac{1}{C_2(\alpha,\beta)} \int_{-\infty}^{z} [4 - 8\alpha z + 8\alpha^2 z^2 - 4z^3\left(\alpha^3 + 2\beta\right) + \alpha z^4\left(\alpha^3 + 16\beta\right) - 12\alpha^2\beta z^5 + 4\beta z^6\left(\alpha^3 + 2\beta\right) - 12\alpha\beta^2 z^7 + 6\alpha^2\beta^2 z^8 - 4\beta^3 z^9 + 4\alpha\beta^3 z^{10} + z^{12}\beta^4] \left(\frac{1}{2}e^{-|z|}\right) dz$$

$$= \frac{1}{C_2(\alpha,\beta)}[4\int_{-\infty}^{z} \left(\frac{1}{2}e^{-|z|}\right) dz - 8\alpha \int_{-\infty}^{z} z\left(\frac{1}{2}e^{-|z|}\right) dz + 8\alpha^2 \int_{-\infty}^{z} z^2\left(\frac{1}{2}e^{-|z|}\right) dz - 4\left(\alpha^3 + 2\beta\right) \int_{-\infty}^{z} z^3\left(\frac{1}{2}e^{-|z|}\right) dz + \alpha\left(\alpha^3 + 16\beta\right) \int_{-\infty}^{z} z^4\left(\frac{1}{2}e^{-|z|}\right) dz - 12\alpha^2\beta \int_{-\infty}^{z} z^5\left(\frac{1}{2}e^{-|z|}\right) dz + 4\beta\left(\alpha^3 + 2\beta\right) \int_{-\infty}^{z} z^6\left(\frac{1}{2}e^{-|z|}\right) dz - 12\alpha\beta^2 \int_{-\infty}^{z} z^7\left(\frac{1}{2}e^{-|z|}\right) dz + 6\alpha^2\beta^2 \int_{-\infty}^{z} z^8\left(\frac{1}{2}e^{-|z|}\right) dz - 4\beta^3 \int_{-\infty}^{z} z^9\left(\frac{1}{2}e^{-|z|}\right) dz + 4\alpha\beta^3 \int_{-\infty}^{z} z^{10}\left(\frac{1}{2}e^{-|z|}\right) dz + \beta^4 \int_{-\infty}^{z} z^{12}\left(\frac{1}{2}e^{-|z|}\right) dz] \quad (10)$$

Now, we have

$$\int_{-\infty}^{z}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{e^{z}}{2} & z\leq 0\\ \frac{1}{2}e^{-z}\left(2e^{z}-1\right) & z>0\end{cases},\ \int_{-\infty}^{z}z\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}(z-1) & z\leq 0\\ -\frac{1}{2}e^{-z}(z+1) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{2}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{2}-2z+2\right) & z\leq 0\\ \frac{1}{2}e^{-z}\left(-z^{2}-2z+4e^{z}-2\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{3}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{3}-3z^{2}+6z-6\right) & z\leq 0\\ -\frac{1}{2}e^{-z}\left(z^{3}+3z^{2}+6z+6\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{4}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{4}-4z^{3}+12z^{2}-24z+24\right) & z\leq 0\\ \frac{1}{2}e^{-z}\left(-z^{4}-4z^{3}-12z^{2}-24z+48e^{z}-24\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{5}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{5}-5z^{4}+20z^{3}-60z^{2}+120z-120\right) & z\leq 0\\ -\frac{1}{2}e^{-z}\left(z^{5}+5z^{4}+20z^{3}+60z^{2}+120z+120\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{6}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{6}-6z^{5}+30z^{4}-120z^{3}+360z^{2}-720z+720\right) & z\leq 0\\ \frac{1}{2}e^{-z}\left(-z^{6}-6z^{5}-30z^{4}-120z^{3}-360z^{2}-720z+1440e^{z}-720\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{7}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}\left(z^{7}-7z^{6}+42z^{5}-210z^{4}+840z^{3}-2520z^{2}+5040z-5040\right) & z\leq 0\\ -\frac{1}{2}e^{-z}\left(z^{7}+7z^{6}+42z^{5}+210z^{4}+840z^{3}+2520z^{2}+5040z+5040\right) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{8}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}(z^{8}-8z^{7}+56z^{6}-336z^{5}+1680z^{4}-6720z^{3}+\\ 20160z^{2}-40320z+40320) & z\leq 0\\ \frac{1}{2}e^{-z}(-z^{8}-8z^{7}-56z^{6}-336z^{5}-1680z^{4}-6720z^{3}-\\ 20160z^{2}-40320z+80640e^{z}-40320) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{9}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}(z^{9}-9z^{8}+72z^{7}-504z^{6}+3024z^{5}-15120z^{4}+60480z^{3}-\\ 181440z^{2}+362880z-362880) & z\leq 0\\ -\frac{1}{2}e^{-z}(z^{9}+9z^{8}+72z^{7}+504z^{6}+3024z^{5}+15120z^{4}+60480z^{3}+\\ 181440z^{2}+362880z+362880) & z>0\end{cases},$$

$$\int_{-\infty}^{z}z^{10}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}(z^{10}-10z^{9}+90z^{8}-720z^{7}+5040z^{6}-30240z^{5}+151200z^{4}-\\ 604800z^{3}+1814400z^{2}-3628800z+3628800) & z\leq 0\\ \frac{1}{2}e^{-z}(-z^{10}-10z^{9}-90z^{8}-720z^{7}-5040z^{6}-30240z^{5}-151200z^{4}-\\ 604800z^{3}-1814400z^{2}-3628800z+7257600e^{z}-3628800) & z>0\end{cases}$$

and

$$\int_{-\infty}^{z}z^{12}\left(\frac{1}{2}e^{-|z|}\right)dz=\begin{cases}\frac{1}{2}e^{z}(z^{12}-12z^{11}+132z^{10}-1320z^{9}+11880z^{8}-95040z^{7}+665280\\ z^{6}-3991680z^{5}+19958400z^{4}-79833600z^{3}+239500800z^{2}-\\ 479001600z+479001600) & z\leq 0\\ \frac{1}{2}e^{-z}(-z^{12}-12z^{11}-132z^{10}-1320z^{9}-11880z^{8}-95040z^{7}-665280\\ z^{6}-3991680z^{5}-19958400z^{4}-79833600z^{3}-239500800z^{2}-\\ 479001600z+958003200e^{z}-479001600) & z>0\end{cases}.$$

Putting these values in (10), we get the desired result in (9). □

The cdf is plotted in Figure 2 for studying variation in its shape with respect to the parameters α and β.

