STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, June 2023, pp 740–754.
Published online in International Academic Press (www.IAPress.org)

On Past Extropy and Negative Cumulative Extropy Properties of Ranked Set Sampling and Maximum Ranked Set Sampling with Unequal Samples

M. R. Irshad[1], R. Maya [2,*], K. Archana [1], S.Tahmasebi [3]

[1] *Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India*
[2] *Department of Statistics, Govt. College for Women, Trivandrum 695 014, Kerala, India*
[3] *Department of Statistics, Persian Gulf University, Bushehr, Iran*

Abstract Ranked set sampling is considered as an alternative to simple random sampling and maximum ranked set sampling is a very useful modification of ranked set sampling. In this paper we focused on information content of ranked set sampling and maximum ranked set sampling with unequal samples in terms of past extropy measure and also considered the information content of negative cumulative extropy and its dynamic version based on maximum ranked set sampling and simple random sampling designs. We also compare ranked set sampling data, maximum ranked set sampling data with simple random sampling and with each other. Also here we obtained a new discrimination information measure among simple random sampling data, ranked set sampling data and maximum ranked set sampling data for past extropy measure.

Keywords Past extropy, Negative cumulative extropy, Ranked set sampling, Maximum ranked set sampling, Discrimination information

AMS 2010 subject classifications 62B10, 62D05

DOI: 10.19139/soic-2310-5070-1453

1. Introduction

Ranked set sampling (RSS) has been used widely since its introduction by [11]. It cover the entire gamut of values in the population. It also merges several other sources of information like auxiliary information, professional knowledge etc besides simple random sampling (SRS). It is a representative part of the population than the same number of observations obtained via SRS.

The method of RSS as suggested by [11] as follows: From the population, the first set which contains n samples are chosen and the unit which is in the lowest position is taken for the actual measurement of variable of interest and the remaining items are discarded. Similarly a second set is chosen and unit corresponding to the second lowest position is chosen. This is continued until the n^{th} unit is selected. The cycle can be repeated according to our choice. Without loss of generality, throughout the article we assume that only one cycle. Now, these n units selected under one cycle comprises a ranked set sample (rss) and are usually denoted by $X^n_{RSS} = \{X_{i:n}\ ;\ i = 1, 2, ..., n\}$. In the case of perfect judgement ranking, $X_{i:n}$ is distributed as the i^{th} order statistic of a random sample of size n, its probability density function (pdf) and cumulative distribution function (cdf) are respectively given by

$$f_{i:n}(x) = \frac{1}{\beta(i, n-i+1)}(F(x))^{i-1}(1-F(x))^{n-i}f(x), \tag{1}$$

*Correspondence to: R. Maya (Email: publicationsofmaya@gmail.com). Department of Statistics, Govt. College for Women, Trivandrum 695 014, Kerala, India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

$$F_{i:n}(x) = \sum_{j=i}^{n} (F(x))^j (1 - F(x))^{n-j} = \frac{\beta_{F(t)}(i, n-i+1)}{\beta(i, n-i+1)}, \tag{2}$$

where $f(.)$ and $F(.)$ are the pdf and cdf of the parent population and $\beta(.,.)$ is the beta function. In judgement ranking process errors occurs in the ranking process and i^{th} judgement order statistic are not the i^{th} order statistic. For imperfect judgement ranking, $X^*_{RSS} = \left\{X_{[i]i} \ ; \ i = 1, 2, ..., n\right\}$, the pdf of $X_{[i]i}$ is $f_{[i]i}(x) = \sum p_{ir} f_i(x)$ where p_{ir} is the probability that r^{th} item is judgement ranked. For more idea of RSS, one can go through the book by [3].

The two important modifications of RSS are maximum RSS with unequal samples ($maxRSSU$) and minimum RSS with unequal samples ($minRSSU$). Some inferential aspects based on $maxRSSU$ and $minRSSU$, one can refer [2]. Here, we focused only on $maxRSSU$.

In one cycle $maxRSSU$, n random sample are drawn and then the observations in the i^{th} sample is ranked from smallest to largest. In the next step i^{th} order statistic is taken from the i^{th} sample of size i. Then the one cycle $maxRSSU$ is represented by $X^n_{maxRSSU} = \left\{X_{(i)i} \ ; \ i = 1, 2, ...n\right\}$. Based on the assumption of perfect judgement ranking, $X_{(i)i}$ is distributed as the i^{th} order statistic based on simple random sample (srs) from X of size i. The pdf and cdf of $X_{(i)i}$ are respectively given by

$$f_{(i)i}(x) = i(F(x))^{i-1} f(x), -\infty \leq x \leq \infty \tag{3}$$

and

$$F_{(i)i}(x) = (F(x))^i, -\infty \leq x \leq \infty. \tag{4}$$

A very important approach to quantify the information, in statistical framework is based on the concept of Fisher information (FI). [1] investigated that FI in usual rss data is always larger than that of srs data. But, in information theory, Shannon entropy (see, [16]) or its generalizations should be considered as appropriate measures to quantify the information content of a sample than the FI. [7] explained about the information content of rss in certain uncertainty measures and made a comparison with srs data.

[6], bestowed a new measure of uncertainty called 'extropy'as a complementary dual to Shannon entropy. For a non-negative absolutely continuous random variable (rv) X, it is given by

$$J(X) = -\frac{1}{2} \int_0^\infty f^2(x) dx = -\frac{1}{2} \int_0^1 f(F^{-1}(u)) du, \tag{5}$$

where $F^{-1}(u)$ is the quantile function. [14] studied the information content of RSS in terms of extropy function. [13], thoroughly discussed information properties of extropy based on $maxRSSU$ and $minRSSU$. Also analogy to the developments of Shannon entropy, some researchers started to study extensions/generalizations of extropy. Analogous to past entropy ([4]), [10] introduced the concept past extropy for the random variable ${}_tX = [t - X | X < t]$. It is given by,

$$J({}_tX) = -\frac{1}{2} \int_0^\infty f^2_{{}_tX}(x) = -\frac{1}{2F^2(t)} \int_0^t f^2(x) dx, \tag{6}$$

where $f_t X(x) = \frac{f(t-x)}{F(t)}$ for $x \epsilon (0, t)$.

