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A New Two-Sided Class of Lifetime Distributions: Applications to Complete and Right Censored Data

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Abstract In this article, we first define a new two-sided distribution called the two-sided Kumaraswamy distribution and then we propose a generalized class of lifetime distributions via compounding two-sided Kumaraswamy and a baseline distribution. One of the advantages of this class of new distributions is that they can be unimodal or bimodal. The general model is specified by taking the exponential distribution as the baseline distribution. Some basic properties of the proposed distribution are derived. The model parameters are estimated by means of maximum likelihood method. In addition, parametric and non-parametric bootstrap procedures are used to obtain point estimates and confidence intervals of the parameters of the model. A simulation study has been conducted to examine the bias and the mean square error of the maximum likelihood estimators. We illustrate the performance of the proposed distribution by means of two real data sets (one is complete data set and other is right censored data set) and both the data sets show that the new distribution is more appropriate as compared to Weibull, gamma, weighted exponential, generalized two-sided exponential, generalized transmuted two-sided exponential and generalized exponential distributions.

Keywords Hazard rate function, Bootstrap, Right censored, Maximum likelihood estimator, Two-sided power distribution

AMS 2010 subject classifications 60Exx, 60E05

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1. Introduction

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, it is necessary to extract data and develop appropriate models. One of the most important models in the statistical theory is the change point models. In the distribution theory, the change point distributions are used in different branches of science such as economics, engineering and agriculture to name a few. Statistical distributions with change point models are rare and interesting. The theory of change point distribution has been less developed by researchers in comparison with non-change point models. This drawback is due to the complex structure of this type of models. Van Dorp and Kotz [19] introduced a family of the change point distributions called two-sided power distribution (TSP) with the

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probability density function (pdf),

$$f(x,\alpha,\beta)=\begin{cases}\alpha\left(\frac{x}{\beta}\right)^{\alpha-1}, & 0<x\leq\beta,\\ \alpha\left(\frac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta\leq x<1,\end{cases} \tag{1}$$

and the corresponding cumulative distribution function (*cdf*),

$$F(x,\alpha,\beta)=\begin{cases}\beta\left(\frac{x}{\beta}\right)^{\alpha}, & 0<x\leq\beta,\\ 1-(1-\beta)\left(\frac{1-x}{1-\beta}\right)^{\alpha}, & \beta\leq x<1,\end{cases} \tag{2}$$

where $0\leq\beta\leq 1$ and $\alpha>0$ are parameters. The parameter β is the location parameter called "turning point" and α is the shape parameter that controls the shape of the distribution on the left and right of β.

The TSP distribution for $\alpha>3$ can be used for modeling unimodal phenomena on a bounded domain when a peak in data is observed. Van Dorp and Kotz [16] further introduced an extension of the three-parameter triangular distribution which was utilized in risk analysis. They also extended the TSP distribution to four-parameter case. Later, Van Dorp and Kotz [17] considered families of continuous distributions on a bounded interval generated by convolutions of TSP distributions. In recent years, a number of researchers have studied the generalization of the TSP distribution, namely, Nadarajah [14], Oruc and Bairamov [15], Vicari et al. [20], Herrerías-Velasco et al. [5] and Soltani and Homei [18]. In recent past, Korkmaz and Genç [12] considered the log transformation of the TSP distribution and introduced a generalization of the exponential distribution. Further, Korkmaz and Genç [13] extended the idea of two-sidedness to other ordinary distributions like normal distribution which is called the two-sided generalized normal distribution. Kharazmi et al. [9] introduced generalized odd transmuted two-sided class of distributions. A general class of two-sided distributions has proposed by Kharazmi et al. [8]. In this regard, see also Korkmaz [10] and Korkmaz and Genç [11].

One of the interesting methods for constructing new distributions is that of Alzaatreh et al. [2] wherein they introduced a new technique of generating new lifetime distributions, which is a special case of $generalized-G$ distribution. It is based on the combination of one arbitrary *cdf* F of a continuous random variable X with the baseline *cdf* G. The integration form of its *cdf* is

$$H(x) \quad = \quad \int_{-\infty}^{G(x)}\frac{f(t)}{F(1)}dt, \tag{3}$$

where f is the corresponding *pdf* of F and $F(1)=P(X\leq 1)$. This interesting method attracted the attention of many researchers, which resulted in the generation of several new flexible statistical models.

In the present paper, we introduce a new family of change point lifetime distributions based on the *cdf* of a parent distribution G for a new two-sided model which is constructed by mixtures of truncated Kumaraswamy distributions. These new models are called generalized two-sided Kumaraswamy-G distribution ($GTSKW-G(or\ GTSKWG)$) and two-sided Kumaraswamy ($TSKW$) distribution, respectively. Our main motivation to introduce this new category of distributions is to provide more flexibility for fitting real data sets compared to the other well-known classic distributions. Moreover, new proposed models $TSKW$ and $GTSKWG$ contain TSP and generalized two-sided-G ($GTSG$) as sub-models, which were introduced by Van Dorp and Kotz [19] and Korkmaz and Genç [13], respectively.

In summary, we first propose a new two-sided distribution based on the mixtures of two truncated Kumaraswamy models and then we introduce a general family of change point models called $GTSKWG$. Some statistical properties of the $GTSKWG$ are studied in a general setting. Thereafter, we consider a special case of this model by taking exponential distribution as the parent distribution G. It is referred as $GTSKWE$ distribution. We provide a comprehensive discussion of statistical and reliability properties of the new $GTSKWE$ model. Furthermore, we consider maximum likelihood and bootstrap estimation procedures for estimating the unknown parameters of the new model for complete and right-censored real data sets. In addition, a simulation study is performed to investigate $MLEs$ consistency.

This paper is organized as follows. In Section 2, we introduce a new distribution called two-sided Kumaraswamy distribution. In Section 3, we propose a generalization of the two-sided Kumaraswamy distribution and consider the moments, hazard function, quantiles and order statistics in a general setting. We consider the exponential distribution as a parent distribution and introduce generalized two-sided Kumaraswamy exponential distribution, in Section 4. In the same section, we plot the shape of density and hazard functions. In Section 5, the estimation of the parameters of the generalized two-sided Kumaraswamy distribution are obtained via maximum likelihood and bootstrap methods. Also, we study the performance of the maximum likelihood estimates of the parameters of the generalized two-sided Kumaraswamy exponential distribution via a simulation study. In Section 6, the superiority of the new model to some competitive models is demonstrated through different criteria of model selection by analyzing complete and right-censored real data sets . Finally, the paper is concluded in Section 7.

2. A new two-sided model based on Kumaraswamy distribution

In this section, first we introduce the two-sided Kumaraswamy distribution and then we propose a generalized family of distributions called generalized two-sided Kumaraswamy distribution. Let the random variable X_1 follow $Kumaraswamy(\alpha,\theta)$ distribution with *pdf*,

$$f_{X_1}(x)=\theta\alpha x^{\alpha-1}(1-x^{\alpha})^{\theta-1},\qquad 0<x<1,\alpha>0,\theta>0. \tag{4}$$

Consider a new transformed variable $X_2=1-X_1$. The *pdf* of X_2 is

$$f_{X_2}(x)=\theta\alpha(1-x)^{\alpha-1}(1-(1-x)^{\alpha})^{\theta-1},\qquad 0<x<1,\alpha>0,\theta>0. \tag{5}$$

By applying the following joint transformation

$$\begin{cases} Y_1 =^D X_2|0<X_2<\beta, \\ Y_2 =^D X_1|\beta<X_1<1, \end{cases} \tag{6}$$

where $=^D$ denotes equality in law, a new two-sided lifetime distribution based on the *pdf*s of the two random variables Y_1 and Y_2 is defined as

$$f(y;\alpha,\beta,\theta)=\begin{cases} \beta f_{Y_1}(y), & 0<y\le\beta, \\ (1-\beta)f_{Y_2}(y), & \beta\le y<1. \end{cases} \tag{7}$$

The *pdf* of the new distribution is given in the following definition.

A random variable X is said to have a two-sided Kumaraswamy distribution if its *pdf* and *cdf* are given, respectively, by

$$f(x;\alpha,\beta,\theta)=\begin{cases} \alpha\,\theta\,\frac{\beta\,(1-x)^{\alpha-1}}{1-(1-\beta)^{\alpha}}\left(\frac{(1-(1-x)^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta-1}, & 0<x\le\beta, \\ \alpha\,\theta\,\frac{(1-\beta)x^{\alpha-1}}{1-\beta^{\alpha}}\left(\frac{1-x^{\alpha}}{1-\beta^{\alpha}}\right)^{\theta-1}, & \beta\le x<1, \end{cases} \tag{8}$$

and

$$F(x;\alpha,\beta,\theta)=\begin{cases} \beta\left(\frac{1-(1-x)^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta}, & 0\le x\le\beta, \\ 1-(1-\beta)\left(\frac{1-x^{\alpha}}{1-\beta^{\alpha}}\right)^{\theta}, & \beta\le x\le 1, \end{cases} \tag{9}$$

where α and θ are shape parameters and β is a location parameter.

