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Moments of Generalized Order Statistics from Doubly Truncated Power Linear Hazard Rate Distribution

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Abstract This paper is concerned with some recurrence relations for single and product moments of doubly truncated power linear hazard rate distribution via generalized order statistics. Some deductions and related results are also considered. The characterization result is provided at the end.

Keywords Generalized order statistics, Power-liner hazard rate distribution, Single moments, Product moments, Characterization

AMS 2010 subject classifications 62E10, 62G30

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1. Introduction

The generalized order statistics (gos) is a scheme. It provides a broad structure for models of ordered random variables. [18] introduced this concept. It has been steadily growing and bringing out more consideration among researchers since its inception. Generalized order statistics is described in the following.
Let $n \geq 2$ be a given integer and, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j \geq 0 \text{ for } 1 \leq r \leq n-1.$$

Suppose $X(1, n, \tilde{m}, k), \cdots, X(n, n, \tilde{m}, k)$, are n *gos* from an absolutely continuous distribution function *df* $F()$ with probability density function *pdf* $f()$, if their joint *pdf* is given in Equation (1)

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} \left[\bar{F}(x_i) \right]^{m_i} f(x_i) \right) \left[\bar{F}(x_n) \right]^{k-1} f(x_n), \tag{1}$$

for $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$.
At the different values of m_i, γ_j and k, Equation (1) reduces to several models. For. e.g., order statistics ($m = 0$, $k = 1$), k^{th} record values ($m = -1$ [19]) and ($m = -1$, $k = 1$) corresponds to upper record values Chandler [10]. In reliability theory, these models play an important role. [11] and [14] provided the detail discussions on *gos*. In statistical analysis, the contribution of doubly truncated distributions is of importance. It covers many areas of study. It is applied in biostatistics, reliability (left truncation), survival analysis (right truncation) and cosmology (double truncation), and non-truncated case.

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Several publications are appeared on doubly truncated distributions in the literature. Detailed surveys are found in ([1, 2, 3, 4, 5, 7, 8, 12, 15, 16, 17]) and among others.
[6] introduced the power-linear hazard rate distribution (PLHRD) as follows.
A random variable (r.v.) $X \sim$ PLHRD(α, γ, δ), if its *cdf* and *pdf* are given respectively by

$$F(x) = 1 - e^{-\left\{\frac{\gamma}{2}x^2 + \frac{\alpha}{\delta+1}x^{\delta+1}\right\}}, \quad x > 0,\ \alpha, \gamma \geq 0,\ \delta > -1,\ \text{and}\ \delta \neq -1. \tag{2}$$

$$f(x) = (\alpha x^{\delta} + \gamma x)e^{-\left\{\frac{\gamma}{2}x^2 + \frac{\alpha}{\delta+1}x^{\delta+1}\right\}}, \quad x > 0. \tag{3}$$

The exponential, Rayleigh, Weibull, Linear hazard rate (LHR), Power hazard rate (PHR), and Quadratic hazard rate (QHR) are the special case of (4). The given distribution has tremendous applications in life testing, reliability, and other fields due to its desirable properties of the different hazard rate features. For more details see ([6]).
For given P_1 and Q_1

$$\int_0^{Q_1} f(x)dx = Q \qquad \text{and} \qquad \int_0^{P_1} f(x)dx = P.$$

Then, *pdf* of doubly truncated PLHRD is,

$$f_1(x) = \frac{(\alpha x^{\delta} + \gamma x)e^{-\left\{\frac{\gamma}{2}x^2 + \frac{\alpha}{\delta+1}x^{\delta+1}\right\}}}{P - Q}, \quad x \in (Q_1, P_1),\ \alpha, \gamma \geq 0, \delta > -1, \tag{4}$$

and the corresponding *df* $F_1(x)$ is

$$\bar{F}_1(x) = -P_2 + \frac{1}{\alpha x^{\delta} + \gamma x} f_d(x), \quad x \in (Q_1, P_1),\ \alpha, \gamma \geq 0, \delta > -1, \tag{5}$$

or

$$f_1(x) = (\alpha x^{\delta} + \gamma x)[P_2 + \bar{F}_1(x)] \tag{6}$$

where

$$P_2 = \frac{1-p}{p-Q}, \qquad Q_2 = \frac{1-Q}{p-Q}$$

$$P = = 1 - e^{-\left\{\frac{\gamma}{2}P_1^2 + \frac{\alpha}{\delta+1}P_1^{\delta+1}\right\}}, \quad Q = 1 - e^{-\left\{\frac{\gamma}{2}Q_1^2 + \frac{\alpha}{\delta+1}Q_1^{\delta+1}\right\}}.$$

The key intent of this research is to present the moments features of doubly TP-LHR distribution and characterization results.