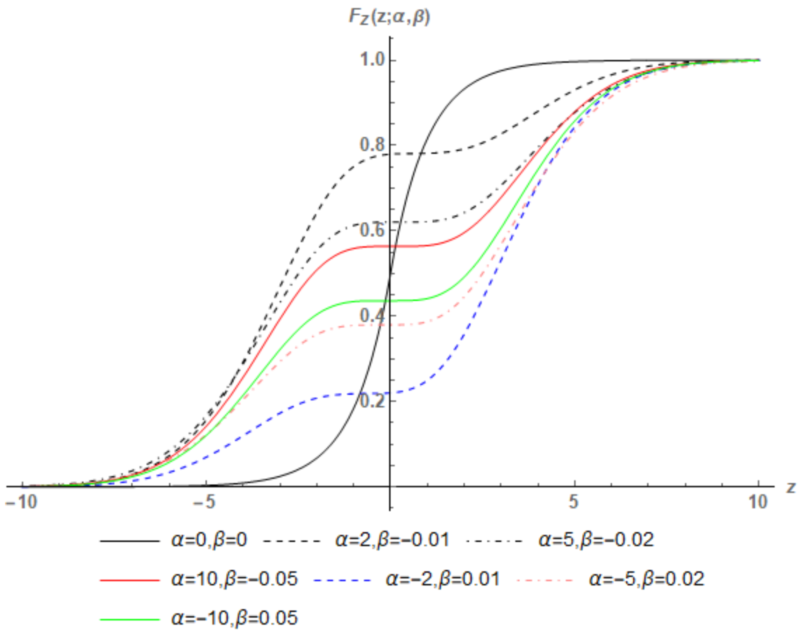

Figure 2. Plots of the cdf of *BABSLa*$_2$*(*α *,*β *)*

Theorem 3
The moment generating function (mgf) of *BABSLa*$_2$*(*α *,*β *)* distribution is given by

$$M_Z(t)=\frac{[M_X(t)]^{13}}{C_2(\alpha,\beta)}4[\left(t^2-1\right)^{12}+4\alpha t\left(t^2-1\right)^{12}+4\alpha^2c_1\left(t^2-1\right)^{10}+720\alpha^2\beta c_2 t\left(t^2-1\right)^7+ 120960\alpha\beta^2c_3\left(t^2-1\right)^5+60480\alpha^2\beta^2c_4\left(t^2-1\right)^4+725760\beta^3c_5(t-1)^3t(t+1)^3+ 3628800\alpha\beta^3c_6\left(t^2-1\right)^2+119750400\beta^4c_7+24c_8\left(t^2-1\right)^9\left(t^3+t\right)+720\beta c_9c_8 \left(t^2-1\right)^6+6\alpha c_{10}c_{11}\left(t^2-1\right)^8] \quad (11)$$

where $M_X(t)$ is the mgf of $X\sim La(0,1)$ and $c_1=3t^2+1$, $c_2=3t^4+10t^2+3$,

$c_3=t\left(t^6+7t^4+7t^2+1\right)$, $c_4=9t^8+84t^6+126t^4+36t^2+1$, $c_5=5t^8+60t^6+126t^4+60t^2+5$,

$c_6=11t^{10}+165t^8+462t^6+330t^4+55t^2+1$, $c_7=13t^{12}+286t^{10}+1287t^8+1716t^6+715t^4+78t^2+1$,

$c_8=\alpha^3+2\beta$, $c_9=7t^6+35t^4+21t^2+1$, $c_{10}=5t^4+10t^2+1$ and $c_{11}=\alpha^3+16\beta$.

Proof
We have,

$$M_Z(t)=E(e^{tz})=\int_{-\infty}^{\infty}e^{tz}\frac{\left[\left(1-\alpha z-\beta z^3\right)^2+1\right]^2}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right)dz$$

$$= \frac{1}{C_2(\alpha,\beta)} \int_{-\infty}^{\infty} e^{tz} [\beta^4 z^{12} + 4\alpha\beta^3 z^{10} - 4\beta^3 z^9 + 6\alpha^2\beta^2 z^8 - 12\alpha\beta^2 z^7 + 4\beta z^6 \left(\alpha^3 + 2\beta\right) - 12\alpha^2\beta z^5 + \alpha z^4 \left(\alpha^3 + 16\beta\right) - 4z^3 \left(\alpha^3 + 2\beta\right) + 8\alpha^2 z^2 - 8\alpha z + 4] \left(\frac{1}{2} e^{-|z|}\right) dz$$

$$= \frac{1}{C_2(\alpha,\beta)} [\beta^4 \int_{-\infty}^{\infty} z^{12} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + 4\alpha\beta^3 \int_{-\infty}^{\infty} z^{10} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz - 4\beta^3 \int_{-\infty}^{\infty} z^{9} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + 6\alpha^2\beta^2 \int_{-\infty}^{\infty} z^{8} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz - 12\alpha\beta^2 \int_{-\infty}^{\infty} z^{7} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + 4\beta \left(\alpha^3 + 2\beta\right) \int_{-\infty}^{\infty} z^{6} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz - 12\alpha^2\beta \int_{-\infty}^{\infty} z^{5} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + \alpha \left(\alpha^3 + 16\beta\right) \int_{-\infty}^{\infty} z^{4} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz - 4 \left(\alpha^3 + 2\beta\right) \int_{-\infty}^{\infty} z^{3} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + 8\alpha^2 \int_{-\infty}^{\infty} z^{2} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz - 8\alpha \int_{-\infty}^{\infty} z e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz + 4 \int_{-\infty}^{\infty} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz] \quad (12)$$

Now, we have

$\int_{-\infty}^{\infty} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = \frac{-1}{t^2-1} = M_X(t)$, $\int_{-\infty}^{\infty} z e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = -2t \left[M_X(t)\right]^2$,

$\int_{-\infty}^{\infty} z^2 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 2\left(3t^2+1\right)\left[M_X(t)\right]^3$, $\int_{-\infty}^{\infty} z^3 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = -24t\left(t^2+1\right)\left[M_X(t)\right]^4$,

