STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. 10, June 2022, pp 873–889.
Published online in International Academic Press (www.IAPress.org)

A Weighted-path Following Interior-point Algorithm for Convex Quadratic Optimization Based on Modified Search Directions

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Abstract Getting a perfectly centered initial point for feasible path-following interior-point algorithms is a hard practical task. Therefore, it is worth to analyze other cases when the starting point is not necessarily centered. In this paper, we propose a short-step weighted-path following interior-point algorithm (IPA) for solving convex quadratic optimization (CQO). The latter is based on a modified search direction which is obtained by using the technique of algebraically equivalent transformation (AET) introduced by a new univariate function to the Newton system which defines the weighted-path. At each iteration, the algorithm uses only full-Newton steps and the strategy of the central-path for tracing approximately the weighted-path. We show that the algorithm is well-defined and converges locally quadratically to an optimal solution of CQO. Moreover, we obtain the currently best known iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ which is as good as the bound for linear optimization analogue. Some numerical results are given to evaluate the efficiency of the algorithm.

Keywords Convex quadratic optimization, Interior-point methods, Short-step method, Polynomial complexity

AMS 2010 subject classifications 90C20, 90C51, 90C60

DOI: 10.19139/soic-2310-5070-1385

1. Introduction

Consider the quadratic optimization (CQO) problem in its primal standard format:

$$(\mathcal{P}) \quad \min_{x} \left\{ \frac{1}{2}x^T Qx + c^T x : Ax = b,\, x \geq 0 \right\},$$

and its dual problem

$$(\mathcal{D}) \quad \max_{(x,y,\,z)} \{b^T y - \frac{1}{2}x^T Qx : A^T y + z - Qx = c,\, z \geq 0\},$$

where $Q \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$ with rank $(A) = m$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Here $x,\, z \geq 0$ says that x and z are nonnegative vectors in $\mathbb{R}^n$.

The CQO problems are an interesting class of nonlinear convex programming which have been proven to be useful in many domains of applied mathematics and engineering. Also the CQO includes the

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

standard linear optimization (LO) [5]. Further, it can be seen as a special case of general symmetric conic optimization problems (see e.g.,[13, 14]). There are many solution methods for solving CQOs. Among them, feasible path-following interior-point methods (IPMs) gained much more attention then other methods due to their numerical efficiency and their polynomial complexity (see e.g.,[18, 21, 22]). These methods used the central-path as a guideline to follow the central-path for reaching an optimal solution of CQO. However, their derived algorithms require that the starting point must be strictly feasible and on the central-path. Still, it is uneasy task to find a perfectly centered initial point. Therefore, it is worth to analyze other cases when starting points are not centered. It is well-known that with every algorithm which follows the central-path, we can associate a target sequence on the central-path. This idea leads to the concept of target-following methods introduced earlier by Jansen et al [15]. Weighted-path following IPMs can be viewed as a special case of them and which are used as an alternative to remedy this drawback [2, 4, 7, 11, 18]. In [10], Darvay proposed a weighted-path following IPMs for solving LO. The modified Newton search direction is obtained by using the technique of algebraically equivalent transformation (AET) based on the univariate function $\sqrt{t}$ to the Newton's system which defines the weighted-path. The corresponding short-step interior-point algorithm (IPA) deserves the best known iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$. Later, Achache [3], Mansouri et al. [16] and Wang et al. [17, 18] extended successfully Darvay's algorithm for solving monotone LCP, $P_*(\kappa)-$LCP over symmetric cones, monotone mixed and horizontal LCP, respectively. The relevance of the technique of AETs for the central-path equation has been treated in [9, 17] and subsequently in references [1, 8]. We also mentioned that some new type of search directions are obtained from the application of so-called kernel functions (see e.g.,[13, 24]).

In this paper, our main purpose is to investigate a new weighted-path following IPA for solving the CQOs based on a new type of Newton search direction. The latter is obtained via the AET technique induced by the new univariate function $t^{\frac{3}{2}}$ applied to the Newton system which defines the weighted-path. At each iteration, only full-Newton steps and the strategy of the central-path are used for getting an ϵ-approximated optimal solution for CQO. For its analysis, specific choices of defaults of the threshold τ which defines the size of the neighborhood of the weighted-path and of the parameter θ which determines the rate of decrease of the barrier parameter are offered. Under these two defaults, the short-step IPA is well-defined and converges locally quadratically to an optimal solution of CQO. Moreover, the currently best known iteration bound is obtained, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. This iteration bound is as good as the bound for LO analogue. Finally, some numerical results are reported to evaluate the efficiency of our algorithm. Moreover, in order to improve our numerical results, some changes are imported on the original version of our algorithm where the obtained results are totally ameliorated.

The notation used in the paper is as follows. $\mathbb{R}^n$ denotes the space of real n-dimensional vectors and $\mathbb{R}^n_{++}$ stand for all positive vectors of $\mathbb{R}^n$. Given $x, z \in \mathbb{R}^n$, $x^Tz = \sum_{i=1}^n x_iz_i$ denotes their usual inner product. Meanwhile, xz denotes the component-wise product of these vectors, i.e. $xz = (x_1z_1, \cdots, x_nz_n)^T$. Let $x,\, z \in \mathbb{R}^n_{++}, \sqrt{x} = (\sqrt{x}_1, \cdots, \sqrt{x}_n)^T, x^{-1} = (x_1^{-1}, \cdots, x_n^{-1})^T$ and $\frac{x}{z} = \left(\frac{x_1}{z_1}, \cdots, \frac{x_n}{z_n}\right)^T, z \neq 0$. Let $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^Tx}$ and $\|x\|_\infty = \max_i |x_i|$ denote its Euclidean and maximum norm, respectively. Furthermore, $\mathrm{diag}(x)$ denotes the diagonal matrix obtained via the components of the vector x. Let $g(x)$ and $f(x)$ be two positive real valued functions, then $g(x) = \mathcal{O}(f(x))$ if $g(x) \leq kf(x)$ for some positive constant k. Finally, e denotes the vector of ones in $\mathbb{R}^n$.

