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A Novel Two-Parameter Compound G Family of Probability Distributions With Some Copulas, Statistical Properties and Applications

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Abstract In this work, we introduce a new G family with two-parameter called the compound reversed Rayleigh-G family. Several relevant mathematical and statistical properties are derived and analyzed. The new density can be heavy tail and right skewed with one peak, symmetric density, simple right skewed density with one peak, asymmetric right skewed with one peak and a heavy tail and right skewed with no peak. The new hazard function can be "upside-down-constant", "constant", "increasing-constant", "revised J shape", "upside-down", "J shape" and "increasing". Many bivariate types have been also derived via different common copulas. The estimation of the model parameters is performed by maximum likelihood method. The usefulness and flexibility of the new family is illustrated by means of two real data sets.

Keywords Poisson Family; Compound Models; Moments; Farlie Gumbel Morgenstern Copula; Ali–Mikhail–Haq Copula.

AMS 2010 subject classifications 0E05; 62N01; 62G05; 62N02; 62N05; 62E10; 62P30

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1. Introduction and motivation

In the last few years, huge efforts have been paid to derive many new G families using the well knowm methods. These new G families have been used for modeling non-censored and censored real data sets in many applied studies such as finance, econometrics, value at risk applications, insurance, biology, engineering, forecasting, medicine and environmental sciences see , for example, Marshall and Olkin [48] (Marshall-Olkin-G (MO-G) family), Eugene et al. [19] (beta generalized-G (B-G) family), Yousof et al. [68] (transmuted exponentiated generalized (TEG) family), Rezaei et al. [57] (Topp Leone generated (TLG) family), Merovci et al. [50] (exponentiated transmuted-G (ET-G) family), Aryal and Yousof [13] (exponentiated generalized-G Poisson (EGGP) family), Brito et al. [14] (Topp-Leone odd log-logistic-G (TLOLL-G) family), Yousof et al. [70] (Burr of the type X G (BX-G) family), Hamedani et al. [30] (type I general exponential-G (TIGE-G) family), Korkmaz et al. [40] (exponential Lindley odd log-logistic-G (ELOLL-G) family), Cordeiro et al. (2018) (Burr XII-G (BXII-G) family), Hamedani et al. [28] (extended-G (Ex-G) family), Korkmaz et al. [41] (Marshall-Olkin generalized-G Poisson (MOGGP) family), Yousof et al. [73] (Burr-Hatke-G (BH-G) family), Nascimento et al. [55] (Nadarajah-Haghighi-G (NH-G) family), Hamedani et al. [29] (type II general exponential-G (TIIGE-G) family), Yousof et al. [75] (Weibull G Poisson (WGP) family), Merovci et al. [51] (Poisson Topp Leone G (PTL-G) family), Karamikabir et al. [38] (Weibull Topp-Leone generated (WTL-G) family), Korkmaz et al. [39] (Hjorth-G (Hj-G) family),

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Alizadeh et al. [9] (flexible Weibull generated (FWG) family) and Alizadeh et al. [10] (transmuted odd log-logistic-G (TOLL-G) family), El-Morshedy et al. [22] (Poisson generalized exponential G (PGE-G) family) among others.

In this paper we propose and study a new family of distributions using the zero truncated Poisson (ZTP) distribution with a strong physical motivation. Suppose that a system has N subsystems functioning independently at a given time where N has ZTP distribution with parameter $\boldsymbol{\sigma}$. It is the conditional probability distribution of a Poisson-distributed random variable (R.V), given that the value of the R.V is not zero (see Maurya and Nadarajah [49]). The probability mass function (PMF) of N is given by

$$P\left(N=n\right)=\left[\boldsymbol{\sigma}^{n}\exp\left(-\boldsymbol{\sigma}\right)\right]/\left(n!\boldsymbol{\sigma}_{\bullet}\right)|_{(n=1,2,\ldots)},\tag{1}$$

where

$$\boldsymbol{\sigma}_{\bullet}=1-\exp\left(-\boldsymbol{\sigma}\right).$$

Suppose that the failure time of each subsystem has the inverse Rayleigh ("IR-G($\underline{\boldsymbol{\Omega}}$)" for short) defined by the cumulative distribution function (CDF) and probability density function (PDF) given by

$$G_{\underline{\boldsymbol{\Omega}}}(z)=\exp\left[-\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^{2}\left(z\right)\right]|_{z\in\mathbb{R}},\tag{2}$$

where

$$\varkappa_{\underline{\boldsymbol{\Omega}}}\left(z\right)=\frac{W_{\underline{\boldsymbol{\Omega}}}(z)}{\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)}|_{z\in\mathbb{R}},$$

and $\underline{\boldsymbol{\Omega}}$ is the parameter vector of the baseline model and

$$\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)=1-W_{\underline{\boldsymbol{\Omega}}}(z)$$

is the survival function of the baseline model and

$$g_{\underline{\boldsymbol{\Omega}}}(z)=2\beta w_{\underline{\boldsymbol{\Omega}}}(z)\frac{\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)}{W_{\underline{\boldsymbol{\Omega}}}(z)^{3}}\exp\left[-\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^{2}\left(z\right)\right],z\in\mathbb{R},\tag{3}$$

respectively, Let Y_i denote the failure time of the ith subsystem and let

$$Z=\min\{Y_1,Y_2,\cdots,Y_N\}.$$

Due to Aryal and Yousof [13], Korkmaz et al. [41], Yousof et al. [69], Abouelmagd et al. [2], Alizadeh [8] and Abouelmagd et al. et al. [3], the conditional CDF of Z given N is

$$F\left(z|N\right)=1-\Pr\left(Z>z|N\right)=1-\left[1-G_{\underline{\boldsymbol{\Omega}}}(z)\right]^{N}.\tag{4}$$

Therefore, the CDF of the compound inverse Rayleigh (CIR-G) family can be expressed as

$$F_{\underline{\boldsymbol{\eta}}}(z)=\boldsymbol{\sigma}_{\bullet}^{-1}\left(1-\exp\left\{-\boldsymbol{\sigma}\exp\left[-\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^{2}\left(z\right)\right]\right\}\right)|_{z\in\mathbb{R}},\tag{5}$$

where $\underline{\boldsymbol{\eta}}=(\boldsymbol{\sigma},\beta,\underline{\boldsymbol{\Omega}})$. The corresponding PDF as

$$f_{\underline{\boldsymbol{\eta}}}(z)=2\boldsymbol{\sigma}\beta\frac{w_{\underline{\boldsymbol{\Omega}}}(z)\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)}{\boldsymbol{\sigma}_{\bullet}W_{\underline{\boldsymbol{\Omega}}}(z)^{3}}\exp\left[-\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^{2}\left(z\right)\right]\exp\left\{-\boldsymbol{\sigma}\exp\left[-\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^{2}\left(z\right)\right]\right\}|_{z\in\mathbb{R}}.\tag{6}$$

In this work, a special attention is paid to two special members called the compound inverse Rayleigh exponential and the compound inverse Rayleigh inverse Weibull distributions. The new density of the compound inverse Rayleigh exponential model can be "asymmetric right skewed with one peak and a heavy tail", "symmetric" and "left skewed with one peak". The new hazard rate function (HRF) of the can heavy tail and right skewed with one peak, symmetric density, simple right skewed density with one peak, asymmetric right skewed with one peak and a heavy tail and right skewed with no peak. The new hazard function can be "upside-down-constant HRF", "constant HRF", "increasing-constant HRF", "J-shape HRF", "revised J-shape HRF", "upside-down HRF" and "increasing HRF".

In fact, the statistical literature has many new families, but most researchers study these families from traditional aspects, and also provide certain theoretical aspects on a repetitive basis. In this work, we have tried to introduce a new family and we have tried and studied it from different sides and in an interesting way in presenting the theoretical and practical results. This work also includes a special Section on the copula and its uses in the derivation of bivariate and multivariate distributions. The new family is better than the odd Lindley family, Marshall-Olkin family, the Burr-Hatke family, generalized Marshall-Olkin family, Beta family, Marshall-Olkin Kumaraswamy family, Kumaraswamy family, the Burr of the type X family and Kumaraswamy Marshall-Olkin family in modeling the bimodal right skewed relief times data set so the new family could be considered as a good alternative to these families. The new family is better than the Marshal-Olkin family, Generalized Marshal-Olkin family, Kumaraswamy family, beta family, Kumaraswamy Marshal-Olkin family and Marshal-Olkin Kumaraswamy family in modeling the gauge lengths data set so the new family could be considered as a good alternative to these families.

We are motivated to introduce the CIR-G family for the following reasons:

- Creating new PDFs that can be "symmetric PDF" or "asymmetric and right skewed with a heavy tail PDF" We can evaluate many environmental data sets using the family since it is so adaptable for every new model.
- Introducing a few new unique models with various HRFs, including, decreasing, upside-down and monomaniacally decreasing. With more distinct failure rate kinds, the distribution's elasticity increases. Thanks to these forms, which facilitate their work, many practitioners may use the new distribution in statistical modelling and mathematical analysis. For this particular reason, we pay close attention to the problem of monitoring the failure rate function.
- How flexible the new distribution is depends on both the kurtosis and skew coefficients. Furthermore, the probability distribution's effectiveness and use in statistical modelling are crucial in this context. We examined the innovative PDF and discovered that, among other things, it was quite adaptable. This motivated us to investigate this probability distribution in great detail.
- Modelling the "bimodal and right skewed heavy tail data" and "the bimodal and left skewed heavy tail data" using new continuous models.
  The new family has demonstrated a significant advantage in modelling various forms of data, whether "bimodal and left skewed heavy tail data" or "the bimodal and right skewed heavy tail data," as is demonstrated in this study. Whether modelling "bimodal and left skewed heavy tail data" or "the bimodal and right skewed heavy tail data," the novel model has demonstrated a significant superiority in this work.
- Whether modelling "bimodal and left skewed heavy tail data" or "the bimodal and right skewed heavy tail data," the novel system has demonstrated a significant superiority in this research. Additionally, the new distribution demonstrated a great improvement in simulating actual data, which includes outliers. The outliers family of distributions, one of the most extensively employed families in such situations, and the new distribution unquestionably belong to the outliers family.

