STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, June 2023, pp 694–708.
Published online in International Academic Press (www.IAPress.org)

Analysis of Dependent Variables Following Marshal- Olkin Bivariate Distributions in the Presence of Progressive Type II Censoring

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Abstract In this paper, the likelihood function under progressive Type II censoring is generalized for Marshal-Olkin bivariate class of distributions and applied it on the bivariate Dagum distribution. Maximum likelihood estimation is considered for the model unknown parameters. Asymptotic and bootstrap confidence intervals for the unknown parameters are evaluated under progressive Type II censoring. Bayesian estimation is also considered in both complete and progressive Type II censored samples; moreover, the Bayes estimators are obtained explicitly with respect to square error loss function in both cases.

Keywords Bivariate Dagum Distribution, Prior Distribution, Bayesian Estimation, Maximum Likelihood Estimation, Bootstrap Confidence Intervals, Marshal - Olkin Copula

AMS 2010 subject classifications 94A08, 68U10

DOI; 10.19139/soic-2310-5070-1308

1. Introduction

Analyzing dependent variables is of great importance. For example, In Economic studies; Study the relation between (years of education and personal income, personal income and expenditure and inflation and unemployment), in Biological studies; Study of (blindness in the left and right eye, the age at death of parent and child in a genetic study, the relation between blood pressure and body weight for a patient and the failure time of the left and right kidney) in engineering studies ; analyzing the lifetime of a twine-engine plane, also warranty polices based on failure time and warranty servicing time, as well as, different applications like Shock model, competing risks model, stress model, maintenance model and longevity model.

Bivariate Marshal-Olkin family is of great importance for understanding and analyzing the failure time of two variables interacting together, because it takes into consideration all different cases of the random variables (i.e. the first random variable is smaller, greater or equal to the second random variable).

Failure times are usually not observed for all units. Those units for which the exact failure time is unknown are called censored data. This data contributes valuable information and should not be omitted from the analysis. There are different censoring schemes like Type I, Type II, random, hybrid and progressive censoring. Type I, Type II, random and hybrid censoring do not allow any unit to be randomly removed during the experiment. Progressive censoring deals with this disadvantage by allowing units to be randomly removed from the experiment, which results in reducing the cost and time of the experiment.

Let $(X_1, X_2, \ldots,\ X_n)$ be a random sample from a probability distribution with absolutely continuous cdf F. these units are placed on a test at time *t = 0*. At this time of the i^{th} failure, $R_i,\ 1 \leq i \leq m$, number of surviving units are randomly withdrawn from the experiment. Thus, if n failures are observed then $R_1 + R_2 + \cdots + R_m$ number

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ISSN 2310-5070 (online) ISSN 2311-004X (print)
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of units that progressively censored; hence $n = m + R_1 + R_2 + \cdots + R_m$. The censoring scheme is denoted by the vector $R = (R_1, R_2, \ldots, R_m)$ and $X_{i:m:n},\ i = 1, \ldots, m$. Is the i^{th} failure time, and called progressive Type II censoring order statistic. The above steps can be extended in bivariate case as follows:

Suppose that there are n independent pairs of components (X_{1i}, X_{2i}), $i = 1 \ldots n$ under experiment, and each of them has bivariate life time distribution. During the experiment, immediately after the i^{th} failure is observed, R_i functioning items are randomly removed from the test. The m complete (ordered) lifetimes thus observed are denoted by $X_{1i:m:n},\ i = 1, \ldots, m$ with corresponding concomitants variables denoted by $X_{[2i:m:n]},\ i = 1, \ldots, m$ and hence $X_{[2i:m:n]}$ is called concomitants of progressively Type II censored order statistics. Further details on progressive censoring and concomitants can be found by Balakrishnan and Aggarwala (2000) and Nagaraja and Abo-Eleneen(2008).

In the presence of progressive Type-II censored samples, El-Sherpieny et al. (2019) discussed estimation for the FGM bivariate Weibull model, Muhammed and Almetwally (2020) introduced Bayesian and Non-Bayesian estimation for the bivariate inverse Weibull model uder progressive Type-II censoring. Moreover, El-Sherpieny et al. (2021) obtained Bayesian and non-Bayesian estimation for the parameters of bivariate generalized Rayleigh model based on clayton copula under progressive Type-II censoring schemes with random removal .This paper deals with generalizing the likelihood function in case of two dependent variables follow Marshal- Olkin bivariate models based on progressive Type II censoring.

The rest of the paper is organized as follows. In Section 2, Marshal – Olkin bivariate models and their properties are described, the data description is also provided in Section 3. The Maximum likelihood estimation for the model parameters is discussed in Section 4, asymptotic and Bootstrap confidence intervals for the model parameters are introduced in Section 5, Bayesian estimation for the model parameters is proposed in Section 6. Finally, simulation studies and concluding remarks are discussed in Sections 7 and 8 respectively.

2. Marshal-Olkin Bivariate Models

According to Marshal and Olkin (1967) and using the maximization process. A class of bivariate Marshal- Olkin distributions will be discussed in this section.

If (X_1, X_2) is distributed as bivariate Marshal- Olkin model denoted by $MOB(\alpha_1, \alpha_2, \alpha_3, \Theta)$. Then, the joint cdf, pdf and the conditional probability density function for bivariate Marshal-Olkin (MOB) models are as follows

$$F_{X_1,X_2}(x_1, x_2) = F_B(x_1; \alpha_1; \Theta) F_B(x_2; \alpha_2; \Theta) F_B(x_3; \alpha_3; \Theta), \tag{1}$$

where $x_3 = \min\{x_1, x_2\}$and $F_B(.;\theta)$is the baseline function Note that (1) can be written as

$$F_{X_1,X_2}(x_1, x_2) = \begin{cases} F_B(x_1; \alpha_{13}; \Theta) \cdot F_B(x_2; \alpha_2; \Theta) & if \quad x_1 < x_2 \\ F_B(x_1; \alpha_1; \Theta) \cdot F_B(x_2; \alpha_{23}; \Theta) & if \quad x_2 < x_1 \\ F_B(x; \alpha_{123}; \Theta) \quad if \quad x_1 = x_2. \end{cases} \tag{2}$$

Where $\alpha_{i3} = \alpha_i + \alpha_3\ , \quad i = 1,\ 2.$
If (X_1, X_2) distributed as $MOB(\alpha_1, \alpha_2, \alpha_3, \Theta)$, then the joint pdf of (X_1, X_2) is given as

$$f(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ f_3(x) & if \quad 0 < x_1 = x_2 = x. \end{cases} \tag{3}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_B(x_1; \alpha_{13}, \Theta) \cdot f_B(x_2; \alpha_2, \Theta), \\ f_2(x_1, x_2) &= f_B(x_1; \alpha_1, \Theta) \cdot f_B(x_2; \alpha_{23}, \Theta), \\ f_3(x) &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \cdot f_B(x; \alpha_{123}, \Theta), \end{aligned}$$

$\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$ and $f_B(.;\Theta)$ is the baseline function.

It should be mentioned that the MOB models has both an absolute continuous part and a singular part. That given by Marshal and Olkin (1967) by the following theorem.