$\int_{-\infty}^{\infty} z^4 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 24\left(5t^4+10t^2+1\right)\left[M_X(t)\right]^5$,

$\int_{-\infty}^{\infty} z^5 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = -240t\left(3t^4+10t^2+3\right)\left[M_X(t)\right]^6$,

$\int_{-\infty}^{\infty} z^6 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 720\left(7t^6+35t^4+21t^2+1\right)\left[M_X(t)\right]^7$,

$\int_{-\infty}^{\infty} z^7 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = -40320t\left(t^6+7t^4+7t^2+1\right)\left[M_X(t)\right]^8$,

$\int_{-\infty}^{\infty} z^8 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 40320\left(9t^8+84t^6+126t^4+36t^2+1\right)\left[M_X(t)\right]^9$,

$\int_{-\infty}^{\infty} z^9 e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = -725760t\left(5t^8+60t^6+126t^4+60t^2+5\right)\left[M_X(t)\right]^{10}$,

$\int_{-\infty}^{\infty} z^{10} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 3628800\left(11t^{10}+165t^8+462t^6+330t^4+55t^2+1\right)\left[M_X(t)\right]^{11}$ and

$\int_{-\infty}^{\infty} z^{12} e^{tz} \left(\frac{1}{2} e^{-|z|}\right) dz = 479001600\left(13t^{12}+268t^{10}+1287t^8+1716t^6+715t^4+78t^2+1\right)\left[M_X(t)\right]^{12}$.

Putting these values in (12), we get the desired result in (11).

□

Theorem 4
The n^{th} order moment of *BABSLa*$_2(\alpha,\beta)$ distribution is given by

$$E(Z^{2n}) = \frac{1}{C_2(\alpha,\beta)} [4E_{La}(Z^{2n}) + 8\alpha^2 E_{La}(Z^{2n+2}) + \alpha\left(\alpha^3+16\beta\right) E_{La}(Z^{2n+4}) + 4\beta\left(\alpha^3+2\beta\right) E_{La}(Z^{2n+6}) + 6\alpha^2\beta^2 E_{La}(Z^{2n+8}) + 4\alpha\beta^3 E_{La}(Z^{2n+10}) + \beta^4 E_{La}(Z^{2n+12})]$$

and

$$E(Z^{2n-1}) = \frac{1}{C_2(\alpha,\beta)}[-8\alpha E_{La}(Z^{2n}) - 4\left(\alpha^3+2\beta\right)E_{La}(Z^{2n+2}) - 12\alpha^2\beta E_{La}(Z^{2n+4}) - 12\alpha\beta^2 E_{La}(Z^{2n+6}) - 4\beta^3 E_{La}(Z^{2n+8})]$$

where $E_{La}(Z^n) = \frac{[1+(-1)^n]}{2}\Gamma(n+1), n>0$ is the moment of standard Laplace distribution ([14]).

Proof
We have,

$$E(Z^{2n}) = \int_{-\infty}^{\infty} z^{2n}\frac{\left[\left(1-\alpha z-\beta z^3\right)^2+1\right]^2}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right)dz$$

$$= \int_{-\infty}^{\infty}\frac{1}{C_2(\alpha,\beta)}[4Z^{2n} - 8\alpha Z^{2n+1} + 8\alpha^2 z^{2n+2} - 4\left(\alpha^3+2\beta\right)z^{2n+3} + \alpha\left(\alpha^3+16\beta\right)z^{2n+4} - 12\alpha^2\beta z^{2n+5} + 4\beta\left(\alpha^3+2\beta\right)z^{2n+6} - 12\alpha\beta^2 z^{2n+7} + 6\alpha^2\beta^2 z^{2n+8} - 4\beta^3 z^{2n+9} + 4\alpha\beta^3 z^{2n+10} + z^{2n+12}\beta^4]\left(\frac{1}{2}e^{|z|}\right)dz$$

$$= \frac{1}{C_2(\alpha,\beta)}[4E_{La}(Z^{2n}) - 8\alpha E_{La}Z^{2n+1} + 8\alpha^2 E_{La}(z^{2n+2}) - 4\left(\alpha^3+2\beta\right)E_{La}(z^{2n+3}) + \alpha\left(\alpha^3+16\beta\right) E_{La}(z^{2n+4}) - 12\alpha^2\beta E_{La}(z^{2n+5}) + 4\beta\left(\alpha^3+2\beta\right)E_{La}(z^{2n+6}) - 12\alpha\beta^2 E_{La}(z^{2n+7}) + 6\alpha^2\beta^2 E_{\mathrm{La}}(z^{2n+8}) - 4\beta^3 E_{La}(z^{2n+9}) + 4\alpha\beta^3 E_{La}(z^{2n+10}) + E_{La}(z^{2n+12})\beta^4]$$

$$= \frac{1}{C_2(\alpha,\beta)}[4E_{La}(Z^{2n}) + 8\alpha^2 E_{La}(Z^{2n+2}) + \alpha\left(\alpha^3+16\beta\right)E_{La}(Z^{2n+4}) + 4\beta\left(\alpha^3+2\beta\right)E_{La}(Z^{2n+6}) + 6\alpha^2\beta^2 E_{La}(Z^{2n+8}) + 4\alpha\beta^3 E_{La}(Z^{2n+10}) + \beta^4 E_{La}(Z^{2n+12})]$$

where $E_{La}(Z^n) = \frac{[1+(-1)^n]}{2}\Gamma(n+1), n>0$ is the moment of standard Laplace distribution and using this formula into the last equation we get the required result.
Similarly, $E(Z^{2n-1})$ can be proved in the same line which can be omitted for the sake of brevity.

