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E-Bayesian Estimations and Its E-MSE for Compound Rayleigh Progressive Type-II Censored Data

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Abstract Over the past decades, various methods to estimate the unknown parameter, the survival function, and the hazard rate of a statistical distribution have been proposed from the availability of type-II censored data. They are all differing in terms of how the progressive type-II censored data of the underlying distribution are available. In this study, we estimate the parameter, the survival function, and the hazard rate of the compound Rayleigh distribution by using the E-Bayesian estimation when the progressive type-II censored data are available. The resulting estimators are evaluated based on the asymmetric general entropy and the symmetric squared error loss functions. In addition, the E-Bayesian estimators under the different loss functions have been compared through a real data analysis and Monte Carlo simulation studies by calculating the E-MSE of the resulting estimators.

Keywords E-Bayesian estimation, Compound Rayleigh distribution, Progressive Type-II censored data, Survival functions, General Entropy loss function, Monte Carlo simulation

AMS 2010 subject classifications 62F15, 62N05

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1. Introduction

The Compound Rayleigh distribution (CRD) is a very useful distribution for modeling and analyzing in different aspects of statistics, such as lifetime studies and reliability theory, especially in medical science. For instance, Mostert et al. [17] used CRD for modeling lifetimes of cancer patients with characteristics of a random hazard rate. Barot and Patel [8] developed Bayesian and Posterior risk approach based on progressive type-II censored data from a CRD under the balanced loss functions. Algarni et al. [2] used two CR lifetime models to analyze the jointly type-II censoring data. They obtained classical estimators such as Maximum Likelihood for CRD.

Reliability and hazard functions have many applications in statistics, engineering and lifetime testing. For example, Asgharzadeh et al. [4] studied the stress-strength reliability of the two-parameter generalized exponential records via the Bayes and maximum likelihood methods. Diamoutene et al. [11] used reliability analysis and proportional hazard rate models (PHM) in aeronautics?. Demiray and K?z?laslan [10] studied classical and Bayesian approaches to estimate the stress¨Cstrength reliability in consecutive k-out of-n systems when the two variables are belonging to PHM. Nabeel et al. [18] proposed robust PHM under monitoring schemes to evaluate the reliability.

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The censored data and new distributions are usually used to study reliability and lifetime functions. For instance, Kayal et al. [15] studied the inference for Chen distribution under type-I progressive hybrid censored data. Al-Zahrani and Ali [3] obtained the recurrence formulas of moments under multiple type-II censored sample of order statistics for Lindley distribution. Valiollahi et al. [23] investigated the stress-strength reliability for two Weibull independent random variables with the same scale and different shape parameters, under progressive type-II censored data. Ma et al. [16] studied the stress-strength reliability under inverted exponential Rayleigh distribution and progressive type-II censored samples?. Ren and Gui [21] investigated the inference of competing risks model for generalized Rayleigh distribution under progressive type-II censored data when the latent lifetime parameters are common or different. Feroze et al. [12] explored Bayesian estimation of reliability characteristics for the Topp¨CLeone distribution based on progressive type-II censored data.

Alternatives to Bayesian estimation, E-Bayesian estimation method has been proposed to deal with the parameter estimation from a statistical distribution Han [13]; Piriaei et al. [20]; Athirakrishnan and Abdul-Sathar [5]). For example, Han [13] derived the E-Bayesian estimation and its expected mean squared error (E-MSE) for exponential distribution based on the scaled squared error loss function. Athirakrishnan and Abdul-Sathar [5] estimated the scale parameter and reversed hazard rate of inverse Rayleigh distribution via Hierarchical Bayesian and E-Bayesian methods under the squared error, precautionary and entropy loss functions. Also, Piriaei et al. [20] estimated the cumulative hazard rate and mean residual life of generalized inverted Exponential distribution via E-Bayesian method and under type-II censored samples. Further, Piriaei et al. [19] derived the E-Bayesian estimations of hazard and reliability functions for Exponential record data.

This paper focuses on the E-Bayesian method for estimating the unknown parameter, the survival function, and the hazard function of CRD. These estimates are obtained based on the asymmetric general entropy (GE) and the symmetric squared error loss functions (SELF). Specifically, we use the E-Bayesian method based on the progressive type-II censored data because it is shown that the E-Bayesian estimation is more precise than the Bayesian estimation, and also the progressive type-II censoring is a more cost-effective plan. Further, we display that the limiting behaviors of the E-Bayesian estimators based on different priors are the same. Moreover, to compare the E-Bayesian estimators, we use the E-MSE of the resulting estimators, which is a new measure to comparison of E-Bayesian estimators (Han [13]).

Our main motivations for writing this paper are as follows:

1. As shown in previous researches, the hazard function of CRD has a zero asymptote as time goes to infinity and it has a lot of flexibility and application for analyzing cancer data.
2. In some survival studies, the CRD has a better level of fit than the other distributions.
3. It is proven that the E-Bayesian estimator has a higher precision than the older estimators including maximum likelihood (ML) and Bayesian estimators.

The rest of the paper is organized as follows. The specifications of the CRD are given in the next section. In sections 3 and 4, the Bayesian and E-Bayesian estimations for the parameter and characteristics of interest, respectively, are presented. We discuss the relationships among E-Bayesian estimators in section 5. Calculations and evaluations the resulting E-Bayesian estimators have been carried out via real data analysis and Monte Carlo simulation in section 6. Eventually, concluding remarks are given in section 7.