The paper is organized as follows. In Section 2, the weighted-path, the modified Newton search direction and the proximity measure are stated. The generic IPA for CQO is also described. In Section 3, detailed proofs of the convergence of the algorithm are given. The iteration bound with short-step method is derived. In Section 4, some numerical results are reported. A conclusion and future remarks end the paper in Section 5.

2. A weighted-path following IPA for CQO

In this section, we study first the existence and the uniqueness of the weighted-path of CQO and the new modified search directions. Finally, we state the generic weighted-path following full-Newton step IPA for CQO.

2.1. The weighted-path of CQO

Throughout the paper, we assume that both problems $(\mathcal{P})$ and $(\mathcal{D})$ satisfy the following conditions.

- Interior-Point-Condition (IPC). There exists a triplet of vectors (x^0, y^0, z^0) such that:

$$Ax^0 = b, x^0 > 0, A^T y^0 + z^0 - Qx^0 = c, z^0 > 0.$$

- Symmetric and positive semi-definiteness. The matrix Q is symmetric positive semidefinite, i.e. $Q = Q^T$ and $v^T Q v \geq 0$, for all $v \in \mathbb{R}^n$.

Getting an optimal solution for both problems $(\mathcal{P})$ and $(\mathcal{D})$ is equivalent to solving the following system of optimality conditions:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ xz = 0. \end{cases} \tag{1}$$

Similar to the standard central-path methods, the basic idea behind weighted-path following IPMs is to replace the third equation (complementarity condition) in (1) by the parametrized equation $xz = \omega^2$; where $\omega \in \mathbb{R}^n_{++}$. Thus we consider the following parametrized system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ xz = \omega^2. \end{cases} \tag{2}$$

Under our assumptions, system (2) has a unique solution denoted by $(x(\omega), y(\omega), z(\omega))$ for all fixed $\omega \in \mathbb{R}^n_{++}$. The set

$$\{(x(\omega), y(\omega), z(\omega)) : \omega > 0\}$$

is called the weighted-path of both problems $(\mathcal{P})$ and $(\mathcal{D})$. If ω tends to zero then the limit of the weighted-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for CQO. Similar to the monotone LCP [3], the existence and the uniqueness of the weighted-path can be derived in the following way. Consider the following log-barrier problem

$$\min_{(x,y,z)} x^T z - \sum_{i=1}^{n} \omega_i^2 \ln x_i z_i \quad \text{s.t.} \quad Ax = b,\ A^T y + z - Qx = c,\ x > 0,\ z > 0.$$

Therefore, the necessary and sufficient optimality conditions of the log-barrier problem are characterized by the solutions of system (2). In other word, the existence and the uniqueness of the weighted-path is equivalent to the existence of unique minimizers for the log-barrier problem for each weight $\omega > 0$. It is easy to verify by the IPC, that the objective function of the log-barrier problem is strictly convex for each $\omega > 0$. Now, by the application of Newton's method for system (2), we get the classical Newton search directions [16]. Note that if $\omega = \sqrt{\mu} e$ with μ is a positive scalar, then the weighted-path reduces to the classical central-path. The relevance of the central-path has been discussed in the monographs, (see, e.g.,[21, 22]).

2.2. The new modified search direction

Following [3, 10], we turn now to describe the new modified Newton search direction for CQO. The AET based directions for CQO is simply based in replacing the weighted equation $xz = \omega^2$ by the new equation

$$\psi(xz) = \psi(\omega^2)$$

where $\psi(.) : (0, +\infty) \to \mathbb{R}$ is continuously differentiable and invertible function. Then, system (2) is converted to the following system

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0 \\ \psi(xz) = \psi(\omega^2), \end{cases} \tag{3}$$

where ψ is applied coordinate-wisely. As system (2) has a unique solution so is the system (3). Applying Newton's method to system (3) for a given strictly feasible point (x, y, z), i.e. the IPC holds, we obtain the following system:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \dfrac{\psi(\omega^2) - \psi(xz)}{\psi'(xz)}, \end{cases} \tag{4}$$

where ψ' denotes the derivative of ψ.
To simplify matters, we define the vectors

$$v := \sqrt{xz} \quad \text{and} \quad d := \sqrt{xz^{-1}}.$$

The vector d is used to scale the vectors x and z to the same vector v as:

$$d^{-1}x = dz = v. \tag{5}$$

Due to (5), the scaling directions are given by

$$d_x = d^{-1}\Delta x \text{ and } d_z = d\Delta z. \tag{6}$$

In addition, we have

$$x\Delta z + z\Delta x = v(d_x + d_z). \tag{7}$$

Now, since Q is positive semidefinite matrix, it follows that

$$d_x^T d_z = (\Delta x)^T(\Delta z) = (\Delta x)^T Q\Delta x \geq 0. \tag{8}$$

Hence, from (5), (7) and (8), system (4) can be written as

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T \Delta y + d_z - \bar{Q}d_x = 0, \\ d_x + d_z = p_v \end{cases} \tag{9}$$

where

$$p_v = \frac{\psi(\omega^2) - \psi(v^2)}{v\psi'(v^2)} \tag{10}$$

with $\bar{A} = AD$, $\bar{Q} = DQD$ and $D :=$diag(d).
Next, substituting $\psi(t) = t^{\frac{3}{2}}$ in (10) and in (4), yields

$$p_v = \frac{2}{3}v^{-2}(\omega^3 - v^3) \tag{11}$$

and

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \frac{2}{3}\frac{(\omega^3 - \sqrt{(xz)^3})}{\sqrt{xz}}. \end{cases} \tag{12}$$

Therefore, the new unique modified search directions $(\Delta x, \Delta y, \Delta z)$ are obtained by solving system (12). Moreover, the new iterate is computed by taking a full-Newton step as follows:

$$x_+ := x + \Delta x; \quad y_+ := y + \Delta y; \quad z_+ := z + \Delta z.$$

We end this subsection with this remark. By choosing function $\psi(t)$ appropriately, the system (12) can be used to define a class of new search directions. For example:

- $\psi(t) = t$ yields $p_v = v^{-1}(\omega^2 - v^2)$, we recuperate the standard weighted search directions (see [4, 5, 18]).
- $\psi(t) = \sqrt{t}$ yields $p_v = 2(\omega - v)$, we get Darvay's weighted search directions [10].