□

From the above equation of n^{th} order moment, we get

$$E(Z) = -\frac{16\left(6\alpha^3+540\alpha^2\beta+30240\alpha\beta^2+\alpha+12\left(75600\beta^3+\beta\right)\right)}{C_2(\alpha,\beta)}$$

$$E(Z^2) = \frac{8\left[6\left\{15\alpha^4+3360\alpha^3\beta+\alpha^2\left(453600\beta^2+4\right)+240\alpha\left(166320\beta^3+\beta\right)+6720\left(270270\beta^4+\beta^2\right)\right\}+1\right]}{C_2(\alpha,\beta)}$$

$$E(Z^3) = \frac{-192\left(15\alpha^3+2520\alpha^2\beta+226800\alpha\beta^2+\alpha+30\left(332640\beta^3+\beta\right)\right)}{C_2(\alpha,\beta)}$$

$$E(Z^4) = \frac{96\left[60\left\{7\alpha^4 + 5040\left(99\alpha^2+1\right)\beta^2 + 56\left(45\alpha^2+2\right)\alpha\beta + \alpha^2 + 60540480\alpha\beta^3 + 3632428800\beta^4\right\} + 1\right]}{C_2(\alpha,\beta)}$$

and variance can be obtained as

$$\text{Var}(Z) = \frac{8}{[C_2(\alpha,\beta)]^2}[-32\{6\alpha^3 + 540\alpha^2\beta + 30240\alpha\beta^2 + \alpha + 12\left(75600\beta^3+\beta\right)\}^2 + C_2(\alpha,\beta)\{6(15\alpha^4 + 3360\alpha^3\beta + \alpha^2\left(453600\beta^2+4\right) + 240\alpha\left(166320\beta^3+\beta\right) + 6720\left(270270\beta^4+\beta^2\right)) + 1\}]$$

By numerically optimizing $E(Z)$ and $Var(Z)$ with respect to α and β , we get the following approximate bounds for mean and variance as $-2.689 \leq E(Z) \leq 2.689$ and $2 \leq Var(Z) \leq 228.033$. The plots of the mean and the variance are given respectively, in Figure 3 and Figure 4 to study their behaviors. These plots also verify these bounds.

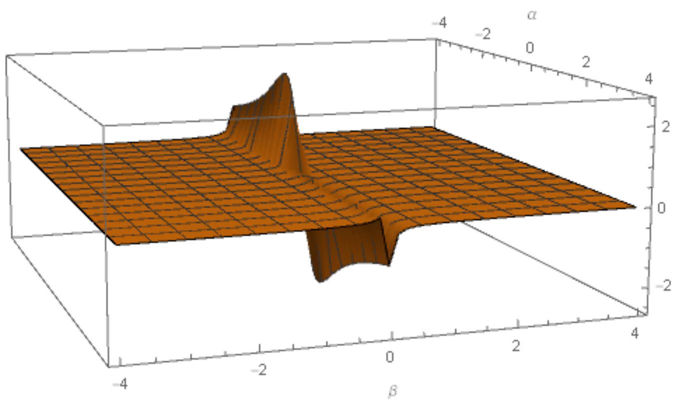

Figure 3. Plots of Mean

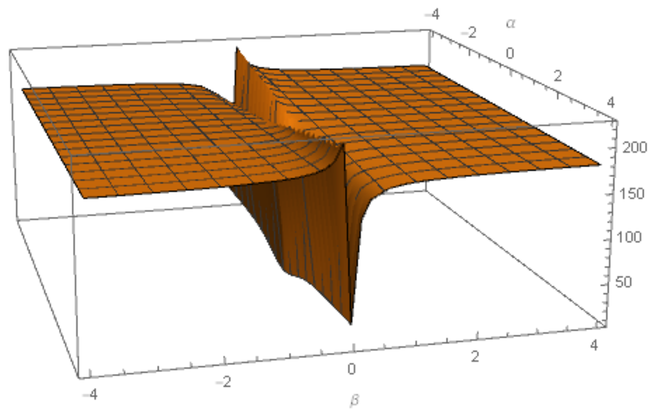

Figure 4. Plots of Variance

Remark 1
By taking the limit $\alpha \to \pm\infty$ in the moments of *BABSLa*$_2(\alpha,\beta)$ distribution, we can derive the moments of *BLa(4)* distribution as $E(Z) = 0$ and $Var(Z) = 30$. Again, by taking the limit $\beta \to \pm\infty$ in the moments of *BABSLa*$_2(\alpha,\beta)$ distribution, we can derive the moments of *BLa(12)* distribution as $E(Z) = 0$ and $Var(Z) = 182$.

2.1. *Skewness and Kurtosis*

The skewness (β_1) and kurtosis (β_2) of *BABSLa*$_2(\alpha, \beta)$ distribution are respectively, given by

$$\beta_1 = \frac{8\left[128d_1^3 - 6C_2(\alpha,\beta)d_1(6d_2+1) + 3C_2(\alpha,\beta)^2 d_3\right]^2}{\left[-32d_1^2 + C_2(\alpha,\beta)d_2\right]^3}$$

Where $d_1 = 6\alpha^3 + 540\alpha^2\beta + 30240\alpha\beta^2 + \alpha + 12\left(75600\beta^3 + \beta\right)$,
$d_2 = 90\alpha^4 + 20160\alpha^3\beta + 24\alpha^2\left(113400\beta^2+1\right) + 1440\alpha\left(166320\beta^3+\beta\right) + 10897286400\beta^4 + 40320\beta^2 + 1$ and
$d_3 = 15\alpha^3 + 2520\alpha^2\beta + 226800\alpha\beta^2 + \alpha + 30\left(332640\beta^3+\beta\right)$.
In particular, when $\alpha = \beta = 1$, the value of $\beta_1 = 0.000027$ and when $\alpha = \beta = 0$, then $\beta_1 = 0$ which is the skewness of standard Laplace distribution and the distribution is symmetric Laplace.
And,

$$\beta_2 = \frac{3\left[-2048d_1^4 + 128C_2(\alpha,\beta)d_1^2(6d_2+1) - 128C_2(\alpha,\beta)^2 d_1 d_3 + C_2(\alpha,\beta)^3(60d_4+1)\right]^2}{\left[-32d_1^2 + C_2(\alpha,\beta)d_2\right]^3}$$

where $d_4 = 420\alpha^4 + 151200\alpha^3\beta + 60\alpha^2\left(498960\beta^2+1\right) + 6720\alpha\left(540540\beta^3+\beta\right) + 217945728000\beta^4 + 302400\beta^2 + 1$.
In particular, when $\alpha = \beta = 1$, the value of $\beta_2 = 1.3227$ and when $\alpha = \beta = 0$, then $\beta_2 = 6$, which is the kurtosis of standard Laplace distribution. By numerically optimizing β_1 and β_2 with respect to α and β , we get the following approximate bounds for skewness and kurtosis as $0 \leq \beta_1 \leq 2.471$ and $1.243 \leq \beta_2 \leq 37.324$. The plots of the skewness and kurtosis are respectively given in Figure 5 and Figure 6 to study their behaviors. These plots also verify these bounds.

