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A New Weighted Half-Logistic Distribution:Properties, Applications and Different Method of Estimations

Majid Hashempour [1], Morad Alizadeh [2,*]

[1] *Department of Statistics, School of Sciences, University of Hormozgan, Bandar Abbas, Iran*
[2] *Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, 75169, Iran*

Abstract In this paper, we introduce a new two-parameter lifetime distribution based on arctan function which is called weighted Half-Logistic (WHL) distribution. Theoretical properties of this model including quantile function, extreme value, linear combination for pdf and cdf, moments, conditional moments, moment generating function and mean deviation are derived and studied in details. The maximum likelihood estimates of parameters are compared with various methods of estimations by conducting a simulation study. Finally, two real data sets show that this model p[rovide better fit than other competitive known models.

Keywords Entropy; Half-Logistic distribution ; Maximum likelihood estimation; Moments

AMS 2010 subject classifications 60Exx, 60E05

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1. Introduction

Modelling and analysing lifetimes are important in engineering, medicine, economics, etc. In many applied areas such as lifetime analysis, finance and insurance, we need extended forms of distributions. So, several methods for generating new families of distributions have been proposed in literature. Some attempts have been made to define new families of probability distributions that extend well-known families of distributions and with great flexibility in modeling data in practice. Among them, the generalized G-classes of distributions say G are used in which one or more parameter(s) are added to a baseline distribution.

Many distribution defined based on trigonometry function. For example, new distribution using sine function due to Kumar et al. (2015), Hyperbolic cosine-f family due to Kharazmi et al. (2016), new distribution using sine function due to Kumar et al. (2015), new class of probability distributions via cosine and sine functions due to Chesneau et al. (2019).

Balakrishnan (1985) proposed the standard half-logistic (SHL) distribution as a lifetime increasing hazard rate function. The cumulative distribution function (cdf) of SHL is given by

$$\Pi(t) = \frac{1 - \mathrm{e}^{-\mathrm{t}}}{1 + \mathrm{e}^{-\mathrm{t}}}, \ |_{t>0} \tag{1}$$

*Correspondence to: Morad Alizadeh(Email: m.alizadeh@pgu.ac.ir). Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, 75169, Iran.

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The probability density function (pdf) of SHL distribution is

$$\pi(t) = \frac{2\mathrm{e}^{-\mathrm{t}}}{(1+\mathrm{e}^{-\mathrm{t}})^2}, \ |_{t>0}. \tag{2}$$

Many extensions of half-logistic distribution has been introduced by several authors. For example, type I half-logistic family of distribution by Cordeiro et al (2016), type II half-logistic family by Soliman et al (2017), new odd log-logistic by Alizadeh et al. (2019), new type I half-logistic by Alizadehg et al. (2020), New two-parameter Modified Half-logistic Distribution by Shaheed (2021) and Complementary Poisson Generalized Half Logistic Distribution by Muhammad and Liu (2021).

In this paper we introduce a new two-parameter lifetime distribution. The cdf of new distribution is given by

$$F(t) = \frac{1}{\arctan(\alpha)} \times \arctan(\frac{\alpha\,\Pi(t)^{\beta}}{1+Pi(t)^{\beta}}), \ |_{t>0,\alpha>0,\beta>0} \tag{3}$$

The corresponding pdf and hazard rate function (hrf) are given by

$$f(t) = \frac{\alpha\beta\,\pi(t)\Pi(t)^{\beta-1}\left[1+\bar{\Pi}(t)^{\beta-1}\right]}{\arctan(\alpha)\left[(1+\bar{\Pi}(t)^{\beta})^2+\alpha^2\,\Pi(t)^{2\beta}\right]} \tag{4}$$

and

$$\psi(t) = \frac{\alpha\beta\,g(t)G(t)^{\beta-1}\left[1+\bar{G}(t)^{\beta-1}\right]}{\left[\arctan(\alpha)-\arctan(\frac{\alpha\,G(t)^{\beta}}{1+\bar{G}(t)^{\beta}}\right]\left[(1+\bar{G}(t)^{\beta})^2+\alpha^2\,G(t)^{2\beta}\right]} \tag{5}$$

The plots of pdf and hrf for some selected value of parameters are given in figures 1 and 2. These graphs show that the pdf of WHL(α,β) is unimodal, right skew, left skew or almost symmetric. The hrf of WHL(α,β) can be decreasing, increasing, upside-down and bathtub shape. These properties provide strong motivation for defining the WHL(α,β) distribution. Also, results of application, show that the WHL(α,β) provide the better fit than other competitive models.

The rest of this paper is organized as follows: In the above, new family of distributions was proposed. Various properties of the proposed distribution are explored in Section 2. These properties include quantile function, extreme value, linear combination for cdf and pdf, moments, conditional moments, moment generating function and mean deviation. The maximum likelihood estimation of parameters are compared with various methods of estimations by conducting simulation study in section 3. Real data sets are analysed to show the performance of the new family in Section 4. In Section 5, some concluding remarks are considered.

2. Basic properties

2.1. Quantile function

If $U \sim U(0,1)$, the solution of equation $u = \frac{1}{\arctan(\alpha)} \times \arctan(\frac{\alpha\,G(t)^{\beta}}{1+\bar{G}(t)^{\beta}}$ have cdf (3).