2.3. The proximity measure

For any positive vector v and according to (11), we define the norm-based proximity measure $\delta(v;\omega)$ as follows:

$$\delta(v;\omega) = \frac{3\|p_v\|}{2\omega_i} = \frac{\|v^{-2}(\omega^3 - v^3)\|}{\omega_i}, \forall i = 1, \cdots, n. \tag{13}$$

It is clear that

$$\delta(v;\omega) = 0 \Leftrightarrow \omega^3 = v^3 \Leftrightarrow xz = \omega^2.$$

So $\delta(v;\omega)$ is to measure the distance of a point (x, y, z) to the weighted-path $(x(\omega), y(\omega), z(\omega))$. Let us define another measure $\sigma_C(\omega)$ as follows:

$$\sigma_C(\omega) = \frac{\max(\omega)}{\min(\omega)} \geq 1. \tag{14}$$

The quantity $\sigma_C(\omega)$ is to measure the closeness of ω to the central-path. Here,

$$\min(\omega) = \min_i(\omega_i)$$

and likewise

$$\max(\omega) = \max_i(\omega_i).$$

Note that in (14), $\sigma_C(\omega) = 1$ if ω is on the central-path.

2.4. The generic weighted-path full-Newton step IPA for CQO

The weighted-path following IPA for CQO works as follows. First, we use a suitable threshold (default) value $\tau > 0$ with $0 < \tau < 3$ and we suppose that a strictly feasible initial point $(x^0 > 0, y^0, z^0 > 0)$ such that $\delta(x^0z^0;\omega_0) \leq \tau$ for some known vector ω_0. Using the obtained search directions from (12) and taking a full Newton-step the algorithm produces a new iterate $(x + \Delta x, y + \Delta y, z + \Delta z)$. Then, the vector ω is reduced by the factor $(1 - \theta)$ with $0 < \theta < 1$ and solves system (12), and so target a new iterate and so on. This procedure is repeated until the stopping criterion $n\max(\omega^2) \leq \epsilon$ is satisfied for a given accuracy parameter $\epsilon > 0$. The generic IPA is stated in Algorithm 1 as follows.

Input:
A threshold parameter $\tau \leq 3$ (default $\tau = 1$);
an accuracy parameter $\epsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{36\sqrt{2n}\sigma_c(\omega_0)}$);
a starting point (x^0, y^0, z^0) and ω_0 s.t. $\delta(x^0 z^0; \omega_0) \leq 1$;
begin
Set $x := x^0; y := y^0; z := z^0; \omega := \omega_0$;
while $n \max(\omega^2) \geq \epsilon$ do
begin
• $\omega := (1 - \theta)\omega$;
• Solve system (12) to obtain the direction $(\Delta x, \Delta y, \Delta z)$;
• Update $x := x + \Delta x, y := y + \Delta y, z := z + \Delta z$;
endwhile
end

Algorithm 1.

3. Convergence analysis

In this section, we will show across our new defaults that Algorithm 1 is well-defined and solves the CQO in polynomial complexity.

We first quote the following technical results which will be used later in the analysis of Algorithm 1.

Lemma 3.1
Let $(d_x, \Delta y, d_z)$ be a solution of (9) and $\omega > 0$. If $\delta := \delta(v;\omega) > 0$, then one has

$$0 \leq d_x^T d_z \leq \frac{2}{9}\delta^2 \omega_i^2, \ \forall i \tag{15}$$

and

$$\|d_x d_z\|_\infty \leq \frac{\delta^2}{9}\omega_i^2, \ \|d_x d_z\| \leq \frac{2\delta^2}{9}\omega_i^2, \ \forall i. \tag{16}$$

Proof
For the first claim, we have

$$0 \leq \|d_x\|^2 + 2d_x^T d_z + \|d_z\|^2 = \|d_x + d_z\|^2 = \|p_v\|^2.$$

But since $d_x^T d_z \geq 0$ by (8), it follows that

$$d_x^T d_z \leq \frac{1}{2}\|p_v\|^2 = \frac{2}{9}\delta^2 \omega_i^2, \ \forall i.$$

For the second claim, as

$$d_x d_z = \frac{1}{4}((d_x + d_z)^2 - (d_x - d_z)^2)$$

then, we have

$$\begin{aligned}\|d_x d_z\|_\infty &= \frac{1}{4}(\|(d_x + d_z)^2 - (d_x - d_z)^2\|_\infty) \\ &\leq \frac{1}{4}\max(\|d_x + d_z\|_\infty^2, \|d_x - d_z\|_\infty^2) \\ &\leq \frac{1}{4}\max(\|d_x + d_z\|^2, \|d_x - d_z\|^2).\end{aligned}$$

Since $d_x^T d_z \geq 0$, we have $\|d_x + d_z\|^2 \geq \|d_x - d_z\|^2$, then we obtain

$$\|d_x d_z\|_\infty \leq \frac{1}{4}\|d_x + d_z\|^2 = \frac{1}{4}\|p_v\|^2 = \frac{\delta^2}{9}\omega_i^2, \ \forall i.$$