2. Compound Rayleigh Distribution (CRD)

The probability density function $f(x)$ and the cumulative distribution function $F(x)$ of CRD are, respectively

$$f(x) = 2\alpha\beta^{\alpha}x(\beta + x^2)^{-(\alpha+1)}\ , \quad x>0,\ \alpha>0,\ \beta>0 \tag{1}$$

and

$$F(x) = 1 - \beta^{\alpha}(\beta + x^2)^{-\alpha}\ , \quad x>0 \tag{2}$$

where β is the scale and α is the shape parameters. In this study, β is considered to be known. As can be seen from (1) and (2), the CRD is belonging to the PHM. The survival and hazard functions of CRD at mission time t is given, respectively, by

$$S(t) = (\frac{\beta}{\beta + t^2})^{\alpha}, \quad t>0, \tag{3}$$

$$h(t) = 2\alpha t(\beta + t^2)^{-1}, \; t>0. \tag{4}$$

It should be noted that Bhattacharya and Tyagi [9] used the Rayleigh distribution, with the hazard function $h(t) = 2\theta t$, to analyze data from cancer patients. Now, suppose that θ to be a random variable (that means the patients are sampled from populations with variable hazard functions), then the lifetimes distribution will be a compound Rayleigh distribution. In particular, if θ to be a Gamma random variable with the shape and scale parameters α and β, respectively, then we arrive at the CRD with the probability distribution function (1).

The compound Rayleigh distribution, with its unimodal hazard function, makes it attractive for modeling lifetimes of patients with characteristics of a random hazard rate. The hazard function has one critical (maximum) point at $(t, h(t)) = (\beta^{\frac{1}{2}}, \alpha\beta^{-\frac{1}{2}})$. Thus, one can control the height of $h(t)$ at the peak time $\beta^{\frac{1}{2}}$ by controlling the parameter α. Figure 1 shows the plot of $h(t)$ for a fixed value of β and different values of α. In contrast, it is easy to observe that for a fixed value of α, the maximum point of $h(t)$ shifts to the right, for an increasing scale parameter β. For large values of t, $h(t)$ is a decreasing function of t, so the compound Rayleigh model is not reasonable as lifetime model in most situations. However, when very large values of t are not of interest, the compound Rayleigh model is usually suitable for the lifetimes data, see for example Abdel-Ghaly and Attia [1]. Samples are usually censored in survival and

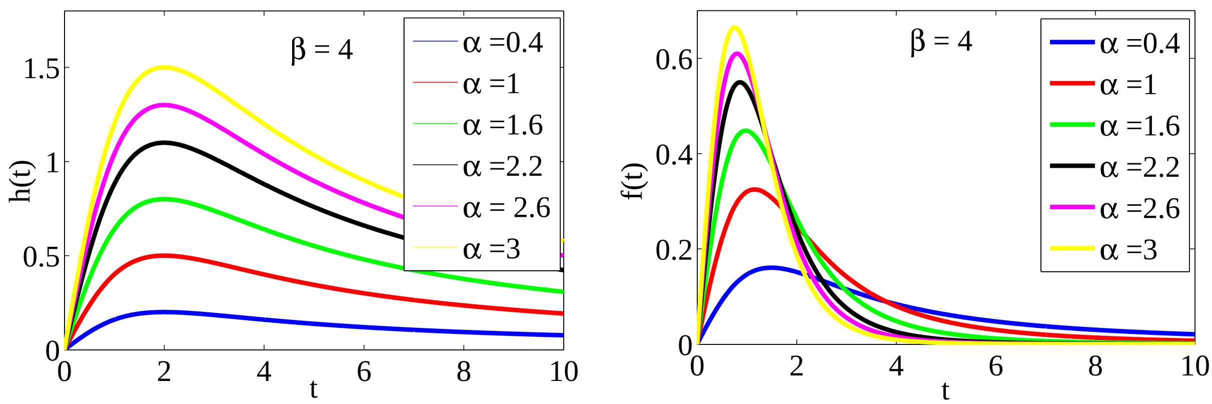

Figure 1. Hazard rate function of CRD (left); and Probability density function of CRD (right).

lifetime analysis. Assume that n independent components are put on a test and the lifetime distribution of each component is the $f(x)$ of (1). The ordered m-failures, denoted by $x_{(1)} < x_{(2)} < \cdots < x_{(m)}$ (for convenience notation are denoted by $x_1 < x_2 < ... < x_m$), are observed based on the progressive type-II censoring scheme $(R_1, \ldots, R_m)$ where each $R_i \geq 0$ and $\sum_{j=1}^{m} R_j + m = n$. Based on these observation, the likelihood function is

$$L(\alpha, \beta) = c \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i}, \tag{5}$$

where $c = n(n - 1 - R_1) \ldots (n - R_1 - \cdots - R_{(m-1)} - m + 1)$. For more details, please see Balakrishnan et al. [6] and Balakrishnan and Cramer [7].

Substituting (1) and (2) in (5), the latter expression can be obtained as follows

$$L(\alpha,\beta) = c\ 2^{m}(\alpha\beta)^{m} e^{-\alpha T} e^{\sum_{i=1}^{m}\ln x_i} e^{-\sum_{i=1}^{m}\ln\left(\beta+x_i^2\right)}\ , \tag{6}$$

where $T=\sum_{i=1}^{m}\left[\ln(\beta+x_i^2)(1+R_i)-R_i\ln\beta\right]$.

3. Bayes Estimation

We consider the following conjugate family of prior

$$\pi(\alpha) = \frac{b^{a}}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}. \tag{7}$$

Combining (6) and (7), we obtain the posterior function as

$$\pi\left(\alpha\mid x\right) = \frac{(T+b)^{m+a}}{\Gamma\left(m+a\right)}\alpha^{m+a-1}e^{-\alpha(T+b)}. \tag{8}$$

Under the SELF, $\left(L(\hat{\alpha}-\alpha)=(\hat{\alpha}-\alpha)^2\right)$, the Bayes estimator of α, can be driven by (8) as

$$\hat{\alpha}_S = \frac{m+a}{T+b}\ . \tag{9}$$

The Bayes estimator of the survival function under the SELF is obtained from (3) and (8) as

$$\hat{S}_S = \left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a}. \tag{10}$$