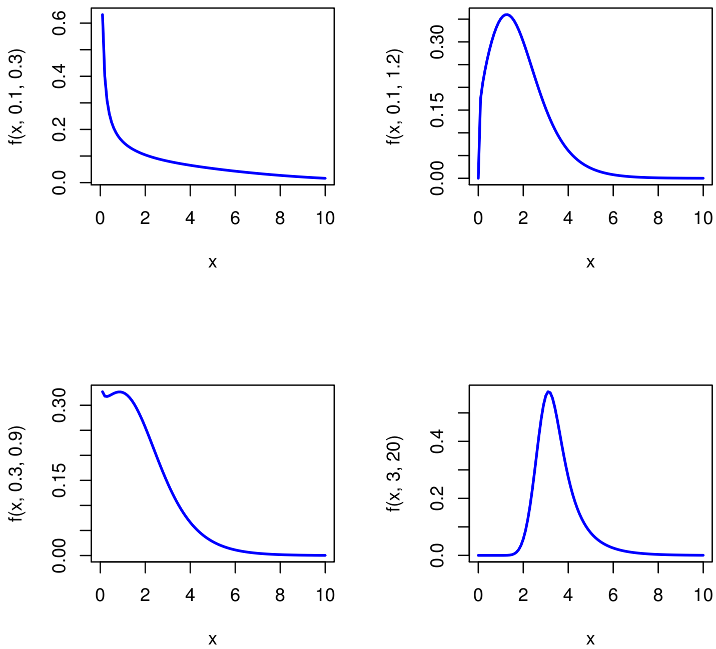

Figure 1. Plots of pdf of WHL(α, β) for some selected value of parameters.

2.2. Linear Combination for cdf and pdf

First note that we can write $\arctan(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{2\,i+1}\, z^{2\,i+1}$ Then

$$F(t) = \frac{1}{\arctan(\alpha)} \times \arctan(\frac{\alpha\,\Pi(t)^{\beta}}{1+\bar{\Pi}(t)^{\beta}} = \sum_{j_1=0}^{\infty} \frac{(-1)_1^{j}}{(2\,j_1+1)\arctan(\alpha)}\,\frac{\Pi(t)^{\beta(2\,j_1+1)}}{\left[1+\bar{\Pi}(t)^{\beta}\right]^{2\,j_1+1}}$$

$$= \sum_{j_1,j_2=0}^{\infty} \frac{(-1)_1^{j}\binom{-2\,j_1-1}{j_2}}{(2\,j_1+1)\arctan(\alpha)}\,G(t)^{\beta(2\,j_1+1)}\bar{\Pi}(t)^{\beta\,j_2}$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(-1)^{j_1+j_3}\binom{-2\,j_1-1}{j_2}\binom{\beta\,j_2}{j_3}}{(2\,j_1+1)\arctan(\alpha)}\,\Pi(t)^{\beta(2\,j_1+1)+j_3} = \sum_{j_1,j_3=0}^{\infty} a_{j_1,j_3}\Pi(t)^{\beta(2\,j_1+1)+j_3} \tag{6}$$

where

$$a_{j_1,j_3} = \sum_{j_2=0}^{\infty} \frac{(-1)^{j_1+j_3}\binom{-2\,j_1-1}{j_2}\binom{\beta\,j_2}{j_3}}{(2\,j_1+1)\arctan(\alpha)}$$

and $\Pi(t)^{\beta(2\,j_1+1)+j_3}$ denote the cdf of Exponentiated standard Half-Logistic with power parameter $\beta(2\,j_1+1)+j_3$. The pdf of T follows by differentiating (6) as

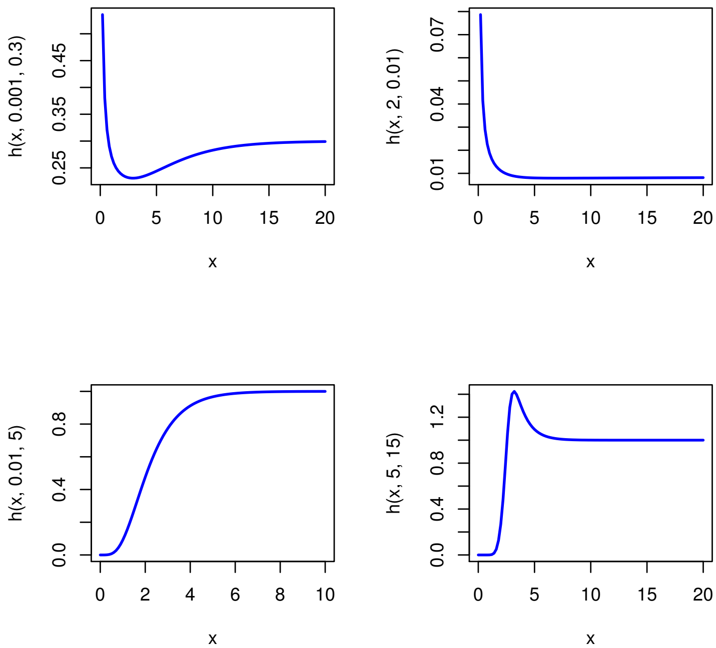

Figure 2. Plots of hrf of WHL(α, β) for some selected value of parameters.

$$f(t) = \sum_{j_1, j_3=0}^{\infty} a_{j_1,j_3}(\beta(2\,j_1+1)+j_3)\pi(x)\Pi(t)^{\beta(2\,j_1+1)+j_3-1} \tag{7}$$

Equation (7) show that the WHL density function is a linear combination of Exponentiated Half-Logistic (EHL) densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the EHL distributions.

2.3. Moments

.Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). Here, we give lemma, which will be used later.

Lemma 1
For $\alpha_1, \alpha_2, \alpha_4 > 0$ and $\alpha_3 > -1$, let

$$A(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^{\infty} \frac{z^{\alpha_1}\, \mathrm{e}^{-\alpha_2\, z}\,(1-\mathrm{e}^{-\,z})^{\alpha_3}}{(1+\mathrm{e}^{-\,z})^{\alpha_4}}\, dz.$$

Then after using some algebra, we obtain

$$
\begin{aligned}
A(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \int_0^\infty \frac{z^{\alpha_1}\,\mathrm{e}^{-\alpha_2\,\mathrm{z}}\,(1-\mathrm{e}^{-\,\mathrm{z}})^{\alpha_3}}{(1+\mathrm{e}^{-\,\mathrm{z}})^{\alpha_4}}\,dz \\
&= \sum_{i_1=0}^{\infty}\binom{-\alpha_4}{i_1}\int_0^\infty z^{\alpha_1}\,\mathrm{e}^{-(\alpha_2+\mathrm{i}_1)\,\mathrm{z}}\,(1-\mathrm{e}^{-\,\mathrm{z}})^{\alpha_3}\mathrm{dz} \\
&= \sum_{i_1,i_2=0}^{\infty}(-1)^{i_2}\binom{-\alpha_4}{i_1}\binom{\alpha_3}{i_2}\int_0^\infty z^{\alpha_1}\,\mathrm{e}^{-(\alpha_2+\mathrm{i}_1+\mathrm{i}_2)\,\mathrm{z}}\mathrm{dz} \\
&= \sum_{i_1,i_2=0}^{\infty}(-1)^{i_2}\binom{-\alpha_4}{i_1}\binom{\alpha_3}{i_2}\frac{\Gamma(\alpha_1+1)}{(i_1+i_2+\alpha_2)^{\alpha_1+1}}.
\end{aligned}
$$

where $\Gamma(.)$ denote the gamma function.