For the last claim, we have

$$\begin{aligned}\|d_x d_z\|^2 &= e^T(d_x d_z)^2 = \frac{1}{16}e^T((d_x + d_z)^2 - (d_x - d_z)^2)^2 \\ &= \frac{1}{16}(\|(d_x + d_z)^2 - (d_x - d_z)^2)\|^2) \leq \frac{1}{16}(\|d_x + d_z\|^2 + \|d_x - d_z\|^2)^2 \\ &\leq \frac{1}{8}(\|d_x + d_z\|^4 + \|d_x - d_z\|^4) = \frac{1}{4}\|d_x + d_z\|^4 = \frac{1}{4}\|p_v\|^4 = \frac{4}{81}\delta^4\omega_i^4.\end{aligned}$$

Hence,

$$\|d_x d_z\| \leq \frac{2}{9}\delta^2\omega_i^2, \ \forall i.$$

This completes the proof. □

The next lemma investigates the feasibility of a full-Newton step.

Lemma 3.2
Let (x, y, z) be a strictly feasible point and assume $\delta := \delta(v; \omega) < 3$, then $x_+ = x + \Delta x > 0$ and $z_+ = z + \Delta z > 0$, i.e., x_+ and z_+ are strictly feasible.

Proof. Let $\alpha \in [0, 1]$, we define $x(\alpha) = x + \alpha\Delta x$ and $z(\alpha) = z + \alpha\Delta z$. Then, we have

$$x(\alpha)z(\alpha) = xz + \alpha(x\Delta z + z\Delta x) + \alpha^2\Delta x\Delta z.$$

Using (7) and (8), we get

$$x(\alpha)z(\alpha) = (1 - \alpha)v^2 + \alpha(v^2 + vp_v + \alpha d_x d_z). \tag{17}$$

Hence $x(\alpha)z(\alpha) > 0$ if $v^2 + vp_v + \alpha d_x d_z > 0$. By Lemma 3.1 (16), and from (11) and let $\delta < 3$, it follows that

$$\begin{aligned} v^2 + vp_v + \alpha d_x d_z &\geq v^2 + vp_v - \alpha \|d_x d_z\|_\infty e \\ &\geq v^2 + vp_v - \alpha\frac{\delta^2}{9}\omega_i^2 \\ &> \frac{1}{3}v^2 + \frac{2}{3}\omega_i^3 v^{-1} - \omega_i^2, \ \forall i \end{aligned}$$

Clearly, $x(\alpha)z(\alpha) > 0$ if

$$\frac{1}{3}v^2 + \frac{2}{3}\omega_i^3 v^{-1} - \omega_i^2 \geq 0, \ \forall i.$$

Letting

$$g(t) = \frac{1}{3}t^2 + \frac{2\omega_i^3}{3}t^{-1} - \omega_i^2, t > 0, \ \forall i.$$

g is a strictly convex function and has a minimum at $t = \omega_i$, and so $g(t) \geq g(\omega_i) = 0$. Hence,

$$\frac{1}{3}v^2 + \frac{2\omega_i^3}{3}v^{-1} - \omega_i^2 \geq 0, \ \forall i.$$

Therefore, $\forall \alpha \in [0, 1]$, $x(\alpha)z(\alpha) > 0$. Since x and z are positive which implies that $x(\alpha) > 0$ and $z(\alpha) > 0$ for all $\alpha \in [0, 1]$. So by continuity the vectors $x(1) = x_+$ and $z(1) = z_+ > 0$. This implies the lemma. □

For the new iterates x_+ and z_+, we define the vector $v_+ = \sqrt{x_+ z_+}$.

Lemma 3.3
Assume $\delta < 3$, then

$$(v_+)_i \geq \frac{\omega_i}{3}\sqrt{9-\delta^2}, \ \ \forall i.$$

Proof. In (17), setting $\alpha = 1$, then from (11), we have

$$(v_+^2)_i = v^2 + vp_v + d_x d_z = \frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 + d_x d_z, \ \forall i.$$

From the proof of Lemma 3.2, we have $\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 - \omega_i^2 \geq 0$ if $\delta < 3$, $\forall i$. From which we deduce that $\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 \geq \omega_i^2$, $\forall i$. Consequently,

$$(v_+^2)_i \geq \omega_i^2 + d_x d_z, \ \forall i.$$

Now, due to (16), we deduce that

$$\omega_i^2 + d_x d_z \geq \left(\omega_i^2 - \|d_x d_z\|_\infty e\right) \geq \frac{\omega_i^2}{9}(9-\delta^2), \ \forall i.$$

Hence,

$$(v_+)_i \geq \frac{\omega_i}{3}\sqrt{9-\delta^2}, \ \forall i.$$

This proves the lemma. □

Next, we prove that the iterate across the proximity measure is locally quadratically convergent during the Newton process.

Lemma 3.4
Assume $\delta < 3$, then

$$\delta_+ := \delta(v_+) := \delta(x_+ z_+; \omega) \leq \frac{5}{9}\left(\frac{9}{9-\delta^2} + \frac{3}{3+\sqrt{9-\delta^2}}\right)\delta^2.$$

In addition, if $\delta \leq 1$ then

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2,$$

which means the local quadratic convergence of the full-Newton step.