Theorem 1. If (X_1, X_2) is distributed as $MOB(\alpha_1, \alpha_2, \alpha_3, \Theta)$, then

$$F_{X_1,X_2}(x_1, x_2) = \frac{\alpha_{12}}{\alpha_{123}} F_a(x_1, x_2) + \frac{\alpha_3}{\alpha_{123}} F_s(x_1, x_2), \tag{4}$$

where $x_3 = \min\{x_1, x_2\}$,

$$F_s(x_1, x_2) = F_B(\min\{x_1, x_2\}; \alpha_{123}; \Theta).F_s(x_1, x_2) = F_B(\min\{x_1, x_2\}; \alpha_{123}; \Theta).$$

and

$$F_{\mathrm{a}}(x_1, x_2) = \frac{\alpha_{123}}{\alpha_{23}} F_B(x_1; \alpha_1; \Theta) F_B(x_2; \alpha_2; \Theta) F_B(x_3; \alpha_3; \Theta) - \frac{\alpha_3}{\alpha_{12}} F_B(x_3; \alpha_{123}; \Theta).$$

Here $F_s(.,.)$and $F_a(.,.)$ are the singular and the absolutely continuous part respectively.

Along the same line Muhammed (2017) introduced the Marshal-Olkin bivariate Dagum (MOBD) distribution as follows

The baseline cdf and pdf following univariate Dagum distribution are given as

$$f_B\ (x; \lambda, \delta, \beta) = \ \lambda\, \delta\, \beta\, x^{-\delta-1}\ (1 + \lambda\, x^{-\delta}\ )^{-\beta-1}, F_B\ (x; \lambda, \delta, \beta) = (1 + \lambda\, x^{-\delta}\ )^{-\beta}$$

Respectively, where $\lambda > 0$ is a scale parameter and $\delta > 0$ and $\beta > 0$ are shape parameters respectively.

Then, according to (1) the joint cdf of (X_1, X_2) follows MOBD distribution is given as follows

$$F_{\mathrm{MOBD}}(x_1, x_2) = F_B(x_1; \beta_1; \Theta) F_B(x_2; \beta_2; \Theta) F_B(x_3; \beta_3; \Theta) \tag{5}$$

where $x_3 = \min(x_1, x_2)$and $\Theta = (\lambda, \delta)$.

The corresponding joint pdf of (X_1, X_2) is given as

$$f_{MOBD}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ f_3(x) & if \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{6}$$

Where

$$f_1\ (x_1,\ x_2) = f_B\ (x_1; \beta_{13}, \Theta)\ f_B\,(x_2; \beta_2, \Theta)\,,$$

$$f_2\ (x_1,\ x_2) = f_B\,(x_1; \beta_1, , \Theta)\ f_B\,(x_2; \beta_{23}, \Theta)\,,$$

and

$$f_3(x) = \frac{\beta_3}{\beta_{123}} \cdot f_B(x; \beta_{123}, \Theta).$$

.

The absolute continuous part and the singular part of the MOBD distribution function are given as

$$F_{X_1,X_2}(x_1, x_2) = \frac{\beta_{12}}{\beta_{123}} F_a(x_1, x_2) + \frac{\beta_3}{\beta_{123}} F_s(x_3), \tag{7}$$

where $x_3 = \min\{x_1, x_2\}$, $F_{\mathrm{s}}(x_3) = (1 + \lambda {x_3}^{-\delta})^{-\beta_{123}}$,

and $F_{\mathrm{a}}(x_1, x_2) = \frac{\beta_{123}}{\beta_{23}}\ (1 + \lambda {x_1}^{-\delta})^{\beta_1} (1 + \lambda {x_2}^{-\delta})^{\beta_2} (1 + \lambda {x_3}^{-\delta})^{\beta_3}\ -\ \frac{\beta_3}{\beta_{12}}\ (1 + \lambda {x_3}^{-\delta})^{-\beta_{123}}$.

Muhammed (2017) introduced an absolutely continuous bivariate Dagum (BVD_{ac}) distribution as following
A random vector (Y_1, Y_2) follows a BVD_{ac} distribution if its pdf is given by

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} c\ f_1(y_1,y_2) & if \quad y_1 < y_2 \\ c\ f_2(y_1,y_2) & if \quad y_1 > y_2 \end{cases} \\ = c. \begin{cases} f_D(y_1;\beta_{13}) \cdot f_D(y_2;\beta_2) & if \quad y_1 < y_2 \\ f_D(y_1;\beta_1) \cdot f_D(y_2;\beta_{23}) & if \quad y_1 > y_2 \end{cases}, \tag{8}$$

Where c is the normalizing constant and $c = \frac{\beta_{123}}{\beta_{12}}$.

3. Data Description and Likelihood Function

Suppose that there are n independent pairs of components (X_{1i}, X_{2i}), $i = 1 \ldots n$ under experiment, and each of them has $MOB(\beta_1, \beta_2, \beta_3, \Theta)$ lifetime distribution.

Based on a Type II progressive censoring scheme $(n, m, R_1, \ldots, R_m)$, we have the following observations;

$$D = \left[(x_{11:m:n}, x_{[21:m:n]}),\ (x_{12:m:n}, x_{[22:m:n]}\right), \ldots, (x_{1m:m:n}, x_{[2m:m:n]})],$$

Where $X_{1i:m:n}$ be the i^{th} order statistic of $X_1$and $X_{[2i:m:n]}$be its concomitant of X_2, $i = 1 \ldots m$
Then the joint probability of $(X_{1i:m:n}, X_{[2i:m:n]})$, $i = 1 \ldots m$ is given by

$$L(\theta) = \prod_{i=1}^{m} f_{(X_{1i:m:n}, X_{[2i:m:n]})}(x_{1i:m:n}, x_{[2i:m:n]})[S_{X_1}(x_{1i:m:n})]^{R_i}$$

$$= C \prod_{i=1}^{m} [f_1(x_{1i:m:n}, x_{[2i:m:n]})]^{\delta_{1i}} \left[f_2(x_{1i:m:n}, x_{[2i:m:n]})\right]^{\delta_{2i}} \left[f_3(x_{1i:m:n}, x_{[2i:m:n]})\right]^{\delta_{3i}} [S_{X_1}(x_{1i:m:n})]^{R_i}. \tag{9}$$

Where $C = n\,(n - R_1 - 1) \ldots (n - R_1 - R_2 - \cdots - m + 1)\,, f_1\,(.)\,,\ f_2\,(.)\,,\ f_3(.)$ are as given in (3) and $S_{X_1}(.)$ is the survival function of X_1. Also $\delta_{ji},\ j = 1, 2, 3$ are event indicators such that

$$\delta_{1i} = \begin{cases} 1, & X_{1i:m:n} < X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

$$\delta_{2i} = \begin{cases} 1, & X_{1i:m:n} > X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

$$\delta_{3i} = \begin{cases} 1, & X_{1i:m:n} = X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

That produce $m_1 = \sum_{i=1}^{m} \delta_{1i}$, $m_2 = \sum_{i=1}^{m} \delta_{2i}$ and $m_3 = \sum_{i=1}^{m} \delta_{3i}$ such that $m = m_1 + m_2 + m_3$.
Throughout this paper it is assumed that $n, m, R_1, \ldots, R_m$are predetermined and fixed.

Follows are special cases from Progressive Type II censoring that applied on bivariate Marshal- Olkin family of Distributions:

I) Complete Case: If $R_1 = \cdots = R_m = 0$ and $n = m$ Then, (9) reduced to

$$L(\theta) = \prod_{i=1}^{n} [f_1(x_{1i}, x_{2i})]^{\delta_{1i}} \left[f_2(x_{1i}, x_{2i})\right]^{\delta_{2i}} \left[f_3(x_{1i}, x_{2i})\right]^{\delta_{3i}}.$$

where $n_1 = \sum_{i=1}^{n} \delta_{1i}$, $n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $n_3 = \sum_{i=1}^{n} \delta_{3i}$ such that $n = n_1 + n_2 + n_3$.