The Bayes estimator of the survival function under the SELF is obtained from (4) and (8) as

$$\hat{h}_S = \frac{2t(m+a)}{(\beta+t^2)(T+b)}. \tag{11}$$

The GELF is asymmetric and has the following form for α

$$L(\delta) = \delta^{q}-q\ln(\delta)-1; q\neq 0, \tag{12}$$

where $\delta=\frac{\hat{\alpha}}{\alpha}$ and $\hat{\alpha}$ is an estimator of α (Varian [24]). The Bayes estimator of α, under the GELF is

$$\hat{\alpha}_G = \left[E(\alpha^{-q})\right]^{-1/q}. \tag{13}$$

Under GELF, we obtain Bayes estimators of α, $S(t)$ and $h(t)$ by combining (8) and (12), respectively, as follows:

$$\hat{\alpha}_G = \left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}, \tag{14}$$

$$\hat{S}_G = \left(\frac{T+b}{T+b+q\ln(\frac{\beta}{\beta+t^2})}\right)^{-\frac{m+a}{q}}, \tag{15}$$

$$\hat{h}_G = \frac{2t}{\beta+t^2}\left(\frac{\Gamma\left(m+a-q\right)}{\Gamma\left(m+a\right)}\right)^{-\frac{1}{q}}(T+b)^{-1}. \tag{16}$$

4. E-Bayesian Estimation

Assume that the parameters a and b of conjugate family of prior for α is unknown. Thus the Bayes estimators in (9)-(?11) and (14)-(?16), are seen that they are related to a and b. In this case, we can use the Bayes estimation to obtain the E-Bayesian estimators. According to Han [14], a and b must be chosen to insure that $g(\alpha)$ is a decreasing function in α. The derivative of $g(\alpha)$ with respect to α is

$$\frac{dg(\alpha)}{d(\alpha)}=\frac{b^a}{\Gamma(a)}\alpha^{a-2}e^{-b\alpha}[(a-1)-b\alpha].$$

Note that $a>0$, $b>0$, and $\alpha>0$, it follows $0<a<1$, $b>0$ due to $\frac{dg(\alpha)}{(\alpha)}<0$; and therefore $g(\alpha)$ is a decreasing function of α. Suppose that a and b are independent with density function

$$\pi\left(a,b\right)=\pi_1\left(a\right)\pi_2\left(b\right),$$

then, the E-Bayesian estimator of α is

$$\hat{\alpha}_{EB}=E(\hat{\alpha}|X)=\int\int\hat{\alpha}_B(a,b)\pi(a,b)dbda \tag{17}$$

where $\hat{\alpha}_B(a,b)$ is the Bayes estimator of α given by (9) and (14). Also, we used the E-MSE, proposed by Han [13], to evaluate the precision of the estimators.

$$\mathrm{E-MSE}\left(\hat{\alpha}_{EB}\right)=\int\int_D MSE[\hat{\alpha}_B(a,b)]\pi(a,b)dbda$$

Here, the E-Bayesian estimations are derived under three different distributions of the hyperparameters a and b to check the effect of different prior distributions on the E-Bayesian estimations. The following distributions of a and b may be used.

$$\begin{cases}\pi_1\left(a,b\right)=\frac{2(c-b)}{c^2}, & 0<a<1, \quad 0<b<c\\ \pi_2\left(a,b\right)=\frac{1}{c}, & 0<a<1, \quad 0<b<c\\ \pi_3\left(a,b\right)=\frac{2b}{c^2}, & 0<a<1, \quad 0<b<c.\end{cases} \tag{18}$$

4.1. E-Bayesian estimations with SELF

The E-Bayesian estimations of α is derived from (9) and (18) as

$$\hat{\alpha}_{EBS1}=\int\int_D\hat{\alpha}_S(a,b)\pi_1(a,b)dbda=\int_0^1\int_0^c\frac{m+a}{T+b}\frac{2(c-b)}{c^2}dbda=\frac{2m+1}{c^2}[(T+c)\ln(\frac{T+c}{T})-c], \tag{19}$$

$$\hat{\alpha}_{EBS2}=\int\int_D\hat{\alpha}_S(a,b)\pi_2(a,b)dbda=\frac{2m+1}{2c}\ln(\frac{T+c}{T}), \tag{20}$$

and

$$\hat{\alpha}_{EBS3}=\int\int_D\hat{\alpha}_S(a,b)\pi_3(a,b)dbda=\frac{2m+1}{c^2}[c-T\ln(\frac{T+c}{T})]. \tag{21}$$

Also, the E-Bayesian estimations of S from (10) and (18) can be obtained as follows

$$\hat{S}_{EBS1}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a}\frac{2(c-b)}{c^2}dbda, \tag{22}$$

$$\hat{S}_{EBS2}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a}\frac{1}{\mathrm{c}}dbda, \tag{23}$$

and

$$\hat{S}_{EBS3}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a}\frac{2b}{c^2}dbda. \tag{24}$$

Further, the E-Bayesian estimations of h from (11) and (18) can be given as follows

$$\hat{h}_{EBS1}=\frac{(2m+1)}{c^2}\frac{2t}{\beta+t^2}[(T+c)\ln(\frac{T+c}{T})-c], \tag{25}$$

$$\hat{h}_{EBS2}=\frac{(2m+1)}{2\mathrm{c}}\frac{2t}{\beta+t^2}\ln(\frac{T+c}{T}), \tag{26}$$

and

$$\hat{h}_{EBS3}=\frac{(2m+1)}{c^2}\frac{2t}{\beta+t^2}[c-T\ln(\frac{T+c}{T})]. \tag{27}$$

4.2. E-Bayesian estimations with GELF

Under the GELF, the E-Bayesian estimation of α, S and h is computed from (14)-(16), respectively, and (18) . The estimators are as follows:

$$\hat{\alpha}_{EBG1}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2(c-b)}{c^2}dbda, \tag{28}$$

$$\hat{\alpha}_{EBG2}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{1}{\mathrm{c}}dbda, \tag{29}$$

$$\hat{\alpha}_{EBG3}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2b}{c^2}dbda, \tag{30}$$

$$\hat{S}_{EBG1}=\int_0^1\int_0^c\left(\frac{T+b}{T+b+q\ln(\frac{\beta}{\beta+t^2})}\right)^{-\frac{m+a}{q}}\frac{2(c-b)}{c^2}dbda, \tag{31}$$

$$\hat{S}_{EBG2}=\int_0^1\int_0^c\left(\frac{T+b}{T+b+q\ln(\frac{\beta}{\beta+t^2})}\right)^{-\frac{m+a}{q}}\frac{1}{\mathrm{c}}dbda, \tag{32}$$

$$\hat{S}_{EBG3}=\int_0^1\int_0^c\left(\frac{T+b}{T+b+q\ln(\frac{\beta}{\beta+t^2})}\right)^{-\frac{m+a}{q}}\frac{2b}{c^2}dbda, \tag{33}$$

$$\hat{h}_{EBG1}=\frac{2t}{\beta+t^2}\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2(c-b)}{c^2}dbda, \tag{34}$$

$$\hat{h}_{EBG2}=\frac{2t}{\beta+t^2}\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{1}{\mathrm{c}}dbda, \tag{35}$$

$$\hat{h}_{EBG3}=\frac{2t}{\beta+t^2}\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2b}{c^2}dbda. \tag{36}$$