This paper is organized as follows. In Section 2, we derive some of its mathematical properties. Section 3 deals with some bivariate version via copulas. In Section 4 gives some special submodels. The maximum likelihood method is discussed in Section 5. In Section 6, we illustrate the importance of the new family by means of two applications to real data sets. Finally, Section 7 offers some concluding remarks.

2. Mathematical properties

2.1. Useful expansions

Using the power series

$$\exp\left(\frac{\upsilon_1}{\upsilon_2}\right)=\sum_{\upsilon_3=0}^{+\infty}\frac{1}{\upsilon_3!}\left(\frac{\upsilon_1}{\upsilon_2}\right)^{\upsilon 3}, \tag{7}$$

the PDF in (6) can be written as

$$f_{\underline{\boldsymbol{\eta}}}(z)=2\beta\sum_{\hbar_3=0}^{+\infty}\frac{\boldsymbol{\sigma}^{1+\hbar_3}\left(-1\right)^{\hbar_3}}{\hbar_3!\boldsymbol{\sigma}_{\bullet}}\frac{w_{\underline{\boldsymbol{\Omega}}}(z)\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)}{W_{\underline{\boldsymbol{\Omega}}}(z)^3}\exp\left[-\left(1+\hbar_3\right)\beta/\varkappa_{\underline{\boldsymbol{\Omega}}}^2\left(z\right)\right], \tag{8}$$

again applying (7) to (8) we get

$$f_{\underline{\boldsymbol{\eta}}}(z)=2\sum_{\hbar_3,\hbar_1=0}^{+\infty}\frac{\boldsymbol{\sigma}^{1+\hbar_3}\left(-1\right)^{\hbar_3+\hbar_1}\beta^{1+\hbar_1}\left(1+\hbar_3\right)^{\hbar_1}}{\hbar_3!\hbar_1!\boldsymbol{\sigma}_{\bullet}}w_{\underline{\boldsymbol{\Omega}}}(z)\overline{W}_{\underline{\boldsymbol{\Omega}}}(z)^{1+2\hbar_1}W_{\underline{\boldsymbol{\Omega}}}(z)^{-2\hbar_1-3}. \tag{9}$$

If $\left|\frac{\upsilon_1}{\upsilon_2}\right|<1$ and $\upsilon_3>0$ is a real non-integer, the following power series holds

$$\left(1-\frac{\upsilon_1}{\upsilon_2}\right)^{\upsilon_3}=\sum_{\hbar_2=0}^{+\infty}\frac{\left(-1\right)^{\hbar_2}\Gamma\left(1+\upsilon_3\right)}{i!\,\Gamma\left(1+\upsilon_3-\hbar_2\right)}\left(\frac{\upsilon_1}{\upsilon_2}\right)^{\hbar_2}. \tag{10}$$

Applying (10) to (9) we have

$$f_{\underline{\boldsymbol{\eta}}}(z)=2\sum_{\hbar_1,\hbar_2,\hbar_3=0}^{+\infty}\left(-1\right)^{\hbar_1+\hbar_2+\hbar_3}\frac{\boldsymbol{\sigma}^{1+\hbar_3}\left(1+\hbar_3\right)^{\hbar_1}\beta^{1+\hbar_1}\Gamma\left(2+2\hbar_1\right)}{\hbar_1!\hbar_2!\hbar_3!\boldsymbol{\sigma}_{\bullet}\Gamma\left(2+2\hbar_1-\hbar_2\right)}w_{\underline{\boldsymbol{\Omega}}}(z)W_{\underline{\boldsymbol{\Omega}}}(z)^{\hbar_2-2\hbar_1-3},$$

where $\boldsymbol{\varpi}=\hbar_2-2\left(1+\hbar_1\right)$, then

$$f_{\underline{\boldsymbol{\eta}}}(z)=\sum_{\hbar_1,\hbar_2=0}^{+\infty}\boldsymbol{\Delta}_{\hbar_1,\hbar_2}\,\boldsymbol{\pi}_{\boldsymbol{\varpi}}\left(z\right), \tag{12}$$

where

$$\begin{aligned}\boldsymbol{\Delta}_{\hbar_1,\hbar_3} &= 2\frac{\left(-1\right)^{\hbar_1+\hbar_2}\beta^{1+\hbar_1}\left(1+\hbar_3\right)^{\hbar_1}\Gamma\left(2+2\hbar_1\right)}{\hbar!_1\hbar_2!\boldsymbol{\sigma}_{\bullet}}\\ &\quad\times\sum_{\hbar_3=0}^{+\infty}\frac{\boldsymbol{\sigma}^{1+\hbar_3}\left(-1\right)^{\hbar_3}}{\hbar_3!\Gamma\left(2+2\hbar_1-\hbar_3\right)\left[\hbar_3-2\left(1+\hbar_1\right)\right]},\end{aligned}$$

and

$$\boldsymbol{\pi}_{\boldsymbol{\varpi}}\left(z\right)=\boldsymbol{\varpi}w_{\underline{\boldsymbol{\Omega}}}(z)W_{\underline{\boldsymbol{\Omega}}}(z)^{\boldsymbol{\varpi}-1}$$

is the PDF of the exponentiated-G (exp-G) family with power parameter $\boldsymbol{\varpi}$. Equation (12) reveals that the density of Z can be expressed as a linear mixture of exp-G densities. So, several mathematical properties of the new family can be obtained from those of the exp-G distribution. Similarly, the CDF of the CIR-G family can also be expressed as a mixture of exp-G CDFs given by

$$F\left(z\right)=\sum_{\hbar_1,\hbar_3=0}^{+\infty}\boldsymbol{\Delta}_{\hbar_1,\hbar_3}\,\boldsymbol{\Pi}_{\boldsymbol{\varpi}}\left(z\right), \tag{13}$$

where

$$\boldsymbol{\Pi}_{\boldsymbol{\varpi}}(z;\underline{\boldsymbol{\Omega}})=W_{\underline{\boldsymbol{\Omega}}}(z)^{\boldsymbol{\varpi}}$$

is the CDF of the exp-G family with power parameter $\boldsymbol{\varpi}$.

2.2. Moments

Let $Y_{\boldsymbol{\varpi}}$ be a R.V having density $\boldsymbol{\pi}_{\boldsymbol{\varpi}}(z)$. The rth ordinary moment of Z, say $\mu'_{r,Z}$, follows from (12) as

$$\mu'_{r,Z} = \mathbf{E}\left(Z^r\right) = \sum_{\hbar_1,\hbar_3=0}^{+\infty} \boldsymbol{\Delta}_{\hbar_1,\hbar_2}\ \mathbf{E}\left(Y^r_{\boldsymbol{\varpi}}\right), \tag{14}$$

where

$$\mathbf{E}(Y^r_{\boldsymbol{\varpi}}) = \boldsymbol{\varpi} \int_{-\infty}^{+\infty} z^r\ w_{\underline{\boldsymbol{\Omega}}}(z)\, W_{\underline{\boldsymbol{\Omega}}}(z)^{\boldsymbol{\varpi}-1}\ dz$$

can be evaluated numerically in terms of the baseline qf $Q_G(u) = W^{-1}(u)$ as

$$\mathbf{E}(Y^r_{\boldsymbol{\varpi}}) = \boldsymbol{\varpi} \int_0^1 u^{\boldsymbol{\varpi}-1} \left[Q_G(u)\right]^r du.$$

Setting $r = 1$ in (14) gives the mean of Z.

2.3. Incomplete moments and mean deviations

The r^{th} incomplete moment of Z is defined by

$$\mathbf{I}_{r,Z}(z) = \int_{-\infty}^{z} z^r\ f_{\underline{\boldsymbol{\eta}}}(z)dz.$$

We can write from (12)

$$\mathbf{I}_{r,Z}(z) = \sum_{\hbar_1,\hbar_2=0}^{+\infty} \boldsymbol{\Delta}_{\hbar_1,\hbar_2}\ \mathbf{I}_{r,\boldsymbol{\varpi}}(z),$$

where

$$\mathbf{I}_{r,\boldsymbol{\sigma}}(z) = \int_0^{W_{\underline{\boldsymbol{\Omega}}}(z)} u^{\boldsymbol{\sigma}-1}\ \left[Q_G(u)\right]^r du.$$

The integral $m_{r,\boldsymbol{\sigma}}(z)$ can be determined analytically for any special model with closed-form expressions for the $Q_G(u)$ or computed at least numerically for most baseline distributions.

2.4. Moment generating function (MGF)

The MGF of Z, say $M_Z(t) = \mathbf{E}\left(\exp\left(t\,Z\right)\right)$, is obtained from (12) as

$$M_Z(t) = \sum_{\hbar_1,\hbar_2=0}^{+\infty} \boldsymbol{\Delta}_{\hbar_1,\hbar_2}\ M_{\boldsymbol{\varpi},Z}\left(t\right),$$

where $M_{\boldsymbol{\varpi}}(t)$ is the generating function of $Y_{\boldsymbol{\varpi}}$ given by

$$M_{\boldsymbol{\varpi},Z}(t) = \boldsymbol{\varpi}\,\boldsymbol{\pi}^{+\infty}_{-\infty}(z) = \boldsymbol{\varpi}\,\boldsymbol{\pi}^{+1}_{0}(u).$$

where

$$\boldsymbol{\pi}^{+\infty}_{-\infty}(z) = \int_{-\infty}^{+\infty} \exp\left(t\,z\right) w_{\underline{\boldsymbol{\Omega}}}(z) \left[W_{\underline{\boldsymbol{\Omega}}}(z)\right]^{\boldsymbol{\varpi}-1}\ dz$$

and $\boldsymbol{\pi}^{+1}_{0}(u) = \int_0^1 \exp[t\,Q_W(u)]\,u^{\boldsymbol{\varpi}-1}du$. The two integrals $\boldsymbol{\pi}^{+\infty}_{-\infty}(z)$ and $\boldsymbol{\pi}^{+1}_{0}(u)$ can be numerically computed for most base line models.