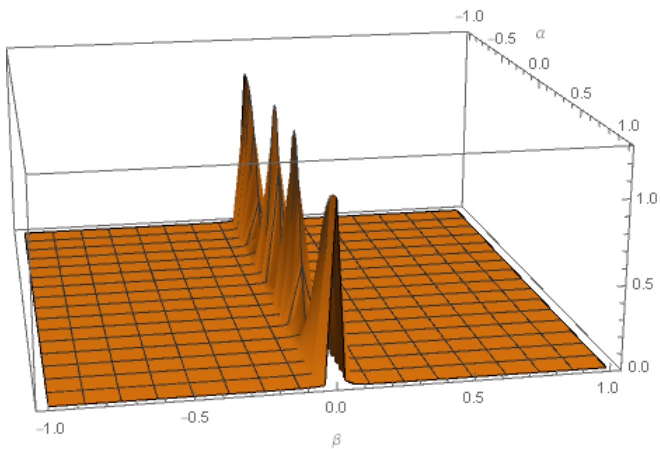

Figure 5. Plots of Skewness

Remark 2
By taking the limit $\alpha \to \pm\infty$ in the results of *BABSLa*$_2(\alpha, \beta)$ distribution, we can derive the skewness and kurtosis of *BLa(4)* distribution as $\beta_1 = 0$, $\beta_2 = 1.8667$. Again, by taking the limit $\beta \to \pm\infty$ in the results of *BABSLa*$_2(\alpha, \beta)$ distribution, we can derive the skewness and kurtosis of *BLa(12)* distribution as $\beta_1 = 0$, $\beta_2 = 1.319$.

3. Random Number Generation

The density function in (6) of model *BABSLa*$_2(\alpha, \beta)$ can be represented as sum of two functions

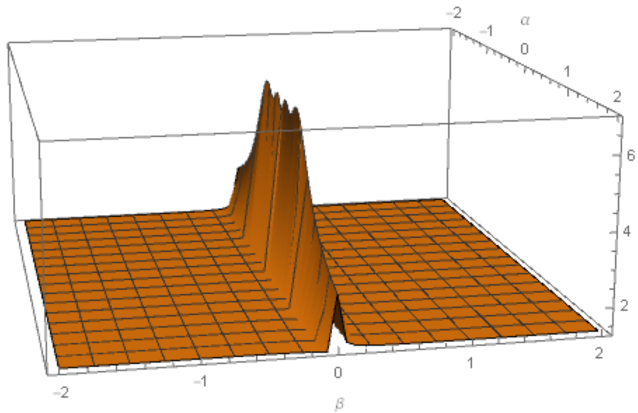

Figure 6. Plots of Kurtosis

$$f_Z(z;\alpha,\beta) = \frac{z^2\left(z^2\left(\alpha+\beta z^2\right)^2+8\right)\left(\alpha+\beta z^2\right)^2+4}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right)+ \frac{-4z\left(\alpha+\beta z^2\right)\left(z^2\left(\alpha+\beta z^2\right)^2+2\right)}{C_2(\alpha,\beta)}\left(\frac{1}{2}e^{-|z|}\right) \tag{13}$$

In (13), the 1st part is symmetric and the second part is asymmetric one and the symmetric part, which is defined below, is symbolically denoted by $SCBABSLa_2(\alpha,\beta)$. For $\alpha=\beta=0$, $Z \sim La(0,1) = \left(\frac{1}{2}e^{|z|}\right)$.
To generate data from $BABSLa_2(\alpha,\beta)$ distribution, we use the acceptance-rejection algorithm which was introduced by [20] as follows:
Let $f(x)$ be the density function of $X \sim BABSLa_2(\alpha,\beta)$ and $f_1(x)$ is the density function of $Z \sim SCBABSLa_2(\alpha,\beta)$, with

$$M = \text{Sup}\left(\frac{f(x)}{f_1(x)}\right) = \frac{1}{3}\left(2\sqrt{2}+3\right)$$

To generate a random variable $X \sim BABSLa_2(\alpha,\beta)$, we shall carry out the following steps:

a) Generate a random variable $Z \sim SCBABSLa_2(\alpha,\beta)$.
b) Generate $U \sim Uniform(0,1)$ independently from Z .
c) If $U < \frac{f(Z)}{M(f_1(Z))} = \frac{3\left[\left(1-\alpha Z-\beta Z^3\right)^2+1\right]^2}{\left(2\sqrt{2}+3\right)\left(Z^2\left(Z^2(\alpha+\beta Z^2)^2+8\right)(\alpha+\beta Z^2)^2+4\right)}$, set $X=Z$ accept; otherwise, go back to step one (reject).

By the acceptance-rejection method, any choice of this random variable will be accepted with probability $\frac{1}{M}$, i.e., $P\left(U < \frac{fZ}{M(f_1 Z)}\right) = \frac{1}{M} = \frac{3}{2\sqrt{2}+3}$. Thus, since the number of trails is geometric with $p=\frac{1}{M}$, the approximated value for this number is $M=\frac{1}{3}\left(2\sqrt{2}+3\right)=1.9428$.

4. Parameter Estimation and Maximum Likelihood Estimation of $BABSLa_2(\alpha,\beta)$ Distribution

Here, we present the problem of parameter estimation of a location and scale extension of $BABSLa_2(\alpha,\beta)$ distribution. If $Z \sim BABSLa_2(\alpha,\beta)$ then $Y=\mu+\sigma Z$ is said to be the location $(\mu \in R)$ and scale $(\sigma > 0)$ extension of Z and has the pdf given by

$$f_Y(y;\mu,\sigma,\alpha,\beta)=\frac{\left[\left(1-\frac{\alpha(y-\mu)}{\sigma}-\beta\left(\frac{y-\mu}{\sigma}\right)^3\right)^2+1\right]^2}{C_2(\alpha,\beta)}\left\{\frac{1}{2\sigma}\text{Exp}\left(-\frac{|y-\mu|}{\sigma}\right)\right\} \tag{14}$$

where $y\in R$, $\alpha\in R$, $\beta\in R$, and $C_2(\alpha,\beta)$ is defined above. We denoted by $Y\sim BABSLa_2(\mu,\sigma,\alpha,\beta)$. The log-likelihood function of the random sample $y_1,y_2,...,y_n$ from $Y\sim BABSLa_2(\mu,\sigma,\alpha,\beta)$ distribution for the parameters $\theta=(\mu,\sigma,\alpha,\beta)$ is given by

$$\begin{aligned} l(\theta)=&2\sum_{i=1}^{n}\log\left[\left(1-\frac{\alpha\left(y_i-\mu\right)}{\sigma}-\beta\left(\frac{y_i-\mu}{\sigma}\right)^3\right)^2+1\right]-n\text{Log}C_2(\alpha,\beta)-n\text{Log}(\sigma)\\ &+n\text{Log}\left(\frac{1}{2}\right)-\sum_{i=1}^{n}\frac{|y_i-\mu|}{\sigma} \end{aligned} \tag{15}$$