5. Asymptotic Behaviours and Relationships Among the E-Bayesian Estimators

Now, we present the relationships among the E-Bayesian estimators. For briefness, we perform theorems only for SELF. For GELF, the results can be easily derived by the relevant relations.

5.1. Relations among $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$

Theorem 1
Let $0 < c < T$ and $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$ be given by (19)-(21), then:

1. $\hat{\alpha}_{EBS3} < \hat{\alpha}_{EBS2} < \hat{\alpha}_{EBS1}$.
2. $\lim_{T\to\infty} \hat{\alpha}_{EBS1} = \lim_{T\to\infty} \hat{\alpha}_{EBS2} = \lim_{T\to\infty} \hat{\alpha}_{EBS3}$.

Proof
The proof is in the appendix A. □

The first part of this Theorem states that with different densities (18), the E-Bayesian estimations $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$ are different too and can be ordered. Part (2) shows that $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$ are asymptotically equal when T is large.

5.2. Relations among $\hat{S}_{EBSi}$ $(i = 1, 2, 3)$

Theorem 2
Let $0 < c < T$ and $\hat{S}_{EBSi}$ $(i = 1, 2, 3)$ be given by (22)-(24), then:

$\lim_{T\to\infty} \hat{S}_{EBS1} < \lim_{T\to\infty} \hat{S}_{EBS2} < \lim_{T\to\infty} \hat{S}_{EBS3}$.

Proof
The proof is in the appendix B. □

The first part of this Theorem states that with different densities (18), the E-Bayesian estimations $\hat{S}_{EBSi}$ $(i = 1, 2, 3)$ are different too and can be ordered.

Remark 1
From (22)-(24), we have

$$\hat{S}_{EBS2} - \hat{S}_{EBS1} = \hat{S}_{EBS3} - \hat{S}_{EBS2} = \frac{1}{c^2}\int_0^1 \int_0^c (2b-c)\left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a} dbda. \quad (37)$$

The equation (37) cannot be calculated in a closed form. Therefore, numerical methods are used for calculating it, by R software. The results (Table 5) implied that the answer of this relation is positive. Or

$$\hat{S}_{EBS1} < \hat{S}_{EBS2} < \hat{S}_{EBS1}.$$

5.3. Relations among $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$

Theorem 3
Let $0 < c < T$ and $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ be given by (25)-(27), then:

1. $\hat{h}_{EBS3} < \hat{h}_{EBS2} < \hat{h}_{EBS1}$.
2. $\lim_{T\to\infty} \hat{h}_{EBS1} = \lim_{T\to\infty} \hat{h}_{EBS2} = \lim_{T\to\infty} \hat{h}_{EBS3}$.

Proof
The proof is in the appendix C. □

The first part of this Theorem states that with different densities (18), the E-Bayesian estimations $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ are different too and can be ordered. Part (2) shows that $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ are asymptotically equal when T is large.

6. Application

To illustrate the application of our results, a real data analysis and simulation study are performed.

6.1. Real data analysis

Stablein et al. [22] reported the survival times (in years) of patients undergoing chemotherapy alone, as below. 0.003, 0.173, 0.288, 0.353, 0.499, 0.592, 0.685, 0.718, 0.825, 0.825, 0.937, 0.970, 0.975, 0.981, 1.041, 1.044, 1.049, 1.049, 1.063, 1.079, 1.118, 1.260, 1.340, 1.367, 1.436, 1.449, 1.466, 1.063, 1.079, 1.118, 1.260, 1.340, 1.367, 1.436, 1.449, 1.466, 2.616, 2.652, 3.233, 3.411, 3.482, 3.499, 3.827, 4.142, 4.162. Mostert et al. [17] showed that the CRD is acceptable for these data. Using the results of equations (19)-(36), different estimates are computed and performed in Table 2. The first column is devoted to censor scheme. For example, $R = (4 * 3, 2, 0 * 10)$ means $(4, 4, 4, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
The estimation steps are:

- For given values of prior parameters, samples a and b from (18), are generated respectively.
- Under above data and generated a and b, the Bayes and E-Bayesian estimations of α, $S(t)$, and $h(t)$ are obtained.

For estimate parameters, we used the following initial values in Table 1.

Table 1. Initial values for real data set.

a	b	β	q	c
0.4962231	0.5934218	2.276	-2	1. 2

Table 2 indicates that E-Bayesian estimates are robust and satisfy the Theorems 1, 2 and 3.

Table 2. Estimates based on real data set.