In view of the theoretical complexities and the fact that the quantile function is not known in a certain closed form, we will use the methods that provide numerical solutions. We will use ready-made programs such as "R" and "MATHCAD" to facilitate numerical operations. The use of the numerical methods has become popular recently for many reasons. The most important of which is the availability of ready-made statistical programs and the presence of lots of mathematically complex distributions and models. The fact that has become recognized and cannot be ignored in the field of statistical analysis and mathematical modeling is that the complexity of models is no longer the real problem facing researchers, because statistical programs and packages have, in fact, contributed a lot in simplifying these complexities by providing numerical solutions. In this paper, we have used numerical methods in the process of estimation (see Section 6).

3. Bivariate version via copulas

3.1. FGM copula

Generally, the copula is a multivariate CDF for which the marginal models is a uniform on the interval $[0,1]$. The Copulas can be used to describe the dependence between the R.Vs. In this Section, we present some new bivariate CIR (B-CIR) type models using the Farlie Gumbel Morgenstern (FGM) copula (Johnson and Kotz [36]), the modified version of the FGM copula, the Renyi's copula, the Clayton copula and also with the Ali-Mikhail-Haq copula. However, many future works could allocated to the study these new models (see also Al-babtain et al. [4], Elgohari et al. [20], Elgohari et al. [21], Mansour et al. ([42],[43],[44],[45],[46] and [47]), Salah et al. [62], Shehata and Yousof ([63], [64], [65], [66]), Ibrahim et al. [34], Ali et al. [5] and Ali et al. [6] as examples).

Consider the joint CDF (J-CDF) of the FGM family with dependence parameter λ, where

$$F_\lambda(h,v) = hv\left(1+\lambda h^\bullet v^\bullet\right)|_{h^\bullet=1-h, v^\bullet=1-v},$$

and the two marginal function $h=F_1$, $v=F_2$, $\lambda\in(-1,1)$ and for every $h,v\in(0,1)$, $F(h,0)=F(0,v)=0$ which is "grounded-minimum" and $F(h,1)=h$ and $F(1,v)=v$ which are "grounded-maximum". Then,

$$F(h_1,v_1)+F(h_2,v_2)-F(h_1,v_2)-F(h_2,v_1)\geq 0.$$

A copula is continuous in h and v; actually, it satisfies the stronger Lipschitz condition, where

$$|F(h_2,v_2)-F(h_1,v_1)|\leq|h_2-h_1|+|v_2-v_1|.$$

For $0\leq h_1\leq h_2\leq 1$ and $0\leq v_1\leq v_2\leq 1$, we have

$$\Pr(h_1\leq h\leq h_2, v_1\leq v\leq v_2)=F(h_1,v_1)+F(h_2,v_2)-F(h_1,v_2)-F(h_2,v_1)\geq 0.$$

Then, setting

$$h^\bullet = 1-F(z_1)|_{[h^\bullet=(1-h)\in(0,1)]}$$

and

$$v^\bullet = 1-F(z_2)|_{[v^\bullet=(1-v)\in(0,1)]},$$

we can esaily obtain the J-CDF of the CIR using the FGM family

$$\begin{aligned} F_\lambda(z_1,z_2) \;=\;& \boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right] \\ & \times\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right] \\ & \times\left[1+\lambda\left(\begin{array}{c}\left\{1-\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right\} \\ \times\left\{1-\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right\}\end{array}\right)\right], \end{aligned}$$

where

$$\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}\left(z_1\right)=\exp\left\{-\boldsymbol{\sigma}_1\exp\left[-\frac{\beta_1}{\varkappa_{\underline{\boldsymbol{\Omega}}}^2\left(z_1\right)}\right]\right\},$$

$$\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}\left(z_2\right)=\exp\left\{-\boldsymbol{\sigma}_2\exp\left[-\frac{\beta_2}{\varkappa_{\underline{\boldsymbol{\Omega}}}^2\left(z_2\right)}\right]\right\}.$$

The joint PDF can then be derived from

$$c_\lambda(h,v)=1+\lambda h^\bullet v^\bullet|_{(h^\bullet=1-2h\text{ and }v^\bullet=1-2v)}$$

or from

$$c_\lambda(z_1,z_2)=f(z_1,z_2)=F\left(F_1,F_2\right)f_1f_2.$$

3.2. The modified FGM

The modified FGM copula is defined as

$$F_\lambda(h,v)=\mathfrak{b}v\left[1+\lambda A^{(h)}C^{(v)}\right]|_{\lambda\in(-1,1)}$$

or

$$F_\lambda(h,v)=\mathfrak{b}v+\lambda\dot{A}_h\dot{C}_v|_{\lambda\in(-1,1)},$$

where

$$\dot{A}_h=hA^{(h)},$$

and

$$\dot{C}_v=vC^{(v)}$$

and $A^{(h)}$ and $C^{(v)}$ are two continuous functions on $(0,1)$ with

$$A^{(0)}=A^{(1)}=C^{(0)}=C^{(1)}=0.$$

3.2.1. Type-I Consider the following functional form for both $A^{(h)}$ and $C^{(v)}$. Then, the B-CIR-FGM (Type-I) can be derived from

$$\begin{aligned}F_\lambda(z_1,z_2)\quad=\quad&\boldsymbol{\sigma}_{\bullet1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}\left(z_1\right)\right]\boldsymbol{\sigma}_{\bullet2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}\left(z_2\right)\right]\\&+\lambda\left(\begin{array}{c}\boldsymbol{\sigma}_{\bullet1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}\left(z_1\right)\right]\\\times\left\{1-\boldsymbol{\sigma}_{\bullet1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}\left(z_1\right)\right]\right\}\\\times\boldsymbol{\sigma}_{\bullet2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}\left(z_2\right)\right]\\\times\left\{1-\boldsymbol{\sigma}_{\bullet2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}\left(z_2\right)\right]\right\}\end{array}\right)|_{\lambda\in(-1,1)}.\end{aligned}$$

3.2.2. Type-II Let $A^{(h)}$ and $C^{(v)}$ be two functional form satisfying all the conditions stated earlier where

$$A^{(h)\bullet}|_{(\lambda_1>0)}=h^{\lambda_1}\left(1-h\right)^{1-\lambda_1}$$

and C

$$^{(v)\bullet}|_{(\lambda_2>0)}=v^{\lambda_2}\left(1-v\right)^{1-\lambda_2}.$$

Then, the corresponding B-CIR-FGM (Type-II) can be derived from

$$F_{\lambda,\lambda_1,\lambda_2}(h,v)=\mathfrak{b}v\left[1+\lambda A\left(h\right)^\bullet\ C^{(v)\bullet}\right].$$

Thus

$$F_{\lambda,\lambda_1,\lambda_2}(z_1,z_2) = \boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right] \times\left[1+\lambda\left(\begin{array}{c}\left\{\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right\}^{\lambda_1}\\ \times\left\{\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right\}^{\lambda_2}\\ \times\left(1-\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right)^{1-\lambda_1}\\ \times\left(1-\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right)^{1-\lambda_2}\end{array}\right)\right]$$

3.2.3. Type-III Let

$$\overline{C^{\bullet}_{(h)}} = h\left[\log\left(1+h^{\bullet}\right)\right]$$

and

$$\overline{D^{\bullet}_{(v)}} = v\left[\log\left(1+v^{\bullet}\right)\right]$$

for all $A^{(h)}$ and $C^{(v)}$ which satisfy all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the B-CIR-FGM (Type-III) from

$$F_{\lambda}(h,v) = \mathfrak{b}v\left(1+\lambda\overline{C^{\bullet}_{(h)}}\;\overline{D^{\bullet}_{(v)}}\right).$$

Then

$$F_{\lambda}(z_1,z_2) = \boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right] \times\left[1+\lambda\left(\begin{array}{c}\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\\ \times\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\\ \times\left[\log\left(2-\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right)\right]\\ \times\left[\log\left(2-\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right)\right]\end{array}\right)\right].$$

3.3. The Clayton copula

The Clayton copula can be considered as

$$F(v_1,v_2) = \left[(1/v_1)^{\lambda}+(1/v_2)^{\lambda}-1\right]^{-\frac{1}{\lambda}}|_{\lambda\in(0,+\infty)}.$$

Setting $v_1 = F(h)$ and $v_2 = F(z)$, the B-CIR type can be derived from $F(v_1,v_2) = F(F(v_1),F(v_2))$. Then

$$F(z_1,z_2) = \left\{\begin{array}{c}\left\{\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right\}^{-\lambda}\\ +\left\{\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right\}^{-\lambda}-1\end{array}\right\}^{-\frac{1}{\lambda}}|_{\lambda\in(0,+\infty)}.$$

3.4. Ali-Mikhail-Haq copula

Under the stronger Lipschitz condition, the Archimedean Ali-Mikhail-Haq copula can expressed as

$$F(h,v) = \mathfrak{b}v\left[1-\lambda h^{\bullet}v\bullet\right]^{-1}|_{\boldsymbol{\tau}\in(-1,1)},$$

then for any

$$F_{\underline{\eta}_1}(z_1) = 1-h^{\bullet}|_{[h^{\bullet}=(1-h)\in(0,1)]}$$

and

$$F_{\underline{\eta}_2}(z_2) = 1-v^{\bullet}|_{[v^{\bullet}=(1-v)\in(0,1)]}$$

we have

$$F(z_1,z_2) = \frac{\left\{\begin{array}{c}\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\\ \times\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\end{array}\right\}}{1-\lambda\left(\begin{array}{c}\left\{1-\boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right]\right\}\\ \times\left\{1-\boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1-\mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right]\right\}\end{array}\right)}.$$

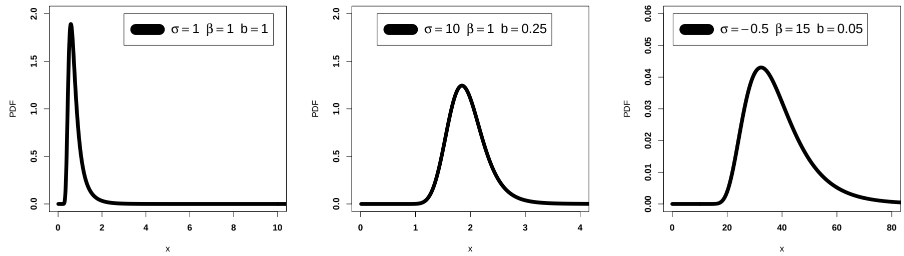

Figure 1. Plots of the PDF of the CIREx model.