II) Type II Censoring Case : if $R_1 = R_2 = \cdots = R_{m-1} = 0,\ and\ R_m = n - m$
Then (9) reduced to

$$L(\theta) = C[S_{X_1}(x_{m:m:n})]^{n-m} \cdot \prod_{i=1}^{m} [f_1(x_{1i:m:n}, x_{[2i:m:n]})]^{\delta_{1i}} \, [f_2(x_{1i:m:n}, x_{[2i:m:n]})]^{\delta_{2i}} \, [f_3(x_{1i:m:n}, x_{[2i:m:n]})]^{\delta_{3i}}.$$

4. Maximum Likelihood Estimation

Assume $D = (x_{11:m:n}, x_{[21:m:n]}) < (x_{12:m:n}, x_{[22:m:n]}) < \ldots (x_{1m:m:n}, x_{[2m:m:n]})$ denote progressively Type II censored sample from MOBD distribution whose pdf and cdf are given as (5) and (6) , for simplicity assume $x_{1i} = x_{i:m:n}$ and $x_{2i} = x_{[2i:m:n]}$.

The log-likelihood function $l(\Theta) = LogL(\Theta)$ without normalized constant is then given as

$$\begin{aligned} l(\Theta) \propto\ & (2m_1 + 2m_2 + m_3)\, log\, \lambda + (2m_1 + 2m_2 + m_3)\ log\, \delta + m_1 log\beta_{13} \\ & + m_1 log\beta_2 + + m_2 log\beta_1 + + m_2 log\beta_{23} + + m_3 log\beta_3 \\ & -(\beta_{13}+1)\sum_{i=1}^{m} \delta_{1i} \log(1+\lambda x_{1i}^{-\delta}) - (\beta_2+1)\sum_{i=1}^{m} \delta_{1i} \log(1+\lambda x_{2i}^{-\delta}) \\ & -(\beta_1+1)\sum_{i=1}^{m} \delta_{2i} \log(1+\lambda x_{1i}^{-\delta}) - (\beta_{23}+1)\sum_{i=1}^{m} \delta_{2i} \log(1+\lambda x_{2i}^{-\delta}) \\ & -(\beta_{123}+1)\sum_{i=1}^{m} \delta_{3i} \log(1+\lambda x_{1i}^{-\delta}) + \sum_{i=1}^{m} R_i \log[1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{123}}]. \end{aligned}$$

Where $\Theta = (\beta_1, \beta_2, \beta_3, \lambda, \delta)$.
The first derivatives of the log- likelihood function with respect to $\beta_1, \beta_2, \beta_3, \lambda$ and δ are as following

$$\frac{\partial l}{\partial \beta_1} = \frac{m_1}{\beta_{13}} + \frac{m_2}{\beta_1} - A_1(\lambda, \delta) + B_1(\beta_{13}, \lambda, \delta),$$

$$\frac{\partial l}{\partial \beta_2} = \frac{m_1}{\beta_2} + \frac{m_2}{\beta_{23}} - A_2(\lambda, \delta),$$

$$\frac{\partial l}{\partial \beta_3} = \frac{m_1}{\beta_{13}} + \frac{m_2}{\beta_{23}} + \frac{m_3}{\beta_3} - A_3(\lambda, \delta) + B_1(\beta_{13}, \lambda, \delta),$$

$$\begin{aligned} \frac{\partial l}{\partial \lambda} = & \frac{2m_1 + 2m_2 + m_3}{\lambda} - (\beta_{13}+1)\, a(x_{1i}, \delta_{1i}\lambda, \delta) - (\beta_2+1)\, a(x_{2i}, \delta_{1i}, \lambda, \delta) \\ & - (\beta_1+1)\, a(x_{1i}, \delta_{2i}, \lambda, \delta) - (\beta_{23}+1)\, a(x_{2i}, \delta_{2i}, \lambda, \delta) \\ & - (\beta_{123}+1)\, a(x_{1i}, \delta_{3i}, \lambda, \delta) + B_2(\beta_{13}, \lambda, \delta)\ . \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \delta} = & \frac{2m_1 + 2m_2 + m_3}{\delta} - - (\beta_{13}+1)\, b(x_{1i}, \delta_{1i}\lambda, \delta) - (\beta_2+1)\, b(x_{2i}, \delta_{1i}, \lambda, \delta) \\ & - (\beta_1+1)\, b(x_{1i}, \delta_{2i}, \lambda, \delta) - (\beta_{23}+1)\, b(x_{2i}, \delta_{2i}, \lambda, \delta) \\ & - (\beta_{123}+1)\, b(x_{1i}, \delta_{3i}, \lambda, \delta) + B_3(\beta_{13}, \lambda, \delta)\ . \end{aligned}$$

Where

$$a(x_{1i}, \delta_{1i}, \lambda, \delta) = \sum_{i=1}^{m} \delta_{1i}\, \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}},\, b(x_{1i}, \delta_{1i}, \lambda, \delta) = \sum_{i=1}^{m} \lambda\delta_{1i}\, \frac{x_{1i}^{-\delta} log x_{1i}}{1+\lambda x_{1i}^{-\delta}},$$

$$A_1(\lambda,\delta)=\sum_{i=1}^{m}\delta_{1i}\log(1+\lambda x_{1i}^{-\delta})+\delta_{2i}\log\left(1+\lambda x_{1i}^{-\delta}\right)\ +\delta_{3i}\log\left(1+\lambda x_{1i}^{-\delta}\right),$$

$$A_2(\lambda,\delta)=\sum_{i=1}^{m}\delta_{1i}\log(1+\lambda x_{2i}^{-\delta})+\delta_{2i}\log\left(1+\lambda x_{2i}^{-\delta}\right)\ +\delta_{3i}\log\left(1+\lambda x_{1i}^{-\delta}\right),$$

$$A_3(\lambda,\delta)=\sum_{i=1}^{m}\delta_{1i}\log(1+\lambda x_{1i}^{-\delta})+\delta_{2i}\log\left(1+\lambda x_{2i}^{-\delta}\right)\ +\delta_{3i}\log\left(1+\lambda x_{1i}^{-\delta}\right),$$

$$B_1(\beta_{13},\lambda,\delta)=\sum_{i=1}^{m}R_i\frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}\ log(1+\lambda x_{1i}^{-\delta})}{1-\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}},\ B_2(\beta_{13},\lambda,\delta)=\sum_{i=1}^{m}\beta_{13}.R_i.\frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}\ x_{1i}^{-\delta}}{1-\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}},$$

$$B_3(\beta_{13},\lambda,\delta)=\sum_{i=1}^{m}R_i\ \beta_{13}\ \lambda\frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{123}}x_{1i}^{-\delta}logx_{1i}}{1-\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}}.$$

After equating the above set of equations to zero, it is noted that they have not explicit form; therefore, their solutions are numerically obtained using Newton-Raphson method as will be seen in Section 7. They are solved simultaneously to obtain $\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda}$ and $\hat{\delta}$.

5. Interval Estimation

In this section the confidence intervals for the five unknown parameters for MOBD distribution are proposed in different methods such as asymptotic confidence intervals and bootstrap confidence intervals

5.1. Asymptotic Confidence Intervals

The most common to set confidence intervals for the parameters is to use the asymptotic normal distribution of MLEs. In relation to asymptotic variance – covariance matrix of the MLEs of the parameters, according to Cohen (1965) it can be approximated by numerically inverting the Fisher information matrix F, where it is consist of the negative derivatives of the natural logarithm of the likelihood function evaluated at ($\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}$) the MLEs of the parameters.