[8] proposed a new measure called cumulative residual extropy with the motivation that survival function is more regular than the pdf and is given by

$$J^*(X) = -\frac{1}{2} \int_0^\infty \bar{F}^2(x) dx. \tag{7}$$

[17] considered an information measure termed by negative cumulative residual extropy($NCEX$) and is defined as

$$\mathcal{CJ}(X) = \frac{1}{2} \int_0^\infty [1 - F^2(x)] \mathrm{d}x = \int_0^1 \frac{\phi(u)}{f(F^{-1}(u))} du, \tag{8}$$

where $\phi(u) = \frac{1-u^2}{2},\ 0 < u < 1$. Note that $0 \le \mathcal{CJ}(X) < \infty$.
[9], provide non-parametric estimators of past extropy under α-mixing dependence condition. A study on recursive and non-recursive kernel estimation of negative cumulative extropy under α-mixing conditions has been done by [15]. But not much work is seen in the available literature on analysis of past extropy properties using RSS and its modifications. Similarly is the case of negative cumulative extropy. This motivated us to analyse information content of past extropy based on RSS, and one of its important modification $maxRSSU$. Information content of negative cumulative extropy based on $maxRSSU$ are also considered.

The paper is structured as follows. In Section 2, we obtain the expression for past extropy of RSS. In Section 3, the expression for past extropy of $maxRSSU$ has acquired and also bounds for $maxRSSU$ data are studied. A theorem related to monotone property is also there in this section. In Section 4, a comparison for past extropy measure based on SRS, RSS and $maxRSSU$ has done. In Section 5, we discussed, the discrimination information among SRS, RSS and $maxRSSU$. Section 6 deals with negative cumulative extropy properties of $maxRSSU$ and SRS designs.

2. Past extropy of ranked set sampling

Assume that the rv, X is absolutely continuous with mean μ and variance σ^2 and consider n independent and identically distributed (iid) sample chosen according to SRS. The collection of this samples are represented by $X^n_{SRS} = \{X_i\ ;\ i = 1, 2, ..., n\}$. Then the past extropy of X^n_{SRS} can be defined as

$$J({}_tX^n_{SRS}) = -\frac{1}{2}\int_0^\infty .....\int_0^\infty f^2_{{}_tX}(x_1)......f^2_{{}_tX}(x_n)dx_1dx_2....dx_n = -\frac{1}{2}\prod_{i=1}^{n}\int_0^\infty f^2_{{}_tX}(x_i)dx_i$$
$$= -\frac{1}{2(F(t))^{2n}}\prod_{i=1}^{n}\int_0^t f^2(x_i)dx_i = -\frac{1}{2}[-2J({}_tX)]^n.$$

In case of perfect judgement ranking X^n_{RSS} is the collection of order statistics from the pdf $f(.)$ and are independent. Therefore,

$$J({}_tX^n_{RSS}) = -\frac{1}{2}\prod_{i=1}^{n}[-2J({}_tX_{i:n})], \tag{9}$$

where $J({}_tX_{i:n})$ is the i^{th} order statistic of past extropy. The pdf of i^{th} order statistic is given in equation (1) and $f(x) = \sum_{i=1}^n f_{i:n}(x)/n$ (see, [3]).

Proposition 1
The expression for the past extropy of RSS with pdf $f(.)$ and cdf $F(.)$ is given by

$$J({}_tX^n_{RSS}) = -\frac{1}{2}\prod_{i=1}^{n}\frac{1}{\beta^2_{F(t)}(i, n-i+1)}\int_0^{F(t)} u^{2i-2}(1-u)^{2n-2i}f(F^{-1}(u))du, \tag{10}$$

where

$$\beta_{F(t)}(i, n-i+1) = \int_0^{F(t)} u^{i-1}(1-u)^{n-i}du.$$

Proof
We have the equation for past extropy of i^{th} order statistic as

$$J({}_tX_{i:n}) = -\frac{1}{2}\int_0^t \frac{f^2_{i:n}(x)}{F^2_{i:n}(t)}, \tag{11}$$

where

$$F_{i:n}(t) = \frac{\beta_{F(t)}(i, n-i+1)}{\beta(i, n-i+1)} = I_{F(t)}(i, n-i+1).$$

Now,

$$
\begin{aligned}
J({}_tX^n_{RSS}) &= -\frac{1}{2}\prod_{i=1}^{n}\int_0^t \frac{f^2_{i:n}(x)}{F^2_{i:n}(t)}dx \\
&= -\frac{1}{2}\prod_{i=1}^{n}\frac{1}{I^2_{F(t)}(i,n-i+1)}\int_0^t \frac{(F(x))^{2i-2}(1-F(x))^{2n-2i}f^2(x)}{\beta^2(i,n-i+1)}dx \\
&= -\frac{1}{2}\prod_{i=1}^{n}\frac{1}{\beta^2_{F(t)}(i,n-i+1)}\int_0^{F(t)} u^{2i-2}(1-u)^{2n-2i}f(F^{-1}(u))du.
\end{aligned}
$$

Hence proved. □

Example 1
Let U be uniformly distributed random variable on $(0,1)$ and $f(F^{-1}(u)) = 1$, $0 \le u \le 1$. Then,

$$J({}_tU^n_{RSS}) = -\frac{1}{2}\prod_{i=1}^{n}\frac{\beta_{F(t)}(2i-1,2n-2i+1)}{\beta^2_{F(t)}(i,n-i+1)}.$$

Example 2
Let Z be an exponentially distributed random variable with failure rate λ and $f(F^{-1}(u)) = \lambda(1-u)$. Then,

$$J({}_tZ^n_{RSS}) = -\frac{\lambda^n}{2}\prod_{i=1}^{n}\frac{\beta_{F(t)}(2i-1,2n-2i+2)}{\beta^2_{F(t)}(i,n-i+1)}.$$

3. Past extropy of maximum ranked set sampling

As in the case of past extropy of RSS, the past extropy of $X^n_{maxRSSU}$ can be defined as

$$J({}_tX^n_{maxRSSU}) = -\frac{1}{2}\prod_{i=1}^{n}[-2J({}_tX_{(i)i})]. \tag{12}$$

Proposition 2
Let $X^n_{maxRSSU}$ be the collection of maximum RSS with unequal samples from a population X with *pdf* $f(.)$ and *cdf* $F(.)$. Then the expression for past extropy is as follows:

$$J({}_tX^n_{maxRSSU}) = -\frac{(n!)^2}{2(2n-1)!}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^{F(t)}(2i-1)u^{2i-2}f(F^{-1}(u))du. \tag{13}$$