Proof
We have

$$\begin{aligned}
\delta(v_+;\omega) &= \frac{1}{\omega_i}\left\|v_+^{-2}(\omega^3 - v_+^3)\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{\omega^3 - v_+^3}{v_+^2}\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{(\omega - v_+)(\omega^2 + \omega v_+ + v_+^2)}{v_+^2}\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{(\omega^2 - v_+^2)(\omega^2 + \omega v_+ + v_+^2)}{v_+^2(\omega + v_+)}\right\|.
\end{aligned}$$

For all fixed $\omega \in \mathbb{R}^n_{++}$, i.e. $\omega_i > 0$, $\forall i$, we define the function g by

$$g(t) = \frac{\omega_i^2 + \omega_i t + t^2}{t^2(\omega_i + t)} = \frac{\omega_i}{t^2} + \frac{1}{\omega_i + t}, \ \forall i.$$

Using g, we deduce that

$$\delta_{+} = \frac{1}{\omega_i}\|g(v_+)(\omega^2 - v_+^2)\| \leq \frac{1}{\omega_i}\|g(v_+)\|_\infty \|\omega^2 - v_+^2\|, \forall i,$$

where $g(v_+) = (g_1(v_+)_1, \cdots, g_n(v_+)_n)$. The function g is continuous and monotonically decreasing and positive on $(0, +\infty)$. Hence, by Lemma 3.3

$$0 < |g_i((v_+)_i)| = g_i((v_+)_i) \leq g(v_+) \leq g\left(\frac{\omega_i}{3}\sqrt{9-\delta^2}\right), \forall i.$$

Then

$$\|g(v_+)\|_\infty \leq \frac{9\omega_i}{9\omega_i^2 - \delta^2\omega_i^2} + \frac{3}{\omega_i(3+\sqrt{9-\delta^2})}, \forall i.$$

This implies that

$$\delta^+ \leq \frac{1}{\omega_i^2}\left(\frac{9}{9-\delta^2} + \frac{3}{3+\sqrt{9-\delta^2}}\right)\|\omega^2 - v_+^2\|, \forall i.$$

Next, in (17), setting $\alpha = 1$ and from (11), we have

$$\begin{aligned}\|\omega^2 - v_+^2\| =& \|\omega^2 - (v^2 + vp_v + d_x d_z)\| \\ =& \|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3 - d_x d_z\|.\end{aligned}$$

Then

$$\|\omega^2 - v_+^2\| \leq \|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\| + \|d_x d_z\|.$$

Next, we may write

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| = \left\|\frac{\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3}{v^{-4}(\omega^3 - v^3)^2}.\frac{9p_v^2}{4}\right\|.$$

And after elementary reductions, we get

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| = \left\|\varphi(v).\frac{9p_v^2}{4}\right\|$$

where

$$\varphi(v) = \frac{v^3(v+2\omega)}{3(v^2 + v\omega + \omega^2)^2}.$$

Let us consider for all i, the following function

$$\varphi(t) = \frac{t^3(t+2\omega_i)}{3(t^2 + \omega_i t + \omega_i^2)^2}.$$

$\varphi(t)$ is continuous and monotonically increasing and positive for all $t \in (0, +\infty)$. Then we have

$$0 \leq \varphi(t) < \frac{1}{3} = \lim_{t \mapsto \infty} \varphi(t), \forall t > 0.$$

This yields

$$0 \leq \varphi(v_i) < \frac{1}{3}, \forall i.$$

Consequently,

$$0 < |\varphi(v_i)| = \varphi(v_i) \leq \frac{1}{3}, \forall i. \tag{18}$$

Then, as $\|p_v\|^2 = \frac{4}{9}\omega_i^2\delta^2$, $\|\varphi(v)\|_\infty = \max_i \varphi(v_i) \leq \frac{1}{3}$ and $\|p_v^2\| \leq \|p_v\|^2$, these imply that

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| \leq \|\varphi(v)\|_\infty \frac{9}{4}\|p_v\|^2 = \frac{1}{3}\omega_i^2\delta^2, \ \forall i.$$

Due to (16), it follows

$$\|\omega^2 - v_+^2\| \leq \frac{5}{9}\delta^2\omega_i^2, \ \forall i.$$

Next, for $\delta < 3$, we have

$$\delta_+ \leq \frac{5}{9}\left(\frac{9}{9-\delta^2} + \frac{3}{3+\sqrt{9-\delta^2}}\right)\delta^2.$$

Now, let $\delta \leq 1$, then $\frac{9}{9-\delta^2} \leq \frac{9}{8}$ and $\frac{3}{3+\sqrt{9-\delta^2}} \leq \frac{3}{3+2\sqrt{2}}$. Hence, after some simplifications, we obtain

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2 < \delta^2.$$

This completes the proof. □

The next lemma gives an upper bound for the duality gap after a full-Newton step.

Lemma 3.5
After a full-Newton step it holds

$$x_+^T z_+ \leq 2n\max(\omega^2).$$

Proof
As $v_+^2 = x_+ z_+$, we have

$$\begin{aligned}
(x_+)^T z_+ &= e^T v_+^2 = e^T(v^2 + vp_v + d_x d_z) \\
&= e^T(\omega^2 + v^2 + vp_v - \omega^2) + d_x^T d_z \\
&= e^T\omega^2 + e^T(v^2 + vp_v - \omega^2) + d_x^T d_z \\
&= e^T\omega^2 + d_x^T d_z + e^T(v^2 + \frac{2}{3}v^{-1}\omega^3 - \frac{2}{3}v^2 - \omega^2) \\
&= e^T\omega^2 + d_x^T d_z + e^T\left(\frac{\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega^3 - \omega^2}{v^{-4}(\omega^3 - v^3)^2}\,\frac{9p_v^2}{4}\right).
\end{aligned}$$

Then after some reductions, we get

$$(x_+)^T z_+ = e^T\omega^2 + d_x d_z + \frac{9}{4}e^T\varphi(v)p_v^2$$

where $\varphi(v) = (\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_n))$ with

$$\varphi(v_i) = \frac{v_i^3(v_i + 2\omega_i)}{3(v_i^2 + v_i\omega_i + \omega_i^2)^2}, \text{ for } i = 1, \cdots, n.$$