By differentiating (15) partially with respect to the parameters $\theta=(\mu,\sigma,\alpha,\beta)$, we get the following likelihood equations as follows:

$$\frac{\partial l(\theta)}{\partial\mu}=2\sum_{i=1}^{n}\frac{2b_i\left(\frac{\alpha}{\sigma}+\frac{3\beta(y_i-\mu)^2}{\sigma^3}\right)}{(1+b_i^2)}-\sum_{i=1}^{n}-\frac{|y_i-\mu|'}{\sigma}$$

$$\frac{\partial l\theta}{\partial\sigma}=-\frac{n}{\sigma}-\sum_{i=1}^{n}-\frac{|y_i-\mu|}{\sigma^2}+2\sum_{i=1}^{n}\frac{2b_i\left(\frac{\alpha(y_i-\mu)}{\sigma^2}+\frac{3\beta(y_i-\mu)^3}{\sigma^4}\right)}{(1+b_i^2)}$$

$$\frac{\partial l\theta}{\partial\alpha}=-\frac{8n\left(3\alpha^3+270\alpha^2\beta+15120\alpha\beta^2+\alpha+12\left(37800\beta^3+\beta\right)\right)}{C_2(\alpha,\beta)}+2\sum_{i=1}^{n}-\frac{2b_i\left(y_i-\mu\right)}{\sigma\left(1+b_i^2\right)}$$

$$\frac{\partial l\theta}{\partial\beta}=-\frac{48n\left(15\alpha^3+2520\alpha^2\beta+\alpha\left(226800\beta^2+2\right)+60\left(166320\beta^3+\beta\right)\right)}{C_2(\alpha,\beta)}+2\sum_{i=1}^{n}-\frac{2b_i\left(y_i-\mu\right)^3}{\sigma^3\left(1+b_i^2\right)}$$

where $b_i=\left(1-\frac{\alpha(y_i-\mu)}{\sigma}-\frac{\beta(y_i-\mu)^3}{\sigma^3}\right)$.

Now, the solutions of the above system of likelihood equations by numerical maximization of (15) with respect to the parameters $\theta=(\mu,\sigma,\alpha,\beta)$ gives the maximum likelihood estimator for the parameters $\theta=(\mu,\sigma,\alpha,\beta)$.

5. Real life applications: comparative data fitting

Here we have considered three datasets, first (Dataset 1) is the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003. The data obtained from the website http://www.globalfindata.com and are given as

Dataset 1: 4.4623, 4.363, 4.4743, 4.662, 4.529, 4.3956, 4.3956, 4.4623, 4.8828, 4.4405, 4.1068, 3.73, 3.7736, 3.8956, 4.4843, 4.7506, 4.5725, 4.529, 4.3403, 4.529, 4.6232, 4.9285, 4.985, 4.7893, 4.8614, 4.845, 4.9652, 4.9432, 4.8662, 4.8614, 4.7281, 4.8757, 4.8286, 4.662, 4.7125, 4.8403, 4.7893, 5.0736, 4.89, 4.8497, 4.8239,

Table 1. Summary Statistic of the Datasets

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	SD	Skewness	Kurtosis
Data set-I								
1.16	2.81	4.75	4.12	4.87	11.09	1.38	0.03	5.28
Data set-II								
9.17	19.53	20.84	20.83	23.13	34.28	4.57	-0.43	5.26

4.845, 4.7237, 4.8239, 4.8828, 4.7962, 4.717, 4.89, 4.8054, 4.8239, 4.8614, 4.9068, 4.89, 4.8497, 4.8123, 4.817, 4.845, 4.8473, 4.8709, 4.8638, 4.6168, 4.8444, 6.5025, 7.3914, 11.0905, 7.0621, 6.5208, 6.5086, 6.5701, 5.8583, 5.3958, 5.3105, 5.4563, 5.3773, 5.4686, 5.5051, 5.2132, 5.0062, 4.875, 4.845, 4.845, 4.85, 4.85, 4.855, 4.85, 4.895, 4.85, 4.87, 4.895, 4.84, 4.845, 4.855, 4.885, 4.885, 4.895, 4.91, 4.88, 4.86, 4.855, 4.885, 4.86, 4.875,4.875, 4.855, 4.88, 4.856, 4.84, 4.8425, 4.871, 4.8675, 4.865, 4.87, 4.853, 4.855, 4.8525,4.7362, 4.7556, 4.7512, 4.7581, 3.75, 3.525, 4.2063, 4.6325, 4.3187, 4.7225, 4.8481, 4.8481,4.8762, 4.8488, 4.8744, 4.8516, 3.3775, 3.3275, 5.12, 4.9363, 4.93, 4.9088, 4.9969, 4.6363, 3.9537, 4.035, 4.0325, 4.0325, 4.02, 4.02, 4.025, 4.025, 4.0331, 4.0319, 2.8006, 2.8013, 2.7814, 2.8096, 2.8106, 2.7856, 2.8025, 2.7856, 2.8093, 2.8021, 2.7995, 2.8038, 2.8081, 2.8025, 2.7969,2.79, 2.8025, 2.79, 2.4067, 2.3849, 2.3989, 2.3938, 2.552, 2.348, 2.3225, 2.347, 2.0242, 1.7025, 1.92, 2.0435, 2.2145, 2.395, 1.917, 1.618, 1.4515, 1.158, 1.439, 1.4819, 1.8867, 1.8089, 1.611, 1.932, 1.865, 1.51, 1.4765, 1.566, 1.55, 1.712, 1.647, 1.6539, 1.6176, 1.4957, 1.4541, 1.611, 1.785.