Proof
Using (6), we can write the equation for past extropy of maximum RSS with unequal samples as

$$
\begin{aligned}
J({}_tX^n_{maxRSSU}) &= -\frac{1}{2}\prod_{i=1}^{n}\int_0^{\infty} f^2_{{}_tX_{(i)i}}(x)dx = -\frac{1}{2}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^t i^2(F(x))^{2i-2}f^2(x)dx \\
&= -\frac{(n!)^2}{2(2n-1)!}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^t (2i-1)(F(x))^{2i-2}f^2(x)dx \\
&= -\frac{(n!)^2}{2(2n-1)!}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^{F(t)} (2i-1)u^{2i-2}f(F^{-1}(u))du.
\end{aligned}
$$

Hence proved. □

Example 3
Let U be a random variable having uniform distribution on (0,1) and $f(F^{-1}(u)) = 1$, $0 \leq u \leq 1$. Then,

$$J({}_tU^n_{maxRSSU}) = -\frac{n!^2}{2(F(t))^n(2n-1)!}. \tag{14}$$

Example 4
Let Z be a random variable having exponential distribution with parameter λ, $f(F^{-1}(u)) = \lambda(1-u)$, $0 \leq u \leq 1$. Then,

$$J({}_tZ^n_{maxRSSU}) = -\frac{n!^2}{2}\prod_{i=1}^{n}\frac{\beta_{F(t)}(2i-1,2)}{F^{(2i)}(t)}. \tag{15}$$

Theorem 3
Let $X^n_{maxRSSU}$ be the collection of maximum RSS with unequal samples from a population X with pdf $f(.)$ and cdf $F(.)$. If $f(F^{-1}(u)) \geq 1$ for all $0 < u < 1$, then $J({}_tX^n_{maxRSSU})$ is decreasing in n≥1.

Proof

$$\begin{aligned}
\frac{J({}_tX^{n+1}_{maxRSSU})}{J({}_tX^n_{maxRSSU})} &= \frac{\prod_{i=1}^{n+1}\left[-2J({}_tX_{(i)i})\right]}{\prod_{i=1}^{n}\left[-2J({}_tX_{(i)i})\right]} \\
&= \frac{1}{F^{2n+2}(t)}\int_0^{F(t)}(n+1)^2u^{2n}f(F^{-1}(u))du \\
&\geq \frac{(n+1)^2}{(2n+1)F(t)} \\
&\geq 1.
\end{aligned}$$

Since past extropy is non-positive, we have $J({}_tX^{n+1}_{maxRSSU}) \leq J({}_tX^n_{maxRSSU})$. Thus the theorem is proved. □

Theorem 4
Let $X^n_{maxRSSU}$, X^n_{SRS} be the collection of max$RSSU$ and simple random samples from a common distribution $F(.)$. Then,

$$J({}_tX^n_{maxRSSU}) \geq -(n!)^2J({}_tX^n_{SRS}). \tag{16}$$

Proof
We have

$$J({}_tX^n_{maxRSSU}) = -\frac{(n!)^2}{2}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^{F(t)}(u)^{2i-2}f(F^{-1}(u))du.$$

Since $F^2(x) \geq F^{2i}(x)$ for $i \geq 1$, we can write

$$\begin{aligned}
J({}_tX^n_{maxRSSU}) &\geq -\frac{(n!)^2}{2}\prod_{i=1}^{n}\frac{1}{F^2(t)}\int_0^{F(t)}f(F^{-1}(u))du \\
&= -\frac{(n!)^2}{2}\left[\frac{\int_0^{F(t)}f(F^{-1}(u))du}{F^2(t)}\right]^n \\
&= -(n!)^2J({}_tX^n_{SRS}).
\end{aligned}$$

Hence the result is proved. □

Theorem 5
Let X be a random variable with mode M, that is, f(x)$\leq$ M for all x. Then for all n$\geq$1,

$$J({}_tX^n_{maxRSSU}) \geq -\frac{1}{2}\frac{(n!)^2M^n}{(2n-1)!F^n(t)}. \tag{17}$$

Proof
We have

$$\begin{aligned} J({}_tX^n_{maxRSSU}) &= -\frac{(n!)^2}{2(2n-1)!}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^{F(t)}(2i-1)u^{2i-2}f(F^{-1}(u))du \\ &\geq -\frac{(n!)^2M^n}{2(2n-1)!}\prod_{i=1}^{n}\frac{1}{F^{2i}(t)}\int_0^{F(t)}(2i-1)u^{2i-2}du \\ &= -\frac{1}{2}\frac{(n!)^2M^n}{(2n-1)!F^n(t)}. \end{aligned}$$

Thus the theorem is proved. □

Theorem 6
Let

$$M = -\frac{A^n(n!)^2}{2}\left[\prod_{i=1}^{n}\int_{t-\lambda}^{t}f^2(x)dx\right]$$

and

$$m = -\frac{A^n(n!)^2}{2}\left[\prod_{i=1}^{n}\int_{0}^{\lambda}f^2(x)dx\right]$$

and suppose that f(.) never increases, where $A = \frac{1}{F^2(t)}$ and $x < t$. Then,

1. m $\leq J({}_tX^n_{maxRSSU}) \leq$ M.
2. If f(.) never decreases then the inequality is reversed.

Proof
Using Hayashi inequality (see, [12], p.107-109), we can obtain exact bounds for the past extropy of RSS.
Let

$$\lambda = \frac{1}{A}\int_0^t\left(\frac{F(x)}{F(t)}\right)^{2i-2}\left(\frac{1}{F^2(t)}\right)dx, \;\; 0 \leq \left(\frac{F(x)}{F(t)}\right)^{2i-2}\frac{1}{F^2(t)} \leq A,$$

where A is a constant greater than 0. In case if $f(.)$ never increases, using Hayashi inequality we can write the inequality as

$$\begin{aligned} A\int_{t-\lambda}^{t}i^2f^2(x)dx &\leq \int_0^t i^2\left(\frac{F(x)}{F(t)}\right)^{2i-2}\left(\frac{1}{F^2(t)}\right)f^2(x)dx \\ &\leq A\int_0^{\lambda}i^2f^2(x)dx. \end{aligned}$$