Censor Scheme (n, m , R)	Estimates E-MSE of E-Bayesian estimates EBS1	EBS2	EBS3	EBG3	EBG2	EBG1
	Alpha					
n=45, m=22; R=(5*4, 3, 0*17)	0.473052	0.47108	0.469107	0.479418	0.481434	0.48345
n=45, m=32; R=(5*2, 3, 0*29)	0.578181	0.576139	0.574096	0.582861	0.584935	0.587009
n=45, m=45; R=(0*45)	0.671088	0.66912	0.667152	0.674443	0.676433	0.678422
	Hazard					
n=45, m=22; R=(5*4, 3, 0*17)	0.187273	0.186492	0.185711	0.189794	0.190592	0.19139
n=45, m=32; R=(5*2, 3, 0*29)	0.228892	0.228083	0.227275	0.230745	0.231566	0.232387
n=45, m=45; R=(0*45)	0.265672	0.264893	0.264114	0.267001	0.267788	0.268576
	Survival					
n=45, m=22; R=(5*4, 3, 0*17)	0.951947	0.952142	0.952337	0.952388	0.952193	0.951998
n=45, m=32; R=(5*2, 3, 0*29)	0.941576	0.941776	0.941976	0.942028	0.941828	0.941628
n=45, m=45; R=(0*45)	0.932501	0.932692	0.932883	0.932932	0.932742	0.932551

6.2. Simulation

For simulation study, according to Balakrishnan et al. [6], a progressive Type-II censored sample from CRD generated, as below:

1. Simulate m independent exponential random variables $Z_1, Z_2, \dots, Z_m$. This can be done using inverse transformation $Z_i = -\ln(1-U_i)$ where U_i are independent uniform $(0,1)$ random variables.
2. Set
$$X_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \frac{Z_3}{n-R_1-R_2-2} + \dots + \frac{Z_i}{n-R_1-R_2-\dots-R_{i-1}-i+1}$$
for $i = 1, 2, \dots, m$. This is the standard exponential progressive type-II censored sample.
3. Eventually, set $Y_i = F^{-1}(1-\exp(-X_i))$, for $i = 1, 2, \dots, m$, where $F^{-1}(.)$ is the inverse of cumulative distribution function of CRD. $Y_1, Y_2, ?, Y_m$ is the required progressive type-II censored sample from $F(.)$.
4. The E-Bayesian estimations of parameter α and survival function $S(t)$ and hazard rate $h(t)$, computed respectively, using (19)-(36).
5. The mentioned stages are repeated 5000 times and the E-MSEs of estimates are calculated as:
$$\mathrm{E-MSE}\left(\hat{Q}\right) = \frac{1}{5000}\sum \int \int MSE[\hat{Q}_B\left(a,b\right)]\pi(a,b)dbda,$$
where $\hat{Q}$ is an estimate of parameter Q.

To estimate the parameters via simulation, we used the following values as starting points in Table 3. The

Table 3. Initial values for real data set.

α	β	q	a	b	c	Hazard $(t=0.5)$	Survival $(t=0.5)$
0.8398245	2	-2	0.4962231	0.6034218	1.2	0.3732553	0.9058178

results are displayed in Table 4-Table 6. Further, the corresponding figures are depicted in Figures 2-4. It is clear from Table 4-Table 6 and Figures 2-4 that the E-Bayesian estimators, are robust and satisfy the Theorems 1, 2, and 3.

From Table 4 and Figure 2, the E-MSEs of estimates decrease as sample size increases. The E-Bayesian estimates, based on SELF, have smaller E-MSEs than that under GELF. The E-MSEs of E-Bayesian estimators are ordered as follows:

$$\mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBS3}}) < \mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBS2}}) < \mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBS1}}),$$

$$\mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBG3}}) < \mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBG2}}) < \mathrm{E-MSE}(\hat{\alpha}_{\mathrm{EBG1}}).$$

From Table 5 and Figure 3, the E-MSEs of estimates decrease as sample size increases. The E-Bayesian estimates, based on SELF, have smaller E-MSEs than that under GELF. The E-MSEs of E-Bayesian estimators are ordered as follows:

$$\mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBS3}}) < \mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBS2}}) < \mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBS1}}),$$

$$\mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBG3}}) < \mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBG2}}) < \mathrm{E-MSE}(\hat{\mathrm{S}}_{\mathrm{EBG1}}).$$

From Table 6 and Figure 4, the E-MSEs of estimates decrease as sample size increases. The E-Bayesian estimates, based on SELF, have smaller E-MSEs than that under GELF. The E-MSEs of E-Bayesian estimators are ordered as follows:

$$\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS3}})<\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS2}})<\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS1}}),$$

$$\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG3}})<\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG2}})<\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG1}}).$$

Table 4. Estimates of α and their E-MSE based on CRD and Monte Carlo simulation (Alpha=0.8398245, q= -2).

Censor Scheme	Estimates E-MSE of E-Bayesian estimates					
(n, m , R)	EBS1	EBS2	EBS3	EBG3	EBG2	EBG1
n=18, m=9; R=(4,4,1,0*6)	0.571883	0.564895	0.557906	0.586535	0.593882	0.601228
	0.035862	0.034963	0.034064	0.071698	0.073591	0.075483
n=18, m=14; R=(2,2, 0*12)	0.55393	0.549652	0.545374	0.563866	0.568289	0.572712
	0.021737	0.021395	0.021053	0.043556	0.044263	0.044969
n=18, m=18; R=(0*18)	0.549319	0.546028	0.542736	0.557211	0.560591	0.56397
	0.016663	0.01646	0.016258	0.033393	0.033809	0.034225
n=28, m=14; R=(4*3,2,0*10)	0.556994	0.55267	0.548346	0.566939	0.57141	0.57588
	0.021971	0.021624	0.021277	0.044018	0.044736	0.045454
n=28, m=18; R=(4,4, 2,0*15)	0.54813	0.544853	0.541576	0.556021	0.559385	0.562749
	0.016587	0.016386	0.016185	0.033244	0.033657	0.03407
n=28, m=28; R=(0*28)	0.544321	0.54223	0.540138	0.549533	0.55166	0.553788
	0.010539	0.010457	0.010375	0.021114	0.02128	0.021447
n=38, m=19; R=(5*3,4,0*15)	0.549873	0.546744	0.543615	0.55738	0.560588	0.563796
	0.01583	0.015647	0.015464	0.03172	0.032095	0.03247
n=38, m=28; R=(4,4,2,0*25)	0.543624	0.541538	0.539452	0.548834	0.550957	0.553079
	0.010513	0.010431	0.01035	0.021062	0.021228	0.021394
n=38, m=38; R=(0*38)	0.540358	0.538834	0.537311	0.544244	0.545787	0.547331
	0.007662	0.007618	0.007575	0.015346	0.015435	0.015523
n=48, m=24; R=(5*4, 4,0*19)	0.544558	0.54212	0.539682	0.550586	0.553073	0.55556
	0.012302	0.01219	0.012079	0.02465	0.024878	0.025105
n=48, m=38; R=(4,4,2,0*35)	0.54047	0.538946	0.537422	0.544356	0.5459	0.547445
	0.007666	0.007622	0.007579	0.015354	0.015443	0.015531
n=48, m=48; R=(0*48)	0.53837	0.537171	0.535971	0.541468	0.54268	0.543892
	0.006025	0.005998	0.005971	0.012064	0.012119	0.012174