Due to (18), we have

$$0 < |\varphi(v_i)| = \varphi(v_i) \leq \frac{1}{3}, \ \forall i.$$

Therefore, by Lemma 3.1 and from (13), we have

$$
\begin{aligned}
e^T v_+^2 &\leq e^T\omega^2 + d_x^T d_z + \frac{9}{4}\max_i|\varphi(v_i)|e^T p_v^2\\
&\leq e^T\omega^2 + d_x^T d_z + \frac{3}{4}\|p_v\|^2\\
&\leq n\max(\omega^2) + \frac{2}{9}\delta^2\omega_i^2 + \frac{1}{3}\delta^2\omega_i^2,\ \forall i.\\
&\leq n\max(\omega^2) + \frac{2}{9}\delta^2\max(\omega^2) + \frac{1}{3}\delta^2\max(\omega^2)\\
&\leq \left(n + \frac{5}{9}\delta^2\right)\max(\omega^2) \leq (n+\delta^2)\max(\omega^2).
\end{aligned}
$$

Let $\delta \leq 1$, then $e^T v_+^2 \leq (n+1)\max(\omega^2)$, but since $(n+1) \leq 2n, \forall n \geq 1$, it follows that $e^T v_+^2 \leq 2n\max(\omega^2)$. This gives the required result. □

Next lemma investigates the effect of a full Newton-step on the proximity measure followed by updating the weighted vector ω by a factor $(1-\theta)$, where $0 < \theta < 1$.

Theorem 3.1
Let $\omega_+ = (1-\theta)\omega$ and let $x_+ > 0, z_+ > 0$, then we have

$$
\delta(v_+;\omega_+) \leq \delta_+ + \frac{3\sqrt{2n}\theta}{1-\theta}\sigma_C(\omega).
$$

In addition, let $\delta \leq 1$, $\sigma_C(\omega) \geq 1$, and $\theta = \dfrac{1}{36\sqrt{2n}\sigma_C(\omega)}$, $n \geq 2$, then $\delta(v_+;\omega_+) \leq 1$.

Proof
Let $\delta(x_+z_+;\omega_+)$ and $\omega_+ = (1-\theta)\omega$ where $\theta \in (0,1)$. We have

$$
\begin{aligned}
\delta(v_+;\omega_+) &= \frac{1}{(\omega_+)_i}\|v_+^{-2}(\omega_+^3 - v_+^3)\|\\
&= \frac{1}{(1-\theta)\omega_i}\left\|\frac{\left((1-\theta)^3\omega^3 - (v_+)^3\right)}{v_+^2}\right\|\\
&= \frac{1}{(1-\theta)\omega_i}\|v_+^{-2}\left((1-\theta)^3\omega^3 + (1-\theta)^3 v_+^3 - (1-\theta)^3 v_+^3 - v_+^3\right)\|\\
&\leq \frac{1}{(1-\theta)\omega_i}\left(\|(1-\theta)^3 v_+^{-2}(\omega^3 - v_+^3)\| + \|v_+((1-\theta)^3 - 1)\|\right)\\
&= (1-\theta)^2\delta_+ + \frac{|(1-\theta)^3 - 1|}{1-\theta}\frac{\|v_+\|}{\omega_i}\\
&\leq \delta_+ + \frac{|(1-\theta)^3 - 1|}{1-\theta}\frac{\|v_+\|}{\omega_i}\\
&\leq \delta_+ + \frac{|\theta(\theta^2 - 3\theta + 3)|}{1-\theta}\frac{\|v_+\|}{\min(\omega)}.
\end{aligned}
$$

As $0 < \theta^2 - 3\theta + 3 < 3, \forall\theta \in (0,1)$, we obtain

$$
\delta(v_+;\omega_+) \leq \delta_+ + \frac{3\theta}{1-\theta}\frac{\|v_+\|}{\min(\omega)}.
$$

By Lemma 3.5, we have

$$
\|v_+\| \leq \sqrt{2n}\max(\omega).
$$

Next, Lemma 3.4 implies that

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2.$$

Now, let $\delta \leq 1$, then we get

$$\delta(v_+;\omega_+) \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right) + \frac{3\sqrt{2n}\sigma_C(\omega)\theta}{1-\theta}.$$

Let $\theta = \dfrac{1}{36\sqrt{2n}\sigma_c(\omega)}$, $n \geq 2$ and $\sigma_c(\omega) \geq 1$ so $\theta \in \left[0, \frac{1}{72}\right]$ from which we deduce that

$$\delta(v_+;\omega_+) \leq \xi(\theta)$$

where

$$\xi(\theta) = \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right) + \frac{1}{12(1-\theta)}.$$

As $\xi'(\theta) = \dfrac{1}{12\,(\theta-1)^2} > 0$, then $\xi(\theta)$ is strictly increasing on the interval $\left[0, \frac{1}{72}\right]$. Hence $\xi(\theta) \leq \xi\left(\frac{1}{72}\right) = 0.9966 < 1$. This proves the theorem. □

Theorem 3.1, shows that Algorithm 1 is well-defined since the conditions $x > 0, z > 0$, and $\delta(xz;\omega) \leq 1$ are maintained throughout the algorithm. Also observe that $\sigma_C(\omega) = \sigma_C(\omega_0)$ for all iterates produced by Algorithm 1.

The next lemma derives an upper bound for the total number of iterations produced by Algorithm 1.