3.5. The Renyi entropy copula

Using the theorem of Pougaza and Djafari (2011) where

$$F(h, v) = z_2 h + z_1 v - z_1 z_2,$$

the associated B-CIR can be derived from

$$F(z_1, z_2) = z_2 \boldsymbol{\sigma}_{\bullet 1}^{-1}\left[1 - \mathbf{O}_{\boldsymbol{\sigma}_1,\beta_1,\underline{\boldsymbol{\Omega}}}(z_1)\right] + z_1 \boldsymbol{\sigma}_{\bullet 2}^{-1}\left[1 - \mathbf{O}_{\boldsymbol{\sigma}_2,\beta_2,\underline{\boldsymbol{\Omega}}}(z_2)\right] - z_1 z_2.$$

4. Special submodels

In this Section we will provide many new distributions based on some common base line models namely: Log-logistic (LL), Weibull (W), inverse Weibull (IW), Exponential (Ex), inverse exponential (IE), Lomax (Lx), inverse Lomax (ILx), Rayleigh (R), inverse Rayleigh (IR), Burr XII (BXII), Half-logistic (HL), Standard Gumbel (Gu), Lindley (L), Nadarajah-Haghighi (NH), Dagum (D), inverse flexible Weibull (IFW), Gumbel (Gu), Gompertz (Gz) and inverse Gompertz (RGz) (see Table 1). A special attention is given to the compound inverse Rayleigh exponential (CIREx) and the compound inverse Rayleigh inverse Weibull (CIRIW) distributions.

Figure 1 and Figure 2 give some plots of the PDF of the CIREx and CIRIW distributions. Figure 3 and Figure 4 give some plots of the HRF of the CIREx and CIRIW distributions. Figure 1 gives some plots of the PDF of the CIREx distribution. Based on Figure 1, the new density of the CIREx distribution can be "heavy tail right skewed with one peak", "symmetric" and "simple right skewed with one peak". Figure 2 gives some plots of the PDF of the CIRIW distribution. Based on Figure 2, the new density of the CIRIW model can be "asymmetric right skewed with one peak and a heavy tail" and "right skewed with no peak". Figure 3 provides different plots of the HRF of the CIREx distribution. Based on Figure 3, the new HRF of the can be CIREx model can be "upside-down-constant", "constant", "increasing-constant". Figure 4 provides different plots of the HRF of the CIRIW distribution. Based on Figure 4, the new hazard rate function (HRF) of the can be CIRIW model can be "revised J-shape", "upside-down", "J-shape" and "increasing". Based on Figure 1 and Figure 2, the family may be useful in modeling the "heavy tail right skewed with one peak", "symmetric", "simple right skewed with one peak", "asymmetric right skewed with one peak and a heavy tail" and "right skewed with no peak" real data sets. Based on Figure 3 and Figure 4, the family may be useful in modeling the real data sets which have "upside-down-constant", "constant", "increasing-constant", "revised J-shape", "upside-down", "J-shape" and "increasing" HRFs.

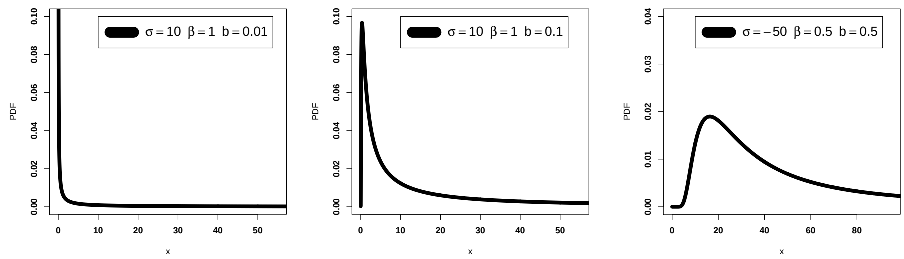

Figure 2. Plots of the PDF of the CIRIW model.

Table 1: New submodels based on the new CIR-G family.

No.	Base line model	$\varkappa_{\underline{\mathbf{\Omega}}}(z)$	Submodel
1	Ex	$\exp(bz)-1\vert_{b>0}$	CIREx
2	W	$\exp\left(z^{b}\right)-1\vert_{b>0}$	CIRW
3	IW	$\left[\exp\left(z^{b}\right)-1\right]^{-1}\vert_{b>0}$	CIRIW
4	IR	$\left[\exp\left(\alpha z^{-2}\right)-1\right]^{-1}\vert_{\alpha>0}$	CIRIR
5	LL	$\frac{\alpha}{b}z^{\alpha}\vert_{b,\alpha>0}$	CIRLL
6	RE	$\left[\exp\left(\alpha z^{-1}\right)-1\right]^{-1}\vert_{\alpha>0}$	CIRIE
7	ILx	$\left[\left(1+z^{-1}\right)^{b}-1\right]^{-1}_{b>0}$	CIRILx
8	NH	$\exp\left[(1+\alpha z)^{b}-1\right]-1\vert_{\alpha,b>0}$	CIRNH
9	Lx	$(1+z)^{\alpha}-1\vert_{\alpha>0}$	CIRLx
8	R	$\exp(\alpha z)^{2}-1\vert_{\alpha>0}$	CIRR
10	IFW	$\left\{\exp\left[\exp\left(\frac{\alpha}{z}-bz\right)\right]-1\right\}^{-1}\vert_{\alpha>0}$	CIRIFW
11	BXII	$\left(1+z^{b}\right)^{\alpha}-1\vert_{b,\alpha>0}$	CIRBXII
12	HL	$\left\{\left[\frac{1-\exp(-z)}{1+\exp(-z)}\right]^{-1}-1\right\}^{-1}$	CIRHL
13	IGz	$\left(\exp\left\{-\frac{b}{\alpha}\left[\exp\left(\frac{z}{\alpha}\right)-1\right]\right\}-1\right)^{-1}\vert_{\alpha,b>0}$	CIRIGz
14	Gu	$\left(\exp\left\{\exp\left[-(z)\right]\right\}-1\right)^{-1}$	CIRGu
15	L	$\exp(\alpha z)\left[\frac{1+\alpha+\alpha z}{1+\alpha}\right]^{-1}-1\vert_{\alpha>0}$	CIRL
16	HL	$\left\{\left[\frac{1-\exp(-\alpha z)}{1+\exp(-\alpha z)}\right]^{-1}-1\right\}^{-1}\vert_{\alpha>0}$	CIRHL
17	Da	$\left\{\left[1+\left(\frac{z}{\lambda}\right)^{-b}\right]^{\alpha}-1\right\}^{-1}\vert_{\alpha,b,\lambda>0}$	CIRDa
18	Gz	$\exp\left\{b\left[\exp(\alpha z)-1\right]\right\}-1\vert_{\alpha>0}$	CIRGz

5. Parameter Estimation

Here, we will consider the estimation of the unknown parameters $(\boldsymbol{\sigma},\beta,\underline{\mathbf{\Omega}})$ of the new G family from complete samples by maximum likelihood method. Let $z_1,\cdots,z_n$ be a random sample (rs) from the

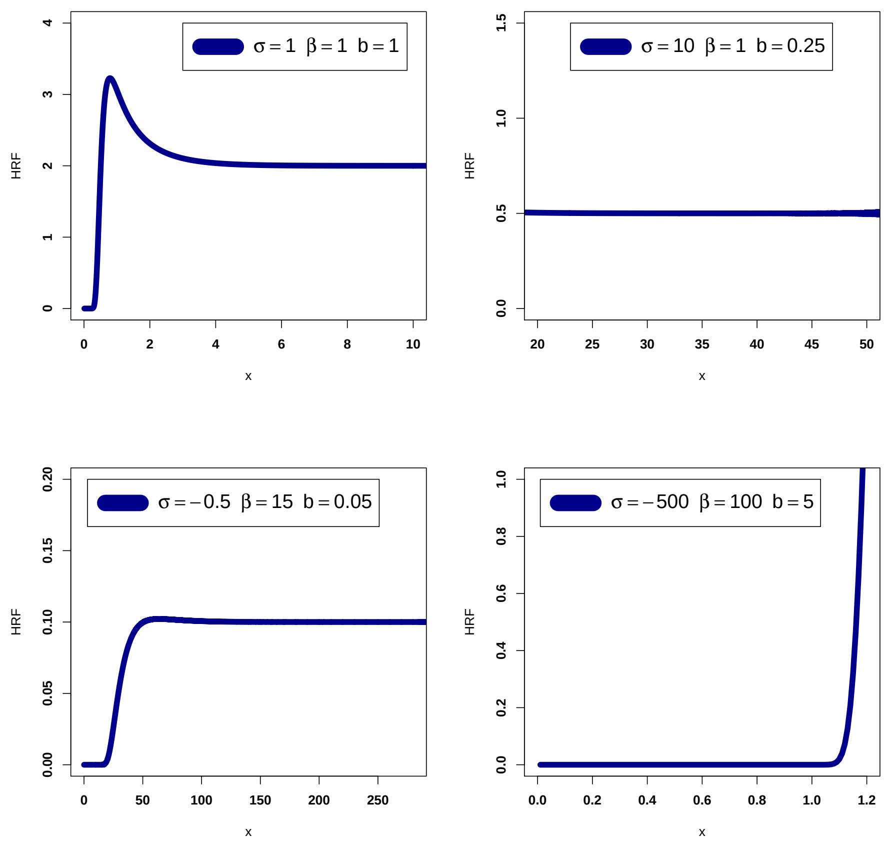

Figure 3. Plots of the HRF of the CIREx model.