Also,

$$\begin{aligned} A^n\prod_{i=1}^{n}\int_{t-\lambda}^{t}i^2f^2(x)dx &\leq \prod_{i=1}^{n}\int_0^t i^2\left(\frac{F(x)}{F(t)}\right)^{2i-2}\left(\frac{1}{F^2(t)}\right)f^2(x)dx \\ &\leq A^n\prod_{i=1}^{n}\int_0^{\lambda}i^2f^2(x)dx. \end{aligned}$$

So we can write the exact bounds for the $maxRSSU$ of past extropy as

$$-\frac{A^n(n!)^2}{2}\prod_{i=1}^{n}\int_0^{\lambda} f^2(x)dx \leq -\frac{(n!)^2}{2}\prod_{i=1}^{n}\int_0^t \left(\frac{F(x)}{F(t)}\right)^{2i-2}\left(\frac{1}{F^2(t)}\right) f^2(x)dx \leq -\frac{A^n(n!)^2}{2}\prod_{i=1}^{n}\int_{t-\lambda}^{t} f^2(x)dx. \tag{18}$$

Hence the proof of result1 is complete. The proof of the result2 is similar to that of result1 and the only difference is that the inequality is reversed. □

4. Comparison

In this section we made a comparison between the expression for past extropy based on SRS, RSS and $maxRSSU$ in case of uniform and exponential distribution.

Example 5
Let X be uniformly distributed on $(0, b)$. Then for $n = 2$, the expression for past extropy based on SRS, RSS and $maxRSSU$ are respectively given by

$$J({}_tX^n_{SRS}) = -\frac{1}{2t^2},$$
$$J({}_tX^n_{RSS}) = -\frac{t^3(\frac{t}{b^2} - \frac{t^2}{b^3} + \frac{t^3}{3b^4})}{6b^4(\frac{t^3}{3b^3} - \frac{t^4}{4b^4})^2(\frac{t^2}{2b^2} - \frac{2t^3}{3b^3} + \frac{t^4}{4b^4})^2} \quad \text{and}$$
$$J({}_tX^n_{maxRSSU}) = -\frac{2}{3t^2}.$$

For $n = 2$, the difference between the above three equations are given by,

$$D_1(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{maxRSSU}) = \frac{1}{6t^2}, \tag{19}$$

$$D_2(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{RSS}) = -\frac{1}{2t^2} + \frac{1152b^8(3b^2 - 3bt + t^2)}{(4b-3t)^2t^6(6b^2 - 8bt + 3t^2)}, \tag{20}$$

and

$$D_3(t) = J({}_tX^n_{maxRSSU}) - J({}_tX^n_{RSS}) = -\frac{2}{3t^2} + \frac{1152b^8(3b^2 - 3bt + t^2)}{(4b-3t)^2t^6(6b^2 - 8bt + 3t^2)}. \tag{21}$$

The figures 1, 2, 3 shows the plot of $D_1(t)$, $D_2(t)$ and $D_3(t)$ for $0 < t < 1$. Here we assume that $b > 0$.

Figure 1. Plot of $D_1(t)$ for $0 < t < 1$ for $U(0, b)$

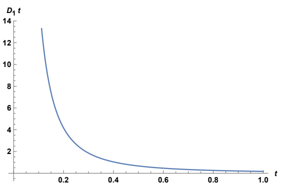


Figure 2. Plot of $D_2(t)$ for $0 < t < 1$ for $U(0, b)$

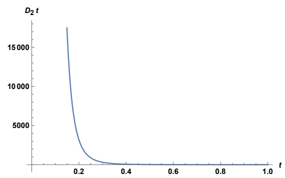
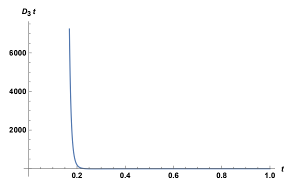


Figure 3. Plot of $D_3(t)$ for $0 < t < 1$ for $U(0, b)$



From the above equations (19), (20) and (21) we can conclude that in case of $U(0, b)$, $b > 0$ and for $n = 2$,

$$J({}_tX^n_{RSS}) < J({}_tX^n_{maxRSSU}) < J({}_tX^n_{SRS}). \tag{22}$$

4.1. Simulation

A comparison between SRS, RSS and $maxRSSU$ has been done by simulation for different values of n. Here we take $b = 1$ and $t = 1$. Let

$$D_1(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{maxRSSU}), \tag{23}$$

$$D_2(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{RSS}) \tag{24}$$

and

$$D_3(t) = J({}_tX^n_{maxRSSU}) - J({}_tX^n_{RSS}). \tag{25}$$

The values of $D_1(t)$, $D_2(t)$ and $D_3(t)$ for different values of n are presented in the table 1. From table 1 it is clear

Table 1. Values of $D_1(t)$,$D_2(t)$ and $D_3(t)$ for $U(0, 1)$

n	$D_1(t)$	$D_2(t)$	$D_3(t)$
2	3.378×10^{-1}	3.383×10^{2}	3.045×10^{2}
3	3.714×10^{2}	1.707×10^{2}	1.670×10^{2}
4	2.812×10^{3}	5.768×10^{2}	5.768×10^{2}
5	9.504×10^{1}	2.562×10^{1}	1.612×10^{1}
6	3.4644×10^{1}	9.392×10^{1}	5.928×10^{1}
7	1.559×10^{4}	5.370×10^{4}	5.370×10^{4}
8	1.720×10^{3}	3.719×10^{4}	3.547×10^{4}
9	1.772×10^{2}	1.329×10^{2}	1.329×10^{2}
10	3.971×10^{1}	3.061×10^{1}	3.061×10^{1}
11	3.943×10^{7}	1.031×10^{1}	1.031×10^{1}
12	1.044×10^{1}	3.969×10^{3}	3.969×10^{3}
13	8.613×10^{3}	1.342×10^{2}	1.342×10^{2}
14	4.456×10^{2}	9.277×10^{2}	9.277×10^{2}
15	1.056×10^{2}	7.513×10^{2}	7.513×10^{2}
16	1.437×10^{3}	7.854×10^{2}	7.710×10^{2}
17	1.499×10^{1}	1.0312×10^{2}	1.396×10^{2}
18	3.963×10^{4}	1.157×10^{4}	1.157×10^{4}
19	4.365×10^{3}	2.82×10^{3}	2.82×10^{3}
20	2.93×10^{2}	7.367×10^{2}	7.368×10^{2}

that the simulation result 4.1 matches with the theoretical result 22. Hence past extropy of RSS is less than that of SRS and $maxRSSU$.