Lemma 3.6
Let x^k and z^k be the k-th iteration produced by Algorithm 1. Then

$$(x^k)^T z^k \leq \epsilon$$

if

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{2n\max(\omega_0)^2}{\epsilon} \right\rceil.$$

Proof
After k iterations, we have $\omega_k = (1-\theta)^k \omega_0$. By Lemma 3.6, we get that

$$(x^k)^T z^k \leq 2n(1-\theta)^{2k}\max(\omega_0)^2.$$

Thus the inequality $(x^k)^T z^k \leq \epsilon$ holds if

$$2n(1-\theta)^{2k}\max(\omega_0)^2 \leq \epsilon.$$

Taking logarithms, we find

$$2k\log(1-\theta) \leq \log\epsilon - \log 2n\max(\omega_0)^2.$$

Using the inequality $-\log(1-\theta) \geq \theta$ where $0 < \theta < 1$, so the above inequality holds if

$$k\theta \geq \frac{1}{2}\log\frac{2n\max(\omega_0)^2}{\epsilon}.$$

This completes the proof. □

Theorem 3.2
Suppose that (x^0, y^0, z^0) is a strictly feasible starting point, $\omega_0 = \frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$ and $\delta(x^0 z^0; \omega_0) \leq 1$. Let $\theta = \frac{1}{36\sqrt{2n}\sigma_c(\omega_0)}$, then Algorithm 1, requires at most

$$O\left(\sqrt{n}\sigma_C(\omega_0)\log\frac{n}{\epsilon}\right)$$

iterations for getting an ϵ-approximate solution of CQO.

Proof
Let $\theta = \frac{1}{36\sqrt{2n}\sigma_C(\omega_0)}$, Theorem 3.2 follows directly from Lemma 3.6. □

Corollary 3.1
If we take $\omega_0 = \frac{1}{\sqrt{3}}e$, then Algorithm 1, requires at most $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ iterations which is the currently best known iteration bound for short-update method.

Proof
The proof is an immediate consequence of Theorem 3.2. □

4. Numerical results

In this section, we implement Algorithm 1 on some examples of CQO with different size by using Matlab R2010a and run on a PC with CPU 2.00 GHz and 4.00 G RAM memory and double precision format. Here our accuracy is set to $\epsilon = 10^{-4}$. The strictly feasible initial point $(x^0 > 0, y^0, z^0 > 0)$ is taken such that $\delta(x^0 z^0, \omega_0) \leq 1$. The optimal primal-dual solution is denoted by $(x^\star, y^\star, z^\star)$. Here, we display the following notations: the "Iter" denotes the number of iterations produced by the algorithm to obtain an approximated optimal solution. The "CPU" denotes the time (in second) required to obtain an approximate optimal solution for CQO. Also to improve our numerical results we have relaxed the barrier vector $\omega_0 = \{\frac{1}{\sqrt{2}}e, \sqrt{x^0 z^0}, \frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}\}$, with the update barrier $\theta = \frac{1}{36\sqrt{2n}\sigma_C(\omega_0)}, n \geq 2$. We also display a table for the number of iterations and the elapsed time for each example.

Example 1. We consider the convex quadratic optimization, where

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}, \; Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$c = (1.3333, 1.3333, 1.3333, 1.3333)^T$, $b = (0.33, 2)^T$. The strictly feasible starting point x^0 and z^0 are chosen according to each value of ω_0 and such that $\delta(x^0 z^0; \omega_0) \leq 1$. For this example, we take

$$x^0 = (0.3333, 0.3333, 0.3333, 0.3333)^T,\; y^0 = (-2, -2)^T,\; z^0 = (2, 2, 2, 2)^T.$$

$\omega_0 \rightarrow$					
$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
Iter	CPU	Iter	CPU	Iter	CPU
1862	0.53163	1876	0.137857	1862	0.139970

Table 1. Numerical results for Example 1.

A primal-dual optimal solution of Example 1 is:

$$\begin{aligned} x^\star &= (0.2000, 0.5333, 0.0000, 0.0000)^T, \\ y^\star &= (-2.0800, -1.1733)^T, \\ z^\star &= (0.0000, 0.0000, 1.4133, 0.5067)^T. \end{aligned}$$

Example 2. The data of the following convex quadratic problem is given by

$$A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix},$$

$c = (2.1, 3.45, 4.25, 2.5, 2.75, 4, 4.7, 2.95, 1.4, 4)^T$, $b = (0.7660, 1.1770, 1.21)^T$. The initial point is taken as:

$$\begin{aligned} x^0 &= (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)^T, \\ y^0 &= (-1, -1, -1)^T, \\ z^0 &= (6, 6, 6, 6, 6, 6, 6, 6, 6, 6)^T. \end{aligned}$$

$\omega_0 \rightarrow$					
$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0z^0}$		$\frac{x^0z^0}{\sqrt{2}\max(x^0z^0)}$	
Iter	CPU	Iter	CPU	Iter	CPU
3022	0.293683	3037	0.293173	3022	0.294073

Table 2. Numerical results for Example 2.

A primal-dual optimal solution for this example is:

$$\begin{aligned} x^\star &= (0.0973, 0.0000, 0.0236, 0.1725, 0.0810, 0.2061, 0.0000, 0.1124, 0.0730, 0.0000)^T, \\ y^\star &= (-1.5354, 2.0035, 0.5461)^T, \\ z^\star &= (0.0000, 1.5550, 0.0000, 0.0000, 0.0000, 0.0000, 0.5015, 0.0000, 0.0000, 3.8707)^T. \end{aligned}$$

Example 3. We consider the CQO, where $n = 2m$ and

$$A[i,j] = \begin{cases} 0 & \text{if} \quad i \neq j \text{ or } (i+1) \neq j \\ 1 & \text{if} \quad j = i + m. \end{cases}$$

$$Q[i,j] = \begin{cases} 2j-1 & \text{if} \quad i > j \quad \text{or } i = j = n \\ 2i-1 & \text{if} \quad i < j \\ i(i+1) & \text{if} \quad i = j. \end{cases} \qquad c = \begin{cases} 0 & \text{if} \qquad i = i + m \\ 1 & \text{else if,} \end{cases}, \; b[i] = 0.5.$$

For this example, we take $x^0 = 0.5e$, $y^0 = 0_{\mathbb{R}^m}$ and $z^0 = e$ as the strictly feasible initial point. The obtained numerical results with $\theta = \frac{1}{36\sqrt{2n}\sigma_C(\omega_0)}$ are showed in Table 3.