Next, the n-th moment of the WHL distribution is given as follows:

$$
E(T^n) = 2\sum_{j_1,j_3=0}^{\infty} a_{j_1,j_3}\,[\beta(2\,j_1+1)+j_3]\,A(n,1,\beta(2\,j_1+1)+j_3-1,\beta(2\,j_1+1)+j_3+1). \tag{8}
$$

For integer values of n, Let $\mu_n^{'} = E(T^n)$ and $\mu = \mu_1^{'} = E(T)$, then one can also find the n-th central moment of the WHL distribution as

$$
\mu_n = E(T-\mu)^n = \sum_{j_1=0}^{n}\binom{n}{j_1}\mu_{j_1}^{'}(-\mu)^{n-j_1}. \tag{9}
$$

Using (9), the measures of skewness and kurtosis of the WHL distribution can be obtained as

$$
Skewness(X) = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2{\mu_1^{'}}^3}{\left(\mu_2^{'} - {\mu_1^{'}}^2\right)^{\frac{3}{2}}}, \tag{10}
$$

and

$$
Kurtosis(X) = \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6{\mu_1^{'}}^2\mu_3^{'} - 3{\mu_1^{'}}^4}{\mu_2^{'} - {\mu_1^{'}}^2}, \tag{11}
$$

respectively. Additionally, the moment generating function of WHL distribution can be written as

$$
E[\mathrm{e}^{\mathrm{s\,T}}] = 2\sum_{\mathrm{j}_1,\mathrm{j}_3=0}^{\infty}\mathrm{a}_{\mathrm{j}_1,\mathrm{j}_3}\,[\beta(2\,\mathrm{j}_1+1)+\mathrm{j}_3]\,\mathrm{A}(0,1-\mathrm{t},\beta(2\,\mathrm{j}_1+1)+\mathrm{j}_3-1,\beta(2\,\mathrm{j}_1+1)+\mathrm{j}_3+1). \tag{12}
$$

Figures 3 shows the behaviour of mean, variance, skewness and kurtosis of WHL distribution. These figures show that the WHL distribution is right skew with positive kurtosis.

2.4. Conditional moments

Here, we intend to determine the conditional moments of the new family. Let

$$
B(\beta_1,\beta_2,\beta_3,\beta_4,y) = \int_0^y \frac{z^{\beta_1}\,\mathrm{e}^{-\beta_2\,\mathrm{z}}\,(1-\mathrm{e}^{-\,\mathrm{z}})^{\beta_3}}{(1+\mathrm{e}^{-\,\mathrm{x}})^{\beta_4}}\,dz,
$$

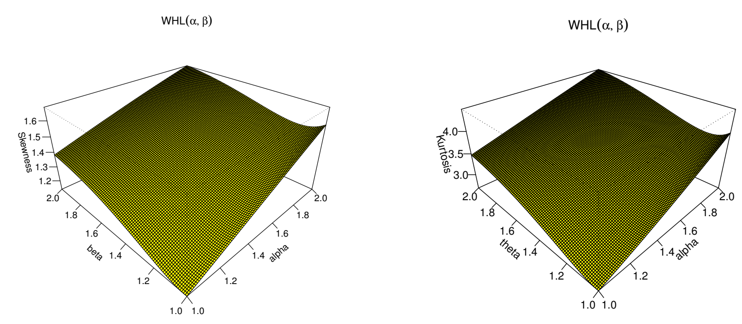

Figure 3. Skewness and kurtosis of WHL(α,β)

For $\beta_1,\beta_2,\beta_4>0$ and $\beta_3>-1$. Then, we obtain

$$B(\beta_1,\beta_2,\beta_3,\beta_4,y)=\sum_{i_1,i_2=0}^{\infty}(-1)_2^{i}\binom{-\beta_4}{i_1}\binom{\beta_3}{j}\frac{\gamma(y(\beta_2+i_1+i_2),\beta_1+1)}{(i_1+i_2+\beta_2)^{\beta_1+1}},$$

where $\gamma(z,a)=\int_0^z t^{a-1}\,\mathrm{e}^{-\mathrm{t}}\,\mathrm{dt}$ is the lower incomplete gamma function. So, the n-th conditional moments of T can be expressed as

$$E(T^n|T>t)=\frac{2}{1-F(t)}\sum_{j_1,j_3=0}^{\infty}a_{j_1,j_3}\,[\beta(2\,j_1+1)+k]\,\{C_{j_1,j_3,\beta}-Q_{j_1,j_3,\beta}\} \tag{13}$$

where

$$C_{j_1,j_3,\beta}=A(n,1,\beta(2\,j_1+1)+j_3-1,\beta(2\,j_1+1)+j_3+1)$$
$$Q_{j_1,j_3,\beta}=B(n,1,\beta(2\,j_1+1)+j_3-1,\beta(2\,j_1+1)+j_3+1,t)$$

Therefore

$$E(T^n|T\le t)=\frac{2}{F(t)}\sum_{j_1,j_3=0}^{\infty}a_{j_1,j_3}\,[\beta(2\,j_1+1)+j_3]\,B(n,1,\beta(2\,j_1+1)+j_3-1,\beta(2\,j_1+1)+j_3+1,t). \tag{14}$$