Example 6

Suppose X follows exponential distribution with mean $=\frac{1}{\theta}$, $\theta > 0$. Then for $n = 2$, the expression for past extropy based on SRS, RSS and $maxRSSU$ are respectively given by

$$J({}_tX^n_{SRS}) = -\frac{(-2+w)^2\theta^2}{8w^2},$$

$$J({}_tX^n_{maxRSSU}) = \frac{(4-3w)(-2+w)\theta^2}{12w^2} \quad \text{and}$$

$$J({}_tX^n_{RSS}) = \frac{(4-3w)(-2+w)w^4(2+(-2+w)w)\theta^2}{96(\frac{w^3}{3}-\frac{w^4}{4})^2(\frac{w^2}{2}-\frac{2w^3}{3}+\frac{w^4}{4})^2},$$

where $w = (1 - e^{-t\theta})$ and $0 < w < 1$.

For $n = 2$, the difference between the above three equations are given by

$$D_1(w) = J({}_tX^n_{SRS}) - J({}_tX^n_{maxRSSU}) = \frac{(-2+w)(-2+3w)\theta^2}{24w^2}, \tag{26}$$

$$D_2(w) = J({}_tX^n_{SRS}) - J({}_tX^n_{RSS}) = \frac{(-2+w)(2-w+\frac{1728(2-2w+w^2)}{w^4(-4+3w)(6-8w+3w^2)^2})\theta^2}{8w^2} \tag{27}$$

and

$$D_3(w) = J({}_tX^n_{maxRSSU}) - J({}_tX^n_{RSS}) = \frac{(4-3w)(-2+w)(8-\frac{20736(2-2w+w^2)}{(4-3w)^2w^4(6-8w+3w^2)^2})\theta^2}{96w^2}. \tag{28}$$

The following plot 4, 5,6 represents the value of $D_1(w)$, $D_2(w)$ and $D_3(w)$ against w.

Figure 4. Plot of $D_1(w)$ for $0 < w < 1$ for exponential distribution

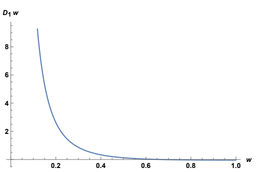


Figure 5. Plot of $D_2(w)$ for $0 < w < 1$ for exponential distribution

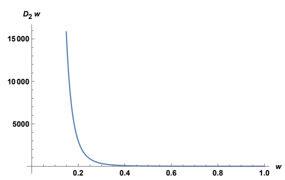
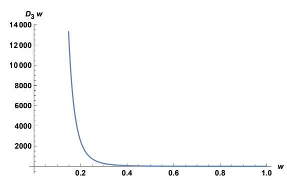


Figure 6. Plot of $D_3(w)$ for $0 < w < 1$ for exponential distribution



From the above equations (26), (27) and (28) it can be easily verified that for n=2,

$$J({}_tX^n_{RSS}) < J({}_tX^n_{SRS}) < J({}_tX^n_{maxRSSU}). \tag{29}$$

4.2. Simulation

In this case we take $\theta = 1$ and $t = 1$. We have

$$D_1(w) = J({}_tX^n_{SRS}) - J({}_tX^n_{maxRSSU}), \tag{30}$$

$$D_2(w) = J({}_tX^n_{SRS}) - J({}_tX^n_{RSS}), \tag{31}$$

$$D_3(w) = J({}_tX^n_{maxRSSU}) - J({}_tX^n_{RSS}). \tag{32}$$

where $w = 1 - e^{-t}$. Table 2 gives the values of $D_1(w)$,$D_2(w)$ and $D_3(w)$ for $n = 2, ..., 20$. The result from simulation 4.2 is same as the theoretical result 29 which is clear from table 2. This shows that in case of exponential distribution RSS for past extropy measure is less than SRS for past extropy measure and both are less than $maxRSSU$ for past extropy measure.

Example 7

Let X follows standard normal distribution with pdf $\phi(x)$ and cdf $\Phi(x)$. Then the expression for past extropy in case of SRS, RSS and $maxRSSU$ are ,

$$J({}_tX^n_{SRS}) = -\frac{1}{2(\Phi(t))^{2n}} \prod_{i=1}^{n} \int_0^t \phi^2(x_i)dx_i \tag{33}$$

$$J({}_tX^n_{RSS}) = -\frac{1}{2} \prod_{i=1}^{n} \frac{1}{\beta^2_{\Phi(t)}(i, n-i+1)} \int_0^t \Phi(x)^{2i-2}(1-\Phi(x))^{2n-2i}\phi^2(x)du \tag{34}$$

$$J({}_tX^n_{maxRSSU}) = -\frac{(n!)^2}{2(2n-1)!} \prod_{i=1}^{n} \frac{1}{\Phi^{2i}(t)} \int_0^t (2i-1)(\Phi(x))^{2i-2}\phi^2(x)dx \tag{35}$$

Furthur simplifications of the equations 33, 34, 35 are not possible since those doesnot have a closed form. But comparison between SRS, RSS and $maxRSSU$ for standard normal distribution is done by simulation.

Table 2. Values of $D_1(w)$,$D_2(w)$ and $D_3(w)$ for $exp(1)$

n	$D_1(w)$	$D_2(w)$	$D_3(w)$
2	-3.482×10^{-2}	6.631×10^{-2}	1.011×10^{-2}
3	-4.581×10^{-2}	5.678×10^{-1}	5.678×10^{-1}
4	-4.261×10^{-5}	5.202×10^{-3}	5.202×10^{-3}
5	-9.56×10^{-3}	3.076×10^{-1}	3.172×10^{-1}
6	-7.489×10^{-3}	1.472×10^{-1}	1.547×10^{-1}
8	-4.259×10^{-3}	2.098×10^{-3}	2.140×10^{-1}
9	-3.068×10^{-2}	3.641×10^{-2}	3.641×10^{-2}
10	-7.489×10^{-3}	1.472×10^{-1}	1.547×10^{-1}
11	-4.156×10^{-2}	4.401×10^{-2}	8.563×10^{-2}
12	-7.888×10^{-3}	8.562×10^{1}	8.562×10^{1}
13	-2.338×10^{-4}	4.176×10^{-5}	2.338×10^{-5}
14	-3.062×10^{-5}	3.634×10^{-4}	3.634×10^{-4}
15	-2.929×10^{-3}	6.054×10^{-1}	6.083×10^{-1}
16	-7.479×10^{-2}	1.574×10^{1}	1.574×10^{1}
17	-8.246×10^{-3}	5.986×10^{1}	5.986×10^{1}
19	-4.259×10^{-3}	2.098×10^{-1}	2.140×10^{-1}
20	-1.915×10^{-4}	4.613×10^{1}	4.613×10^{1}