CIR-G models parameter vector $\underline{\boldsymbol{\eta}} = (\boldsymbol{\sigma}, \beta, \underline{\boldsymbol{\Omega}}^{\intercal})^{\intercal}$. The log-likelihood function for $\underline{\boldsymbol{\eta}}$ is given by

$$\begin{aligned}\ell_n(\underline{\boldsymbol{\eta}}) \quad = \quad & n\log 2 + n\log\beta + n\log\boldsymbol{\sigma} - n\log\boldsymbol{\sigma}_{\bullet} + \sum_{i=1}^{n}\log w_{\underline{\boldsymbol{\Omega}}}(z_i) + \sum_{i=1}^{n}\log\overline{W}_{\underline{\boldsymbol{\Omega}}}(z_i) \\ & - \sum_{i=1}^{n}\frac{\beta}{\varkappa^2_{z_i,\underline{\boldsymbol{\Omega}}}} - 3\sum_{i=1}^{n}\log W_{\underline{\boldsymbol{\Omega}}}(z_i) - \boldsymbol{\sigma}\sum_{i=1}^{n}\exp\left(-\frac{\beta}{\varkappa^2_{z_i,\underline{\boldsymbol{\Omega}}}}\right).\end{aligned}$$

The above log-likelihood function can be maximized numerically by using R (optim), SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS), among others. For confidence interval (C.I.) estimation of the parameters, the elements of the observed information matrix $J(\underline{\boldsymbol{\eta}})$ can be evaluated numerically, where

$$U_{\boldsymbol{\sigma}} = \frac{\partial}{\partial\boldsymbol{\sigma}}\ell_n(\underline{\boldsymbol{\eta}}), U_{\beta} = \frac{\partial}{\partial\beta}\ell_n(\underline{\boldsymbol{\eta}}),$$

and

$$U_{\underline{\boldsymbol{\Omega}}_{\mathsf{P}}} = \frac{\partial}{\partial\underline{\boldsymbol{\Omega}}_{\mathsf{P}}}\ell_n(\underline{\boldsymbol{\eta}}),$$

where P is the number of parameters of the base line model. Setting the nonlinear system of equations $U_{\boldsymbol{\sigma}} = U_{\beta} = U_{\underline{\boldsymbol{\Omega}}_{\mathsf{P}}} = 0$ and solving them simultaneously yields the maximum likelihood estimations (MLEs)

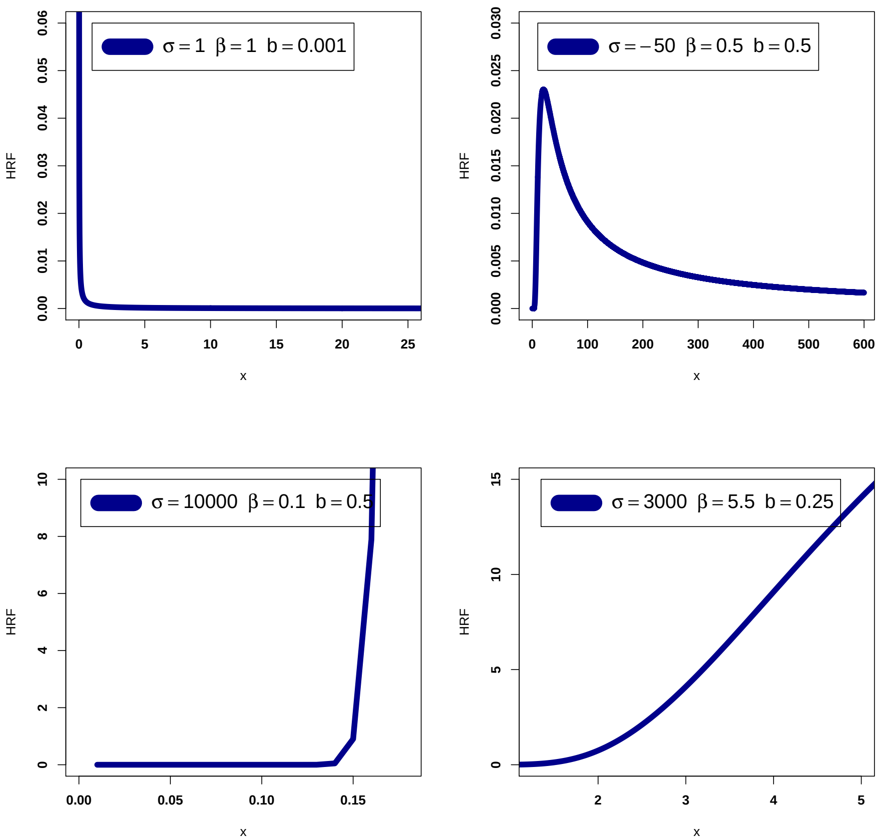

Figure 4. Plots of the HRF of the CIREIW model.

of $\underline{\boldsymbol{\eta}} = (\boldsymbol{\sigma}, \beta, \underline{\boldsymbol{\Omega}}^{\intercal})^{\intercal}$. These equations can be solved numerically using convenient iterative method such as the Newton-Raphson type algorithms. For interval estimation of these parameters, we can obtain the observed information matrix

$$\mathbf{J}\left(\widehat{\underline{\boldsymbol{\eta}}}\right) = \frac{\partial^2 \ell_n(\underline{\boldsymbol{\eta}})}{\partial m \partial n} (\forall m, n = \boldsymbol{\sigma}, \beta, \underline{\boldsymbol{\Omega}})$$

which can be computed numerically.

Under standard regularity conditions when $n \to +\infty$, the distribution of $\left(\widehat{\underline{\boldsymbol{\eta}}}\right)$ can be approximated by a multivariate normal

$$N_{(\mathsf{P}+1)}\left(0, \mathbf{J}\left(\widehat{\underline{\boldsymbol{\eta}}}\right)^{-1}\right)$$

distribution to construct approximate confidence intervals for the parameters. Here, $\mathbf{J}\left(\widehat{\underline{\boldsymbol{\eta}}}\right)$ is the total observed information matrix evaluated at $\left(\widehat{\underline{\boldsymbol{\eta}}}\right)$. Large sample theory for these estimators delivers simple approximations that work well in finite samples.

The normal approximation for the MLEs is easily handled numerically. Likelihood ratio tests can be performed for the proposed family in the usual way. Theoretically, the issue of identifiability is not easy to ensure for new models. Ideally, to ensure and prove this property, we have to prove that, for any $q \in \mathbb{R}$, the equality

$$F_{\underline{\boldsymbol{\eta}}^*}(q)=F_{\underline{\boldsymbol{\eta}}}(q),$$

implies that $\underline{\boldsymbol{\eta}}^*=\underline{\boldsymbol{\eta}}$ which means $\boldsymbol{\sigma}^*, \beta^*, \underline{\boldsymbol{\Omega}}^* = \boldsymbol{\sigma}, \beta, \underline{\boldsymbol{\Omega}}$. This equality is clear for $q = 0$. However, for the other cases, the complexity of $F_{\underline{\boldsymbol{\eta}}^*}(q)$ is a significant barrier to demonstrating that in full rigor. Our practical investigations, however, have revealed no problem of this kind, but the rigorous proof remains a strong mathematical challenge.

6. Real data applications

In this Section we analyze two real data sets. For data the first data set (the first application), we considered the standard one-parameter exponential distribution as our base line model and then we compare the fits of the CIREx distribution with other competitive exponential extensions such as the odd Lindley exponential (OLEx) model, Marshall-Olkin exponential (MOEx) model, the generalized Marshall-Olkin exponential (GMOEx) model, Marshall-Olkin Kumaraswamy exponential (MOKumEx) model, the moment exponential (MEx) model, the Burr-Hatke exponential (BrHEx) model, Kumaraswamy exponential (KumEx), beta exponential (BEx) model, Kumaraswamy Marshall-Olkin exponential (KumMOEx) model and standard exponential model (see Almamy et al. [7], Dara and Ahmad [18], Ghitany et al. [23], Yousof et al. [73]).