2.5. Asymptotic

The asymptotic of equations (3), (4) and (5) as $t\to 0^+$ are given by

$$F(t)\sim\frac{\alpha\,t^{\beta}}{2^{\beta+1}\arctan(\alpha)}$$
$$f(t)\sim\frac{\alpha\beta\,t^{\beta-1}}{2^{\beta+1}\arctan(\alpha)}$$
$$\psi(t)\sim\frac{\alpha\beta\,t^{\beta-1}}{2^{\beta+1}\arctan(\alpha)-t^{\beta}} \tag{15}$$

The asymptotic of equations (3), (4) and (5) as $t\to\infty$ are given by

$$F(t)\sim\frac{2\alpha\beta\,\mathrm{e}^{-\mathrm{t}}}{\arctan(\alpha)}$$
$$f(t)\sim\frac{2\alpha\beta\,\mathrm{e}^{-\mathrm{t}}}{\arctan(\alpha)}$$
$$\psi(t)\sim 1. \tag{16}$$

2.6. EXTREME VALUE

If $\bar{T} = (T_1 + ... + T_n)/n$ denotes the sample mean, then by the usual central limit theorem, $\sqrt{n}(\bar{T} - E(T))/\sqrt{\mathrm{Var}(T)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(T_1, ..., T_n)$ and $m_n = \min(T_1, ..., T_n)$. Let $\tau(s) = \frac{1}{\beta}$, we obtain following equations for the CDF in of WHL(α, β) as

$$\lim_{s \to 0} \frac{\Pi(s\, t)}{\Pi(s)} = t^{\beta} \tag{17}$$

and

$$\lim_{s \to \infty} \frac{1 - \Pi(s + t\, \tau(s))}{1 - \Pi(s)} = \mathrm{e}^{-t}. \tag{18}$$

Thus, from Leadbetter et al. (2012), there must be norming constants $a_n > 0, \quad b_n, \quad c_n > 0$ and d_n such that

$$Pr\left[a_n(M_n - b_n) \leq t\right] \to \mathrm{e}^{-\mathrm{e}^{-\mathrm{t}}},$$

and

$$Pr\left[c_n(m_n - d_n) \leq t\right] \to 1 - \mathrm{e}^{-\mathrm{t}^{\beta}},$$

as $n \to \infty$. The form of the norming constants can also be determined.

3. Estimation

3.1. Maximum-likelihood estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the WHL distribution from complete samples only. Let $T_1, \ldots, T_n$ be a random sample of size n from the WHL(α, β) distribution. The log-likelihood function for the vector of parameters $= (\alpha, \beta)^T$ can be written as

$$\begin{aligned} l() \quad &= \quad n \log\left(\frac{2\alpha\beta}{\arctan(\alpha)}\right) - \sum_{i=1}^{n} t_i + (\beta - 1) \sum_{i=1}^{n} \log(1 - \mathrm{e}^{-\mathrm{t_i}}) - (\beta + 1) \sum_{\mathrm{i}=1}^{\mathrm{n}} \log(1 - \mathrm{e}^{-\mathrm{t_i}}) \\ &+ \quad \sum_{i=1}^{n} \log\left[1 + \bar{\Pi}(t_i)^{\beta - 1}\right] - \sum_{i=1}^{n} \log\left[(1 + \bar{\Pi}(t_i)^{\beta})^2 + \alpha^2\, \Pi(t_i)^{2\beta}\right]. \end{aligned} \tag{19}$$

The log-likelihood can be maximized either directly by using the SAS (Procedure NLMixed) or by solving the non-linear likelihood equations obtained by differentiating (19). The components of the score vector $U()$ are given by

$$U_{\alpha}() \quad = \quad \frac{n}{\alpha} - \frac{n}{(1 + \alpha^2) \arctan(\alpha)} - 2\alpha \sum_{i=1}^{n} \frac{\Pi(t_i)^{2\beta}}{(1 + \bar{\Pi}(t_i)^{\beta})^2 + \alpha^2\, \Pi(t_i)^{2\beta}},$$

and

$$\begin{aligned} U_{\beta}() \quad &= \quad \frac{n}{\beta} + \sum_{i=1}^{n} \log(\Pi(t_i)) - \sum_{i=1}^{n} \frac{\bar{\Pi}(t_i)^{\beta - 1} \log[\bar{\Pi}(t_i)]}{1 + \bar{\Pi}(t_i)^{\beta - 1}} \\ &- \quad 2 \sum_{i=1}^{n} \frac{\bar{\Pi}(t_i)^{\beta}(1 + \bar{\Pi}(t_i)^{\beta}) \log[\bar{\Pi}(t_i)] + \alpha^2\, \Pi(t_i)^{2\beta} \log[G(t_i)]}{(1 + \bar{\Pi}(t_i)^{\beta})^2 + \alpha^2\, \Pi(t_i)^{2\beta}}. \end{aligned}$$

where $\Pi(t_i) = \frac{1 - \mathrm{e}^{-\mathrm{t_i}}}{1 + \mathrm{e}^{-\mathrm{t_i}}}$ and $\bar{\Pi}(t_i) = 1 - \Pi(t_i)$.

3.2. *The other estimation methods*

There are several approaches to estimate the parameters of distributions that each of them has its characteristic features and benefits. In this subsection five of those methods are briefly introduced and will be numerically investigated in the simulation study. Here $\{s_i; i = 1, 2, ..., n\}$ and $\{s_{i:n}; i = 1, 2, ..., n\}$ is the random sample and associated order statistics and F is the distribution function of WHL distribution.