We denote this new two-sided distribution by $TSKW(\alpha,\beta,\theta)$. The density shapes of the $TSKW$ distribution for different choices of the parameters are plotted in Figure 1.

Remark 1
For $\alpha = 1$, we have the pdf and cdf of TSP distribution, respectively.

Remark 2
It is worthwhile to note that the proposed method in (7) is applicable for any arbitrary Unit-interval distribution such as Beta distribution.

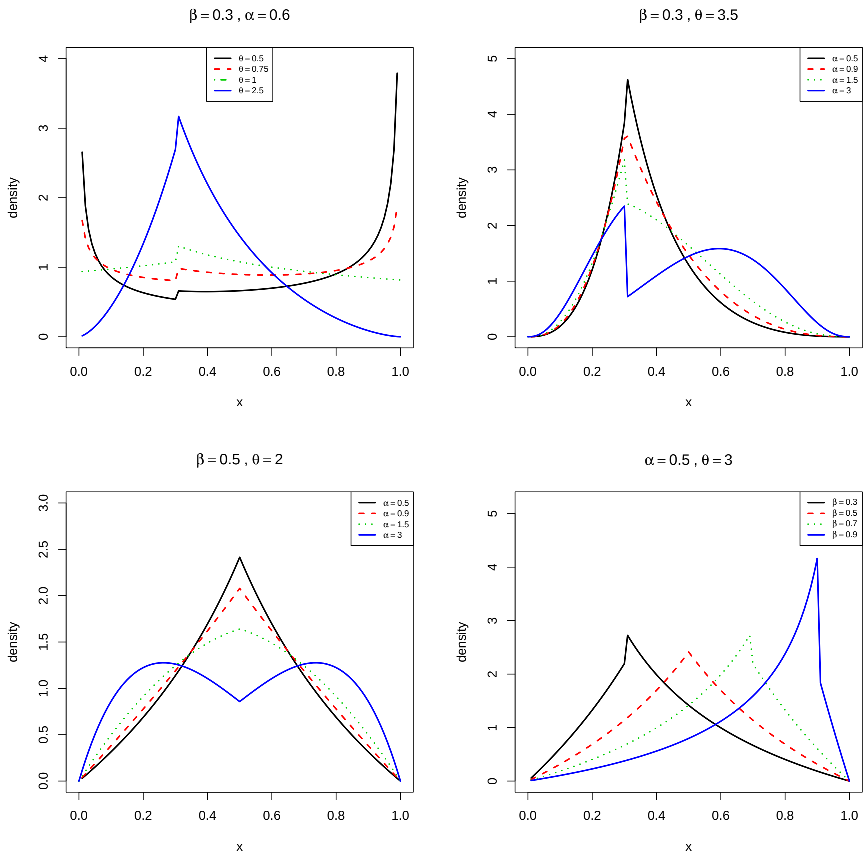

Figure 1. The graphs of the density of the TSKW distribution for selected parameter values.

In the following lemma, we investigate the relation between TSP and $TSKW$ models.

Lemma 1
Suppose that the random variable X has $TSKW(\alpha, \beta, \theta)$ distribution, then the transformed variable $Y = \frac{\beta}{1-(1-\beta)^\alpha}[1-(1-X)^\alpha]$ has $TSP(\theta, \beta)$ with *cdf* (2).

Proof
Using the method of distribution function, we have

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(\frac{\beta}{1-(1-\beta)^\alpha}[1-(1-X)^\alpha] \leq y) \\
&= \begin{cases} \beta \left(\frac{y}{\beta}\right)^\theta, & 0 < y \leq \beta, \\ 1-(1-\beta)\left(\frac{1-y}{1-\beta}\right)^\theta, & \beta \leq y < 1, \end{cases}
\end{aligned}
$$

the proof is completed. □

3. Generalized two-sided Kumaraswamy-G distribution

In this section, first we propose a new two-sided lifetime distributions and then we obtain main statistical and reliability properties of the proposed family in a general setting. Let X be a continuous random variable with *cdf* $G(x;\xi)$ and *pdf* $g(x;\xi)$. Using definition 2 and relation (3), we define a new general two-sided class of distributions as follows. A random variable X is said to have a generalized two-sided Kumaraswamy distribution if its *pdf* is given by

$$f(x;\alpha,\beta,\theta,\xi)=\begin{cases}\alpha\theta\dfrac{\beta g(x;\xi)(1-G(x;\xi))^{\alpha-1}}{1-(1-\beta)^{\alpha}}\left(\dfrac{1-(1-G(x;\xi))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta-1}, & -\infty<x\leq G^{-1}_{(x;\xi)}(\beta),\\ \alpha\theta\dfrac{(1-\beta)g(x;\xi)G^{\alpha-1}(x;\xi)}{1-\beta^{\alpha}}\left(\dfrac{1-G^{\alpha}(x;\xi)}{1-\beta^{\alpha}}\right)^{\theta-1}, & G^{-1}_{(x;\xi)}(\beta)\leq x<\infty,\end{cases} \tag{10}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\theta,\xi)=\begin{cases}\beta\left(\dfrac{1-(1-G(x;\xi))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta}, & -\infty<x\leq G^{-1}_{(x;\xi)}(\beta),\\ 1-(1-\beta)\left(\dfrac{1-G(x;\xi)^{\alpha}}{1-\beta^{\alpha}}\right)^{\theta}, & G^{-1}_{(x;\xi)}(\beta)\leq x<\infty,\end{cases} \tag{11}$$

where ξ is a parameter vector of the *cdf* $G(x;\xi)$ and $G^{-1}_{(x;\xi)}(.)$ is its inverse. We denote this generalized two-sided distribution by $GTSKWG(\alpha,\beta,\theta,\xi)$. Hereafter for simplicity, we write $G(x)$, $G^{-1}(x)$ and $g(x)$ instead of $G(x;\xi)$, $G^{-1}(x;\xi)$ and $g(x;\xi)$, respectively.

In the following lemma, we give an extension of lemma 1 demonstrating the relation between $GTSG$ and $GTSKWG$ models.

Lemma 2
Suppose that the random variable X has $GTSKWG(\alpha,\beta,\theta,\xi)$ distribution, then the transformed variable $Y=\frac{\beta}{1-(1-\beta)^{\alpha}}[1-(1-G(X))^{\alpha}]$ has $GTSG(\theta,\beta,\xi)$ distribution introduced by Korkmaz and Genç (2017).

Proof
Using the method of distribution function, we have

$$\begin{aligned}F_Y(y)&=P(Y\leq y)\\&=P(\frac{\beta}{1-(1-\beta)^{\alpha}}[1-(1-G(X))^{\alpha}]\leq y)\\&=\begin{cases}\beta\left(\frac{G(y)}{\beta}\right)^{\theta}, & -\infty<y\leq G^{-1}(\beta),\\ 1-(1-\beta)\left(\frac{1-G(y)}{1-\beta}\right)^{\theta}, & G^{-1}(\beta)\leq y<\infty,\end{cases}\end{aligned}$$

the proof is completed. □

3.1. Hazard function and moments

The hazard rate is a fundamental tool in reliability modeling for evaluation of the aging process. Knowing the shape of the hazard rate is important in reliability theory, risk analysis and other disciplines. The concepts of increasing, decreasing, bathtub shaped (first decreasing and then increasing) and upside down bathtub shaped (first increasing and then decreasing) hazard rate functions are very useful in reliability analysis. The lifetime distributions with these aging properties are designated as the IFR, DFR, BUT and UBT distributions, respectively. The hazard function of the $GTSKWG$ distribution is given by

$$
\begin{aligned}
r(x) &= \frac{f(x)}{1-F(x)} \\
&= \begin{cases} \dfrac{\alpha\,\theta\,\beta\,g(x)\,(1-G(x))^{\alpha-1}\,[1-(1-G(x))^{\alpha}]^{\theta-1}}{(1-(1-\beta)^{\alpha})^{\theta}-\beta[1-(1-G(x))^{\alpha}]^{\theta}}, & -\infty < x \leq G^{-1}(\beta), \\[2ex] \dfrac{\alpha\,\theta\,g(x)\,G^{\alpha-1}(x)}{1-G^{\alpha}(x)}, & G^{-1}(\beta) \leq x < \infty. \end{cases}
\end{aligned} \tag{12}
$$

The rth moment of the $GTSKWG$ distribution is

$$
\begin{aligned}
E(X^r) &= \frac{\alpha\theta\beta}{(1-\beta^{\alpha})^{\theta}} \int_0^{\beta} \left(G^{-1}(y)\right)^r (1-y)^{\alpha-1} \left(1-(1-y)^{\theta-1}\right) dy \\
&\quad + \frac{\alpha\theta(1-\beta)}{(1-\beta^{\alpha})^{\theta}} \int_{\beta}^{1} \left(G^{-1}(y)\right)^r y^{\alpha-1} (1-y^{\alpha})^{\theta-1} dy \\
&= E_Y[(G^{-1}(Y))^r],
\end{aligned}
$$

where the random variable Y has $TSKW(\alpha, \beta, \theta)$ distribution.