However, in the statistical literature there are several useful exponential versions which can be used in comparison such as the transmuted exponentiated generalized E (TEGE) model and the Burr of the type X exponential (BXEx) model (Yousof et al. [72]), the Poisson-E (PEx) distribution (Cancho et al. [15]), Burr type XII exponential (BXIIEx) model (Cordeiro et al. [17]), the Burr of the type X exponentiated exponential (BXEEx) model (see Khalil et al. [37]), generalized odd log-logistic exponentiated Ex (GOLLEEx) distribution , quasi Poisson Burr of the type X exponentiated exponential (QPBXEEx) distribution (Mansour et al. [42], [43] and [44]), among others. For data set (the second application), we considered the standard two-parameters inverse Weibull model as our base line model and then we compare the fits of the CIRIW model with other competitive inverse Weibull models such as the Marshal-Olkin inverse Weibull (MOIW) model, standard inverse Weibull (IW) model, Generalized Marshal-Olkin inverse Weibull (GzMOIW) model, beta inverse Weibull (BIW) model, Kumaraswamy Marshal-Olkin inverse Weibull (KumMOIW) model, Kumaraswamy inverse Weibull (KumIW) model and Marshal-Olkin Kumaraswamy inverse Weibull (MOKumIW) distribution. Some details related to the these copetitive model are available in Ibrahim et al. [33].

For comparing models, we consider the Cramér-Von Mises (CM) and the Anderson-Darling (AD) and the Kolmogorov-Smirnov (KS) statistic (and its corresponding P-value), the Akaike Information Criterion ($C_{1,\widehat{\ell}}$), Bayesian Information Criterion ($C_{2,\widehat{\ell}}$), consistent Akaike Information Criterion ($C_{3,\widehat{\ell}}$) and Hannan-Quinn Information Criterion ($C_{4,\widehat{\ell}}$). However, other potential goodness-of-fit statistic tests for validation such as Nikulin-Rao-Robson statistic test and the modified Nikulin-Rao-Robson statistic test may be used (see Goual et al. [24], Ibrahim et al.[35], Ibrahim et al.[32], Goual et al. [26], Goual and Yousof [25], Yadav et al. [67] and Yousof et al. [71]).

6.1. First application (failure times data)

The failure times data set (see Gross and Clark [27]). The data represent the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. Table 2 lists the MLEs, standard errors (SEs) and $(1-\varepsilon)\,\%$ confidence intervals ($(1-\varepsilon)\,\%$ C.I.s) by providing the lower bound (LB) and the upper bound (LB). Table 3 lists the $C_{1,\widehat{\ell}}$, $C_{2,\widehat{\ell}}$, $C_{3,\widehat{\ell}}$, $C_{4,\widehat{\ell}}$, AD, CM, K.S. and its p-value. Figure 5 gives the total time in test (TTT) plot for the relief times data along with the corresponding quantile-quantile (QQ) plot , box plot and the nonparametric Kernel density estimation (N-KDE) plot. Based on Figure 5, the HRF of the relief times is "monotonically increasing HRF" (top right plot) and this data has an extreme value (see top right and bottom right plots) and its density is right skewed and bimodal. Figure 6 gives the estimated PDF (E-PDF), estimated CDF (E-CDF), estimated HRF (E-HRF) and P-P plot for relief times data. Based on results of Table 3 and Table 4, it is concluded that the CIREx model is much better than the exponential, odd Lindley exponential, Marshall-Olkin exponential, moment exponential, the logarithmic Burr-Hatke exponential, generalized Marshall-Olkin exponential, Beta exponential, Marshall-Olkin Kumaraswamy exponential, Kumaraswamy exponential, the Burr of the type X exponential and Kumaraswamy Marshall-Olkin exponential models with $C_{1,\widehat{\ell}}$=36.75, $C_{2,\widehat{\ell}}$=39.74, $C_{3,\widehat{\ell}}$=38.25, $C_{4,\widehat{\ell}}$=37.34, AD= 0.15, CM=0.027, K.S=0.090 and p-value=0.996 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 6, the CIREx distribution provides adequate fits to the empirical functions.

Table 2: MLEs, SEs, C.I.s (in parentheses) values for the relief times data.

Models		Estimates
Ex(b)	MLE	0.52639
	SE	(0.11715)
	95%(LB, UB)	(0.290, 0.75)
OLEx(b)	MLE	0.60443
	SE	(0.0535)
	95%(LB, UB)	(0.501, 0.73)
MEx(b)	MLE	0.95034
	SE	(0.15012)
	95%(LB, UB)	(0.6611, 1.245)
BrHEx(b)	MLE	0.52633
	SE	(0.11823)
	95%(LB, UB)	(0.431, 0.633)
MOEx(β, b)	MLE	54.474, 2.316
	SE	(35.5821), (0.3742)
	95%(LB, UB)	(0, 124.212), (1.58, 3.044)
GMOEx($\boldsymbol{\sigma}, \beta, b$)	MLE	0.5192, 89.4623, 3.1688
	SE	(0.256), (66.278), (0.772)
	95%(LB, UB)	(0.02, 1.02), (0, 219.37), (1.66, 4.68)
KumEx($\boldsymbol{\sigma}, \beta, b$)	MLE	83.75623, 0.5681, 3.3302
	SE	(42.3612), (0.3262), (1.1881)
	95%(LB, UB)	(0.729, 166.8), (0, 1.213), (1, 5.66)
BEx($\boldsymbol{\sigma}, \beta, b$)	MLE	81.6332, 0.5421, 3.5142
	SE	(120.4122), (0.3266), (1.40114)
	95%(LB, UB)	(0, 317.63), (0, 1.177), (0.75, 6.28)
MOKumEx($\alpha, \boldsymbol{\sigma}, \beta, b$)	MLE	0.13323, 33.232, 0.5713, 1.6743
	SE	(0.3322), (57.837), (0.721), (1.814)
	95%(LB, UB)	(0, 0.783, (0, 146.6), (0, 1.99), (0, 5.2)
KwMOE($\alpha, \boldsymbol{\sigma}, \beta, b$)	MLE	8.8744, 34.8265, 0.2992, 4.8993
	SE	(9.1463), (22.3128), (0.24), (3.1767)
	95%(LB, UB)	(10.9, 46.8), (0, 78.6), (0, 0.8), (0, 11.12)
EIBXEx($\boldsymbol{\sigma}, \beta, b$)	MLE	2.09125, 2.89979, 0.5307
	SE	2.83243, 1.3667, 0.53723
	95%(LB, UB)	(0, 7.69), (0, 5.5), (0, 1.56)

6.2. Second application (gauge lengths data)

The gauge lengths data set (Kundu and Raqab (2009)) consists of 74 observations. Table 5 lists the MLEs, SEs confidence intervals (C.I.s) for the gauge lengths data. Table 6 lists the $C_{1,\widehat{\ell}}$, $C_{2,\widehat{\ell}}$, $C_{3,\widehat{\ell}}$, $C_{4,\widehat{\ell}}$, AD, CM, K.S. and p-value. Figure 7 gives the total time test (TTT) plot (Aarset (1987)) for the relief times data along with the corresponding box plot, QQ plot and the N-KDE plot. Based on Figure 7, the HRF of the gauge lengths data is "monotonically increasing HRF" (see top right plot) and this data has no extreme observation (see top right and bottom right plots) and its density is semi-bimodal. Figure 6 gives the E-PDF, E-CDF, E-HRF and P-P plot for gauge lengths data.

Based on results Figure 8 and Table 7, it is concluded that the CIRIW model is much better than the inverse Weibull, Marshal-Olkin inverse Weibull, Generalized Marshal-Olkin inverse Weibull, Kumaraswamy inverse Weibull, beta inverse Weibull, Kumaraswamy Marshal-Olkin inverse Weibull and Marshal-Olkin Kumaraswamy inverse Weibull models with $C_{1,\widehat{\ell}}$=108.82, $C_{2,\widehat{\ell}}$=115.74, $C_{3,\widehat{\ell}}$=109.17, $C_{4,\widehat{\ell}}$=111.58, AD=0.28, CM=0.039, K.S=0.055 and p-value=0.977 so the new lifetime model is a good alternative to these models in modeling gauge lengths data set. According to Figures 6, the CIRIW distribution provides adequate fits to the empirical functions.

Table 3: $C_{1,\widehat{\ell}}$, $C_{2,\widehat{\ell}}$, $C_{3,\widehat{\ell}}$, $C_{4,\widehat{\ell}}$ for relief data.

Models	$C_{1,\widehat{\ell}}$	$C_{2,\widehat{\ell}}$	$C_{3,\widehat{\ell}}$	$C_{4,\widehat{\ell}}$
OLEx	49.12	50.11	49.32	49.30
KumMOEx	42.83	46.84	45.55	43.60
Ex	67.67	68.67	67.89	67.87
MEx	54.32	55.31	54.54	54.50
BEx	43.48	46.45	44.98	44.02
BrHEx	67.67	68.67	67.89	67.87
MOEx	43.51	45.51	44.22	43.90
GMOEx	42.75	45.74	44.25	43.34
KumEx	41.78	44.75	43.28	42.32
MOKumEx	41.58	45.54	44.25	42.30
CIREx	36.75	39.74	38.25	37.34

Table 4: AD,CM,K.S. and p-value for relief times data.

Models	AD	CM	K.S.	p-value
Ex	4.602	0.960	0.446	< 0.001
BEx	0.705	0.120	0.166	0.801
MOKumEx	0.602	0.111	0.143	0.872
MEx	2.764	0.536	0.329	0.073
BrHEx	0.629	0.105	0.444	< 0.001
KumEx	0.458	0.078	0.146	0.863
MOEx	0.847	0.144	0.184	0.553
GMOEx	0.512	0.088	0.153	0.788
KumMOEx	1.081	0.199	0.154	0.863
OLEx	1.339	0.228	0.854	< 0.001
CIREx	0.15	0.027	0.090	0.996

Table 5: MLEs, SEs, C.I.s (in parentheses) values for the gauge lengths data.