Now, the second derivatives of the log- likelihood function are as follows

$$I_{11}=\frac{\partial^2 l}{\partial\beta_1^2}=\frac{-m_1}{\beta_{13}^2}-\frac{m_2}{\beta_1^2}-B_4(\beta_{13},\lambda,\delta),$$

$$I_{22}=\frac{\partial^2 l}{\partial\beta_2^2}=\frac{-m_1}{\beta_2^2}-\frac{m_2}{\beta_{23}^2},\ I_{12}=I_{21}=0,$$

$$I_{33}=\frac{\partial^2 l}{\partial\beta_3^2}=\frac{-m_1}{\beta_{13}^2}-\frac{m_2}{\beta_{23}^2}-\frac{m_3}{\beta_3^2}-B_4(\beta_{13},\lambda,\delta),$$

$$I_{13}=\frac{\partial^2 l}{\partial\beta_1\partial\beta_3}=\frac{-m_1}{\beta_{13}^2}-B_4(\beta_{13},\lambda,\delta),$$

$$I_{23}=\frac{\partial^2 l}{\partial\beta_2\partial\beta_3}=\frac{-m_2}{\beta_{23}^2},$$

$$I_{41}=\frac{\partial^2 l}{\partial\lambda\partial\beta_1}=B_5(\beta_{13},\lambda,\delta)-A_4(\lambda,\delta),$$

$$I_{43}=\frac{\partial^2 l}{\partial\lambda\partial\beta_3}=B_5(\beta_{13},\lambda,\delta)-A_5(\lambda,\delta),$$

$$I_{42} = \frac{\partial^2 l}{\partial\lambda\partial\beta_2} = -A_6(\lambda,\delta),$$

$$I_{44} = \frac{\partial^2 l}{\partial\lambda^2} = -\frac{2m_1+2m_2+m_3}{\lambda^2} + (\beta_{13}+1)\, d(x_{1i},\delta_{1i}\lambda,\delta) + (\beta_2+1)\, d(x_{2i},\delta_{1i},\lambda,\delta)$$
$$+(\beta_1+1)\, d(x_{1i},\delta_{2i},\lambda,\delta) + (\beta_{23}+1)\, d(x_{2i},\delta_{2i},\lambda,\delta),$$
$$+(\beta_{123}+1)\, d(x_{1i},\delta_{3i},\lambda,\delta) + B_6(\beta_{13},\lambda,\delta),$$

$$I_{45} = \frac{\partial^2 l}{\partial\lambda\partial\delta} = (\beta_{13}+1)\, g(x_{1i},\delta_{1i}\lambda,\delta) + (\beta_2+1)\, g(x_{2i},\delta_{1i},\lambda,\delta),$$
$$+(\beta_1+1)\, g(x_{1i},\delta_{2i},\lambda,\delta) + (\beta_{23}+1)\, g(x_{2i},\delta_{2i},\lambda,\delta),$$
$$+(\beta_{123}+1)\, g(x_{1i},\delta_{3i},\lambda,\delta) + B_7(\beta_{13},\lambda,\delta),$$

$$I_{51} = \frac{\partial^2 l}{\partial\delta\partial\beta_1} = B_8(\beta_{13},\lambda,\delta) + g(x_{1i},\delta_{1i},\lambda,\delta) + g(x_{1i},\delta_{2i},\lambda,\delta) + g(x_{1i},\delta_{3i},\lambda,\delta),$$

$$I_{52} = \frac{\partial^2 l}{\partial\delta\partial\beta_2} = g(x_{2i},\delta_{1i},\lambda,\delta) + g(x_{2i},\delta_{2i},\lambda,\delta) + g(x_{1i},\delta_{1i},\lambda,\delta),$$

$$I_{53} = \frac{\partial^2 l}{\partial\delta\partial\beta_3} = g(x_{2i},\delta_{1i},\lambda,\delta) + g(x_{2i},\delta_{2i},\lambda,\delta) + g(x_{1i},\delta_{3i},\lambda,\delta) + B_9(\beta_{13},\lambda,\delta).$$

Where

$$d(x_{1i},\delta_{1i},\lambda,\delta) = \sum_{i=1}^{m} \delta_{1i}\, \frac{x_{1i}^{-\delta}}{\left(1+\lambda x_{1i}^{-\delta}\right)^2},\ g(x_{1i},\delta_{1i},\lambda,\delta) = \sum_{i=1}^{m} \delta_{1i}\, \frac{x_{1i}^{-\delta} log x_{1i}}{\left(1+\lambda x_{1i}^{-\delta}\right)^2}$$

$$A_4(\lambda,\delta) = \sum_{i=1}^{m} \delta_{1i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}} + \delta_{2i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}} + \delta_{3i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}},$$

$$A_5(\lambda,\delta) = \sum_{i=1}^{m} \delta_{1i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}} + \delta_{2i} \frac{x_{2i}^{-\delta}}{1+\lambda x_{2i}^{-\delta}} + \delta_{3i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}},$$

$$A_6(\lambda,\delta) = \sum_{i=1}^{m} \delta_{1i} \frac{x_{2i}^{-\delta}}{1+\lambda x_{2i}^{-\delta}} + \delta_{2i} \frac{x_{2i}^{-\delta}}{1+\lambda x_{2i}^{-\delta}} + \delta_{3i} \frac{x_{1i}^{-\delta}}{1+\lambda x_{1i}^{-\delta}},$$

$$B_4(\beta_{13},\lambda,\delta) = \sum_{i=1}^{m} R_i \frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}} \left(log(1+\lambda x_{1i}^{-\delta})\right)^2}{\left(1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}\right)^2},$$

$$B_5(\beta_{13},\lambda,\delta) = \sum_{i=1}^{m} R_i \left(1-\beta_{13}\, .\frac{\log\left(1+\lambda x_{1i}^{-\delta}\right)}{1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}}\right) \frac{x_{1i}^{-\delta}(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}-1}}{1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}},$$

$$B_6(\beta_{13},\lambda,\delta) = \beta_{13} \sum_{i=1}^{m} R_i \frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-2\beta_{13}-2} x_{1i}^{-2\delta}.\left[(\beta_{13}+1).\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}} - 1\right]}{\left(1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}\right)^2},$$

$$B_7(\beta_{13},\lambda,\delta) = \beta_{13} \sum_{i=1}^{m} R_i \frac{x_{1i}^{-\delta}.\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}-1} log x_{1i}[1-(\beta_{13}+1)\lambda x_{1i}^{-\delta}] - \beta_{13}\lambda x_{1i}^{-\delta}.\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}-1}}{\left(1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}\right)^2},$$

$$B_8(\beta_{13},\lambda,\delta) = \lambda \sum_{i=1}^{m} R_i \frac{x_{1i}^{-\delta} log x_{1i}[1-\beta_{13}\log\left(1+\lambda x_{1i}^{-\delta}\right)[1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}] + \beta_{13}(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}\log\left(1+\lambda x_{1i}^{-\delta}\right)]}{\left(1-(1+\lambda x_{1i}^{-\delta})^{-\beta_{13}}\right)^2},$$

$$B_9\left(\beta_{13},\lambda,\delta\right)=\beta_{13}\sum_{i=1}^{m}R_i\frac{\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}-1}}{1-\left(1+\lambda x_{1i}^{-\delta}\right)^{-\beta_{13}}}\ .$$

Therefore, the asymptotic variance –covariance matrix can be written as follows

$$F^{-1}=\left.\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15}\\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25}\\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35}\\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45}\\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55}\end{bmatrix}^{-1}\right|_{\Theta=\hat{\Theta}}$$

Now, the asymptotic normality results will be stated to obtain the asymptotic confidence intervals of $\beta_1,\beta_2,\beta_3,\lambda$ and δ. It can be stated as follows

$\sqrt{n}\ [(\hat{\lambda}-\lambda),(\hat{\delta}-\delta),(\hat{\beta}_1-\beta_1),(\hat{\beta}_2-\beta_2),(\hat{\beta}_3-\beta_3)]\rightarrow N_5(0\ ,\ F^{-1})$ as $n\rightarrow\infty$

Where F^{-1} is the variance-covariance matrix, $\hat{\Theta}=(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta})$. and $\Theta=(\beta_1,\beta_2,\beta_3,\lambda,\delta)$.Since Θ is unknown , then $F^{-1}(\Theta)$ is estimated by $F^{-1}(\hat{\Theta})$; the asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of $\beta_1,\beta_2,\beta_3,\lambda$ and δ.