4.3. Simulation

We have,

$$D_1(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{maxRSSU}), \tag{36}$$

$$D_2(t) = J({}_tX^n_{SRS}) - J({}_tX^n_{RSS}), \tag{37}$$

$$D_3(t) = J({}_tX^n_{maxRSSU}) - J({}_tX^n_{RSS}). \tag{38}$$

Table 3 gives the values of equations 36, 37 and 38 by simulation for $n = 2, 3, ..., 20$ in case of standard normal distribution. From the above table 3 we get the result for standard normal distribution as

$$J({}_tX^n_{RSS}) < J({}_tX^n_{maxRSSU}) < J({}_tX^n_{SRS}). \tag{39}$$

From the above result 39 it is clear that in case of standard normal distribution past extropy of RSS is always less than that of both $maxRSSU$ and SRS. This shows that RSS is always better than both SRS and $maxRSSU$.

5. Discrimination information

This section deals with discrimination information between distribution of RSS, $maxRSSU$ statistic and underlying data. [14] defined discrimination information between rv's X and Y with pdf's f and g respectively as

$$R(X,Y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} f(x)dx.$$

[5] has given a measure of discrimination between past lifetime random variables. So in this case in order to deal with the discrimination information between the past lifetime random variables ${}_tX$ and ${}_tY$ with pdf's $f_{{}_tX}(x)$ and $g_{{}_tX}(x)$, cdf's $F_{{}_tX}(t)$ and $G_{{}_tX}(t)$, can be defined as

$$D_t({}_tX, {}_tY) = \frac{1}{2F_{{}_tX}(t)} \int_0^t f_{{}_tX}(x) \frac{f_{{}_tX}(x)/F_{{}_tX}(t)}{g_{{}_tX}(x)/G_{{}_tX}(t)} dx. \tag{40}$$

Table 3. Values of $D_1(t)$,$D_2(t)$ and $D_3(t)$ for $N(0,1)$

n	$D_1(t)$	$D_2(t)$	$D_3(t)$
2	1.336×10^{2}	4.911×10^{1}	8.455×10^{1}
3	1.984×10^{-1}	9.637×10^{-2}	1.025×10^{-1}
4	1.321×10^{-2}	2.197×10^{-2}	8.768×10^{-1}
5	1.004×10^{2}	5.587×10^{2}	1.005×10^{2}
6	4.597×10^{4}	5.641×10^{0}	4.596×10^{4}
7	7.728×10^{-3}	8.555×10^{-1}	7.725×10^{-3}
8	2.412×10^{3}	3.550×10^{3}	2.143×10^{3}
9	1.048×10^{-2}	1.726×10^{-2}	6.7811×10^{-3}
10	2.101×10^{3}	6.498×10^{3}	2.102×10^{3}
11	2.744×10^{3}	7.631×10^{4}	2.745×10^{4}
12	1.554×10^{5}	6.610×10^{1}	1.553×10^{5}
13	6.824×10^{5}	1.057×10^{5}	6.823×10^{5}
14	1.608×10^{4}	1.421×10^{5}	1.609×10^{4}
15	9.810×10^{4}	6.054×10^{4}	9.810×10^{4}
16	6.880×10^{-1}	9.775×10^{-2}	5.902×10^{-1}
17	9.324×10^{4}	2.389×10^{4}	9.324×10^{4}
18	1.9365×10^{6}	1.526×10^{3}	1.934×10^{1}
19	2.363×10^{5}	2.551×10^{5}	2.363×10^{5}
20	4.018×10^{-5}	5.854×10^{-2}	5.850×10^{1}

Proposition 7
The discrimination information between the distribution of i^{th} RSS statistic $f_{i:n}$ and the data distribution f can be expressed as

$$D_t(f_{(i:n)}(x), f(x)) = \frac{F(t)}{2}\frac{\beta_{F(t)}(2i-1, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}. \tag{41}$$

Proof
It follows from equation (40) that

$$D_t(f_{i:n}(x), f(x)) = \frac{1}{2F_{i:n}(t)}\int_0^t f_{i:n}(x)\frac{f_{i:n}(x)}{F_{i:n}(t)}\frac{F(t)}{f(x)}dx. \tag{42}$$

Using equation (1) and (2), the equation (42) can be written as

$$\begin{aligned}
D_t(f_{i:n}(x), f(x)) &= \frac{1}{2\beta^2_{F(t)}(i, n-i+1)}\int_0^t (F(x))^{2i-2}(1-F(x))^{2n-2i}f(x)F(t)dx \\
&= \frac{F(t)}{2\beta^2_{F(t)}(i, n-i+1)}\int_0^{F(t)} u^{2i-2}(1-u)^{2n-2i}du \\
&= \frac{F(t)}{2}\frac{\beta_{F(t)}(2i-1, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}.
\end{aligned}$$

Hence the result is attained. □

Proposition 8
The discrimination information between the distribution of i^{th} $maxRSSU$ statistic $f_{(i)i}$ and the data distribution f

can be expressed as

$$D_t(f_{(i)i}(x), f(x)) = \frac{i^2}{2(2i-1)}. \tag{43}$$

Proof
Using equation (40),(3) and (4), we can write the equation for discrimination information as

$$\begin{aligned} D_t(f_{(i)i}(x), f(x)) &= \frac{1}{2F_i(t)} \int_0^t f_{(i)i}(x) \frac{f_{(i)i}(x)}{F_i(t)} \frac{F(t)}{f(x)} dx \\ &= \frac{1}{2(F(t))^{2i-1}} \int_0^{F(t)} i^2 u^{2i-2} du \\ &= \frac{i^2}{2(2i-1)}. \end{aligned}$$