Models		Estimates
IW(β, b)	MLE	4.11022, 2.16921
	SE	(0.32320), (0.0648)
	95%(LB, UB)	(3.48,4.75), (2.04, 2.3)
MOIW($\boldsymbol{\sigma}, \beta, b$)	MLE	80.346, 8.03, 1.426
	SE	(62.01), (0.7634), (0.11005)
	95%(LB, UB)	(0, 201.93),(6.5, 9.53),(1.2, 1.66)
GzMOIW($\alpha, \boldsymbol{\sigma}, \beta, b$)	MLE	3.7024, 63.707, 5.9181, 1.577
	SE	(2.687), (38.666), (0.951), (0.144)
	95%(LB, UB)	(0, 8.967), (0,139.53), (4.06, 7.8), (1.30, 2)
KumIW($\alpha, \boldsymbol{\sigma}, \beta, b$)	MLE	3.218444, 217.0311, 1.00518, 4.3844
	SE	(1.0359), (268.6), (0.22), (1.01)
	95%(LB, UB)	(1.2, 5.30), (0, 743.41), (0.6,1.446), (2.40, 6.4)
BIW($\alpha, \boldsymbol{\sigma}, \beta, b$)	MLE	2.039301, 5.8572, 0.2424, 37.17889
	SE	(1.0153), (1.813), (0.38), (33.51)
	95%(LB, UB)	(0.04,4.023),(2.30,9.4),(0,0.98),(0,102.9)
KumMOIW($\alpha, a, \boldsymbol{\sigma}, \beta, b$)	MLE	0.01553, 0.827, 16.9853, 0.894, 25.1274
	SE	(0.0234), (0.789), (24.975), (0.396), 19.688)
	95%(LB, UB)	(0,0.06), (0,2.4), (0,65.9), (0.11,1.7), (0,63.7)
MOKumIW($\alpha, a, \boldsymbol{\sigma}, \beta, b$)	MLE	7.995, 2.933,35.707,1.221,2.415
	SE	(13.0634),(0.825),(41.500),(0.412),(1.032)
	95%(LB, UB)	(0,33.6), (1.3,4.6), (0,18), (0.4,2.03), (0.39,4.4)
CIRIW($\boldsymbol{\sigma}, \beta, b$)	MLE	2991.518, 5.50003, 0.240797
	SE	3.88119, 0.460513, 0.004561
	95%(LB, UB)	(2983.76,2999.28), (4.6, 6.42), (0.23,0.25)

Table 6: $C_{1,\widehat{\ell}}$, $C_{2,\widehat{\ell}}$, $C_{3,\widehat{\ell}}$, $C_{4,\widehat{\ell}}$ for the gauge lengths data.

Models	$C_{1,\widehat{\ell}}$	$C_{2,\widehat{\ell}}$	$C_{3,\widehat{\ell}}$	$C_{4,\widehat{\ell}}$
IW	142.02	146.63	142.19	143.86
GzMOIW	112.80	122.00	113.37	126.48
KumIW	113.68	122.82	114.25	117.36
KumMOIW	113.30	124.82	114.18	117.90
MOKumIW	113.19	124.68	114.07	117.78
BIW	112.63	121.84	113.21	116.30
MOIW	115.06	121.96	115.40	117.81
CIRIW	108.82	115.74	109.17	111.58

Table 7: AD,CM,K.S. and p-value for the lengths data.

Models	AD	CM	K.S.	p-value
IW	2.931	0.468	0.151	0.096
GzMOIW	0.399	0.071	0.065	0.946
KumIW	0.370	0.079	0.065	0.924
KumMOIW	0.363	0.055	0.066	0.953
MOKumIW	0.369	0.055	0.063	0.941
BIW	0.446	0.073	0.064	0.931
MOIW	0.788	0.123	0.072	0.862
CIRIW	0.28	0.039	0.055	0.977

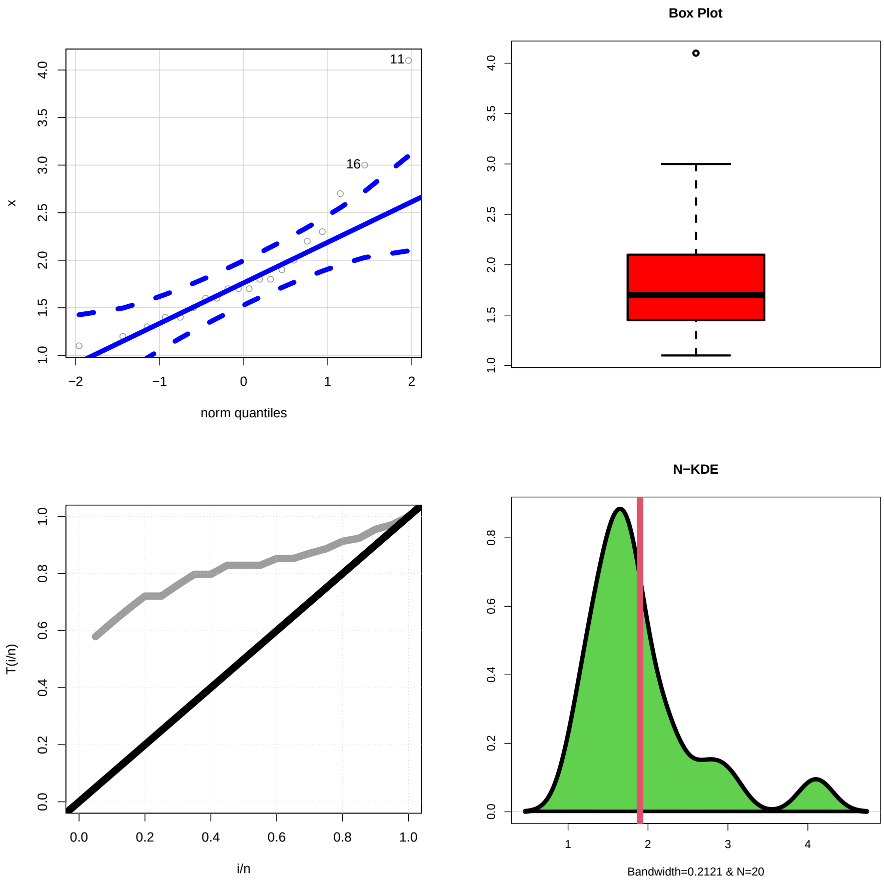

Figure 5. QQ, Box, TTT and N-KDE plots for the relief times data.

7. Concluding remarks

In this work, a new one-parameter compound G family of continuous distributions is derived and studied. Relevant statistical properties such as moments, incomplete moments and moment generating function are derived. The density of the new family is re-expressed in terms of the exponentiated G family. The new density can be ”heavy tail right skewed with one peak”, ”symmetric”, ”simple right skewed with one peak”, ”asymmetric right skewed with one peak and a heavy tail” and ”right skewed with no peak”. The corresponding hazard function can be ”upside-down-constant”, ”constant”, ”increasing-constant”, ”revised J-shape”, ”upside-down”, ”J-shape” and ”increasing”. Many bivariate types have been also derived via different common copulas. The estimation of the model parameters is performed by the maximum likelihood method. The usefulness and flexibility of the new family is illustrated by means of two real data sets. The new family is better than the odd Lindley family, Marshall-Olkin family, the Burr-Hatke family, generalized Marshall-Olkin family, Beta family, Marshall-Olkin Kumaraswamy family, Kumaraswamy family, the Burr of the type X family and Kumaraswamy Marshall-Olkin family in modeling the bimodal right skewed relief times data set with with $C_{1,\widehat{\ell}}$=36.75, $C_{2,\widehat{\ell}}$=39.74, $C_{3,\widehat{\ell}}$=38.25,

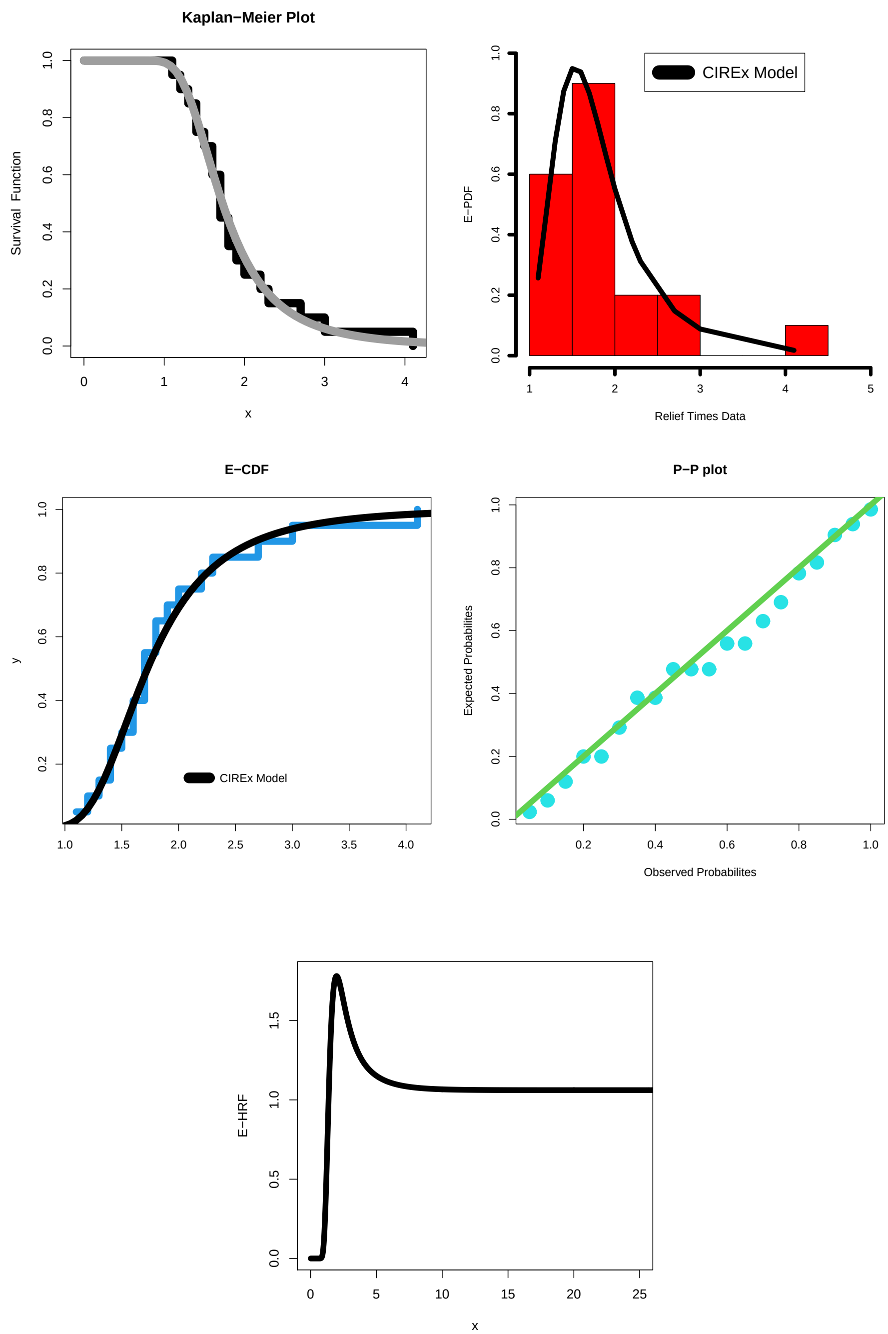

Figure 6. KM, E-PDF, E-CDF, P-P and E-HRF for relief times data.