2. Single Moments

Case I: $m_i = m_j = m$
In view of (1), the *pdf* of a single *gos*, is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)], \quad -\infty < x < \infty \tag{7}$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i,$$

and

$$g_m(x) = \begin{cases} \frac{1}{m+1}\left[1 - (1-x)^{m+}\right], & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases} \qquad x \in [0, 1).$$

Case II: $\gamma_i \neq \gamma_j,\, i \neq j$.
The *pdf* of $r-th$ *gos* is

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}\sum_{i=1}^{r} a_i(r) f(x)\left[\bar{F}(x)\right]^{\gamma_i - 1} \tag{8}$$

where

$$a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i},\ 1 \leq i \leq r \leq n$$

In the following, we derive the single moments based on doubly TP-LHR distribution and deduced many earlier results. The single moments of *gos* have a pivotal role in calculating mean, variance of record values and order statistics.

Theorem 2.1. For TP-LHR distribution given in Equation (4) and for $n \in N$, $m \in \Re$, $2 \leq r \leq n$,

$$\begin{aligned} E[X^j(r,m,n,k)] \;=\;& P_2 B\left[\left\{\frac{\alpha}{j+\delta+1} E\left[X^{j+\delta+1}(r,n-1,m,k+m)\right] - E\left[X^{j+\delta+1}(r-1,n-1,m,k+m)\right]\right\}\right. \\ &\left.+\left\{\frac{\gamma}{j+2} E\left[X^{j+2}(r,n-1,m,k+m)\right] - E\left[X^{j+2}(r-1,n-1,m,k+m)\right]\right\}\right] \\ &+\frac{\gamma}{j+\delta+1}\left\{\gamma_r E\left[X^{j+\delta+1}(r,n,m,k)\right] - E\left[X^{j+\delta+1}(r-1,n,m,k)\right]\right\} \\ &+\frac{\gamma}{j+2}\left\{\gamma_r E\left[X^{j+2}(r,n,m,k)\right] - E\left[X^{j+2}(r-1,n,m,k)\right]\right\} \end{aligned} \tag{9}$$

where

$$B = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1}\left(\frac{\gamma_i}{\gamma_i - 1}\right), \qquad C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1}\gamma_i^{(n-1,k+m)}$$
$$\gamma_i^{n-1,k+m} = (k+m) + (n-1-i)(m+1) = \gamma_i - 1.$$

Proof: The below mention expressions have been obtained from the Equations (6–7)

$$E\left[X^j(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j [\bar{F}_1(x)]^{\gamma_r - 1}\left\{(\alpha x^\delta + \gamma x)[P_2 + \bar{F}_1(x)]\right\} g_m^{r-1}[F_1(x)]dx$$

which can be written as

$$\begin{aligned} =\;& \frac{C_{r-1}}{(r-1)!}P_2\alpha\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r-1} g_m^{r-1}[F_1(x)]dx + \frac{C_{r-1}}{r-1)!}P_2\gamma\int_{Q_1}^{P_1} x^{j+1}[\bar{F}_1(x)]^{\gamma_r-1} g_m^{r-1}[F_1(x)]dx \\ &+\frac{C_{r-1}}{(r-1)!}\alpha\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r} g_m^{r-1}[F_1(x)]dx + \frac{C_{r-1}}{(r-1)!}\gamma\int_{Q_1}^{P_1} x^{j+1}[\bar{F}_1(x)]^{\gamma_r} g_m^{r-1}[F_1(x)]dx \end{aligned}$$

$$\begin{aligned} E[X^j(r,n,m,k)] \;=\;& \frac{C_{r-1}}{(r-1)!}\left[P_2\left\{\alpha\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}[F_1(x)]dx\right.\right. \\ &\left.+\gamma\int_{Q_1}^{P_1} x^{j+1}[\bar{F}_1(x)]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}[F_1(x)]dx\right\} \\ &\left.+\alpha\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r} g_m^{r-1}[F_1(x)]dx + \gamma\int_{Q_1}^{P_1} x^{j+1}[\bar{F}_1(x)]^{\gamma_r} g_m^{r-1}[F_1(x)]dx\right] \end{aligned}$$