3.2. Random variate generation

One of the important quantity for each probabilistic model is to have the data generator function based on an explicit form.For generating random variables from the $GTSKWG$ distribution, we use the inverse transformation method. The quantile of order q of the $GTSKWG$ distribution is

$$
x_q = F^{-1}(q; \alpha, \beta, \theta, \xi) = \begin{cases} G^{-1}\left[1-\left(1-\left(\frac{q}{\beta}\right)^{\frac{1}{\theta}}(1-(1-\beta)^{\alpha})\right)^{\frac{1}{\alpha}}\right], & 0 < q \leq \beta, \\[2ex] G^{-1}\left[\left(1-\left(\frac{1-q}{1-\beta}\right)^{\frac{1}{\theta}}(1-\beta^{\alpha})\right)^{\frac{1}{\alpha}}\right], & \beta \leq q < 1. \end{cases} \tag{13}
$$

Let U be a random variable with a uniform distribution on $(0, 1)$, then

$$
X = \begin{cases} G^{-1}\left[1-\left(1-\left(\frac{U}{\beta}\right)^{\frac{1}{\theta}}(1-(1-\beta)^{\alpha})\right)^{\frac{1}{\alpha}}\right], & 0 < U \leq \beta, \\[2ex] G^{-1}\left[\left(1-\left(\frac{1-U}{1-\beta}\right)^{\frac{1}{\theta}}(1-\beta^{\alpha})\right)^{\frac{1}{\alpha}}\right], & \beta \leq U < 1. \end{cases} \tag{14}
$$

is a random variable with the $GTSKWG$ distribution by the probability integral transform.

3.3. Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from $GTSKWG$ distribution. Let $X_{i:n}$ denote the ith order statistic. Then, the pdf of $X_{i:n}$ is given by

$$
f_{i:n}(x) = A \begin{cases} \beta^2 g(x)(1-G(x))^{\alpha-1}\left(\frac{(1-(1-G(x))^{\alpha})^{i\theta-1}}{(1-(1-\beta)^{\alpha})^{i\theta}}\right)\left(1-\beta\left(\frac{1-(1-G(x))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)\right)^{n-i}, & x \leq G(x)^{-1}(\beta), \\[2ex] (1-\beta)^2 g(x)G(x)^{-1}\left(\frac{(1-G(x)^{\alpha})^{\theta(n-i+1)-1}}{(1-\beta^{\alpha})^{\theta(n-i+1)}}\right)\left(1-(1-\beta)\left(\frac{1-G(x)}{1-\beta^{\alpha}}\right)^{\theta}\right)^{i-1}, & G^{-1}(\beta) \leq x, \end{cases}
$$

where $A = \frac{n!\alpha\theta}{(i-1)!(n-i)!}$.

The rth moment of the ith order statistic is

$$\begin{aligned} E(X_{i:n}^r) &= A\left\{\sum_{t=0}^{n-i}\sum_{k=0}^{i\theta-1}\frac{\binom{i\theta-1}{k}\binom{n-i}{t}}{(1-(1-\beta)^{\alpha})^{i\theta+t}}\beta^{t+2}(-1)^k(-1)^t\right. \\ &\times\int_0^{\beta}\left(G^{-1}(y)\right)^r(1-y)^{\alpha(k+1)-1}\left(1-(1-y)^{\alpha}\right)^t dy \\ &+\sum_{t=0}^{i-1}\sum_{k=0}^{\theta(n-i+1)-1}\frac{\binom{\theta(n-i+1)-1}{k}\binom{i-1}{t}}{(1-\beta^{\alpha})^{\theta(t+n-i+1)}}(1-\beta)^{t+2}(-1)^k(-1)^t \\ &\left.\times\int_{\beta}^{1}\left(G^{\alpha-1}(y)\right)^r y^{\alpha\theta(n-i+1)-\alpha-1}(1-y)^{t\theta}dy\right\}. \end{aligned}$$

4. A generalized two-sided Kumaraswamy exponential distribution

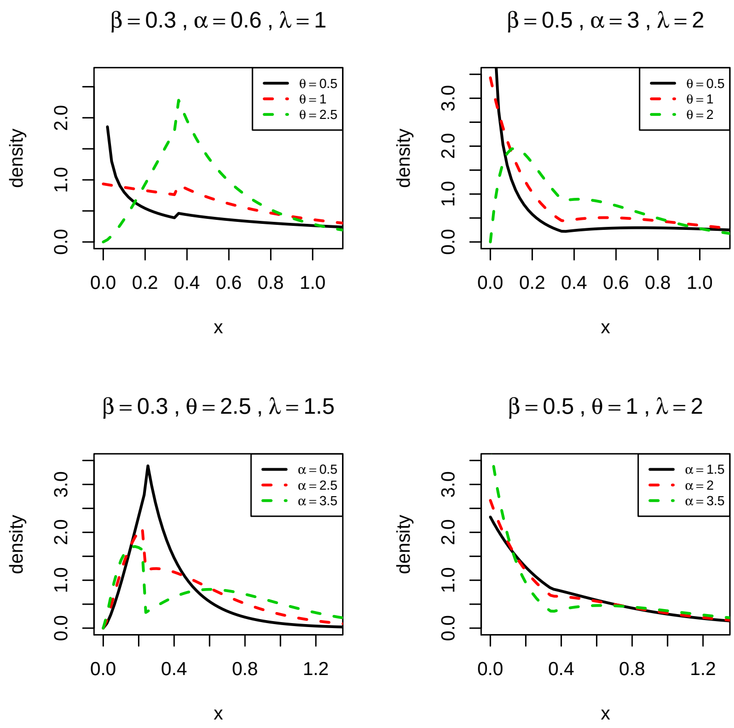

Figure 2. The graphs of the density of the GTSKWE distribution for selected parameter values.

Here, we consider the exponential distribution as a parent distribution with *cdf* and inverse *cdf* functions $G(x;\lambda) = 1 - e^{-\lambda x}, x > 0, \lambda > 0$ and $G^{-1}(x;\lambda) = -\frac{1}{\lambda}\log(1-x)$, respectively. The *pdf* of the parent distribution is $g(x;\lambda) = \lambda e^{-\lambda x}$. By considering this distribution, the *pdf* of the generalized two-sided Kumaraswamy

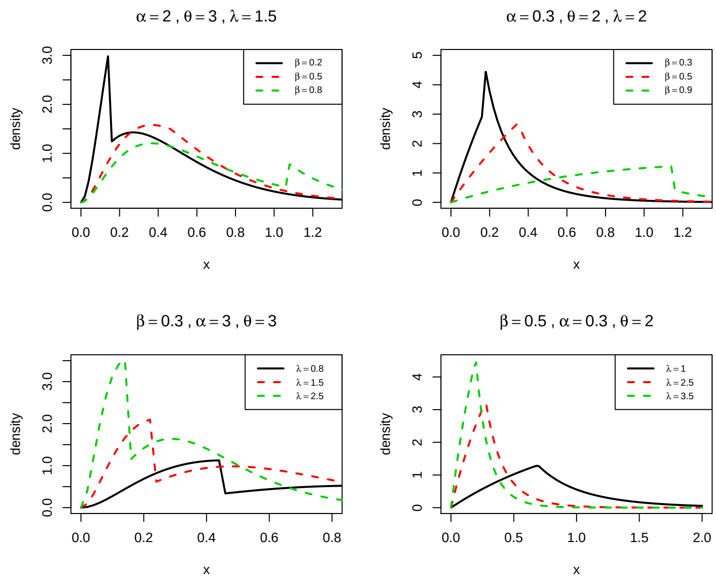

Figure 3. The graphs of the density of the GTSKWE distribution for selected parameter values.