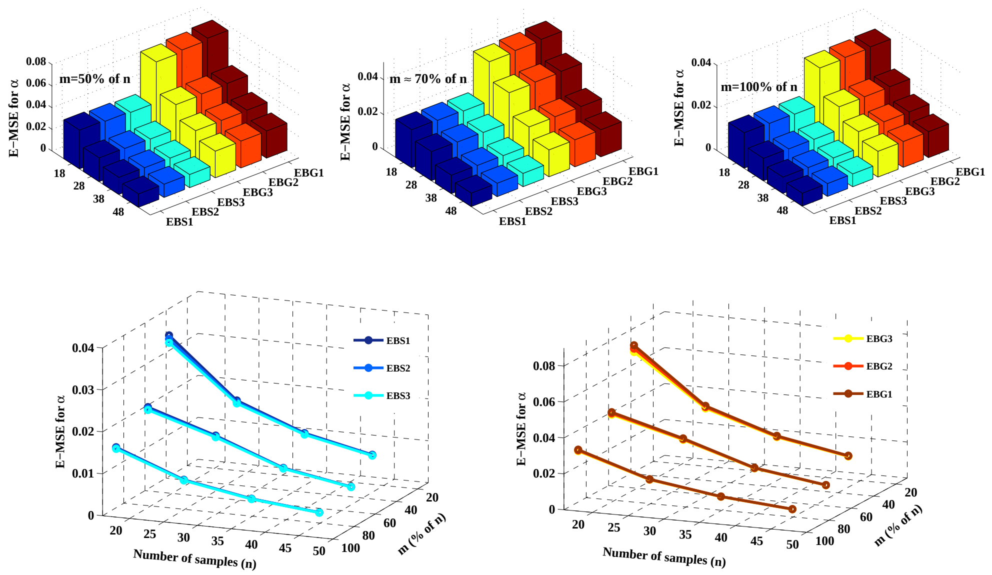

Figure 2. E-MSEs of estimates of α under (50, 70 and 100)% of samples.

Table 5. Estimates of survival and their E-MSE based on CRD and Monte Carlo simulation (S= 0.9058178, t=0.5, q=-2).

Censor Scheme	Estimates E-MSE of E-Bayesian estimates					
(n, m , R)	EBS1	EBS2	EBS3	EBG3	EBG2	EBG1
n=18, m=9; R=(4,4,1,0*6)	0.935181	0.935941	0.936702	0.936919	0.936164	0.935409
	0.000424	0.000414	0.000404	0.000798	0.000817	0.001238
n=18, m=14; R=(2,2, 0*12)	0.937035	0.937503	0.937972	0.938107	0.937641	0.937174
	0.000261	0.000257	0.000253	0.000501	0.000509	0.000857
n=18, m=18; R=(0*18)	0.937499	0.93786	0.938221	0.938326	0.937966	0.937606
	0.000201	0.000198	0.000196	0.000337	0.00039	0.000394
n=28, m=14; R=(4*3,2,0*10)	0.936698	0.937171	0.937645	0.937781	0.93731	0.936839
	0.000263	0.000259	0.000255	0.000506	0.000514	0.001019
n=28, m=18; R=(4,4, 2,0*15)	0.937629	0.937989	0.938349	0.938453	0.938094	0.937736
	0.0002	0.000198	0.000195	0.000388	0.000392	0.000654
n=28, m=28; R=(0*28)	0.937995	0.938225	0.938455	0.938523	0.938293	0.938063
	0.000128	0.000127	0.000126	0.0002	0.00025	0.000252
n=38, m=19; R=(5*3,4,0*15)	0.937431	0.937774	0.938118	0.938218	0.937875	0.937533
	0.000191	0.000189	0.000187	0.000371	0.000375	0.000673
n=38, m=28; R=(4,4,2,0*25)	0.938072	0.938302	0.938531	0.938598	0.938369	0.93814
	0.000127	0.000126	0.000125	0.00025	0.000252	0.000327
n=38, m=38; R=(0*38)	0.938407	0.938575	0.938743	0.938792	0.938625	0.938457
	9.31E-05	9.26E-05	9.21E-05	1.98E-04	1.84E-04	1.85E-04
n=48, m=24; R=(5*4, 4,0*19)	0.937986	0.938254	0.938522	0.9386	0.938332	0.938065
	0.000149	0.000147	0.000146	0.000291	0.000293	0.000535
n=48, m=38; R=(4,4,2,0*35)	0.938395	0.938563	0.938731	0.93878	0.938613	0.938445
	9.31E-05	9.26E-05	9.21E-05	1.84E-04	1.85E-04	2.41E-04
n=48, m=48; R=(0*48)	0.938612	0.938745	0.938877	0.938916	0.938784	0.938651
	7.33E-05	7.30E-05	7.27E-05	9.72E-05	1.45E-04	1.46E-04