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\left[P_2\alpha H_{j+\delta}^{(n-1,k+m)}(x) + \gamma H_{j+1}^{(n-1,k+m)}(x) + \alpha H_{j+\delta}^{(n,k)}(x) + \gamma H_{j+1}^{(n,k)}(x)\right] \tag{10}$$

where

$$\begin{aligned} H_{t+\delta}^{(n-1,k+m)}(x) &= \int_{Q_1}^{P_1} x^{t+\delta}[\bar{F}_1(x)]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}[F_1(x)]dx, \\ H_{t+\delta}^{(n,k)}(x) &= \int_{Q_1}^{P_1} x^{t+\delta}[\bar{F}_1(x)]^{\gamma_r} g_m^{r-1}[F_1(x)]dx. \end{aligned}$$

By implementing integration by parts method, we get

$$H_{t+\delta}^{(n-1,k+m)}(x) = \frac{(r-1)!}{(t+\delta+1)C_{r-2}^{(n-1,k+m)}}\left\{E[X^{t+\delta+1}(r,n-1,m,k+m) - E[X^{t+\delta+1}(r-1,n-1,m,k+m)\right\}$$

and

$$H_{t+1}^{(n-1,k+m)}(x) = \frac{(r-1)!}{(t+2)C_{r-2}^{(n-1,k+m)}}\left\{E[X^{t+2}(r,n-1,m,k+m) - E[X^{t+2}(r-1,n-1,m,k+m)]\right\}.$$

Similarly

$$H_{t+\delta}^{(n,k)}(x) = \frac{(r-1)!}{(t+\delta+1)C_(r-2)}\left\{E[X^{t+\delta+1}(r,n,m,k)] - E[X^{t+\delta+1}(r-1,n,m,k)]\right\}$$

and

$$H_{t+1}^{(n,k)}(x) = \frac{(r-1)!}{(t+2)C_{r-2}}\left\{E[X^{t+2}(r,n,m,k)] - E[X^{t+2}(r-1,n,m,k)]\right\}.$$

Inserting the value of $H_{t+\delta}^{(n-1,k+m)}(x)$, $H_{t+1}^{(n-1,k+m)}(x)$, $H_{t+\delta}^{(n,k)}(x)$ and $H_{t+1}^{(n,k)}(x)$ in Equation (10) and solving the resulting terms, the Equation (9) is determined.

Corollary 2.1. For Case II ($\gamma_i \neq \gamma_j$), replacing m by $\tilde{m}$, results may be obtained.

Remark 2.1.

(i) For order statistics ($m = 0, k = 1$), Equation (9) obtained as

$$\begin{aligned} E[X_{r:n}^j] &= P_2\left[\left\{\frac{\alpha}{j+\delta+1}E[X_{r:n-1}^{j+\delta+1}] - E[X_{r-1:n-1}^{j+\delta+1}]\right\} + \left\{\frac{\gamma}{j+2}E[X_{r:n-1}^{j+2}] - E[X_{r-1:n-1}^{j+2}]\right\}\right] \\ &+\frac{\alpha}{j+\delta+1}\left\{(n-r+1)E[X_{r:n}^{j+\delta+1}] - E[X_{r-1:n}^{j+\delta+1}]\right\} \\ &+\frac{\gamma}{j+2}\left\{(n-r+1)E[X_{r:n}^{j+2}] - E[X_{r-1:n}^{j+2}]\right\}. \end{aligned}$$

(ii) For $k-th$ records values ($m = -1$), Equation (9) reduced as.