Size $(m,n)\downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
$(10,20)$	4356	0.819771	4356	0.797447	4356	0.806954
$(50,100)$	10162	98.984386	10163	134.182516	10163	148.585150
$(100,200)$	14624	577.192434	14625	694.052203	14625	708.916735
$(1000,2000)$	51120	1481.75235	51120	1482.24890	51120	1481.06895

Table 3. Numerical results for Example 3 with different sizes.

A primal-dual optimal solution of Example 3 is:

$$x^\star = (0.0000, \cdots, 0.0000, 0.4185)^T,$$
$$y^\star = (0.0388, \cdots, 0.0388)^T,$$
$$z^\star = (0.3876, \cdots, 0.3876, 0.0000)^T.$$

Comment. Across the obtained numerical results, we see that Algorithm 1 computes a primal-dual optimal solution for CQOs but in a large number of iterations and with a significant elapsed time. This means that the algorithm converges slowly to an optimal solution while using θ stated in our analysis. The cause is due to the fact that θ becomes very small for problems with a large size n. Consequently, the rate of decrease $(1-\theta)$ in the sequence of barrier vectors $\{\omega_k\}$ approaches to one.

4.1. A numerical amelioration of Algorithm 1

In this subsection, based on our comment and in order to improve our numerical results, we import some changes on the original version of Algorithm 1, where instead of using the updating θ provided by our analysis, we take it as a constant belongs to the set $\{0.1, ..., 0.9\}$. Moreover, to guarantee that the iterates remain interior, we introduce a step-size $\alpha_{\max} > 0$ such that $x + \rho\alpha_{\max}\Delta x > 0$ and $z + \rho\alpha_{\max}\Delta z > 0$ with $\alpha_{\max} = \min\{\alpha_{\mathcal{P}}, \alpha_{\mathcal{D}}\}$ and $\rho \in (0,1)$ where $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{D}}$ are given by

$$\alpha_{\mathcal{P}} = \begin{cases} \min_i\left(-\frac{x_i}{\Delta x_i}\right) & \text{if} \quad \Delta x_i < 0 \\ 1 & \text{if} \quad \Delta x_i \geq 0, \end{cases} \qquad \alpha_{\mathcal{D}} = \begin{cases} \min_i\left(-\frac{z_i}{\Delta z_i}\right) & \text{if} \quad \Delta z_i < 0 \\ 1 & \text{if} \quad \Delta z_i \geq 0. \end{cases}$$

Based, on the imported changes our new obtained numerical results for the same examples are stated in tables below.

$\theta\downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	175	0.139169	176	0.193436	175	0.117749
0.2	84	0.083488	83	0.081854	83	0.062218
0.3	52	0.067883	52	0.068502	52	0.046201
0.4	36	0.063592	36	0.060969	36	0.039543
0.5	27	0.054806	27	0.051760	27	0.039456
0.6	20	0.054244	20	0.054322	20	0.032333
0.7	19	0.048853	19	0.055084	19	0.031448
0.8	19	0.051142	19	0.048136	19	0.036626
0.9	18	0.051417	18	0.047807	18	0.035654

Table 1. Numerical results for Example 1.

$\theta \downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	179	0.187614	180	0.201249	179	0.190970
0.2	85	0.146417	86	0.115407	85	0.262326
0.3	53	0.092770	53	0.081417	53	0.069546
0.4	37	0.077989	37	0.067789	37	0.052182
0.5	27	0.066829	27	0.064044	27	0.045523
0.6	21	0.057692	21	0.057452	21	0.042347
0.7	21	0.057708	21	0.054687	21	0.042074
0.8	21	0.063730	21	0.058077	21	0.039714
0.9	21	0.059037	21	0.059576	21	0.036237

Table 2. Numerical results for Example 2.

$\omega_0 \downarrow$	Size $(m, n) \downarrow$	$\theta \rightarrow$							
		0.1		0.3		0.5		0.9	
		Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\frac{1}{\sqrt{2}}e$	$(10, 20)$	182	0.428436	53	0.087912	28	0.060326	18	0.047816
	$(50, 100)$	190	2.366860	55	0.648054	29	0.340302	18	0.224515
	$(100, 200)$	193	8.695862	57	2.523648	29	1.421619	18	0.817562
	$(1000, 2000)$	205	5240.8705	59	2237.279743	31	835.91287	19	502.69953
$\sqrt{x^0 z^0}$	$(10, 20)$	182	0.255420	53	0.090728	28	0.061706	18	0.045603
	$(50, 100)$	190	2.404953	55	0.694129	29	0.400725	18	0.464692
	$(100, 200)$	193	8.743116	57	2.542191	29	1.338784	18	0.806318
	$(1000, 2000)$	204	5239.9845	59	2236.4251	31	803.36214	19	472.19425
$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	$(10, 20)$	182	0.274023	53	0.086902	28	0.059837	18	0.048521
	$(50, 100)$	190	2.003090	55	0.640520	29	0.365868	18	0.223728
	$(100, 200)$	193	8.822089	57	2.344394	29	1.343666	18	0.973272
	$(1000, 2000)$	204	5239.5095	59	2237.26837	31	784.092613	19	486.44412

Table 3. Numerical results for Example 3.

5. Conclusion

In this paper, we presented a new weighted full-Newton step path-following interior-point method for CQO based on a new modified Newton search direction obtained by the application of the AET technique introduced by the new univariate function $\psi(t) = t^{\frac{3}{2}}$ for the Newton system which defines the weighted-path. New appropriate choices of the defaults of τ and θ are proposed where the favorable iteration bound of the algorithm with short-step method is achieved, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. This iteration bound is as good as for LO analogue. Meanwhile, for the obtained numerical results by Algorithm 1 for its first version are not good for CQO problems with a large size. But, with the imported changes on Algorithm 1, the obtained numerical results are significantly improved. Finally, the extension of this method for other class of optimization problems deserves a good topic of research in the future.

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