Hence the result is holds. □

Proposition 9
The discrimination information between the distribution of i^{th} RSS statistic $f_{i:n}$ and the i^{th} $maxRSSU$ statistic $f_{(i)i}$ can be expressed as

$$D_t(f_{i:n}(x), f_{(i)i}(x)) = \frac{(F(t))^i}{2i} \frac{\beta_{F(t)}(i, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}. \tag{44}$$

Proof
From equations(40),(1),(2),(3),(4),

$$\begin{aligned} D_t(f_{i:n}(x), f_{(i)i}(x)) &= \frac{1}{2F_{i:n}(t)} \int_0^t f_{i:n}(x) \frac{f_{i:n}(x)}{F_{i:n}(t)} \frac{F_i(t)}{f_{(i)i}(x)} dx \\ &= \frac{(F(t))^i}{2i\beta^2_{F(t)}(i, n-i+1)} \int_0^{F(t)} u^{i-1}(1-u)^{2n-2i} du \\ &= \frac{(F(t))^i}{2i} \frac{\beta_{F(t)}(i, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}. \end{aligned}$$

Hence the proof is completed. □

Theorem 10
For ${}_tX^n_{RSS}$ and ${}_tX^n_{SRS}$ designs, we can define the discrimination information as

$$D_t({}_tX^n_{RSS}, {}_tX^n_{SRS}) = \frac{(F(t))^n}{2} \prod_{i=1}^{n} \frac{\beta_{F(t)}(2i-1, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}. \tag{45}$$

Proof
We have,

$$D_t({}_tX^n_{RSS}, {}_tX^n_{SRS}) = \frac{1}{2} \prod_{i=1}^{n} \frac{1}{F_{i:n}(t)} \int_0^t f_{i:n}(x) \frac{f_{i:n}(x)}{F_{i:n}(t)} \frac{F(t)}{f(x)} dx.$$

Proceeding as Proposition (7), proof can be obtained. □

Theorem 11
For ${}_tX^n_{maxRSSU}$ and ${}_tX^n_{SRS}$ designs, we can define the discrimination information as

$$D_t({}_tX^n_{maxRSSU}, {}_tX^n_{SRS}) = \frac{n!^2}{2(2n-1)!}. \tag{46}$$

Proof

$$D_t({}_tX^n_{maxRSSU}, {}_tX^n_{SRS}) = \frac{1}{2}\prod_{i=1}^{n}\frac{1}{F_i(t)}\int_0^t f_{(i)i}(x)\frac{f_{(i)i}(x)}{F_i(t)}\frac{F(t)}{f(x)}dx.$$

The proof follows by recalling Proposition(8). □

Theorem 12
For ${}_tX^n_{RSS}$ and ${}_tX^n_{maxRSSU}$ designs, we can define the discrimination information as

$$D_t({}_tX^n_{RSS}, {}_tX^n_{maxRSSU}) = \frac{1}{2n!}\prod_{i=1}^{n}(F(t))^i\frac{\beta_{F(t)}(i, 2n-2i+1)}{\beta^2_{F(t)}(i, n-i+1)}. \tag{47}$$

Proof

$$D_t({}_tX^n_{RSS}, {}_tX^n_{maxRSSU}) = \frac{1}{2}\prod_{i=1}^{n}\frac{1}{F_{i:n}(t)}\int_0^t f_{i:n}(x)\frac{f_{i:n}(x)}{F_{i:n}(t)}\frac{F_i(t)}{f_{i(i)}(x)}dx.$$

By referring Proposition (9), proof can be obtained. □

6. NCEX of $maxRSSU$

If the $NCEX$ of X is less than that of another random variable, say Y, that is $\mathcal{CJ}(X) \leq \mathcal{CJ}(Y)$, i.e. X has more uncertainty than Y. Now let $\mathcal{CJ}(X) < +\infty$. Then, for the $maxRSSU$ and SRS designs, we have

$$\begin{aligned}\mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU}) &= \frac{1}{2}\prod_{i=1}^{n}[2\mathcal{CJ}(X_{(i)i})] = \frac{1}{2}\prod_{i=1}^{n}\int_0^1\frac{1-u^{2i}}{f(F^{-1}(u))}du \\ &= \frac{1}{2}\prod_{i=1}^{n}\mathbb{E}\left[\frac{1-U^{2i}}{f(F^{-1}(U))}\right],\end{aligned} \tag{48}$$

and

$$\mathcal{CJ}(\boldsymbol{X}^n_{SRS}) = \frac{1}{2}[2\mathcal{CJ}(X)]^n. \tag{49}$$

To compare the above measures, let us consider the following examples.

Example 8
If $U \sim Uniform(0, \theta)$, then

$$\mathcal{CJ}(\boldsymbol{U}^n_{maxRSSU}) = \frac{1}{2}\prod_{i=1}^{n}\left[1-\frac{1}{(2i+1)\theta^{2i}}\right] \geq \mathcal{CJ}(\boldsymbol{U}^n_{SRS}) = \frac{1}{2}\left(1-\frac{1}{3\theta^2}\right)^n, \quad \theta \geq 1. \tag{50}$$

Example 9
Let Z be a random variable with the cdf $F(z) = z^a,\ 0 < z < 1,\ a > 1$. Then, $f(F^{-1}(u)) = au^{1-\frac{1}{a}}$, $0 < u < 1$, and we have

$$\mathcal{CJ}(\boldsymbol{Z}^n_{maxRSSU}) = \frac{1}{2}\prod_{i=1}^{n}[1-\frac{1}{2ia+1}] > \mathcal{CJ}(\boldsymbol{Z}^n_{SRS}) = \frac{1}{2}[1-\frac{1}{2a+1}]^n. \tag{51}$$

Theorem 13
Let $\boldsymbol{X}^n_{maxRSSU}$ be the $maxRSSU$ from population X with cdf F. Then, $\mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU}) \geq \mathcal{CJ}(\boldsymbol{X}^n_{SRS})$ for $n > 1$.

Proof
Since $1-F^2(x) \leq 1-F^{2i}(x)$ for $i \geq 1$, we have

$$\left(\int_0^\infty [1-F^2(x)]dx\right)^n \leq \prod_{i=1}^n \int_0^\infty [1-F^{2i}(x)]dx.$$

The proof follows by recalling (48) and (49). □

Proposition 14
Let $Y = aX + b$ with $a > 0$ and $b \geq 0$. Then, $\mathcal{CJ}(\boldsymbol{Y}^n_{maxRSSU}) = a^n \mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU})$.