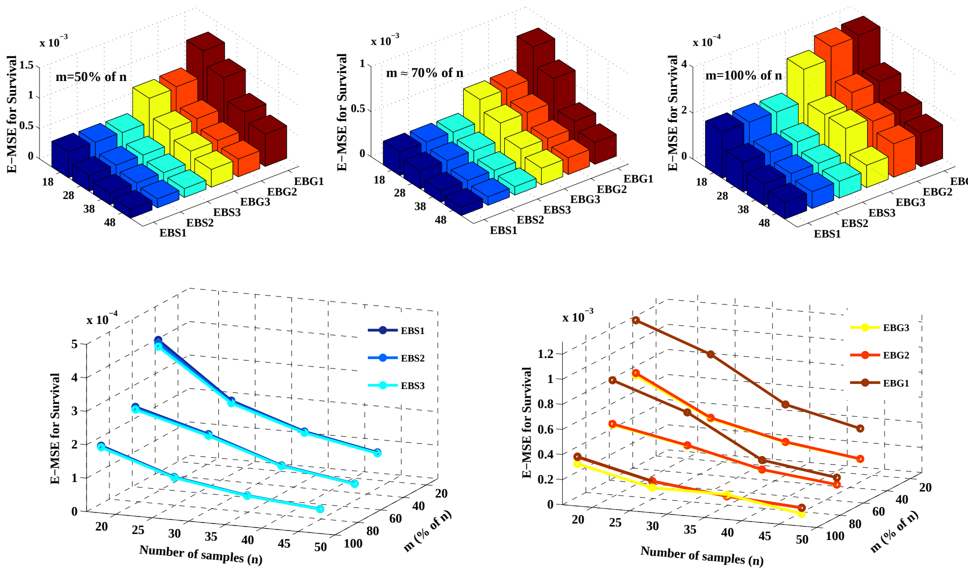

Figure 3. E-MSEs of estimates of survival function under (50, 70 and 100)% of samples.

Table 6. Estimates of $h(t)$ and their E-MSE based on CRD and Monte Carlo simulation (h=0.3732553, t=0.5, q=-2).

Censor Scheme	Estimates E-MSE of E-Bayesian estimates					
(n, m , R)	EBS1	EBS2	EBS3	EBG3	EBG2	EBG1
n=18, m=9; R=(4,4,1,0*6)	0.25417 0.007084	0.251064 0.006906	0.247958 0.006729	0.260682 0.014163	0.263947 0.014536	0.267213 0.01491
n=18, m=14; R=(2,2, 0*12)	0.246191 0.004294	0.24429 0.004226	0.242388 0.004159	0.250607 0.008604	0.252573 0.008743	0.254539 0.008883
n=18, m=18; R=(0*18)	0.244142 0.003291	0.242679 0.003251	0.241216 0.003211	0.24765 0.006596	0.249152 0.006678	0.250653 0.006761
n=28, m=14; R=(4*3,2,0*10)	0.247553 0.00434	0.245631 0.004271	0.24371 0.004203	0.251973 0.008695	0.25396 0.008837	0.255947 0.008979
n=28, m=18; R=(4,4, 2,0*15)	0.243613 0.003277	0.242157 0.003237	0.240701 0.003197	0.24712 0.006567	0.248616 0.006648	0.250111 0.00673
n=28, m=28; R=(0*28)	0.24192 0.002082	0.240991 0.002066	0.240062 0.002049	0.244237 0.004171	0.245182 0.004203	0.246128 0.004236
n=38, m=19; R=(5*3,4,0*15)	0.244388 0.003127	0.242998 0.003091	0.241607 0.003055	0.247724 0.006266	0.24915 0.00634	0.250576 0.006414
n=38, m=28; R=(4,4,2,0*25)	0.241611 0.002077	0.240684 0.00206	0.239756 0.002044	0.243926 0.00416	0.24487 0.004193	0.245813 0.004226
n=38, m=38; R=(0*38)	0.240159 0.001513	0.239482 0.001505	0.238805 0.001496	0.241886 0.003031	0.242572 0.003049	0.243258 0.003066
n=48, m=24; R=(5*4, 4,0*19)	0.242026 0.00243	0.240942 0.002408	0.239859 0.002386	0.244705 0.004869	0.24581 0.004914	0.246915 0.004959
n=48, m=38; R=(4,4,2,0*35)	0.240209 0.001514	0.239532 0.001506	0.238854 0.001497	0.241936 0.003033	0.242622 0.00305	0.243309 0.003068
n=48, m=48; R=(0*48)	0.239276 0.00119	0.238743 0.001185	0.238209 0.001179	0.240653 0.002383	0.241191 0.002394	0.24173 0.002405

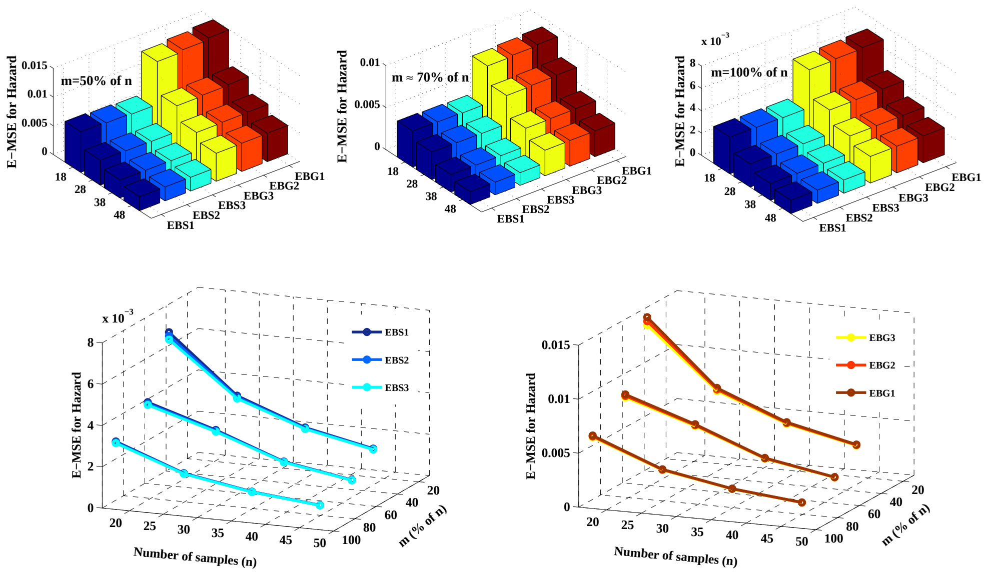

Figure 4. E-MSEs of estimates of $h(t)$ under (50, 70 and 100)% of samples.