•Least squares and weighted least squares estimators

The Least Squares (LSE) and weighted Least Squares Estimators (WLSE) are introduced by Swain et al., (1988). The LSE's and WLSE's are obtained by minimizing the following functions:

$$S_{\text{LSE}}(\alpha, \beta) = \sum_{i=1}^{n} \left(F_{WHL}(s_{i:n}; \alpha, \beta) - \frac{i}{n+1} \right)^2,$$

and

$$S_{\text{WLSE}}(\alpha, \beta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F_{WHL}(s_{i:n}; \alpha, \beta) - \frac{i}{n+1} \right)^2.$$

•Cramér–von–Mises estimator

Cramér–von–Mises Estimator (CME) is introduced by Choi and Bulgren (1968). The CMEs is obtained by minimizing the following function:

$$S_{\text{CME}}(\alpha, \beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F_{WHL}(s_{i:n}; \alpha, \beta) - \frac{2i-1}{2n} \right)^2.$$

•Anderson–Darling and right-tailed Anderson–Darling

The Anderson Darling (ADE) and Right-Tailed Anderson Darling Estimators (RTADE) are introduced by Anderson and Darling (1952). The ADE's and RTADE's are obtained by minimizing the following functions:

$$S_{\text{ADE}}(\alpha, \beta) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1)\{\log F_{WHL}(s_i; \alpha, \beta) + \log \overline{F_{WHL}}(s_{n+1-i}; \alpha, \beta)\},$$

and

$$S_{\text{RTADE}}(\alpha, \beta) = \frac{n}{2} - 2\sum_{i=1}^{n} F_{WHL}(s_i; \alpha, \beta) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \overline{F_{WHL}}(s_{n+1-i}; \alpha, \beta),$$

where $\overline{F_{WHL}}(\cdot) = 1 - F_{WHL}(\cdot)$.

•Method of maximum product of spacings Cheng and Amin (1979 , 1983) introduced the maximum product of spacings (MPS) method as an alternative to MLEs for estimating parameters of continuous univariate distributions. The MPS's are obtained by maximizing the following functions:

$$G(\alpha, \beta) = \left[\prod_{i=1}^{n+1} D_i(\alpha, \beta) \right]^{\frac{1}{n+1}},$$

where $D_i(\alpha, \beta) = F_{WHL}(s_{i:n}; \alpha, \beta) - F_{WHL}(s_{i-1:n}; \alpha, \beta),\ i = 1, ..., n,,$ $F_{WHL}(s_{0:n}; \alpha, \beta) = 0$ and $F_{WHL}(s_{n+1:n}; \alpha, \beta) = 1$.

3.3. *Simulation study*

In order to explore the estimators introduced above we consider the one model that have been used in this section , and investigate MSE of those estimators for different samples. For instance according to what has been mentioned above, for $(\alpha, \beta) = (0.5, 1.5), (1.5, 2), (2, 3)$.

The performance of each method of parameters estimations for the WHL distribution with respect to sample size n is considered. To do this, a simulation study is done based on following steps:

Step 1. Generate ten thousand samples of size n from (3) for Half-Logistic case. This work is done simply by quantile function and generated data from uniform distribution .

Step 2. Compute the estimates for the one thousand samples, say $\left(\hat{\alpha}_i, \hat{\beta}_i\right)$ for $i = 1, 2, ..., 10^4$.

Step 3. Compute the biases and mean squared errors by

$$Bias_\varepsilon(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\varepsilon}_i - \varepsilon),$$

and

$$MSE_\varepsilon(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\varepsilon}_i - \varepsilon)^2.$$

for $\varepsilon = \alpha, \beta$. We repeated these steps for $n = 30(40)500$ with mentioned special case of parameters. So computing $Bias_\varepsilon(n)$ and $MSE_\varepsilon(n)$ for $\varepsilon = \alpha, \beta$ and $n = 30(40)500$. To obtain the value of the estimators, we have used the optim function and the Nelder-Mead method in the statistical package R version 3.4.4. The results are shown in Figures 4-6.

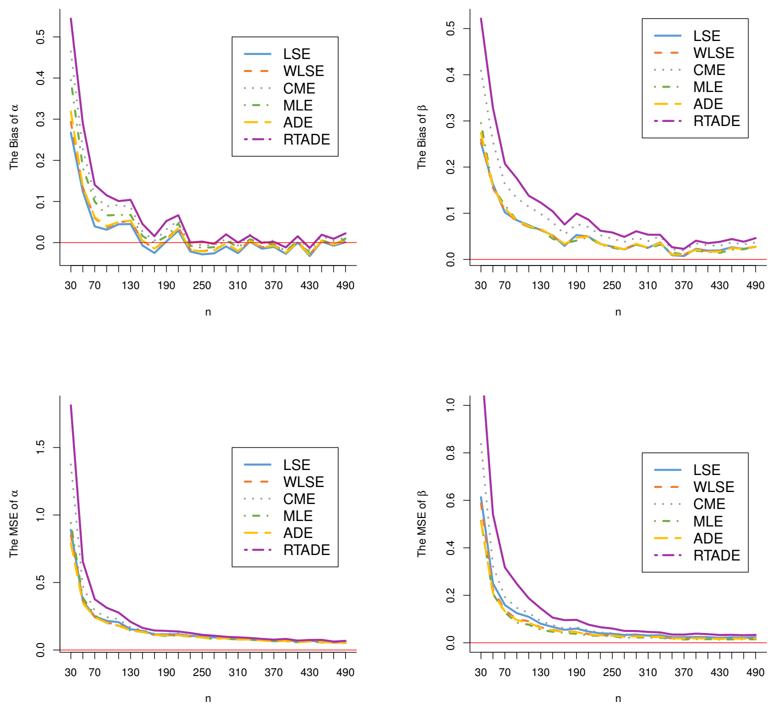

Figure 4. Bias and MSE of estimations for parameter values $(\alpha, \beta) = 0.5, 1.5)$