The problem with applying normal approximation of the MLE is that when the sample size is small, the normal approximation may be poor. However a different transformation of the MLE can be used to correct the inadequate performance of the normal approximation. Meeker and Escobar (1998) suggested the use of the normal approximation for the log transformed MLE. Let $\hat{w}_i,\ i=1,2,3,4$ with ($\hat{w}_1=\widehat{\beta}_1,\hat{w}_2=\widehat{\beta}_2,\ \hat{w}_3=\widehat{\beta}_3,\ \hat{w}_4=\widehat{\lambda}\ and\ \hat{w}_5=\widehat{\delta}$). A two sided 100$(1-\gamma)$ % normal approximation confidence Intervals for $\hat{w}_i,\ i=1,2,3,4,5$, are given by

$$\left[\hat{w}_i\exp\left(\frac{-z_{\frac{\gamma}{2}}\sqrt{var(\hat{w}_i)}}{\hat{w}_i}\right),\ \ \hat{w}_i\exp\left(\frac{z_{\frac{\gamma}{2}}\sqrt{var(\hat{w}_i)}}{\hat{w}_i}\right)\right].$$

Where and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail $\frac{\gamma}{2}$.

5.2. Bootstrap Confidence Intervals

The bootstrap is a resampling method for the statistical inference. It is commonly used to estimate confidence intervals, moreover it can also use to estimate bias and variance of an estimator or calibrate hypothesis tests. The parametric and nonparametric bootstrap have been studied by many authors such as Davison and Hinkley (1997), Efron and Tibshirani (1993) and Kreiss and Paparoditis(2011). It is concluded that the nonparametric bootstrap method does not work well. So, in this section, we use the parametric bootstrap method to construct confidence intervals for the unknown parameters $\beta_1,\beta_2,\beta_3,\lambda$ and δ. two parametric bootstrap methods will be introduced, percentile bootstrap confidence interval (B-PCI) discussed by Efron (1982) and bootstrap-t confidence interval (B-TCI) discussed by Hall (1988). The following steps are followed to obtain samples for both methods:

1. Obtain the MLEs $\widehat{\Theta}$= $(\widehat{\beta}_1,\widehat{\beta}_2,\widehat{\beta}_3,\widehat{\lambda},\widehat{\delta})$ for the unknown parameters $\Theta=(\beta_1,\beta_2,\beta_3,\lambda,\delta)$ based on the original progressively Type II censored sample

$$(x_{1i},x_{2i})=(x_{11:m:n},x_{[21:m:n]})<(x_{12:m:n},x_{[22:m:n]})<\cdots<(x_{1m:m:n},x_{[2m:m:n]})$$

2. By using $\widehat{\Theta}$, generate a bootstrap sample $(x_{1i},x_{2i})^*,\ \ i=1,2,\ldots m$ where $(x_{1i},x_{2i})^*\sim MOBD$ distribution.
3. As in step 1 based on $(x_{1i},x_{2i})^*,\ i=1,2,\ldots m$. Compute the bootstrap sample estimates of $\widehat{\Theta}$ say, $\widehat{\Theta}^*$
4. Repeat the above steps 2 and 3, N=1000 times, then we have N estimate of Θ.
5. Order the bootstrap replications of $\widehat{\Theta}^*$ such that $\widehat{\Theta}_1^*<\widehat{\Theta}_2^*<\cdots<\widehat{\Theta}_N^*$.

Percentile bootstrap confidence interval (B-PCI):
Let $G(\xi) = P(\widehat{\Theta}^* \leq \xi)$ be the cdf of $\widehat{\Theta}^*$. Define $\widehat{\Theta}^* = G^{-1}(\xi)$ for given ξ. The approximate bootstrap $100\,(1-\gamma)\,\%$ confidence interval of $\widehat{\Theta}^*$ is given by

$$(\widehat{\Theta}^*{}_{\frac{\gamma}{2}}\,, \widehat{\Theta}^*_{1-\frac{\gamma}{2}})$$

Bootstrap-t confidence interval (B-TCI): In step 3 get $\widehat{\Theta}^*$ and also calculate $var(\widehat{\Theta}^*)$ using the observed fisher information matrix.

Compute the statistic $T_j^* = \frac{\widehat{\Theta}_j^* - \widehat{\Theta}}{\sqrt{var(\widehat{\Theta}_j^*)}}$, $j = 1, \ldots N$

Arrange the bootstrap replications of T^*such that $T_1^* < T_2^* < \cdots < T_{N.}^*$

Let $H(\xi) = P(T^* \leq \xi)$ be cdf of T^*. For a given ξ define

$$\widehat{\Theta}_{boot-t} = \widehat{\Theta} + \sqrt{var\left(\widehat{\Theta}\right)}\ H^{-1}(\xi)$$

The approximate $100\,(1-\gamma)\,\%$ bootstrap confidence interval of $\widehat{\Theta}$ will be

$$(\widehat{\Theta}_{boot-t}\left(\frac{\gamma}{2}\right),\ \widehat{\Theta}_{boot-t}\left(1-\frac{\gamma}{2}\right)).$$

6. Bayes Estimation

In this section the Bayesian analysis for the MOBD distribution is considered. The explicit Bayes estimators under the squared error loss function are obtained. When the parameters λ and δ are assumed to be fixed known, the same conjugate prior on β_1, β_2 and β_3 is considered as follows:

Assume β_1, β_2 and β_3 are independent, and distributed as gamma as following

$$\pi_i(\beta_i) = \frac{b^{a_i}}{\Gamma(a_i)}\ \beta_i{}^{a_i-1}e^{-b_i\beta_i}\ ,\ i = 1, 2, 3\ , \beta_i > 0$$

The joint prior density of β_1, β_2 and β_3 is given as follows

$$\pi_0(\beta_1, \beta_2, \beta_3) = \prod_{i=1}^{3} \frac{b^{a_i}}{\Gamma(a_i)}\ \beta_i{}^{a_i-1}e^{-b_i\ \beta_i}$$

Posterior Analysis and Bayesian Estimation
Assume we have a bivariate sample from MOBD $(\beta_1, \beta_2, \beta_3)$ under progressive Type II censoring and it is denoted as

$$D = \left[(x_{11:m:n}, x_{[21:m:n]}),\ (x_{12:m:n}, x_{[22:m:n]}\right), \ldots, (x_{1m:m:n}, x_{[2m:m:n]})]$$