exponential distribution $(GTSKWE)$ can be written as

$$f(x;\alpha,\beta,\theta,\lambda)=\begin{cases}\alpha\,\dfrac{\theta\,\lambda\,\beta\,e^{-\alpha\lambda x}}{1-(1-\beta)^{\alpha}}\left(\dfrac{1-e^{-\alpha\lambda x}}{1-(1-\beta)^{\alpha}}\right)^{\theta-1}, & 0<x\leq-\frac{1}{\lambda}\log(1-\beta),\\ \alpha\frac{\theta(1-\beta)\lambda\,e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{(1-\beta^{\alpha})}\left(\dfrac{1-(1-e^{-\lambda x})^{\alpha}}{1-\beta^{\alpha}}\right)^{\theta-1}, & -\frac{1}{\lambda}\log(1-\beta)\leq x<\infty,\end{cases} \tag{15}$$

and its cdf is

$$F(x;\alpha,\beta,\theta,\lambda)=\begin{cases}\beta\left(\dfrac{1-e^{-\alpha\lambda x}}{1-(1-\beta)^{\alpha}}\right)^{\theta}, & 0<x\leq-\frac{1}{\lambda}\log(1-\beta),\\ 1-(1-\beta)\left(\dfrac{1-\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\beta^{\alpha}}\right)^{\theta}, & -\frac{1}{\lambda}\log(1-\beta)\leq x<\infty.\end{cases} \tag{16}$$

The shapes of the $GTSKWE$ distribution for different values of the parameters are plotted in Figures 2 and 3. These plots indicate that the $GTSKWE$ distribution can be bimodal and unimodal.

4.1. Moments of the $GTSKWE$ distribution

In this subsection, moments and related measures including coefficients of variation, skewness and kurtosis are presented. The rth moment of the $GTSKWE$ distribution is denoted by μ'_r and is given by

$$\begin{aligned}\mu'_r=E(X^r)\quad&=\quad\frac{\alpha\theta\beta}{(1-\beta^{\alpha})^{\theta}}\int_0^{\beta}\left(-\frac{1}{\lambda}\log(1-y)\right)^{r}(1-y)^{\alpha-1}\left(1-(1-y)^{\theta-1}\right)dy\\&\quad+\frac{\alpha\theta(1-\beta)}{(1-\beta^{\alpha})^{\theta}}\int_{\beta}^{1}\left(-\frac{1}{\lambda}\log(1-y)\right)^{r}y^{\alpha-1}\left(1-y^{\alpha}\right)^{\theta-1}dy.\end{aligned}$$

The variance, CV, CS and CK are given by

$$\sigma^2=\mu'_2-\mu^2,\quad CV=\frac{\sigma}{\mu}=\frac{\sqrt{\mu'_2-\mu^2}}{\mu}=\sqrt{\frac{\mu\prime_2}{\mu^2}-1}, \tag{17}$$

$$CS = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \tag{18}$$

and

$$CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2} \tag{19}$$

respectively. Table 1 lists the first six moments of the $GTSKWE$ distribution for selected values of the parameters by fixing $\beta = 0.3$ and $\alpha = 0.5$. Table 2 lists the first six moments of the $GTSKWE$ distribution and the corresponding quantities, SD, CV, CS and CK for selected values of the parameters by fixing $\theta = 2$ and $\lambda = 0.5$. These values can be determined numerically using R software. In the next subsection, we consider the shape of the hazard rate function of $GTSKWE$ distribution.

Table 1. Moments of the GTSKWE distribution for some parameter values $\beta = 0.3, \alpha = 0.5$.

μ_r'	$\theta = 1.5,\ \lambda = 0.5$	$\theta = 1.5,\ \lambda = 1$	$\theta = 1,\ \lambda = 1.5$	$\theta = 2,\ \lambda = 2$
μ_1'	1.406269	0.7031344	0.6052671	0.3028411
μ_2'	3.435218	0.8588045	0.7443232	0.1415015
μ_3'	12.78225	1.597782	1.427548	0.0966633
μ_4'	65.30961	4.081851	3.733302	0.0903544
μ_5'	425.0348	13.28234	12.33200	0.1081604
μ_6'	3354.346	52.41166	49.11631	0.1579032
SD	1.2073216	0.6036609	0.6147967	0.2231340
CV	0.8585282	0.8585285	1.0157444	0.7368021
CS	2.189524	2.189612	2.236865	2.134713
CK	10.55565	10.55624	10.58074	10.42958

Table 2. Moments of the GTSKWE distribution for some parameter values $\theta = 2, \lambda = 0.5$.

μ_r'	$\beta = 0.3,\ \alpha = 0.5$	$\beta = 0.3,\ \alpha = 1$	$\beta = 0.5,\ \alpha = 1.5$	$\beta = 0.8,\ \alpha = 2$
μ_1'	1.211373	1.335517	1.65086	1.87327
μ_2'	2.264065	2.825710	4.08400	5.661233
μ_3'	6.189064	8.558456	13.65346	22.72128
μ_4'	23.13038	34.27972	58.07806	110.1745
μ_5'	110.7562	171.4248	301.475	620.6141
μ_6'	646.7746	1028.569	1852.428	3995.993
SD	0.8925472	1.0208351	1.1656163	1.4670012
CV	0.7368062	0.7643745	0.7060661	0.7831232
CS	2.132663	1.880938	1.531526	1.683802
CK	10.42636	8.522673	6.726908	4.788111

4.2. Hazard function of $GTSKWE$ distribution

The hazard function of $GTSKWE$ distribution is

$$
\begin{aligned}
r(x) &= \frac{f(x)}{1-F(x)} \\
&= \begin{cases} \frac{\alpha\theta\lambda\beta(e^{-\lambda x})^{\alpha}(1-(e^{-\lambda x})^{\alpha})^{\theta-1}}{(1-(1-\beta)^{\alpha})^{\theta}-\beta(1-(e^{-\lambda x})^{\alpha})^{\theta}}, & 0 < x \leq -\frac{1}{\lambda}\log(1-\beta), \\ \frac{\alpha\theta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}}{1-(1-e^{-\lambda x})^{\alpha}}, & -\frac{1}{\lambda}\log(1-\beta) \leq x < \infty. \end{cases}
\end{aligned} \tag{20}
$$

In the next section, we consider the shape of the hazard rate function of $GTSKWE$ distribution. Because of complicated form of the hazard function, we couldn't explore this function analytically. Some shapes of hazard function for the selected values of the parameters are given in Figure 4. It can be seen that the proposed distribution has various hazard rate shapes such as IFR, DFR, BUT and UBT.

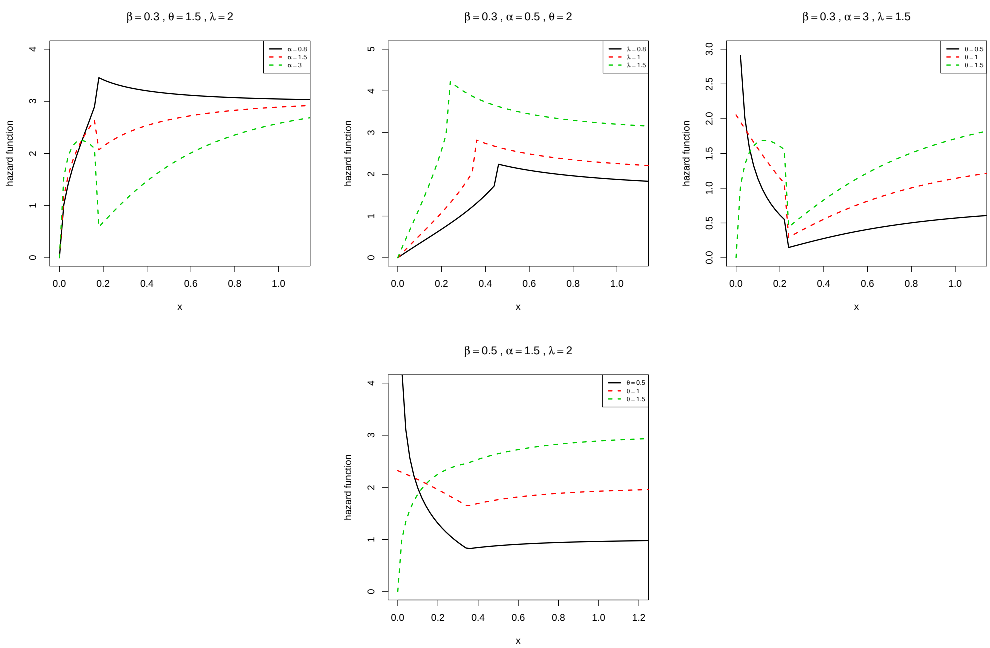

Figure 4. The graphs of the hazard function of the GTSKWE distribution for selected parameter values.

5. Estimation of the parameters of $GTSKWE$

In this section, we obtain the estimates of the parameters of $GTSKWE$ via two methods: maximum likelihood and bootstrap methods. The maximum likelihood procedure is one of the most common methods for obtaining an estimator for an unknown parameter in classical statistical inference. Here, the structure of the likelihood function is expressed for two modes of observations including complete and right-censored data sets.