$$\begin{aligned} E(X_{U(r)}^j)^k &= P_2\left(\frac{k}{k-1}\right)^{r-1}\left[\left\{\frac{\alpha}{j+\delta+1}E(X_{U(r)}^{j+\delta+1})^{k-1} - E(X_{U(r-1)}{}^{j+\delta+1})^{k-1}\right\}\right. \\ &\left.+\left\{\frac{\gamma}{j+2}E(X_{U(r)}^{j+2})^{k-1} - E(X_{U(r-1)}^{j+2})^{k-1}\right\}\right] \\ &+\frac{\alpha}{j+\delta+1}\left\{kE(X_{U(r)}^{j+\delta+1})^k - E(X_{U(r-1)}^{j+\delta+1})^k\right\} + \frac{\gamma}{j+2}\left\{kE(X_{U(r)}^{j+2})^k - E(X_{U(r-1)}^{j+2})^k\right\} \end{aligned}$$

(iii) Setting $P = 1$ and $Q = 0$, i.e., $(P_2 = 0)$ for non- truncated case in Theorem 2.1,

$$E(X^j_{r,n,m,k}) = \frac{\alpha}{j+\delta+1}\left[\gamma_r E(X^{j+\delta+1}_{r,n,m,k}) - E(X^{j+\delta+1}_{r-1,n,m,k})\right] + \frac{\gamma}{j+2}\left[\gamma_r E(X^{j+2}_{r,n,m,k}) - E(X^{j+2}_{r-1,n,m,k})\right]$$

agrees with [13].

(iv) Some doubly truncated distributions are the special case of Theorem 2.1, which is given in Table 1.

Table 1:

S. No.	α	γ	δ	Doubly Truncated Distribution	Author
1	–	0	0	Class of truncated distribution	[1]
2	–	–	0	linear exponential distribution	[16]
3	–	0	α-1	Weibull distribution	[17]
4	–	0	–	power hazard rate distribution	–

3. Product Moments

Case I: $m_i = m_j = m$.
The joint density function of two *gos* is $(1 \le r < s \le n)$,

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1} F(x) \times \\ & [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y), \quad -\infty < x < y < \infty, \end{aligned} \tag{11}$$

Case II: $\gamma_i \ne \gamma_j$, $i \ne j$, $i,j = 1,2,\cdot,n-1$. The joint pdf of the $r-th$ and $s-th$ *gos*, $1 \le r < s \le n$, is

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1}\sum_{j=r+1}^{s} a_j^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)} \tag{12}$$

where

$$a_i^{(r)}(s) = \prod_{j=r+1, j\ne i}^{s}\left(\frac{1}{\gamma_j - \gamma_i}\right), \quad r+1 \le i \le s \le n.$$

In this section, the recurrence relation for product moments of gos from doubly TP-LHR distribution has been presented. The product moments based on *gos* are enabled to compute the covariance of record values and order statistics.

Theorem 3.1. For doubly TP-LHR distribution revealed in Equation (4) and $1 \le r < s \le n-1$, $m \in \Re$, $n \ge 2$ and $i,j \ge 0$

$$\begin{aligned} & E[X^i(r,n,m,k)X^j(s,n,m,k)] = \\ & \quad P_2 B^*\Bigg[\Bigg\{\frac{\alpha}{j+\delta+1}E\left[X^i(r,n-1,m,k+m)X^{j+\delta+1}(s,n-1,m,k+m)\right] \\ & \quad -E\left[X^i(r,n-1,m,k+m)X^{j+\delta+1}(s-1,n-1,m,k+m)\right]\Bigg\} \\ & \quad +\Bigg\{\frac{\gamma}{j+2}E\left[X^i(r,n-1,m,k+m)X^{j+2}(s,n-1,m,k+m)\right] \\ & \quad -E\left[X^i(r,n-1,m,k+m)X^{j+2}(s-1,n-1,m,k+m)\right]\Bigg\}\Bigg] \\ & \quad +\frac{\alpha}{j+\delta+1}\left\{\gamma_s E\left[X^i(r,n,m,k)X^{j+\delta+1}(s,n,m,k)\right] - E\left[X^i(r,n,m,k)X^{j+\delta+1}(s-1,n,m,k)\right]\right\} \\ & \quad +\frac{\gamma}{j+2}\left\{\gamma_s E\left[X^i(r,n,m,k)X^{j+2}(s,n,m,k)\right] - E\left[X^i(r,n,m,k)X^{j+2}(s-1,n,m,k)\right]\right\} \end{aligned} \tag{13}$$

where

$$B^* = \frac{C_{s-2}}{C_{s-2}^{(n-1,k+m)}} = \prod_{i=1}^{s-1}\left(\frac{\gamma_i}{\gamma_i - 1}\right).$$