Proposition 15
If $f(F^{-1}(u)) \geq 1$, $0 < u < 1$, then $\mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU})$ is decreasing in $n \geq 1$.

Proof
From (48), we get

$$\frac{\mathcal{CJ}(\boldsymbol{X}^{n+1}_{maxRSSU})}{\mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU})} = \int_0^1 \frac{u^{2n+2}}{f(F^{-1}(u))}du \leq \frac{1}{2n+3} \leq 1,$$

and the result follows readily. □

Now, we can define a generalized measure of $NCEX$ as

$$\mathcal{CJ}(X;t) = \frac{1}{2}\int_0^t \left[1-\frac{F(x)}{F(t)}\right]^2 dx. \tag{52}$$

Moreover

$$\mathcal{CJ}(\boldsymbol{X}^n_{SRS};t) = \frac{1}{2}[-2\mathcal{CJ}(X,t)]^n. \tag{53}$$

Under the $maxRSSU$ design, it is clear to show that

$$\begin{aligned} \mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU};t) &= \frac{1}{2}\prod_{i=1}^n [2\mathcal{CJ}(X_{(i)i};t)] = \frac{1}{2}\prod_{i=1}^n \int_0^t \left(1-[\frac{F(x)}{F(t)}]^{2i}\right)dx \\ &= \frac{1}{2}\prod_{i=1}^n \mathbb{E}\left[\frac{(1-U^{2i})F(t)}{f(F^{-1}(UF(t)))}\right], \end{aligned} \tag{54}$$

where $U \sim Uniform(0,1)$.

Theorem 16
Let X be a non-negative random variable. Then, for $n > 1$

$$\mathcal{CJ}(\boldsymbol{X}^n_{maxRSSU};t) \geq \mathcal{CJ}(\boldsymbol{X}^n_{SRS};t). \tag{55}$$

Proof
The proof is similar to Theorem 13. □

7. Conclusion

In this paper we have considered the information content of both RSS and $maxRSSU$ data for past extropy and $NCEX$. We also compared these two designs with SRS in uniform and exponential distributions for $n = 2$ and also by simulation for $n = 2, ..., 20$,in the case of past extropy measure. For $U(0, b)$ if $b > 0$ we obtained the result as $J({}_tX^n_{RSS}) < J({}_tX^n_{maxRSSU}) < J({}_tX^n_{SRS})$, and $J({}_tX^n_{RSS}) < J({}_tX^n_{SRS}) < J({}_tX^n_{maxRSSU})$ for exponential distribution. And both the results are exactly the same as the simulation result. In case of standard normal distribution the result acquired is $J({}_tX^n_{RSS}) < J({}_tX^n_{maxRSSU}) < J({}_tX^n_{SRS})$. We obtained several results related to bounds, monotone properties and sharp bounds under some assumptions for $maxRSSU$ data. Results related to discrimination information which is a measure of closeness among SRS, RSS and $maxRSSU$ are also developed. We also compared $maxRSSU$ and SRS schemes for $NCEX$ and its dynamic version.

Acknowledgement

The authors thank the referees for the valuable comments which helps in improving the article to a great extent.

REFERENCES

1. L. Barabesi, and A. El-Sharaawi, *The efficiency of ranked set sampling for parameter estimation*, Statistics and Probability Letters, vol. 53, no. 2, pp. 189–199, 2001.
2. W. Chen, M. Xie, and M. Wu, *Parametric estimation for the scale parameter for scale distributions using moving extremes ranked set sampling*, Statistics and Probability Letters, vol. 83, no. 9, pp. 2060–6, 2013.
3. Z. Chen, Z. D. Bai, and B. K. Sinha, *Ranked Set Sampling: Theory and Applications*, Springer, New York, 2004.
4. A. Di Crescenzo, and M. Longobardi, *Entropy-based measure of uncertainity in past lifetime distributions*, Journal of Applied Probability, vol. 39, pp. 434–440, 2002.
5. A. Di Crescenzo, and M. Longobardi, *A measure of discrimination between past lifetime distributions*, Statistics and Probability Letters, vol. 67, no. 2, pp. 173–182, 2004.
6. L. Frank, G. Sanfilippo, and G. Agro, *Extropy: Complementary dual of entropy*, Statistical Science, vol. 30, no. 1, pp. 40–58, 2015.
7. M. J. Jozani, and J. Ahmadi, *On uncertainity and information properties of ranked set samples*, Information Sciences, vol. 264, pp. 291–301, 2014.
8. S. M. A. Jahanshahi, H. Zarei, and A. H. Khammar, *On cumulative residual extropy*, Probability in the Engineering and Informational Sciences, vol. 264, pp. 1–21, 2019.
9. M. R. Irshad and R. Maya, *Nonparametric estimation of past extropy under α-mixing dependence*, Ricerche di Matematica, doi:10.1007/s11587-021-00570-8, 2021.
10. A. S. Krishnan, S. M. Sunoj, and N. U. Nair, *Some reliability properties of extropy for residual and past lifetime random variables*, J. Korean Stat. Soc, doi:10.1007/s42952-019-00023-x, 2020.
11. G. A. McIntyre, *A method for unbiased selective sampling, using ranked sets*, Australian Journal of Agricultural Research, vol. 3, no. 4, pp. 385–390, 1952.
12. D. S. Mitrinovic, *Analytic inequalities*, Berlin, Springer, 1970.
13. G. Qiu, and A. Eftekharian, *Extropy information of maximum and minimum ranked set samples with unequal samples*, Communication in Statistics- Theory and Methods, doi:10.1080/03610926.2019.1678640, 2020.
14. M. Z. Raqab and G. Qiu, *On extropy properties of ranked set sampling*, Statistics, vol. 53, no. 1, pp. 210–226, 2019.
15. R. Maya, M. R. Irshad, and K. Archana, *Recursive and non-recursive kernel estimation of negative cumulative extropy under α-mixing dependence condition*, Ricerche di Matematica, doi:10.1007/s11587-021-00605-0, 2021.
16. C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J., vol. 27, no. 3, pp. 379–432, pp. 623–656, 1948.
17. S. Tahmasebi, and A. Toomaj, *On negative cumulative extropy with applications* Communication in Statistics- Theory and Methods, doi: 10.1080/03610926.2020.1831541, 2020.