5.1. Maximum likelihood estimation for complete data

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $GTSKWG$ distribution and $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics. The log-likelihood function is given by

$$\begin{aligned}
l(\underline{x}, \alpha, \beta, \theta) &= n \log \alpha\, \theta + r \log \beta + (n-r)\log(1-\beta) + \sum_{i=1}^{n} \log g(x_{i:n}; \xi) \\
&+ (\theta-1)\log\left(\left(1-(1-\beta)^{\alpha}\right)^{r}(1-\beta)^{n-r}\right) - \theta \log\left((1-\beta^{\alpha})^{n-r}(1-(1-\beta)^{\alpha})^{r}\right) \\
&+ (\alpha-1)\log\left(\frac{\prod_{i=1}^{r}(1-G(x_{i:n};\xi))\prod_{i=r+1}^{n} G(x_{i:n};\xi)}{(1-\beta)^{r}\,\beta^{n-r}}\right) \\
&+ (\theta-1)\log\left(\frac{\prod_{i=1}^{r}(1-(1-G(x_{i:n};\xi))^{\alpha})\prod_{i=r+1}^{n}(1-G(x_{i:n};\xi))}{(1-(1-\beta)^{\alpha})^{r}(1-\beta)^{n-r}}\right) \\
&+ (\alpha-1)\log\left((1-\beta)^{r}\beta^{n-r}\right).
\end{aligned}$$

where $x_{r:n} \leq G^{-1}_{(x;\xi)}(\beta) < x_{r+1:n}$ for $r = 1, 2, \ldots, n$ and $X_{0:n} = -\infty$, $X_{n+1:n} = \infty$.

For estimating the parameters, we obtain the partial derivatives of the log-likelihood function with respect to the parameters. At the corner point β, the log-likelihood function for the $GTSKWE$ distribution is not differentiable and we can not find the estimate of β in a regular way. But, we can obtain the maximum likelihood estimates (MLEs) of parameters using Van Dorp and Kotz (2002a) method. For this, we first consider the MLEs of θ and β when the parameters α and ξ are known. Without loss of generality, we assume that $\alpha = 1$. So, the log-likelihood function will become

$$\begin{aligned}
\ell(\underline{x}, \alpha=1, \beta, \theta, \xi) &= n\log\theta + \sum_{i=1}^{n}\log(g(x_i);\xi) + \log\left\{\prod_{i=1}^{r}\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\theta-1}\prod_{i=r+1}^{n}\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\theta-1}\right\} \\
&= n\log\theta + \sum_{i=1}^{n}\log(g(x_i;\xi)) + (\theta-1)\log\left\{\frac{\prod_{i=1}^{r} G(x_{i:n};\xi)\prod_{i=r+1}^{n}(1-G(x_{i:n};\xi))}{\beta^{r}(1-\beta)^{n-r}}\right\}.
\end{aligned}$$

According to Van Dorp and Kotz (2002a) and Korkmaz and Genç (2017), the MLEs of θ and β are as follows

$$\hat{\theta} = -\frac{n}{\log M(\hat{r},\xi)}, \quad \hat{\beta} = G(x_{\hat{r}:n};\xi),$$

where $\hat{r} = argmaxM(r,\xi)$, $r \in \{1, 2, \ldots, n\}$ with

$$M(r,\xi) = \prod_{i=1}^{r-1}\frac{G(x_{i:n};\xi)}{G(x_{r:n};\xi)}\prod_{r+1}^{n}\frac{1-G(x_{i:n};\xi)}{1-G(x_{r:n};\xi)}.$$

By taking the derivative of the log-likelihood function with respect to the parameter vector ξ and parameter α, the MLEs of the parameters ξ and α are obtained by equating it to zero. Thus the partial derivatives of the log-likelihood function with respect to the parameter vector ξ and parameter α are given by

$$\begin{aligned}
\frac{\partial l(\underline{x}, \alpha, \beta, \theta, \xi)}{\partial \alpha} &= \frac{n}{\alpha} + \log\left((1-\beta)^{r}\beta^{n-r}\right) \\
&+ (\theta-1)\sum_{i=1}^{n}\frac{(-\log(1-G(x_{i:n};\xi)))\,(1-G(x_{i:n};\xi))^{\alpha}}{(1-(1-G(x_{i:n};\xi))^{\alpha})} \\
&- (n-r)\theta\,\frac{(-(\log\beta)\beta^{\alpha})}{1-\beta^{\alpha}} - \theta r\frac{(-\log(1-\beta))(1-\beta)^{\alpha}}{1-(1-\beta)^{\alpha}} \\
&+ \frac{\log\left(\prod_{i=1}^{r}(1-G(x_{i:n};\xi))\prod_{i=r+1}^{n}G(x_{i:n};\xi)\right)}{(1-\beta)^{r}\beta^{r}},
\end{aligned}$$

$$\begin{aligned}\frac{\partial l(\underline{x},\alpha,\beta,\theta,\xi)}{\partial \xi_k} &= \sum_{i=1}^{n}\frac{g'(x_{i:n};\xi)}{g(x_{i:n};\xi)}+(\alpha-1)\left(\sum_{i=1}^{\hat{r}}\frac{-g(x_{i:n};\xi)}{1-G(x_{i:n};\xi)}+\sum_{i=\hat{r}+1}^{n}\frac{g(x_{i:n};\xi)}{G(x_{i:n};\xi)}\right)\\ &+(\theta-1)\left(\sum_{i=1}^{\hat{r}}\frac{\alpha g(x_{i:n})(1-G(x_{i:n};\xi))^{\alpha-1}}{1-(1-G(x_{i:n};\xi)^{\alpha})}+\sum_{i=\hat{r}+1}^{n}\frac{-g(x_{i:n};\xi)}{(1-G(x_{i:n};\xi))}\right),\end{aligned}$$

where $g'(t;\xi)=\frac{\partial g(t;\xi)}{\partial \xi_k}$ and $G'(t;\xi)=\frac{\partial G(t;\xi)}{\partial \xi_k}$.

Setting the above two differentiations equal to zero and solving simultaneously for α and ξ, we can obtain the maximum likelihood estimators of the parameters α and ξ.

5.2. Maximum likelihood estimation for right censored data

In real life situation, sometimes it is hard to obtain a complete data set. Often with lifetime data, one encounters censoring. There are different forms of censoring: type I, type II, etc. Let

$$(X_1,\delta_1),\ (X_2,\delta_2),\ldots,(X_n,\delta_n)$$

be a right-censored random sample of size n from $GTSKWG$ distribution. Note that δ_i is a censoring indicator variable, that is, $\delta_i=1$ for an observed survival time and $\delta_i=0$ for a right-censored survival time. The likelihood function based on right censored sample is given by

$$L(\underline{x},\underline{\delta},\alpha,\beta,\theta,\xi) = \begin{cases}\prod_{i=1}^{n}\left[\alpha\theta\beta\left(\frac{g(x_i)(1-G(x_i))^{\alpha-1}}{1-(1-\beta)^{\alpha}}\right)\left(\frac{1-(1-G(x_i))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta-1}\right]^{\delta_i}\left[1-\beta\left(\frac{1-(1-G(x_i))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta}\right]^{1-\delta_i}\\ \prod_{i=1}^{n}\left[\alpha\theta\left(\frac{(1-\beta)g(x_i)(G(x_i))^{\alpha-1}}{1-\beta^{\alpha}}\right)\left(\frac{1-(G(x_i))^{\alpha}}{1-\beta^{\alpha}}\right)\right]^{\delta_i}\left[(1-\beta\left(\frac{1-(G(x_i))^{\alpha}}{1-\beta^{\alpha}}\right))\right]^{1-\delta_i}\end{cases}.$$

The corresponding log-likelihood function is

$$\begin{aligned} l(\underline{x},\underline{\delta},\alpha,\beta,\theta,\xi) &= \log L(\underline{x},\underline{\delta},\alpha,\beta,\theta,\xi)\\ &= \begin{cases}\sum_{i=1}^{n}\delta_i\log\left[\alpha\theta\beta\left(\frac{g(x_i)(1-G(x_i))^{\alpha-1}}{1-(1-\beta)^{\alpha}}\right)\left(\frac{1-(1-G(x_i))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta-1}\right]\\ +\sum_{i=1}^{n}(1-\delta_i)\log\left[1-\beta\left(\frac{1-(1-G(x_i))^{\alpha}}{1-(1-\beta)^{\alpha}}\right)^{\theta}\right]\\ \\ \sum_{i=1}^{n}\delta_i\log\left[\alpha\theta\left(\frac{(1-\beta)g(x_i)(G(x_i))^{\alpha-1}}{1-\beta^{\alpha}}\right)\left(\frac{1-(G(x_i))^{\alpha}}{1-\beta^{\alpha}}\right)\right]\\ +\sum_{i=1}^{n}(1-\delta_i)\left[1-\beta\left(\frac{1-(G(x_i))^{\alpha}}{1-\beta^{\alpha}}\right)\right].\end{cases}\end{aligned}$$

Here, we can derive normal equations of the log-likelihood function in a similar way to complete sample data. In practice, due to the non-linearity of corresponding normal equations, we can use numerical algorithms as well as for complete data case to extract MLE estimators.