The second (Dataset 2) consists of the velocities of 82 distant galaxies, diverging from our own galaxy. The data set is available at http://www.stats.bris.ac.uk/ peter/mixdata and are given as
Dataset 2: 9.172, 9.350, 9.483, 9.558, 9.775, 10.227, 10.406, 16.084, 16.170, 18.419, 18.552, 18.600, 18.927, 19.052, 19.070, 19.330, 19.343, 19.349, 19.440, 19.473, 19.529, 19.541, 19.547, 19.663, 19.846, 19.856, 19.863, 19.914, 19.918, 19.973, 19.989, 20.166, 20.175, 20.179, 20.196, 20.215, 20.221, 20.415, 20.629, 20.795, 20.821, 20.846, 20.875, 20.986, 21.137, 21.492, 21.701, 21.814, 21.921, 21.960, 22.185, 22.209, 22.242, 22.249, 22.314, 22.374, 22.495, 22.746, 22.747, 22.888, 22.914, 23.206, 23.241, 23.263, 23.484, 23.538, 23.542, 23.666, 23.706, 23.711, 24.129, 24.285, 24.289, 24.366, 24.717, 24.990, 25.633, 26.960, 26.995, 32.065, 32.789, 34.279.
The summary statistics of the datasets are given in Table 1.

We then compared the proposed $BABSLa_2(\mu,\sigma,\alpha,\beta)$ distribution with the normal $N\left(\mu,\sigma^2\right)$distribution, the skew normal $SN(\mu,\sigma,\lambda)$ distribution, the Laplace $La(\mu,\sigma)$ distribution, the skew-Laplace $SLa(\mu,\sigma,\lambda)$ distribution of [1], the alpha-skew-Laplace $ASLa(\mu,\sigma,\alpha)$ distribution of [10], the alpha beta skew Laplace $ABSLa(\mu,\sigma,\alpha,\beta)$ distribution of [19] and the Balakrishnan alpha skew Laplace $BASLa_2(\mu,\sigma,\alpha)$ distribution of [18].
Using R software package (see GenSA package version-1.0.3, [21]), the MLE of the parameters are obtained by using numerical optimization routine. AIC and BIC are used for model comparison. Table 2 and 3 shows the MLE's, log-likelihood, AIC and BIC of the above mentioned distributions. The graphical representations of the results taking only the top three competitors for the proposed model are given in Figure 7 and 8.

From Table 2 and 3, it is observe that the proposed Balakrishnan-alpha-beta-skew-Laplace $BABSLa_2(\mu,\sigma,\alpha,\beta)$ distribution provides best fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figures 7 and 8, also confirm our findings.

5.1. Likelihood Ratio Test

Since $La(\mu,\sigma)$, $BASLa_2(\mu,\sigma,\alpha)$ and $BABSLa_2(\mu,\sigma,\alpha,\beta)$ are nested models, the likelihood ratio (LR) test is used to discriminate between between them with the following procedure.

i) To discriminate $BASLa_2(\mu,\sigma,\alpha)$ and $BABSLa_2(\mu,\sigma,\alpha,\beta)$ distributions, we have to test the null hypothesis $H_0:\beta=0$ against the alternative hypothesis $H_1:\beta\neq 0$ and the test statistic is

Table 2. MLE's, log-likelihood, AIC and BIC for the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003

Distributions	μ	σ	λ	α	β	log L	AIC	BIC
$SN(\mu,\sigma,\lambda)$	3.589	1.478	0.501	–	–	-355.217	716.434	726.388
$N(\mu,\sigma^2)$	4.117	1.381	–	–	–	-355.265	714.529	721.165
$La(\mu,\sigma)$	4.754	0.971	–	–	–	-339.315	682.630	689.265
$SLa(\mu,\sigma,\lambda)$	4.855	1.000	1.506	–	–	-311.318	628.636	638.590
$ASLa(\mu,\sigma,\alpha)$	4.861	0.677	–	0.539	–	-301.443	608.885	618.840
$BASLa_2(\mu,\sigma,\alpha)$	4.871	0.603	–	0.297	–	-296.849	599.698	609.652
$ABSLa(\mu,\sigma,\alpha,\beta)$	4.845	0.326	–	-0.076	0.038	-265.372	538.744	552.017
$BABSLa_2(\mu,\sigma,\alpha,\beta)$	4.852	0.204	–	0.038	0.008	-239.984	487.968	501.241

Table 3. MLE's, log-likelihood, AIC and BIC for the velocities of 82 distant galaxies, diverging from our own galaxy

Distributions	μ	σ	λ	α	β	log L	AIC	BIC
$N(\mu,\sigma^2)$	20.832	4.540	–	–	–	-240.417	484.833	489.646
$SN(\mu,\sigma,\lambda)$	24.610	5.907	-1.395	–	–	-239.210	484.420	491.640
$La(\mu,\sigma)$	20.838	2.997	–	–	–	-228.830	461.660	466.474
$SLa(\mu,\sigma,\lambda)$	20.846	2.997	1.002	–	–	-228.829	463.658	470.878
$BASLa_2(\mu,\sigma,\alpha)$	18.419	1.870	–	-0.761	–	-221.549	449.098	456.318
$ASLa(\mu,\sigma,\alpha)$	19.473	1.805	–	-0.842	–	-220.789	447.578	454.798
$ABSLa(\mu,\sigma,\alpha,\beta)$	19.856	1.386	–	-0.484	-0.018	-218.328	444.656	454.283
$BABSLa_2(\mu,\sigma,\alpha,\beta)$	19.343	0.840	–	-0.405	-0.005	-216.838	441.676	451.303