$C_{4,\hat{\ell}}$=37.34, AD= 0.15, CM=0.027, K.S=0.090 and p-value=0.996 so the new family could be considered as a good alternative to these families. The new family is better than the Marshal-Olkin family, Generalized Marshal-Olkin family, Kumaraswamy family, beta family, Kumaraswamy Marshal-Olkin family and Marshal-Olkin Kumaraswamy family in modeling the gauge lengths data set with $C_{1,\hat{\ell}}$ =108.82,

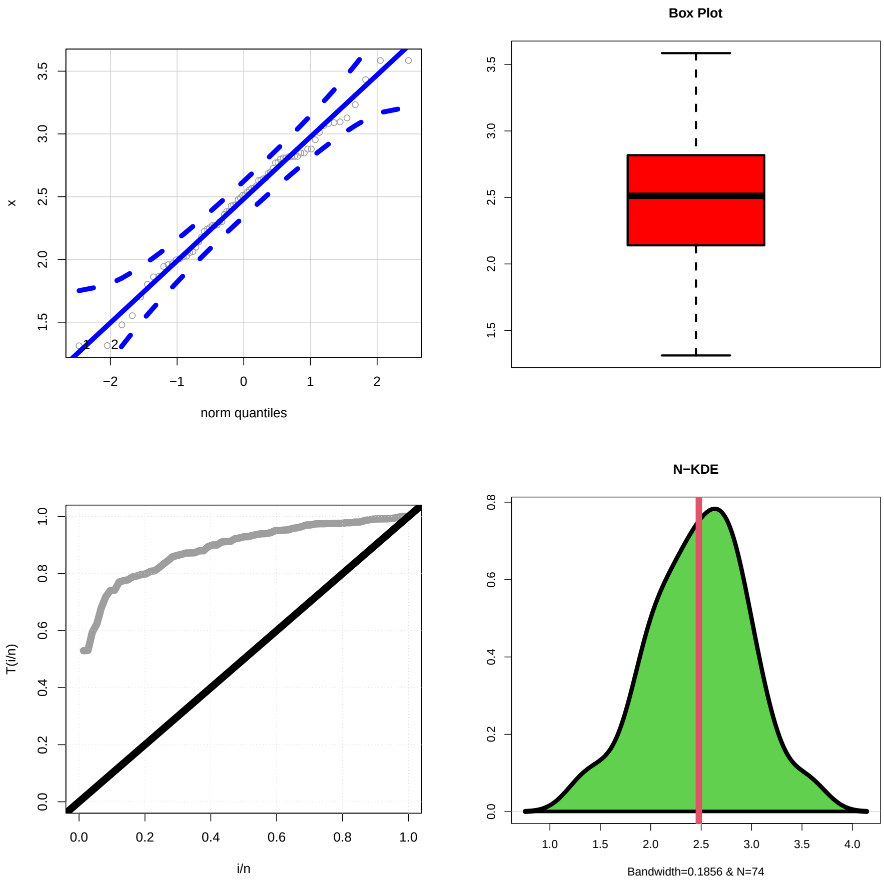

Figure 7. QQ, Box, TTT and N-KDE plots for the gauge lengths data.

$C_{2,\widehat{\ell}}$ =115.74, $C_{3,\widehat{\ell}}$ =109.17, $C_{4,\widehat{\ell}}$ =111.58, AD=0.28, CM=0.039, K.S=0.055 and p-value=0.977 so the new family could be considered as a good alternative to these families.

As a potential future work that can be created based on the new family, we can use the compound inverse Rayleigh family in the field of insurance and actuarial sciences, especially in modeling insurance payments data and calculating life and death rates (see Mohamed et al. [52], [53] and [54]). Moreover, the compound inverse Rayleigh family can be applied in (see Rasekhi et al. [56], Saber and Yousof [58], Saber et al. [59] and Saber et al. [60]). In fact, the new family has great flexibility, which will motivate many researchers to study it from other aspects. The new family can be used to make more new distributions. We hope that this family will take a great deal of interest, research and study, especially in the field of mathematical modeling and application in the field of engineering, medicine and actuarial sciences. Moreover following Chesneau et al. [16], Yousof et al. [74] and Ibrahim et al. [31], this new family can be used to generate new discrete families for count data modeling, and we hope that this family will gain the interest of researchers in the journals of statistics, insurance, actuarial science, and others. The novel G family can also be used for single, double and multiple acceptance sampling plans with many application in quality and risk decisions (see Ahmed et al. [12], Ahmed and Yousof [11]).

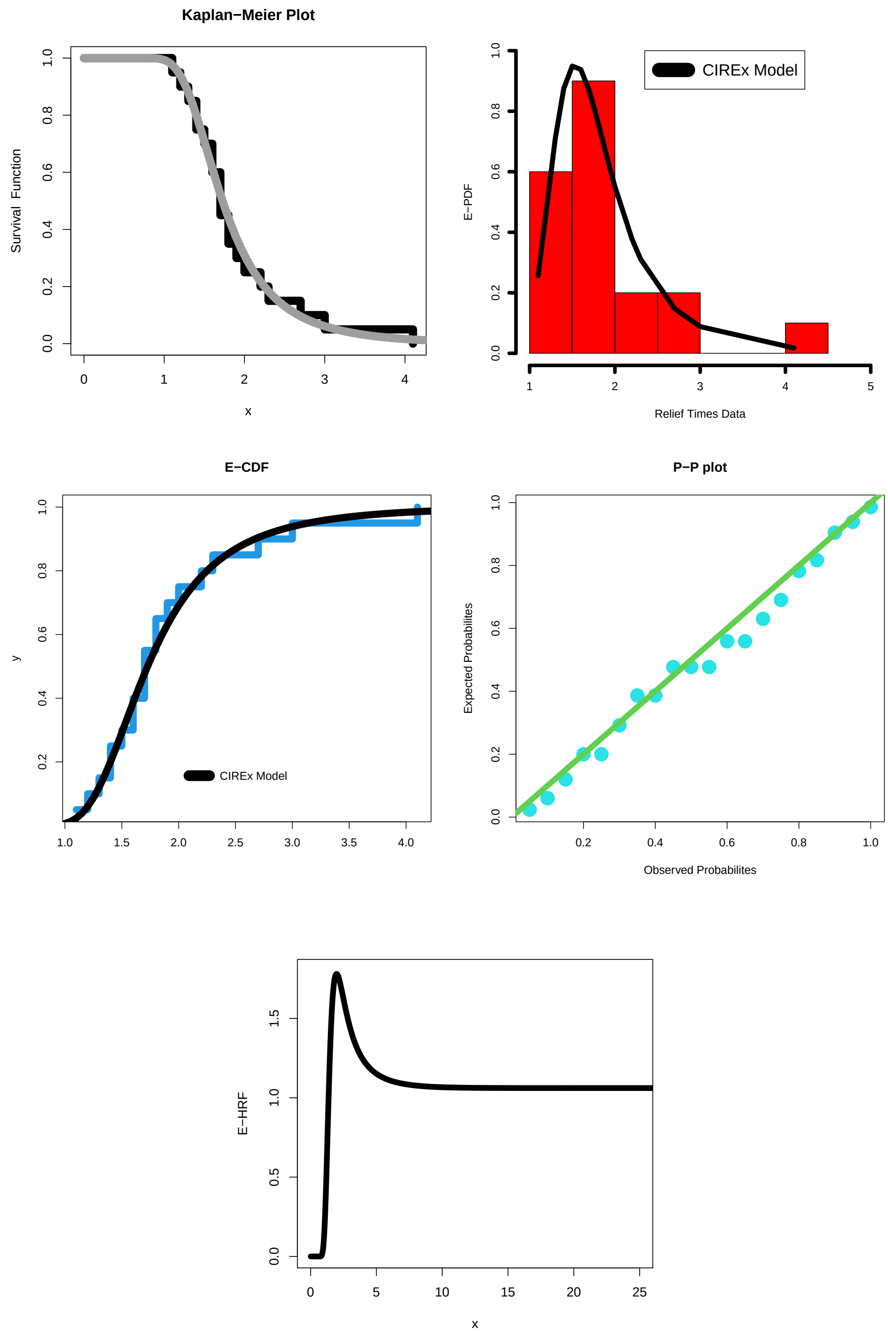

Figure 8. KM, E-PDF, E-CDF, P-P and E-HRF for the gauge lengths data.

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