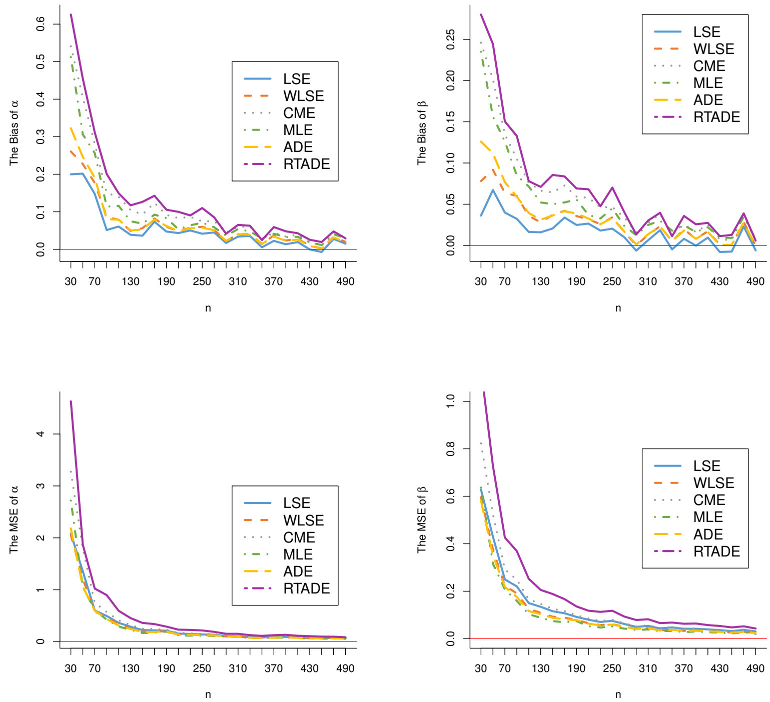

Figure 5. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (1.5, 2)$

One can see that MSE plots for three parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the distribution parameters WHL. Also

- Based on figures 4-6, for estimating α, LSE method has the minimum amount of bias, however for large sample size, all methods have almost same behaviour and converge to zero as expected.
- Based on figures 4-6, for estimating β, LSE method has the minimum amount of bias, however for large sample size, all methods have almost same behaviour and converge to zero as expected.
- Based on figures 4-6, for estimating α, MLE method has the minimum amount of MSE,however for large sample size, all methods have almost same behaviour and converge to zero as expected..
- Based on figures 4-6, for estimating β, MLE method has the minimum amount of MSE, however for large sample size, all methods have almost same behaviour and converge to zero as expected.

4. Applications

In this section, we present two applications by fitting the WHL model and some famous models. The Cramér–von Mises (W^*), Anderson-Darling (A^*) and $p - value$ for Kolmogorow-Smirnow test have been chosen for comparison the models for the first two examples.
The exponentiated half-logistic (ESHL) distribution (Kang and Seo, 2011), Kumaraswamy standard Half-Logistic distribution (KwSHL) (Cordeiro and de Castro, 2011) , the Beta standard Half-Logistic (BSHL) (Jones, 2004), McDonald standard Half-Logistic (McSHL)distributin (Oliveria et.al, 2016), New Odd log-logistic standard Half-Logistic (NOLL-SHL) distribution (Alizadeh et al. , 2019), weibull distrbution (W), Generalized Exponential

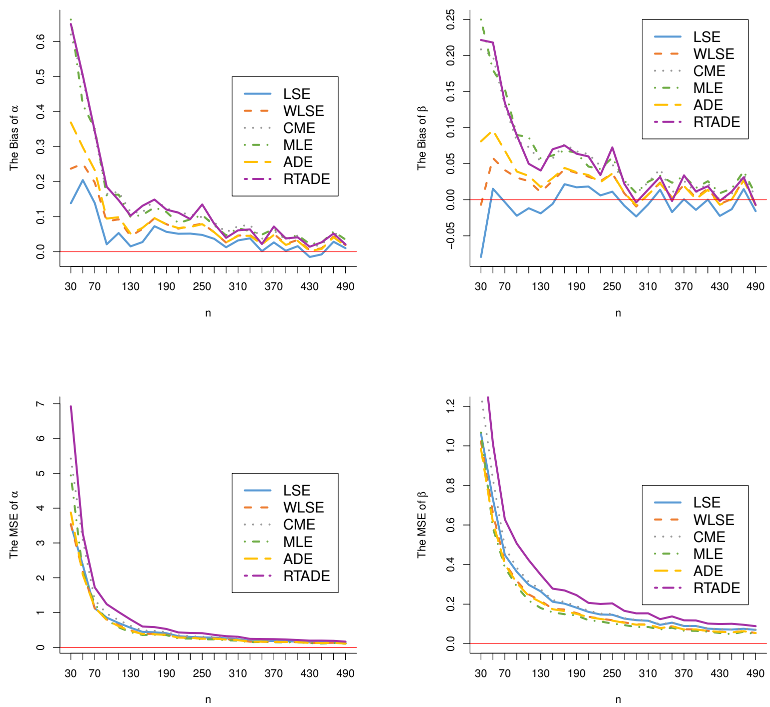

Figure 6. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (2, 3)$

(GE) distribution (Gupta and Kundu, 1998), Log Normal (LN) distribution, Gamma (Ga) distribution, Lindley (Li) distribution (Ghitany et al., 2008), Power Lindley (PL) distribution (Ghitany et al., 2013) and Generalized Lindley (GL) distribution (Nadarajah et al., 2011) have been selected for comparison in two examples. The cdf of these models are given in Appendix. The parameters of models have been estimated by the MLE method.

4.1. Data set I

. The first real data set is failure time of 50 items given by Murty (2004, p195). The data are: 0.008, 0.017, 0.058, 0.061, 0.084, 0.090, 0.134, 0.238, 0.245, 0.353, 0.374, 0.480, 0.495, 0.535, 0.564, 0.681, 0.686, 0.688, 0.921, 0.959, 1.022, 1.092, 1.260, 1.284, 1.295, 1.373, 1.395, 1.414, 1.760, 1.858, 1.892, 1.921, 1.926, 1.933, 2.135, 2.169, 2.301, 2.320, 2.405, 2.506, 2.598, 2.808, 2.971, 3.087, 3.492, 3.669, 3.926, 4.446, 5.119, 8.596 .
In the Table 1, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. One can see , the WHL(α, β) distribution is selected as the best model with more criteria. The histogram of the data set I and the plots of fitted pdf are displayed in Figure 7.