5.3. Bootstrap estimation

The uncertainty in the parameters of the fitted distribution can be estimated by parametric (re-sampling from the fitted distribution) or non-parametric (re-sampling with replacement from the original data set) bootstraps re-sampling methods (Efron and Tibshirani [4]). These two parametric and nonparametric bootstrap procedures are based on maximum likelihood estimation procedure as follows.

Parametric bootstrap procedure:

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, employing the MLE procedure based on a random sample.
2. Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$, using θ and obtain the bootstrap estimate of θ, say $\widehat{\theta^*}$, from the bootstrap sample based on the MLE procedure.
3. Repeat Step 2 $NBOOT$ times.
4. Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In case of the $GTSKWE$ distribution, the parametric bootstrap estimators (PBs) of α, β, λ and θ are $\hat{\alpha}_{PB}, \hat{\beta}_{PB}, \hat{\lambda}_{PB}$ and $\hat{\theta}_{PB}$, respectively.

Nonparametric bootstrap procedure

1. Generate a bootstrap sample $\{X_1^*, \ldots, X_m^*\}$ with replacement from the original data set. Obtain the bootstrap estimate of θ with MLE procedure, say $\widehat{\theta^*}$ using the bootstrap sample.
2. Repeat Step 2 $NBOOT$ times.
3. Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In case of the $GTSKWE$ distribution, the nonparametric bootstrap estimators (NPBs) of α, β, λ and θ, are $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}, \hat{\lambda}_{NPB}$ and $\hat{\theta}_{NPB}$, respectively.

5.4. Simulation

Here, we assess the performance of the MLE's of the parameters with respect to sample size n for the $GTSKWE$ distribution. The assessment of the performance of the MLEs of the parameters is based on a simulation study via the Monte Carlo method. Let $\hat{\beta}, \hat{\alpha}, \hat{\theta}$ and $\hat{\lambda}$ be the MLEs of the parameters β, α, θ and λ, respectively. We calculate the mean square error (MSE) and bias of the MLEs of the parameters β, α, θ and λ based on the simulation results of 2000 independence replications.

Some results associated with simulation performance of maximum likelihood procedure are summarized in Table 3 based on different values of β, α, θ and λ. It is evident from Table 3 that as the sample size increases, the MSE measure decreases which verifies the consistency property of all the estimators.

6. Application of $GTSKWE$ distribution

The goal of this section is to show the importance of the $GTSKWE$ model based on two real data sets. The fits of the $GTSKWE$ distribution will be compared with some competitive models, listed in Tables 6 and 7. Furthermore, in this section, we provide bootstrap analysis of the parameter estimation of $GTSKWE$ model for these data sets. The following data sets contain two modes of real world observations: complete and right-censored.

Complete data set: Bjerkedal [3] provides a data set consists of survival times of 72 guinea pigs injected with different amount of tubercle. This species of Guinea pigs are known to have high susceptibility of human tuberculosis, which is one of the reasons for choosing. For more details, see also Altun et al. [1] and Korkmaz (2017). We consider only the study in which animals in a single cage are under the same regimen. The data represents the survival times of Guinea pigs in days. These data sets are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

Censored data set: the survival times in months of 100 patients who have been infected by HIV were provided by Hosmer and Lemeshow [6], where the plus sign in the data indicates a right-censored time.

5 6+ 8 3 22 1+ 7 9 3 12 2+ 12 1 15 34 1 4 19+ 3+ 2 2+ 6 60+ 7+ 60+ 11 2+ 5 4+ 1+ 13 3+ 2+ 1+ 30 7+ 4+ 8+ 5+ 10 2+ 9+ 36 3+ 9+ 3+ 35 8+ 1+ 5+ 11 56+ 2+ 3+ 15 1+ 10 1+ 7+ 3+ 3+ 2+ 32 3+ 10+ 11 3+ 7+ 5+ 31 5+ 58 1+ 2+ 1 3+ 43 1+ 6+ 53 14 4+ 54 1+ 1+ 8+ 5+ 1+ 1+ 2+ 7+ 1+ 10 24+ 7+ 12+ 4+ 57 1+ 12+.

For this data set, there are 37 uncensored time and 63 right censored time.

Table 3: The MSE and bias (values in parentheses) of the MLE of the parameters α, β, λ and θ.

		$\alpha = 1$	$\beta = 0.3$	$\theta = 0.5$	$\lambda = 0.5$
n	30	3.0795 (1.0528)	0.0104 (0.0328)	0.1031 (0.1098)	0.0722 (0.0801)
	50	1.5408 (0.6387)	0.0093 (0.0240)	0.0575 (0.0416)	0.0402 (0.0395)
	100	0.7733 (0.3617)	0.0129 (0.0143)	0.0273 (0.0115)	0.0207 (0.0151)
	200	0.3954 (0.1970)	0.0145 (0.0086)	0.0151 (0.0057)	0.0114 (0.0074)
		$\alpha = 1$	$\beta = 0.3$	$\theta = 1$	$\lambda = 0.5$
n	30	2.0861 (0.4302)	0.0276 (0.0457)	0.3225 (1.3588)	0.0078 (0.0405)
	50	1.2885 (0.2940)	0.0209 (0.0378)	0.2179 (0.8657)	0.0059 (0.0266)
	100	0.7175 (0.1398)	0.0310 (0.0319)	0.0774 (0.0722)	0.0051 (0.0112)
	200	0.4277 (0.0846)	0.0246 (0.0255)	0.0382 (0.0262)	0.0075 (0.0053)
		$\alpha = 1.5$	$\beta = 0.4$	$\theta = 2$	$\lambda = 1$
n	30	2.0318 (0.4907)	0.0236 (0.0393)	0.9270 (4.6550)	0.0245 (0.0689)
	50	1.2864 (0.3127)	0.0242 (0.0284)	0.5017 (0.9618)	0.0154 (0.0377)
	100	0.7109 (0.1122)	0.0154 (0.0156)	0.2274 (0.3478)	0.0069 (0.0190)
	200	0.3778 (0.1026)	0.0141 (0.0091)	0.1296 (0.1435)	0.0081 (0.0098)
		$\alpha = 0.5$	$\beta = 0.4$	$\theta = 2$	$\lambda = 1$
n	30	1.6179 (0.2953)	0.0247 (0.0310)	0.7691 (7.3403)	0.0467 (0.1384)
	50	1.1465 (0.2263)	0.0235 (0.0212)	0.4172 (1.2764)	0.0351 (0.0804)
	100	0.5732 (0.0607)	0.0177 (0.0116)	0.1566 (0.3544)	0.0213 (0.0421)
	200	0.3119 (-0.0165)	0.0101 (0.0058)	0.0438 (0.1256)	0.0099 (0.0193)
		$\alpha = 0.5$	$\beta = 0.5$	$\theta = 2$	$\lambda = 1$
n	30	1.6948 (0.3312)	0.0199 (0.0337)	0.7881 (4.3240)	0.0612 (0.1253)
	50	1.1680 (0.2592)	0.0117 (0.0223)	0.4704 (1.5170)	0.0369 (0.0778)
	100	0.6202 (0.1243)	0.0073 (0.0128)	0.1998 (0.4140)	0.0168 (0.0390)
	200	0.3619 (0.0699)	0.0069 (0.0069)	0.0979 (0.1765)	0.0142 (0.0207)
		$\alpha = 1.5$	$\beta = 0.6$	$\theta = 2$	$\lambda = 1$
n	30	1.8169 (0.5251)	0.0444 (0.0320)	0.8532 (2.4150)	0.0494 (0.0613)
	50	1.1296 (0.3042)	0.0349 (0.0235)	0.4833 (0.9301)	0.0331 (0.0326)
	100	0.6936 (0.2003)	0.0238 (0.0143)	0.2609 (0.3642)	0.0219 (0.0153)
	200	0.3746 (0.1169)	0.0127 (0.0083)	0.1390 (0.1472)	0.0126 (0.0080)
		$\alpha = 1.5$	$\beta = 0.4$	$\theta = 1$	$\lambda = 1.5$
n	30	2.5836 (1.0492)	0.0384 (0.0581)	0.4278 (0.7292)	0.0317 (0.1922)
	50	1.5645 (0.7943)	0.0453 (0.0487)	0.2448 (0.2002)	0.0120 (0.1057)
	100	0.6930 (0.4503)	0.0486 (0.0362)	0.1219 (0.0574)	0.0055 (0.0462)
	200	0.3639 (0.3187)	0.0362 (0.0255)	0.0731 (0.0238)	0.0083 (0.0224)

6.1. Bootstrap inference for $GTSKWE$ parameters

In this subsection, we obtain point and 95% confidence interval (CI) estimation of the $GTSKWE$ parameters by parametric and non-parametric bootstrap methods. We provide results of bootstrap estimation in Table 4 for complete data set. Also Table 5 shows the non-parametric bootstrap estimation for the right-censored data set. The joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters are shown in Figures 5 and 6.