Let $m = m_1 + m_2 + m_3$, $\beta_{i3} = \beta_1 + \beta_3$, $i = 1, 2$

Then the Likelihood function can be rewritten as follows

$$L(D \backslash \Theta) = \ Exp(\log L(D\backslash\Theta)\ )$$

$$L\,(D\backslash\Theta) = \lambda^{2m_1+2m_2+m_3}\delta^{2m_1+2m_2+m_3}\beta_{13}^{m_1}\ \beta_{23}^{m_2}\ \ \beta_2^{m_1}\beta_1^{m_2}\beta_3^{m_3}$$
$$.Exp\,\{-(\beta_{13}+1)\,Z_1(\lambda,\delta) - (\beta_2+1)\,Z_2(\lambda,\delta) - (\beta_1+1)Z_3(\lambda,\delta)$$
$$-(\beta_{23}+1)\,Z_4(\lambda,\delta) - (\beta_{123}+1)Z_5(\lambda,\delta)\}. \prod_{i=1}^{m}(1-[Z(\lambda,\delta)^{-\beta_{13}}])^{R_i}.$$

$$L\left(D\backslash\Theta\right)\propto\prod_{i=1}^{m}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2}\sum_{l=1}^{R_i}(-1)^l\left(\begin{array}{c}R_i\\l\end{array}\right)\left(\begin{array}{c}m_1\\j\end{array}\right)\left(\begin{array}{c}m_2\\k\end{array}\right)\beta_1^{m\ 2+j}\ \ \beta_2^{m_1+k}\beta_3^{m-j-k}$$

$$.Exp\left(\beta_1T_1+\beta_2T_2+\beta_3T_3\right). \tag{10}$$

Where

$$Z_1\left(\lambda,\delta\right)=\sum_{i=1}^{m}\log\left(1+\lambda x_{1i}^{-\delta}\right)^{\delta_{1i}},Z_2\left(\lambda,\delta\right)=\sum_{i=1}^{m}\log\left(1+\lambda x_{2i}^{-\delta}\right)^{\delta_{1i}},$$

$$Z_3\left(\lambda,\delta\right)=\sum_{i=1}^{m}\log\left(1+\lambda x_{1i}^{-\delta}\right)^{\delta_{2i}},Z_4\left(\lambda,\delta\right)=\sum_{i=1}^{m}\log\left(1+\lambda x_{2i}^{-\delta}\right)^{\delta_{2i}},$$

$$Z_5\left(\lambda,\delta\right)=\sum_{i=1}^{m}\log\left(1+\lambda x_{1i}^{-\delta}\right)^{\delta_{3i}},Z\left(\lambda,\delta\right)=\sum_{i=1}^{m}\left(1+\lambda x_{1i}^{-\delta}\right),$$

$$Z_6\left(\lambda,\delta\right)=l\ \log Z\left(\lambda,\delta\right),T_1=Z_1\left(\lambda,\delta\right)+Z_3\left(\lambda,\delta\right)+Z_5\left(\lambda,\delta\right)+Z_6\left(\lambda,\delta\right),$$

$$T_2=Z_2\left(\lambda,\delta\right)+Z_4\left(\lambda,\delta\right)+Z_5\left(\lambda,\delta\right),$$

and

$$T_3=Z_1\left(\lambda,\delta\right)+Z_3\left(\lambda,\delta\right)+Z_4\left(\lambda,\delta\right)+Z_5\left(\lambda,\delta\right)+Z_6\left(\lambda,\delta\right)$$

.

Since $f\left(D,\Theta\right)=\pi_0\left(\Theta\right)L(D\backslash\Theta)$ and $f\left(D\right)=\int f\left(D\backslash\Theta\right)d\Theta=\int\pi_0\left(\Theta\right)L(D\backslash\Theta)d\Theta$

Hence the joint posterior density function of $\Theta=(\beta_1,\beta_2,\beta_3)$ given the data D, denoted by $\pi_1(\ \Theta\backslash D)$ can be written as

$$\pi_1(\ \Theta\backslash D)\ =\frac{f\left(D,\Theta\right)}{f\left(D\right)}$$

$$\pi_1\left(\ \Theta\backslash D\right)\propto\prod_{i=1}^{m}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2}\sum_{l=1}^{R_i}A_{ijkl}\ Gamma\ \left[\beta_1;\ a_{1j},\ b_1+T_1\right].Gamma\ \left[\beta_2;\ a_{2k},\ b_2+T_2\right]$$

$$.Gamma\ \left[\beta_3;\ a_{3jk},\ b_3+T_3\right]. \tag{11}$$

Where $A_{ijkl}\ =\frac{C_{ijkl}}{\prod_{i=1}^{m}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2}\sum_{l=1}^{R_i}\ C_{ijkl}}$,

and $C_{ijkl}=(-1)^l\left(\begin{array}{c}R_i\\l\end{array}\right)\left(\begin{array}{c}m_1\\j\end{array}\right)\left(\begin{array}{c}m_2\\k\end{array}\right)\cdot\frac{\Gamma(a_{1j})}{[b_1+T_1]^{a_{1j}}}\cdot\ \frac{\Gamma(a_{2k}\ )}{[b_2+T_2]^{a_{2k}}}\cdot\ \frac{\Gamma(a_{3jk})}{[b_3+T_3]^{a_{3jk}}}$.

$a_{1j}=a_1+m_2+j$, $a_{2k}=a_2+m_1+k$ and $a_{3jk}=a_3+m-j-k$.

Therefore, under the assumption of independence of $\beta_1,\beta_2$and β_3 and λ,δ are assumed to be known. It is possible to get the Bayes estimators of $\beta_1,\beta_2$and β_3 explicitly under the square error loss function as follows:

$$\check{\beta}_1=\frac{1}{b_1+T_1}\prod_{i=1}^{m}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2}\sum_{l=1}^{R_i}A_{ijkl}\ \ a_{1j}\ , \tag{12}$$

$$\check{\beta}_2=\frac{1}{b_2+T_2}\prod_{i=1}^{m}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2}\sum_{l=1}^{R_i}A_{ijkl}\ \ a_{2k}\ , \tag{13}$$

and

$$\breve{\beta}_3 = \frac{1}{b_3 + T_3} \prod_{\boldsymbol{i=1}}^{\boldsymbol{m}} \sum_{\boldsymbol{j=1}}^{\boldsymbol{m_1}} \sum_{k=1}^{m_2} \sum_{l=1}^{R_i} A_{ijkl} \;\; a_{3jk} \; , \tag{14}$$

Bayesian Estimation for Complete Case:

The complete case appeared, If $R_1 = \cdots = R_m = 0$ and $n = m$. Hence, the likelihood function can be written as special case of (10) as follows

$$L\left(D\backslash\Theta\right) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} \binom{n_1}{j}\binom{n_2}{k} \beta_1^{n_2+j} \;\; \beta_2^{n_1+k}\beta_3^{n-j-k} \; .Exp\left(\beta_1 T_1 + \beta_2 T_2 + \beta_3 T_3\right)$$

Where $D = \{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$, $\Theta = (\beta_1, \beta_2, \beta_3)$,

$n_1 = \sum_{i=1}^{n} \delta_{1i}$, $n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $n_3 = \sum_{i=1}^{n} \delta_{3i}$ such that $n = n_1 + n_2 + n_3$,

$Z_1\left(\lambda, \delta\right) = \sum_{i=1}^{n} \log\left(1 + \lambda x_{1i}^{-\delta}\right)^{\delta_{1i}}$, $Z_2\left(\lambda, \delta\right) = \sum_{i=1}^{n} \log\left(1 + \lambda x_{2i}^{-\delta}\right)^{\delta_{1i}}$,