7. Conclusions

In this paper, we have investigated the E-Bayesian estimation to estimate the parameter, the survival, and the hazard functions of CRD under the availability of the progressive type-II censored data. Our interest in this stems from the fact that we predicted the different estimators to be differently affected by the availability of the progressive type-II censored samples in a known statistical distribution such as CRD. To obtain the E-MSE of the E-Bayesian estimators, we used both the real data analyzing and Monte Carlo simulation studies with different scenarios to compare the E-MSE of the resulting estimators. In contrast to our anticipation, we found that:

- The E-Bayesian estimations obtained from various prior parameters are more close to each other when the sample size increases. By various priors, the E-Bayesian estimators are robust, and also satisfy the corresponding theorems.
- The E-MSEs of the E-Bayesian estimators decrease when the sample size n increases.
- This paper states that the E-Bayesian estimation in all scenarios, as well as the simplicity of calculations, it also has high efficiency.

Appendix A: Proof of Theorem 1.

1. From (19)-(21), we have:

$$\hat{\alpha}_{EBS2}-\hat{\alpha}_{EBS3}=\hat{\alpha}_{EBS1}-\hat{\alpha}_{EBS2}=\frac{1}{c}\left(m+\frac{1}{2}\right)\left[\frac{c+2T}{c}\ln\left(\frac{c+T}{T}\right)-2\right]. \tag{38}$$

Now, according to Mac laurin series, we have:

$$\ln(1+x) \;=\; x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\sum_{k=1}^{\infty}(-1)^{k-1}x^k/k, \qquad |x|<1. \tag{39}$$

Now, it suffices to replace x by c/T in (39). Also, note that when $0<c<T$, then we have $0<c/T<1$, and:

$$\begin{aligned}\left[\frac{c+2T}{c}\ln\left(\frac{T+c}{T}\right)\;-2\right] &=\frac{c+2T}{c}\left((c/T)-\frac{1}{2}(c/T)^2+\frac{1}{3}(c/T)^3-\frac{1}{4}(c/T)^4+\frac{1}{5}(c/T)^5-\dots\right)-2\\ &=\left((c/T)-\frac{1}{2}(c/T)^2+\frac{1}{3}(c/T)^3-\frac{1}{4}(c/T)^4+\frac{1}{5}(c/T)^5-\dots\right)-2\\ &+\left(2-(c/T)+\frac{2}{3}(c/T)^2-\frac{2}{4}(c/T)^3+\frac{2}{5}(c/T)^4-\dots\right)\\ &=\left(c^2/6T^2-c^3/6T^3\right)+\left(3c^4/6T^4-2c^5/15T^5\right)+\dots\\ &=\frac{c^2}{6T^2}\left(1-c/T\right)+\frac{c^4}{60T^4}\left(9-8c/T\right)+\dots\;. \end{aligned} \tag{40}$$

Now according to (38) and (40), we have:

$$\hat{\alpha}_{EBS2}-\hat{\alpha}_{EBS3}=\hat{\alpha}_{EBS1}-\hat{\alpha}_{EBS2}>0,$$

or

$$\hat{\alpha}_{EBS3}<\hat{\alpha}_{EBS2}<\hat{\alpha}_{EBS1}.$$

2. From (38) and (40), we get:

$$\lim_{T\to\infty}\left(\hat{\alpha}_{EBS2}-\hat{\alpha}_{EBS3}\right)=\lim_{T\to\infty}\left(\hat{\alpha}_{EBS1}-\hat{\alpha}_{EBS2}\right),$$

or

$$\lim_{T\to\infty}\hat{\alpha}_{EBS1}=\lim_{T\to\infty}\hat{\alpha}_{EBS2}=\lim_{T\to\infty}\hat{\alpha}_{EBS3}.\blacksquare$$

Appendix B: Proof of Theorem 2.

From (22)-(24), we have:

$$\lim_{T\to\infty}\left(\hat{S}_{EBS3}-\hat{S}_{EBS2}\right)=\lim_{T\to\infty}\left(\hat{S}_{EBS2}-\hat{S}_{EBS1}\right)=\frac{1}{c^2}\lim_{T\to\infty}\int_0^1\int_0^c(2b-c)\left(\frac{T+b}{T+b-\ln\left(\frac{\beta}{\beta+t^2}\right)}\right)^{m+a}dbda.$$

Here, applying the monotone convergence theorem and equivalence rule:

$$\lim_{T\to\infty}\hat{S}_{EBS1}=\lim_{T\to\infty}\hat{S}_{EBS2}=\lim_{T\to\infty}\hat{S}_{EBS3}\ .\blacksquare$$

Appendix C: Proof of Theorem 3.

1. From (25)-(27), we get:

$$\hat{h}_{EBS2}-\hat{h}_{EBS3}=\hat{h}_{EBS1}-\hat{h}_{EBS2}=\frac{2t}{\beta+t^2}\frac{1}{c}\left(m+\frac{1}{2}\right)\left[\frac{c+2T}{c}\ln\left(\frac{c+T}{T}\right)-2\right]. \tag{41}$$

   Now, according to (41), (38) and (40), we have:

$$\hat{h}_{EBS2}-\hat{h}_{EBS3}=\hat{h}_{EBS1}-\hat{h}_{EBS2}>0,$$

   or:

$$\hat{h}_{EBS3}<\hat{h}_{EBS2}<\hat{h}_{EBS1}.$$

2. According to (41), (38) and (40), we have:

$$\lim_{T\to\infty}(\hat{h}_{EBS2}-\hat{h}_{EBS3})=\lim_{T\to\infty}\left(\hat{h}_{EBS1}-\hat{h}_{EBS2}\right)$$

$$=\frac{2t}{\beta+t^2}\frac{1}{c}\left(m+\frac{1}{2}\right)\lim_{T\to\infty}\left\{\frac{c^2}{6T^2}(1-c/T)+\frac{c^4}{60T^4}(9-8c/T)+\dots\right\}=0.$$

   Or,

$$\lim_{T\to\infty}\hat{h}_{EBS1}=\lim_{T\to\infty}\hat{h}_{EBS2}=\lim_{T\to\infty}\hat{h}_{EBS3}\ .\blacksquare$$

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