Table 4: Bootstrap point and interval estimation of the parameters β, α, λ and θ for survival times of guinea pigs data set.

	parametric bootstrap		non-parametric bootstrap	
	point estimation	CI	point estimation	CI
β	0.946	(0.689,0.986)	0.218	(0.141,0.274)
α	0.349	(0.004,1.409)	0.125	(0.002,0.496)
θ	1.627	(1.061,2.241)	2.151	(1.633,2.750)
λ	0.009	(0.005,0.013)	0.004	(0.003,0.006)

Table 5: Bootstrap point and interval estimation of the parameters β, α, λ and θ for the HIV data set.

	non-parametric bootstrap	
	point estimation	CI
β	0.008	(0.004,0.018)
α	1.482	(1.104,2.029)
θ	8.971	(5.095,30.719)
λ	0.008	(0.003,0.012)

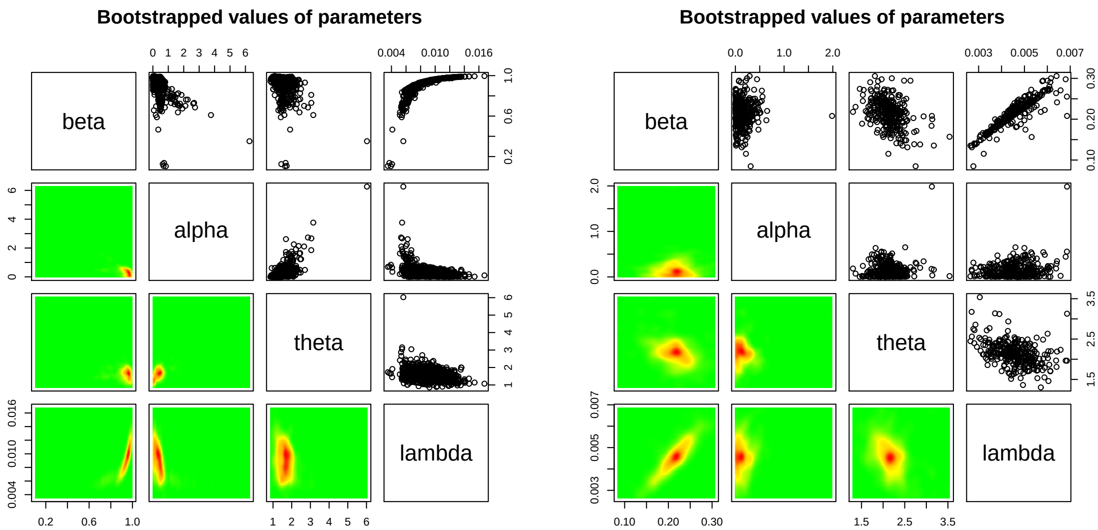

Figure 5. Parametric (left) and non-parametric (right) bootstraped values of parameters of the GTSKWE distribution for the survival times of guinea pigs data set data.

6.2. Model comparisons

We fit the proposed distribution to these two data sets by ML and bootstrap methods and compare the results with the gamma, Weibull, two-sided generalized exponential ($TSGE$) by Korkmaz and Genç [13], transmuted two-sided generalized exponential ($TTSGE$) by Kharazmi and Zargar [7], generalized exponential (GE) and weighted exponential (WE) distributions with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}, \qquad x > 0$$

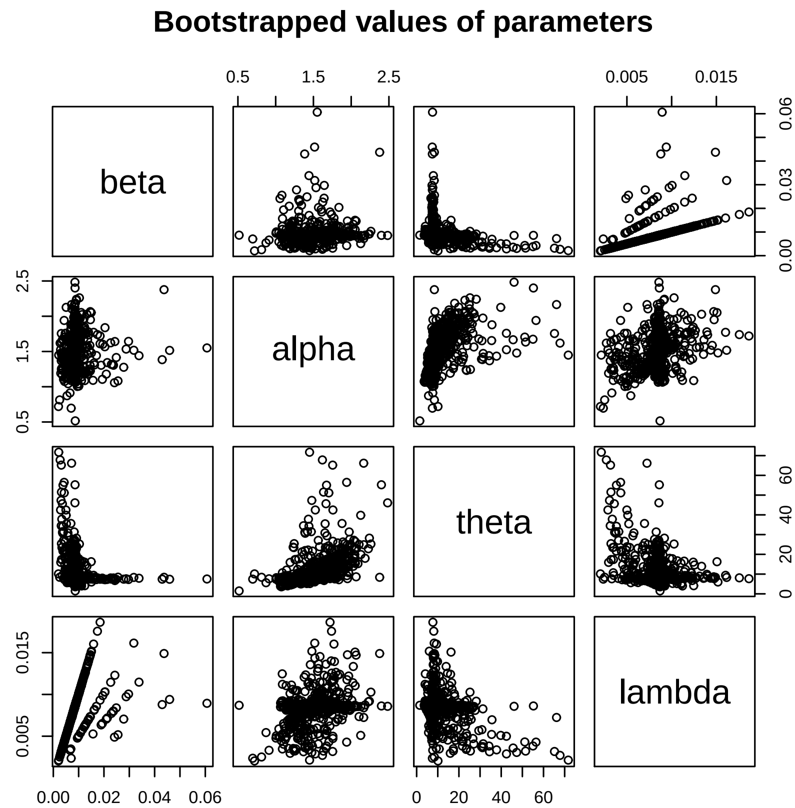

Figure 6. Non-parametric bootstraped values of parameters of the GTSKWE distribution for the HIV data set.

$$f_{Weibull}(x) = \frac{\beta}{\lambda^{\beta}} x^{\beta-1} e^{-(\frac{x}{\lambda})^{\beta}}, \qquad x > 0$$

$$f_{TSGE}(x) = \begin{cases} \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} (\frac{1-e^{\frac{-x}{\theta}}}{\beta})^{\alpha-1}, & 0 < x \leq -\theta \log(1-\beta), \\ \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} (\frac{e^{\frac{-x}{\theta}}}{1-\beta})^{\alpha-1}, & -\theta \log(1-\beta) \leq x < \infty, \end{cases}$$

$$f_{TTSGE}(x) = \begin{cases} \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} \left((1+\lambda) \left(\frac{1-e^{-\frac{x}{\theta}}}{\beta} \right)^{\alpha-1} - 2\lambda \left(\frac{1-e^{-\frac{x}{\theta}}}{\beta} \right)^{2\alpha-1} \right), & 0 < x \leq -\theta \log(1-\beta), \\ \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} \left((1+\lambda) \left(\frac{e^{-\frac{x}{\theta}}}{1-\beta} \right)^{\alpha-1} - 2\lambda \left(\frac{e^{-\frac{x}{\theta}}}{1-\beta} \right)^{2\alpha-1} \right), & -\theta \log(1-\beta) \leq x < \infty, \end{cases}$$

$$f_{GE}(x) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}, \qquad x > 0$$

$$f_{WE}(x) = \frac{\alpha+1}{\alpha} \lambda e^{-\lambda x} (1 - e^{-\alpha \lambda x}), \qquad x > 0.$$

Table 6 shows the MLEs of the parameters, Kolmogorov-Smirnov (KS) distance between the empirical distribution and the fitted model, its corresponding p-value, log-likelihood and Akaike information criterion

(AIC) for the real data set. We fit the $GTSKWE$ distribution to the real data set and compare it with the gamma, generalized exponential (GE), Weibull, two-sided generalized exponential ($TSGE$), transmuted two-sided generalized exponential ($TTSG-E$) and weighted exponential (WE) distributions. The selection criterion is that the lowest AIC and $K-S$ values correspond to the best t model. Thus, the $GTSKWE$ distribution provides the best t for the data set as it shows the lowest AIC and $K-S$ values than the other competing models. The relative histograms, fitted $GTSKWE$, gamma, generalized exponential (GE), Weibull, two-sided generalized exoponential ($TSGE$), transmuted two-sided generalized exponential ($TTSG-E$) and weighted exponential (WE) PDFs for the real data set are plotted in Figure 7. The plots of empirical and fitted survival functions, P-P plots and Q-Q plots for the $GTSKWE$, and other fitted distributions are displayed in Figures 7 and 8, respectively. These plots also support the results in Table 6. Also Table 7 shows the MLEs of the parameters, log-likelihood, Akaike information criterion (AIC) for the censored data set. The $GTSKWE$ distribution provides the best fit for the HIV data set as it provides the lowest AIC than the other considered models. Also, Figure 9 represents the empirical and fitted cumulative probability functions for the right-censored observations.