Proof: From Equation (11), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{Q_1}^{P_1} x^i[\bar{F}_1(x)]^m f_1(x) g_m^{r-1}[F_1(x)]K(x)dx \tag{14}$$

where

$$K(x) = \int_x^{P_1} \gamma^j[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s-1} f_1(y)dy. \tag{15}$$

Now using (6) in (15), we get

$$\begin{aligned} K(x) &= P_2\left\{\alpha\int_x^{P_1} y^{j+\delta}\left[h_m(F_1(y)) - h_m(F_1(x))\right]^{s-r-1}\left[\bar{F}_1(y)\right]^{\gamma_s-1}dy\right. \\ &\quad \left.+\gamma\int_x^{P_1} y^{j+1}\left[h_m(F_1(y)) - h_m(F_1(x))\right]^{s-r-1}\left[\bar{F}_1(y)\right]^{\gamma_s-1}dy\right\} \\ &\quad +\alpha\int_x^{P_1} y^{j+\delta}\left[h_m(F_1(y)) - h_m(F_1(x))\right]^{s-r-1}\left[\bar{F}_1(y)\right]^{\gamma_s}dy \\ &\quad +\gamma\int_x^{P_1} y^{j+1}\left[h_m(F_1(y)) - h_m(F_1(x))\right]^{s-r-1}\left[\bar{F}_1(y)\right]^{\gamma_s}dy \\ &= P_2\left\{\alpha K_{j+\delta}^{(n-1,k+m)}(x) + \gamma K_{j+1}^{(n-1,k+m)}(x)\right\} + \alpha K_{j+\delta}^{(n,k)}(x) + \gamma K_{j+1}^{(n,k)}(x) \end{aligned} \tag{16}$$

where

$$K_{t+\delta}^{(n-1,k+m)}(x) = \int_x^{P_1} y^{t+\delta}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s^{(n-1,k+m)}}dy$$

and

$$K_{t+\delta}^{(n,k)}(x) = \int_x^{P_1} y^{t+\delta}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s}dy.$$

Integrating by parts taking $y^{t+\delta}$ for integration, we attain,

$$\begin{aligned} k_{t+\delta}^{(n,k)}(x) &= \frac{1}{t+\delta+1}\left\{\gamma_s\int_x^{P_1} y^{t+\delta+1}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s-1}f_1(y)dy\right. \\ &\quad \left.(s-r-1)\int_x^{P_1} y^{t+\delta+1}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-2}[\bar{F}_1(y)]^{\gamma_s+m}f_1(y)dy\right\} \end{aligned}$$

and

$$\begin{aligned} K_{t+\delta}^{(n-1,k+m)}(x) &= \frac{1}{t+\delta+1}\left\{\gamma_s\int_x^{P_1} y^{t+\delta+1}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s^{(n-1,k+m)-1}}f_1(y)dy\right. \\ &\quad \left.(s-r-1)\int_x^{P_1} y^{t+\delta+1}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-2}[\bar{F}_1(y)]^{\gamma_{(s-1)}^{(n-1,k+m)-1}}f_1(y)dy\right\}. \end{aligned}$$

Similarly

$$K_{t+1}^{(n,k)}(x) = \frac{1}{t+2}\left\{\gamma_s \int_x^{P_1} y^{t+2}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s - 1} f_1(y)dy \right.$$
$$\left. (s-r-1)\int_x^{P_1} y^{t+2}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-2}[\bar{F}_1(y)]^{\gamma_s + m} f_1(y)dy \right\}$$

and

$$K_{t+1}^{(n-1,k+m)}(x) = \frac{1}{t+2}\left\{\gamma_s \int_x^{P_1} y^{t+2}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-1}[\bar{F}_1(y)]^{\gamma_s^{(n-1,k+m)}-1} f_1(y)dy \right.$$
$$\left. (s-r-1)\int_x^{P_1} y^{t+2}[h_m(F_1(y)) - h_m(F_1(x))]^{s-r-2}[\bar{F}_1(y)]^{\gamma_{s-1}^{(n-1,k+m)}-1} f_1(y)dy \right\}.$$