$Z_3\left(\lambda, \delta\right) = \sum_{i=1}^{n} \log\left(1 + \lambda x_{1i}^{-\delta}\right)^{\delta_{2i}}$, $Z_4\left(\lambda, \delta\right) = \sum_{i=1}^{n} \log\left(1 + \lambda x_{2i}^{-\delta}\right)^{\delta_{2i}}$, $Z_5\left(\lambda, \delta\right) = \sum_{i=1}^{n} \log\left(1 + \lambda x_{1i}^{-\delta}\right)^{\delta_{3i}}$,

$T_1 = Z_1(\lambda, \delta) + Z_3(\lambda, \delta) + Z_5(\lambda, \delta), T_2 = Z_2(\lambda, \delta) + Z_4(\lambda, \delta) + Z_5(\lambda, \delta),$

and $T_3 = Z_1\left(\lambda, \delta\right) + Z_3\left(\lambda, \delta\right) + Z_4\left(\lambda, \delta\right) + Z_5\left(\lambda, \delta\right).$

And the corresponding joint posterior density function reduces to

$$\pi_1(\Theta\backslash D) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk} Gamma[\beta_1; a_{1j}, b_1 + T_1].Gamma[\beta_2; a_{2k}, b_2 + T_2]$$

$$.Gamma\;\;[\beta_3;\;a_{3jk},\;b_3 + T_3]\,.$$

Where $A_{jk} = \frac{C_{jk}}{\sum_{\boldsymbol{j=1}}^{\boldsymbol{n_1}} \sum_{k=1}^{n_2} \; C_{jk}}$, and $C_{jk} = \binom{n_1}{j}\binom{n_2}{k} \cdot \frac{\Gamma(a_{1j})}{[b_1+T_1]^{a_{1j}}} \cdot \frac{\Gamma(a_{2k}\;)}{[b_2+T_2]^{a_{2k}}} \cdot \frac{\Gamma(a_{3jk})}{[b_3+T_3]^{a_{3jk}}}$,

$a_{1j} = a_1 + n_2 + j$, $a_{2k} = a_2 + n_1 + k$ and $a_{3jk} = a_3 + n - j - k$.

Hence, the Bayes estimators of $\beta_1, \beta_2$and β_3 explicitly under the square error loss function based on complete samples are given as follows:

$$\breve{\beta}_1 = \frac{1}{b_1 + T_1} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\; a_{1j},$$

$$\breve{\beta}_2 = \frac{1}{b_2 + T_2} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\; a_{2k},$$

and

$$\breve{\beta}_3 = \frac{1}{b_3 + T_3} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\; a_{3jk}.$$

7. Simulation Study

In this section, some results based on simulations are presented, to compare the performance of MLE for the model parameters for different sample sizes, and censoring schemes. The performance of the resulting estimators have been considered in terms of their Average estimates (AVG), Mean Square Error (MSE) and Estimated Risk (ER). Also, obtained confidence intervals (CIs) are compared by using asymptotic distributions of MLEs and two bootstrap CIs, the comparison of them are made in terms of coverage percentage (CP). For each simulated sample a 95% CI are computed and the estimated CP was computed as the number of CIs that covers the true value divided by repeated times (1000)

In this study, the following censoring schemes (CS) are considered:

Scheme I: $R_m = n - m$, $R_i = 0$ for $i \neq m$.

Scheme II: $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.

Scheme III: $R_{\frac{m+1}{2}} = n - m$, $R_i = 0$ for $i \neq \frac{m+1}{2}$; if m odd and $R_{\frac{m}{2}} = n - m$, $R_i = 0$ for $i \neq \frac{m}{2}$; if m even.

Scheme IV: $R_{\frac{2m-1}{2}+1} = \cdots = R_{\frac{n}{2}} = 1$, other $R_i = 0$.

The population parameter values are ($\beta_1 = 1.8,\ \ \beta_2 = 1.4,\ \ \beta_3 = 1.7,\ \ \lambda = 0.5\ \ \text{and}\ \delta = 1.2$) and the sample sizes are ($n = 20,\ 40 \text{ and } 80$) and number of censoring stages are ($m = 10,\ 15,\ 20, 30,\ 40$).

The Bayes Estimates (BEs) and ER of β_1, β_2 and β_3 are computed based on squared error loss function using equations (12), (13) and (14) at prior distribution parameters as ($a_1 = 1.3$, $a_2 = 1.5$, $a_3 = 1.7$, $b_1 = 0.3$, $b_2 = 0.5$, $b_3 = 0.7$).

8. Concluding Remarks

Based on progressive Type II censored samples, this paper is related to derive the likelihood function and to full Bayesian and non- Bayesian estimation procedures for the analysis of dependent variables using Marshal - Olkin bivariate models in general and considered Marshal– Olkin bivariate Dagum distribution especially. The Bayesian estimates are obtained in explicit forms. But non- Bayesian ones cannot be obtained in explicit form. A simulation study was conducted to examine the performance of the resulting estimators for different sample sizes, different censoring schemes and different parameter values. Simulation results for the MLEs and the BEs are summarized in Table 1 and Table 2. From the results, the following general remarks are observed:

For increasing sample size the MSEs of the considered parameters decreases

As expected, for fixed values of the sample size, scheme II in which the censoring occurs after the first observed failures gives more accurate results through the MSEs and CIs than other schemes. Moreover, results in censoring schemes III and IV are closed to other.

For small sample sizes, the results corresponding to Bayesian procedure are better than those corresponding to non- Bayesian procedure in the sense of ER.

Additionally, it is noted that Bayesian estimators have a closed forms, which it is highly recommended to study its properties as a future work.