4.2. Data set II

. The second real data set is failure time of 50 items given by Murty (2004, p195). The data are: 0.061, 0.073, 0.075, 0.084, 0.086, 0.087, 0.088, 0.089, 0.089, 0.089, 0.099, 0.102, 0.117, 0.118, 0.119, 0.120, 0.123, 0.135, 0.143, 0.168, 0.183, 0.185, 0.191, 0.192, 0.199, 0.203, 0.213, 0.215, 0.257, 0.258, 0.275, 0.297, 0.297, 0.298, 0.299, 0.308, 0.314, 0.315, 0.330, 0.374, 0.388, 0.403, 0.497, 0.714, 0.790, 0.815, 0.817, 0.859, 0.909, 1.286 .
.
Similar to the previous application example, we have Tables 2 . As it is clear, the WHL(α, β) is selected as the best

Table 1. Results for data set I

model	estimated parameters (se)			W^*	A^*	$p-value$
WHL (α,β)	0.863 (0.456)	0.684 (0.098)		**0.027**	**0.182**	**0.975**
NOLL-SHL (α,β)	0.796 (0.147)	0.743 (0.126)		0.041	0.256	0.715
ESHL (α)	0.946 (0.133)			0.035	0.224	0.127
KwSHL (α,β)	0.719 (0.149)	0.668 (0.116)		0.037	230	0.910
BSHL (α,β)	0.738 (0.136)	0.664 (0.119)		0.038	0.239	0.900
McSHL (α,β,c)	91.722 (78.442)	0.677 (0.115)	0.007 (0.005)	0.034	0.218	0.921
Li(α)	0.910 (0.096)			0.049	0.300	0.865
PL(α,β)	0.994 (0.127)	0.882 (0.096)		0.063	0.375	0.818
GE(α,β)	0.560 (0.105)	0.903 (0.162)		0.076	0.448	0.656
GL(α,β)	0.790 (0.122)	0.767 (0.142)		0.046	0.285	0.887
LN(α,β)	−0.123 (0.205)	1.452 (0.145)		0.329	1.902	0.155
Ga(α,β)	0.914 (0.159)	0.546 (0.125)		0.077	0.455	0.661
W(α,β)	0.610 (0.105)	0.976 (0.111)		0.079	0.468	0.723

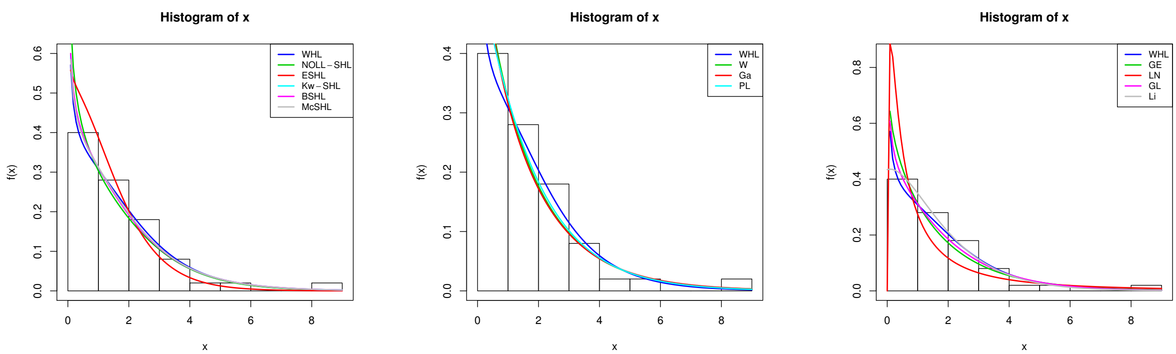

Figure 7. Histogram and fitted pdfs for data set I.

model with more criteria. The histogram of the data set I and the plots of fitted PDF are displayed in Figure 8.

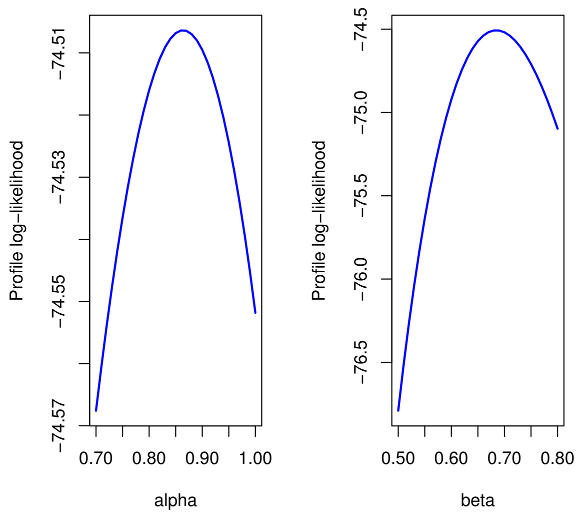

Figure 8. Unimodality of profile likelihood functions of parameters for first data set I.

Table 2. Results for data set II

model	estimated parameters (se)			W^*	A^*	$p-value$
WHL (α, β)	88.986 (49.827)	1.705 (0.227)		$\mathbf{0.107}$	$\mathbf{0.840}$	$\mathbf{0.761}$
NOLL-SHL (α, β)	0.632 (0.094)	8.928 (1.349)		0.420	2.689	0.092
ESHL (α)	0.445 (0.062)			0.192	1.398	$7.745e-06$
KwSHL (α, β)	1.226 (0.145)	8.811 (2.049)		0.261	1.794	0.239
BSHL (α, β)	1.490 (0.271)	8.713 (1.822)		0.240	1.674	0.263
McSHL (α, β, c)	831.317 (183.122)	6.952 (1.392)	0.003 (0.004)	0.191	1.391	0.461
Li(α)	4.060 (0.493)			0.206	1.489	0.048
PL(α, β)	4.830 (0.731)	1.236 (0.130)		0.245	1.706	0.307
GE(α, β)	5.088 (0.845)	1.955 (0.432)		0.178	1.320	0.511
GL(α, β)	5.732 (0.861)	1.896 (0.422)		0.185	1.367	0.488
LN(α, β)	-1.539 (0.109)	0.771 (0.077)		0.091	0.743	0.644
Ga(α, β)	1.718 (0.316)	5.828 (1.242)		0.197	1.436	0.406
W(α, β)	4.189 (0.716)	1.257 (0.129)		0.237	1.664	0.318