Upon substituting for $K_{t+\delta}^{(n-1,k+m)}(x), K_{t+1}^{(n-1,k+m)}(x), K_{t+\delta}^{(n,k)}(x)$ and $K_{t+1}^{(n,k)}(x)$ in Equation (16) and then putting the resulting terms for $K(x)$ in Equation (14). On simplification, Equation (13) yields.

Corollary 3.1. Replacing m by $\tilde{m}$ results may be obtained for Case II ($\gamma_i \neq \gamma_j$).

Remark 3.1.

(i) For order statistics ($m = 0, k = 1$), the Equation (13) reduces as

$$\begin{aligned} E[X_{r:n}^i, X_{s:n}^j] = P_2 &\left[\left\{\frac{\alpha}{j+\delta+1} E[X_{r:n-1}^i, X_{s:n-1}^{j+\delta+1}] - E[X_{r:n-1}^i, X_{s-1:n-1}^{j+\delta+1}]\right\}\right. \\ &\left. + \left\{\frac{\gamma}{j+2} E[X_{r:n-1}^i, X_{s:n-1}^{j+2}] - E[X_{r:n-1}^i, X_{s-1:n-1}^{j+2}]\right\}\right] \\ &+ \frac{\alpha}{j+\delta+1}\left\{(n-s+1)E[X_{r:n}^i, X_{s:n}^{j+\delta+1}] - E[X_{r:n}^i, X_{s-1:n}^{j+\delta+1}]\right\} \\ &+ \frac{\gamma}{j+2}\left\{(n-s+1)E[X_{r:n}^i, X_{s:n}^{j+2}] - E[X_{r:n}^i, X_{s-1:n}^{j+2}]\right\} \end{aligned}$$

(ii) For non- truncated case, Theorem 3.1 reduces as

$$\begin{aligned} E\left(X_{r,n,m,k}^i, X_{s,n,m,k}^j\right) = &\frac{\alpha}{j+\delta+1}\left[\gamma_s E[X_{r,n,m,k}^i, X_{s,n,m,k}^{j+\delta+1}] - E[X_{r,n,m,k}^i, X_{s-1,n,m,k}^{j+\delta+1}]\right] \\ &+ \frac{\gamma}{j+2}\left[\gamma_s E[X_{r,n,m,k}^i, X_{s,n,m,k}^{j+2}] - E[X_{r,n,m,k}^i, X_{s-1,n,m,k}^{j+2}]\right] \end{aligned}$$

as verified by [13].

(iii) Product moments of records can be attained from Equation (13), at $m = -1$.

(iv) Table 2 contains some doubly truncated distributions as a particular case of Theorem 3.1

Table 2:

S. No.	α	γ	δ	Doubly Truncated Distribution	Author
1	–	–	0	linear exponential distribution	[16]
2	–	0	α-1	Weibull distribution	[17]
3	–	0	–	power hazard rate distribution	–

4. Characterization

In this section, doubly TP-LHR distribution is characterized via the single moments of *gos*.

Theorem 4.1: A random variable X is to be distributed with *pdf* given in Equation (4), for which the necessary and sufficient conditions are represented as,

$$\begin{aligned}
&E[X^j(r,m,n,k)] = \\
&\quad P_2 B\left[\left\{\frac{\alpha}{j+\delta+1}E\left[X^{j+\delta+1}(r,n-1,m,k+m)\right]-E\left[X^{j+\delta+1}(r-1,n-1,m,k+m)\right]\right\}\right.\\
&\quad\left.+\left\{\frac{\gamma}{j+2}E\left[X^{j+2}(r,n-1,m,k+m)\right]-E\left[X^{j+2}(r-1,n-1,m,k+m)\right]\right\}\right]\\
&\quad+\frac{\alpha\gamma_r}{j+\gamma+1}\left\{E\left[X^{j+\delta+1}(r,n,m,k)\right]-E\left[X^{j+\delta+1}(r-1,n,m,k)\right]\right\}\\
&\quad+\frac{\gamma\gamma_r}{j+2}\left\{E\left[X^{j+2}(r,n,m,k)\right]-E\left[X^{j+2}(r-1,n,m,k)\right]\right\}.
\end{aligned} \tag{17}$$