Table 1. AVG and (MSE) of MLEs and BEs and (ER) for The Model Parameters

		MLE (MES)					BE (ER)		
(n, m)	CS	β_1**(1.8)**	β_2**(1.4)**	β_3**(1.7)**	λ**(0.5)**	δ**(1.2)**	β_1**(1.8)**	β_2**(1.4)**	β_3**(1.7)**
(20,10)	I	1.051	0.983	1.293	2	0.932	1.834	1.52	1.1
		(0.561)	(0.174)	(0.166)	(2.25)	(0.072)	(0.058)	(0.06)	(0.05)
	II	1.494	1.343	1.576	1.23	1.911	1.843	1.53	1.2
		(0.094)	(0.31)	(0.124)	(0.533)	(0.506)	(0.044)	(0.08)	(0.03)
	III	1.441	1.023	0.653	1.89	0.983	1.854	1.61	1.3
		(0.128)	(0.142)	(1.09)	(1.932)	(0.047)	(0.055)	(0.067)	(0.03)
	IV	1.432	1.044	0.661	1.812	0.99	1.841	1.84	1.3
		(0.135)	(0.127)	(1.079)	(1.721)	(0.044)	(0.057)	(0.09)	(0.03)
(20,15)	I	1.043	0.971	1.193	2.2	0.899	1.832	1.97	1.5
		(0.573)	(0.184)	(0.257)	(2.89)	(0.091)	(0.045)	(0.072)	(0.052)
	II	1.44	1.3	1.3	1.001	1.921	1.831	1.72	1.52
		(0.13)	(0.16)	(0.16)	(0.251)	(0.52)	(0.032)	(0.015)	(0.047)
	III	1.42	1.111	0.651	1.781	0.892	1.844	1.63	1.43
		(0.144)	(0.084)	(1.1)	(1.641)	(0.095)	(0.038)	(0.012)	(0.016)
	IV	1.423	1.052	0.711	1.732	0.897	1.822	1.91	1.61
		(0.142)	(0.121)	(0.978)	(1.518)	(0.092)	(0.049)	(0.08)	(0.019)
(40,20)	I	1.53	1.177	1.97	0.814	1.41	1.845	1.75	1.75
		(0.08)	(0.05)	(0.073)	(0.099)	(0.044)	(0.037)	(0.01)	(0.01)
	II	1.699	1.45	1.871	0.805	1.37	1.824	1.57	1.74
		(0.09)	(0.025)	(0.029)	(0.093)	(0.029)	(0.021)	(0.011)	(0.019)
	III	1.63	1.21	2.01	0.751	1.501	1.8	1.82	1.76
		(0.03)	(0.036)	(0.096)	(0.063)	(0.096)	(0.017)	(0.014)	(0.018)
	IV	1.65	1.26	1.91	0.8	1.503	1.809	1.51	1.77
		(0.023)	(0.02)	(0.044)	(0.09)	(0.092)	(0.012)	(0.016)	(0.017)
(40, 30)	I	1.55	1.221	1.92	0.762	1.39	1.837	1.34	1.72
		(0.06)	(0.032)	(0.048)	(0.068)	(0.036)	(0.013)	(0.013)	(0.0162)
	II	1.667	1.543	1.841	0.702	1.35	1.82	1.37	1.74
		(0.018)	(0.02)	(0.02)	(0.04)	(0.02)	(0.011)	(0.01)	(0.016)
	III	1.621	1.27	1.978	0.762	1.499	1.86	1.39	1.75
		(0.032)	(0.017)	(0.07)	(0.069)	(0.089)	(0.014)	(0.0002)	(0.015)
	IV	1.599	1.25	1.991	0.772	1.512	1.881	1.38	1.76
		(0.04)	(0.022)	(0.085)	(0.074)	(0.097)	(0.015)	(0.004)	(0.014)
(80,30)	I	1.861	1.32	1.77	0.591	1.3	1.801	1.44	1.71
		(0.004)	(0.06)	(0.0049)	(0.0083)	(0.01)	(0.001)	(0.003)	(0.013)
	II	1.842	1.37	1.73	0.521	1.26	1.802	1.42	1.73
		(0.002)	(0.0009)	(0.009)	(0.0004)	(0.0036)	(0.0015)	(0.0021)	(0.003)
	III	1.85	1.307	1.781	0.551	1.29	1.872	1.45	1.72
		(0.003)	(0.0086)	(0.0066)	(0.00261)	(0.0081)	(0.01)	(0.0013)	(0.002)
	IV	1.851	1.309	1.783	0.408	1.27	1.865	1.46	1.721
		(0.003)	(0.00083)	(0.0067)	(0.0084)	(0.0049)	(0.001)	(0.0003)	(0.0012)
(80, 40)	I	1.863	1.381	1.75	0.583	1.29	1.854	1.43	1.732
		(0.0039)	(0.00036)	(0.0025)	(0.0069)	(0.0028)	(0.0023)	(0.0004)	(0.0006)
	II	1.833	1.352	1.722	0.507	1.19	1.898	1.41	1.742
		(0.007)	(0.0018)	(0.0005)	(0.00005)	(0.0001)	(0.0014)	(0.0002)	(0.0004)
	III	1.861	1.334	1.73	0.552	1.281	1.832	1.44	1.764
		(0.0082)	(0.0044)	(0.0009)	(0.0027)	(0.0066)	(0.0019)	(0.0001)	(0.00015)
	IV	1.862	1.324	1.732	0.553	1.284	1.876	1.43	1.783
		(0.0038)	(0.0058)	(0.001)	(0.0028)	(0.007)	(0.002)	(0.0003)	(0.0001)

Table 2. Coverage Probability (CP) of a 95% CIs for The Model Parameters

		ACI					Boot- P					Boot- t				
(n,m)	CS	β_1	β_2	β_3	λ	δ	β_1	β_2	β_3	λ	δ	β_1	β_2	β_3	λ	δ
(20,10)	I	0.894	0.886	0.873	0.9	0.874	0.917	0.902	0.926	0.918	0.908	0.891	0.893	0.882	0.895	0.877
	II	0.933	0.903	0.915	0.942	0.91	0.945	0.954	0.955	0.943	0.923	0.839	0.911	0.881	0.892	0.916
	III	0.916	0.894	0.886	0.891	0.905	0.891	0.909	0.894	0.87	0.905	0.862	0.855	0.887	0.909	0.89
	IV	0.893	0.891	0.865	0.823	0.914	0.902	0.932	0.915	0.94	0.917	0.915	0.846	0.828	0.872	0.831
(20,15)	I	0.932	0.96	0.894	0.925	0.862	0.974	0.952	0.928	0.9	0.898	0.952	0.896	0.931	0.934	0.942
	II	0.941	0.941	0.903	0.937	0.909	0.937	0.929	0.941	0.927	0.919	0.951	0.95	0.938	0.925	0.92
	III	0.929	0.892	0.972	0.894	0.883	0.934	0.941	0.961	0.891	0.901	0.937	0.913	0.901	0.929	0.886
	IV	0.921	0.913	0.941	0.932	0.891	0.914	0.922	0.942	0.879	0.915	0.897	0.925	0.933	0.941	0.899
(40,20)	I	0.898	0.904	0.943	0.884	0.905	0.921	0.939	0.894	0.821	0.864	0.929	0.933	0.942	0.913	0.819
	II	0.956	0.952	0.93	0.963	0.932	0.941	0.973	0.953	0.903	0.924	0.932	0.954	0.931	0.936	0.903
	III	0.92	0.933	0.902	0.932	0.943	0.952	0.942	0.942	0.936	0.945	0.914	0.935	0.892	0.923	0.91
	IV	0.901	0.921	0.943	0.894	0.924	0.914	0.916	0.926	0.953	0.964	0.961	0.953	0.864	0.917	0.889
(40,30)	I	0.914	0.935	0.942	0.972	0.943	0.955	0.951	0.919	0.935	0.953	0.896	0.897	0.829	0.858	0.839
	II	0.942	0.952	0.984	0.961	0.953	0.953	0.924	0.942	0.945	0.936	0.954	0.922	0.921	0.901	0.94
	III	0.932	0.972	0.963	0.952	0.942	0.916	0.926	0.897	0.952	0.982	0.934	0.901	0.983	0.921	0.997
	IV	0.943	0.962	0.957	0.899	0.91	0.92	0.917	0.902	0.942	0.892	0.941	0.934	0.976	0.951	0.932
(80,30)	I	0.952	0.927	0.895	0.942	0.916	0.926	0.973	0.932	0.954	0.973	0.916	0.948	0.939	0.942	0.918
	II	0.975	0.937	0.951	0.985	0.987	0.963	0.952	0.917	0.985	0.943	0.952	0.954	0.935	0.901	0.934
	III	0.952	0.962	0.887	0.992	0.998	0.952	0.946	0.973	0.877	0.921	0.894	0.935	0.962	0.893	0.917
	IV	0.971	0.991	0.894	0.995	0.909	0.934	0.947	0.942	0.895	0.93	0.925	0.946	0.947	0.976	0.949
(80,40)	I	0.979	0.982	0.899	0.921	0.983	0.985	0.981	0.931	0.945	0.891	0.931	0.925	0.929	0.952	0.974
	II	0.977	0.954	0.961	0.953	0.978	0.999	0.96	0.895	0.982	0.874	0.916	0.989	0.992	0.961	0.995
	III	0.981	0.972	0.993	0.945	0.939	0.978	0.976	0.945	0.97	0.951	0.954	0.915	0.975	0.952	0.942
	IV	0.935	0.981	0.903	0.956	0.954	0.966	0.902	0.931	0.991	0.976	0.911	0.922	0.962	0.952	0.953

Acknowledgement

The author is profoundly grateful to the anonymous referees and editor for their valuable suggestions and comments which improved the earlier draft of the article.

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