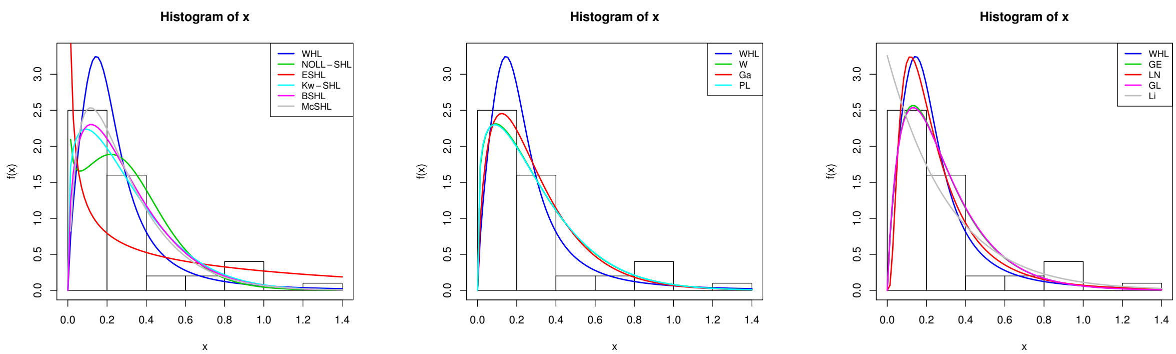

Figure 9. Histogram and fitted pdfs for data set II.

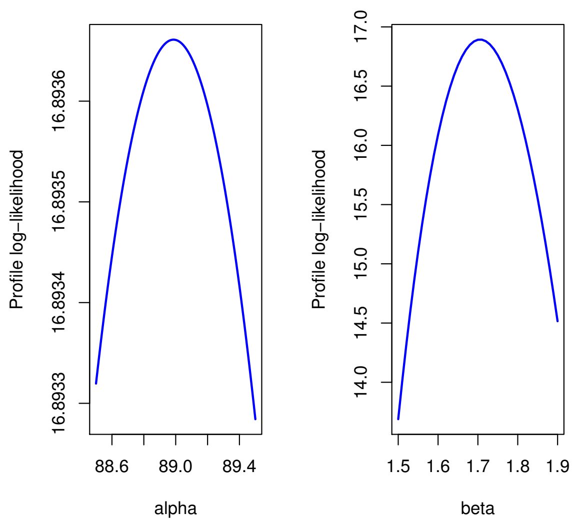

Figure 10. Unimodality of profile likelihood functions of parameters for first data set II.

5. Conclusions

We introduce a new two-parameter extension of half-logistic distributions based on arctan function called the weighted half-logistic (WHL) distribution. Some properties of the new family, such as quantile function, extreme value, linear combination for cdf and pdf, moments, conditional moments, moment generating function and mean deviation are obtained. We estimate the parameters using maximum likelihood and other different methods. The Bias and MSE plots of parameters for all methods, will approach to zero with the increase in the volume of the sample which verifies the validity of the these estimation methods. The flexibility of this distribution is assessed by applying it to real data sets and comparing purpose distribution with others. The results show that the new model provide consistently better fits than other competitive models for these data sets. So Applications demonstrate the importance of the new family.

Appendix: cdf of competitive models in application section

$$F_{ESHL(\alpha)}(t) = \Pi(t)^{\alpha}, \qquad t>0, \alpha>0,$$

$$F_{NOLL-SHL(\alpha,\beta)}(t) = \frac{\Pi(t)^{\alpha}}{\Pi(t)^{\alpha}+\bar{\Pi}(t)^{\alpha}}, t>0, \alpha>0, \beta>0,$$

$$F_{KwSHL(\alpha,\beta)}(t) = 1-\left[1-\Pi(t)^{\alpha}\right]^{\beta}, \qquad t>0, \alpha>0, \beta>0,$$

$$F_{BSHL(\alpha,\beta)}(t) = \frac{1}{Beta(\alpha,\beta)}\int_0^{\Pi(t)} t^{\alpha-1}(1-t)^{\beta-1}, \qquad t>0, \alpha>0, \beta>0,$$

$$F_{McSHL(\alpha,\beta,c)}(t) = \frac{1}{Beta(\alpha,\beta)}\int_0^{\Pi(t)^c} t^{\alpha-1}(1-t)^{\beta-1}, \qquad t>0, \alpha>0, \beta>0, c>0,$$

$$F_{Li(\alpha)}(t) = 1-\frac{1+\alpha+\alpha\, t}{1+\alpha}\,\mathrm{e}^{-\alpha\,\mathrm{t}}, \qquad \mathrm{t}>0, \alpha>0,$$

$$F_{PL(\alpha,\beta)}(t) = 1-\frac{1+\alpha+\alpha\, t^{\beta}}{1+\alpha}\,\mathrm{e}^{-\alpha\,\mathrm{t}^{\beta}}, \qquad \mathrm{t}>0, \alpha>0, \beta>0,$$

$$F_{GE(\alpha,\beta)}(t) = (1-\mathrm{e}^{-\alpha\,\mathrm{t}})^{\beta}, \qquad \mathrm{t}>0, \alpha>0, \beta>0,$$

$$F_{GL(\alpha,\beta)}(t) = \left[1-\frac{1+\alpha+\alpha\, t}{1+\alpha}\,\mathrm{e}^{-\alpha\,\mathrm{t}}\right]^{\beta}, \qquad t>0, \alpha>0,$$

$$F_{LN(\alpha,\beta)}t;\alpha,\beta = \Phi(\frac{\log(t)-\alpha}{\beta}), \qquad t>0, \alpha\in\mathbf{R}, \beta>0, t;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^t z^{\alpha-1}\mathrm{e}^{-\beta\,\mathrm{z}}\mathrm{dz}, \qquad \mathrm{t}>0\mathrm{nonumber}$$

$$F_{Ga(\alpha,\beta)}, \alpha>0, \beta>0,$$

$$F_{W(\alpha,\beta)}(t) = 1-\mathrm{e}^{-\alpha\,\mathrm{t}^{\beta}}, \qquad \mathrm{t}>0, \alpha>0, \beta>0.$$

where $\Phi(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{z}^2/2}\mathrm{dz}$ denote the cdf of standard normal random variable.

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