Proof: The necessary part follows from Equation (9). If the expression in Equation (17) is satisfied, then Equation (17) can be rearranged, as follows

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]f_1(x)dx = \\
&\qquad P_2\left\{\frac{\alpha}{j+\delta+1}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta+1}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-2}[F_1(x)]f(x)A_1(x)dx\right.\\
&\qquad\left.+\frac{\gamma}{j+2}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-2}[F_1(x)]f(x)A_1(x)dx\right\}\\
&\qquad+\frac{\alpha\gamma_r}{j+\delta+1}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta+1}[\bar{F}_1(x)]^{\gamma_r}g_m^{r-2}[F_1(x)]f(x)A_2(x)dx\\
&\qquad+\frac{\gamma}{j+2}\frac{C_{r-1}\gamma_r}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}[\bar{F}_1(x)]^{\gamma_r}g_m^{r-2}[F_1(x)]f(x)A_2(x)dx.
\end{aligned} \tag{18}$$

where,

$$A_1(x)=\frac{(\gamma_r-1)g_m[F_1(x)]}{\bar{F}_1(x)}-(r-1)[\bar{F}_1(x)]^m, \quad \text{and} \quad A_2(x)=\frac{\gamma_r g_m[F_1(x)]}{\bar{F}_1(x)}-(r-1)[\bar{F}_1(x)]^m.$$

Let

$$z_t(x) = -[\bar{F}_1(x)]^t g_m^{r-1}[F_1(x)]. \tag{19}$$

Differentiating Equation (19), w.r.t. x, we get

$$z_t'(x) = [\bar{F}_1(x)]^t g_m^{r-2}[F_1(x)]f(x)\left[\frac{t g_m[F_1(x)]}{[\bar{F}_1(x)]}-(r-1)[\bar{F}_1(x)]^m\right].$$

Thus

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]f_1(x)dx = \\
&\quad P_2\left\{\frac{\alpha}{(j+\delta+1)}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta+1}z'_{\gamma_r-1}(x)dx+\frac{\gamma}{(j+2)}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}z'_{\gamma_r-1}(x)dx\right\}\\
&\quad+\frac{\alpha}{(j+\delta+1)}\frac{C_{r-1}\gamma_r}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta+1}z'_{\gamma_r}(x)dx+\frac{\gamma}{j+2}\frac{C_{r-1}\gamma_r}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}z'_{\gamma_r}(x)dx.
\end{aligned} \tag{20}$$

Integrating R.H.S. in Equation (20) by parts and make use of the values of $z_{\gamma_r}(x)$ and $z_{\gamma_r-1}(x)$ from Equation (18), we get

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]f_1(x)dx =$$
$$P_2\left\{\alpha\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]dx + \gamma\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]dx\right\}$$
$$+\alpha\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+\delta}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]dx + \gamma\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j+2}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)]dx$$

which reduces to,

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[\bar{F}_1(x)]^{\gamma_r-1}g_m^{r-1}[F_1(x)][f_1(x)-(\alpha x^{\delta}+\gamma x)(P_2+\bar{F}_1(x))]dx = 0. \tag{21}$$

The Müntz-Szász generalized theorem [9] has been implemented to the Equation (21), to get the below mentioned result

$$f_1(x) = (\alpha x^{\delta}+\gamma x)[P_2+\bar{F}_1(x)]$$

which is Equation (6) and above relationship holds between pdf and cdf of TP-LHR distribution. Hence the Theorem 4.1 is proved.

5. Conclusion

The power-linear hazard rate distribution was suggested by Tarvirdizade and N. Nematollahi [6]. It can be used where the hazard rate has both forms of power and linear. In this paper, we have derived moments properties based on *gos* from doubly TP-LHR distribution. The doubly truncated distribution is broadly used as it possesses non-truncated, left and right as a particular case.

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