Table 6. Goodness-of-fit statistics and MLEs of the parameters for survival times of guinea pigs data set.

Model	Estimation	Log-likelihood	AIC	K-S	P-value
GTSKWE	$(\hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$= (0.219, 0.125, 2.183, 0.004)	-385.781	779.562	0.09	0.587
TTSGE	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$=(1.950, 0.303, -0.423, 160.67)	-388.063	784.127	0.097	0.508
gamma	$(\hat{\alpha}, \hat{\lambda})$=(2.812, 0.020)	-394.247	792.495	0.138	0.127
Weibull	$(\hat{\beta}, \hat{\lambda})$=(1.392, 110.529)	-397.147	798.295	0.146	0.091
TSGE	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$=(2.561, 0.270, 177.911)	-389.549	785.099	0.130	0.171
WE	$(\hat{\alpha}, \hat{\lambda})$=(1.626, 0.0138)	-393.568	791.138	0.117	0.274
GE	$(\hat{\alpha}, \hat{\lambda})$=(2.476, 0.017)	-393.110	790.220	0.133	0.159

Table 7. AIC and MLEs of the parameters for HIV data.

Model	Estimation	Log-likelihood	AIC
GTSKWE	$(\hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$= (0.008, 1.371, 7.346, 0.008)	-158.427	324.855
TTSGE	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$=(1.485, 0.769, 0.517, 28.006)	-161.786	331.572
gamma	$(\hat{\alpha}, \hat{\lambda})$=(1.306, 0.046)	-162.438	328.876
Weibull	$(\hat{\beta}, \hat{\lambda})$=(1.182, 29.514)	-162.631	329.263
TSGE	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$=(5.661, 0.006, 15.861)	-160.922	327.844
WE	$(\hat{\alpha}, \hat{\lambda})$=(15.767, 0.037)	-162.159	328.319
GE	$(\hat{\alpha}, \hat{\lambda})$=(1.325, 0.042)	-162.416	328.832

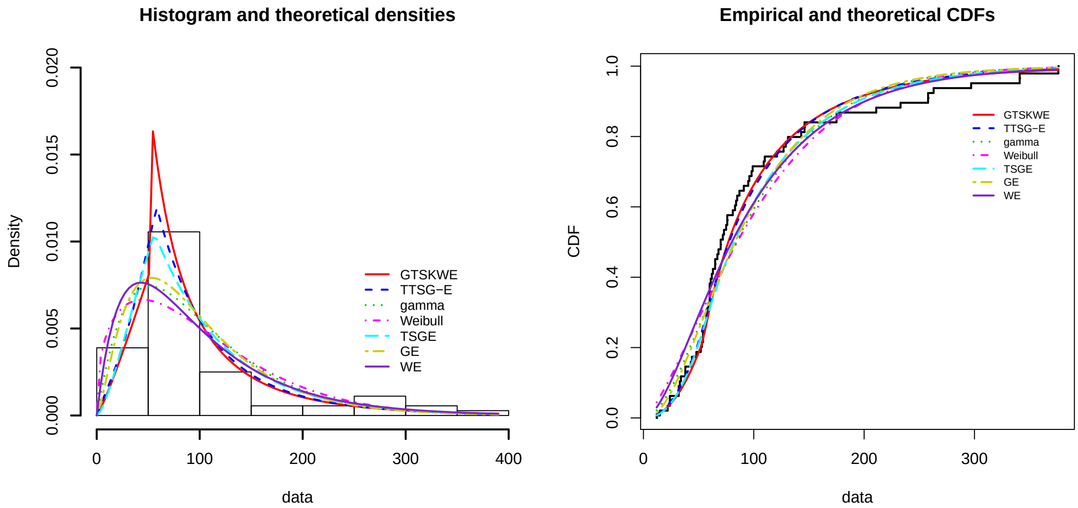

Figure 7. Estimated densities and Empirical and Estimated cdf for the data set.

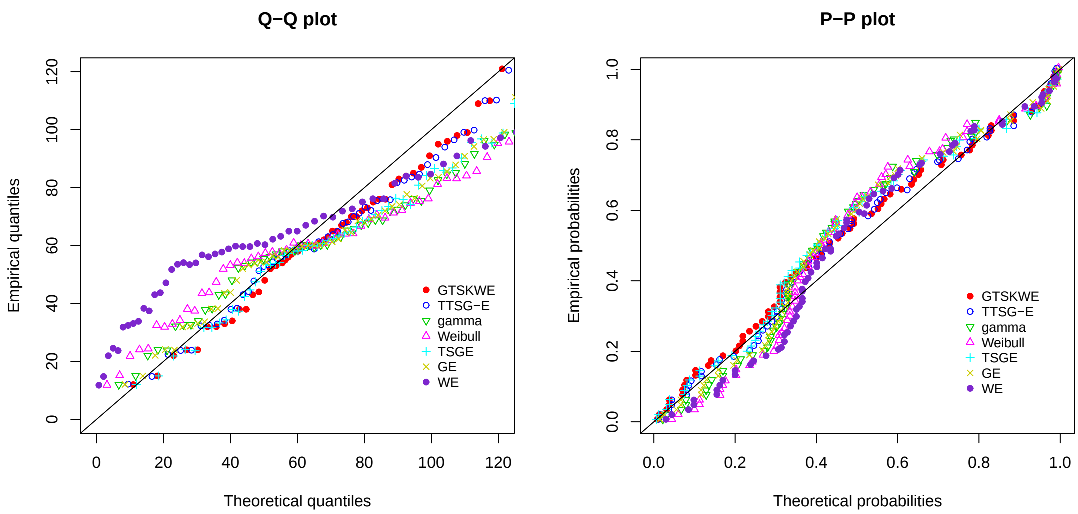

Figure 8. Q-Q and P-P plots for the data set.

7. Conclusion

In this article, a new two-sided family of lifetime distributions is introduced and its some basic properties are derived. One of the interesting and important properties of the proposed family is that, it contains the tow-sided and generalized two-sided distribution, as special cases, when the shape parameters tends to special values. A special example of this family is introduced by considering the exponential distribution as the baseline distribution. We also show that the proposed distribution has various hazard rate shapes such as increasing, decreasing, bathtub shape and upside-down bathtub shapes. Numerical results of maximum likelihood and bootstrap procedures for the

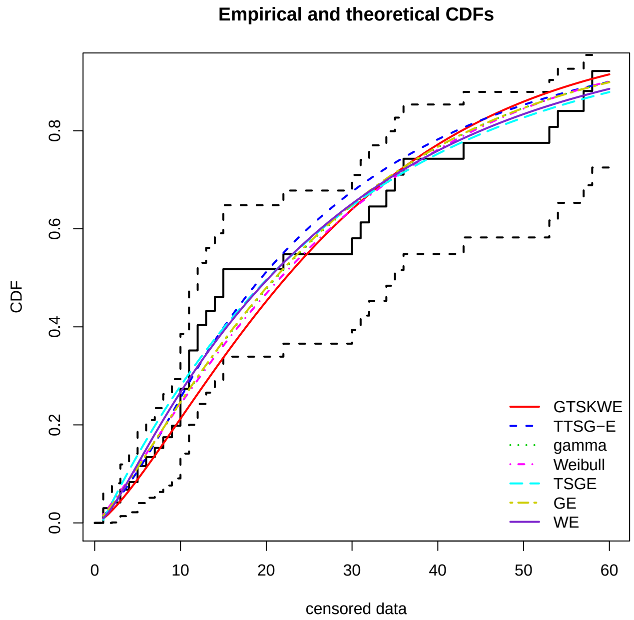

Figure 9. Plots of fitted $cdf\,s$ and $GTSKWE$ distribution for right-censored data set.

two real data sets are presented in separate tables. Data analysis shows that the proposed distribution provides a better fit than its sub-models and other common distributions for both complete and censored data sets. We expect the utility of the newly proposed model in different fields especially in reliability when the hazard rate is decreasing, increasing or unimodal (upside-down bathtub) and bathtub shaped.

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