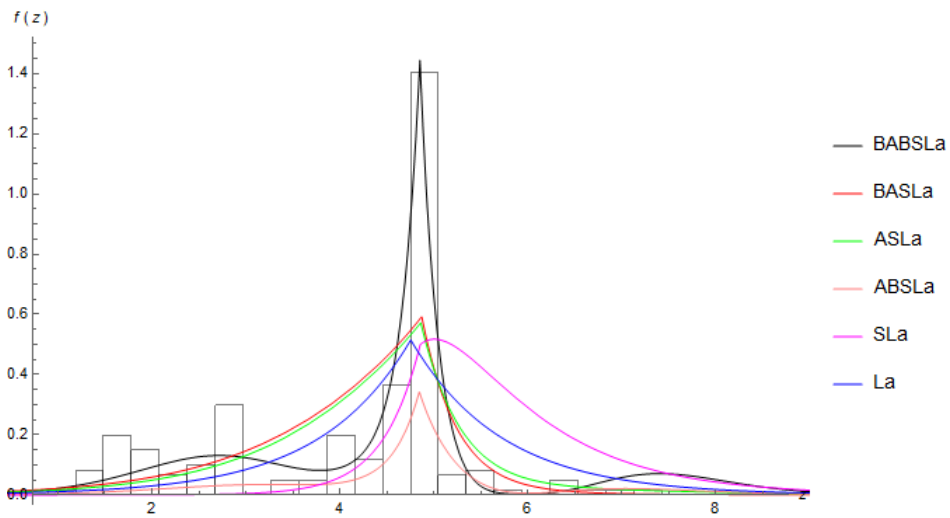

Figure 7. Plots of observed and expected densities for exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003

$$-2log(LR) = -2[logL(\tilde{\mu},\tilde{\sigma},\tilde{\alpha},\beta=0|y) - logL(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\beta}\Big| y)] \sim \chi^2_1$$

,

where $\tilde{\mu},\tilde{\sigma},\tilde{\alpha}$ and $\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\beta}$ are the MLEs of $BASLa_2(\mu,\sigma,\alpha)$ and $BABSLa_2(\mu,\sigma,\alpha,\beta)$ distributions respectively and $r=1$ (difference between the number of parameters). Similarly,

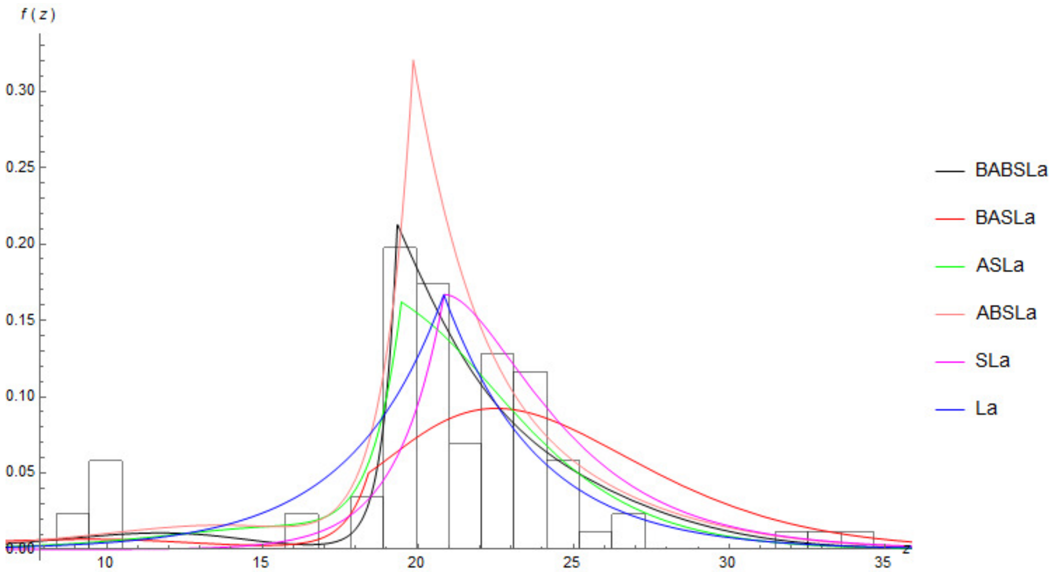

Figure 8. Plots of observed and expected densities for velocities of 82 distant galaxies, diverging from our own galaxy

Table 4. The values of the LR test statistic for different datasets

Hypothesis	LR test statistic values			Degrees of Freedom	Critical values at 5%
	Dataset 1	Dataset 2	Dataset 3		
$H_0 : \beta = 0$ vs $H_1 : \beta \neq 0$	5.096	113.73	9.422	1	3.841
$H_0 : \alpha = 0, \beta = 0$ vs $H_1 : \alpha \neq 0, \beta \neq 0$	19.316	198.662	23.984	2	5.991

ii) to discriminate $La(\mu, \sigma)$ and $BABSLa_2(\mu, \sigma, \alpha, \beta)$ distributions, we have to test the null hypothesis $H_0 : \alpha = 0, \beta = 0$ against the alternative hypothesis $H_1 : \alpha \neq 0, \beta \neq 0$ the test statistic is

$$-2log(LR) = -2[logL(\tilde{\mu}, \tilde{\sigma}, \alpha = 0, \beta = 0|y) - logL(\hat{\mu}, \hat{\sigma}, \hat{\alpha}, \hat{\beta}\Big| y)] \sim \chi^2_1$$

,
where $\tilde{\mu}, \tilde{\sigma}$ and $\hat{\mu}, \hat{\sigma}, \hat{\alpha}, \hat{\beta}$ are the MLEs of $La(\mu, \sigma)$ and $BABSLa_2(\mu, \sigma, \alpha, \beta)$ distributions and $r = 2$ (difference between the number of parameters).

The results of the LR test are shown in Table 4.

Since all the values of LR test statistics for different hypothesis exceed the critical values at 5% level of significance. Thus, we accept the alternative hypothesis. Therefore, we may conclude that the sampled data comes from $BABSLa_2(\mu, \sigma, \alpha, \beta)$ distribution.

6. Conclusions

In this study a new alpha-beta-skew-Laplace distribution is constructed which includes unimodal, bimodal as well as multimodal shapes and some of its properties are studied. The logarithmic extensions of the proposed distribution with some of their special cases are presented. Our findings adequately supported the proposed $BABSLa_2(\mu, \sigma, \alpha, \beta)$ distribution as the better fitted one to the datasets under consideration in terms of AIC and BIC. The plots of

observed and expected densities presented also confirm our findings. Also, the values of LR test statistics for different hypothesis conclude that the sampled data comes from the proposed distribution and not from other distributions considered.
Furthermore, there is scope of extending the present work by considering the Normal and the Logistic distributions. Moreover, logarithmic forms and bivariate generalizations can also be considered as future work.

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