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A New Weighted Topp-Leone Family of Distributions

Gorgees Shaheed Mohammad *

* *University of AL-Qadisiyah, College of Education, Department of Mathematics, IRAQ*

Abstract Based on T-X transform due to Alzaatreh et al. (2013), we propose the new weighted Topp-Leone (NWTL-Π) continuous statistical distributions with two extra shpae parameters .Then we study some basic mathematical properties. Then we study Uniform model as member of the new class with more details. Using a simulation study, we compared some methods of estimation. Finally we analyzed and used lifetime and failure time real data sets to illustrate the purposes.

Keywords Topp-Leone distribution, maximum likelihood estimation, moments, quantile function, weighted distribution.

AMS 2010 subject classifications 60Exx, 60E05

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1. Introduction

Topp and Leone (1955) introduced a family of J-shaped density. The cdf of Topp-Leone (TL) is given by

$$\Pi(y) = \left[-y^2 + 2\,y\right]^{\alpha} = \left[-(1-y)^2 + 1\right]^{\alpha}, \quad 0 < y < 1, 0 < \alpha < 1. \tag{1}$$

The pdf and hrf releted to equation (2) are given by

$$\pi(y) = 2\left[-(1-y)^2 + 1\right]^{\alpha-1} \alpha(1-y), \quad h(y) = \frac{2\left[-(1-y)^2 + 1\right]^{\alpha-1} \alpha(1-y)}{1 - \left[-(1-y)^2 + 1\right]^{\alpha}}. \tag{2}$$

TL distribution is J-shaped density and bathtub hazard rate (hrf) for any $0 < \alpha < 1$. Many extensions of Topp-Leone distributions has been introduced by different authors. For example, Topp-Leone family of distributions by Al-Shomran et al. (2016), Topp-Leone generated family by Rezaei et al. (2017), Topp-Leone odd log-logistic by Brito et al. (2019), type II generalized Topp-Leone family by Hassan et al. (2019), new power Topp-Leone generated family by Bantan et al. (2019) and weibull Topp-Leone generated family by Karamikabir et al. (2020).

In this paper, we obtain a new weighted Topp-Leone class of statistical distributions. The cdf of new class is defined by

$$\begin{aligned} F(y;\alpha,\beta,\Theta) &= \int_0^{\frac{\left[1-(1-\Pi(y;\Theta))^2\right]^{\alpha}}{[1-\Pi(y;\Theta)]^{2\beta}}} \frac{d\,t}{(1+t)^2} \\ &= \left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha}\left\{\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha} + [1-\Pi(y;\Theta)]^{2\beta}\right\}^{-1}, \end{aligned} \tag{3}$$

*Correspondence to: Gorgees Shaheed Mohammad (Email: gorgees.alsalamy@qu.edu.iq). University of AL-Qadisiyah, College of Education, Department of Mathematics, IRAQ.

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where $\alpha > 0, \beta > 0$ are two shape parameters and Θ represent the vector of parameters for parent cdf $\Pi(.)$. We denote it by NWTL-$\Pi(\alpha, \beta, \Theta)$. The pdf and hrf of are obtained as

$$f(y;\alpha,\beta,\Theta) = \frac{2\pi(y;\Theta)\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha-1}\left[1-\Pi(y;\Theta)\right]^{2\beta-1}\left\{\alpha+(\beta-\alpha)\left[1-(1-\Pi(y;\Theta))^2\right]\right\}}{\left\{\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha}+\left[1-\Pi(y;\Theta)\right]^{2\beta}\right\}^2}, \tag{4}$$

and

$$\psi(y;\alpha,\beta,\Theta) = \frac{2\pi(y;\Theta)\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha-1}\left\{\alpha+(\beta-\alpha)\left[-(1-\Pi(y;\Theta))^2+1\right]\right\}}{\left\{\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha}+\left[1-\Pi(y;\Theta)\right]^{2\beta}\right\}}, \tag{5}$$

where $\pi(y) = \frac{d\Pi(y)}{dy}$. Suppose $w(z)$ is a non-negative function with $E(w(Z)) < \infty$, Patil and Rao (1986) suggested the pdf of the weighted random variable by

$$f_w(y) = f(y)w(y)/E(w(Y)),$$

where $E(w(Y))$ represent the expected value of $w(Y)$. Taking $\pi(y) = \frac{d\Pi(y)}{dy}$ as a certain pdf with cdf $\Pi(.)$ and

$$w(y) = \frac{2\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha-1}\left[1-\Pi(y;\Theta)\right]^{2\beta-1}\left\{\alpha+(\beta-\alpha)\left[1-(1-\Pi(y;\Theta))^2\right]\right\}}{\left\{\left[-(1-\Pi(y;\Theta))^2+1\right]^{\alpha}+\left[1-\Pi(y;\Theta)\right]^{2\beta}\right\}^2}$$

equation (4) presents a new weighted version of the Exp-Π family. The main goal of this paper is to induce two extra parameter to class of lifetime distribution, the new family can be used for generating more flexibility Exp-Π ones. Furthermore, the following are the main reasons for employing the NWTL-$\Pi(\alpha, \beta, \Theta)$ family in practice:

- A simple way for making new flexible distributions .
- To induce more flexibility to classical distributions..
- For introducing new extended distributions with closed form for cdf, pdf and hrf.
- For making better fits than other competitive modles.

The remainder of this work is organized as follows: Various properties are explored in Section 2. In section 3, we Lindley special case with more focus. In sections 4 and 5 the many estimation methods are presented and compared via a simulation analysis. The new family's outcomes are demonstrated in Section 6 by analysing lifetime data and failure time data. In Section 7, some interesting remarks are discussed.

2. Basic Properties

2.1. Quantile function

Let $u \sim u(0,1)$ and $Q_\Pi(.)$ represent the quantile function of parent cdf Π, then for $\alpha = \beta$,

$$Y_u = Q_\Pi\left(1 - \sqrt{\frac{(1-u)^{\frac{1}{\alpha}}}{(1-u)^{\frac{1}{\alpha}} + u^{\frac{1}{\alpha}}}}\right), \tag{6}$$

follow cdf (3). For $\alpha \neq \beta$, the root of equation $F(y) = u$ has cdf (3).

2.2. Asymptotic

In this part, we study the asymptotic of cdf, pdf and hrf for NWTL-Π distributions. Tails are important measures for studying ..... Let $c = \min\{y|\Pi(y) > 0\}$, then $\left[-(1-\Pi(y;\Theta))^2+1\right]^\alpha + [1-\Pi(y;\Theta)]^{2\beta} \to 1$ and $-(1-t)^2+1 \sim 2t$ as $t \to 0$, then for $y \to c$,

$$F(y) \sim [2\Pi(y]^\alpha, \quad f(y) \sim \alpha\, 2^\alpha\, \pi(y)\Pi(y)^{\alpha-1}, \quad h(y) \sim \frac{\alpha\, 2^\alpha\, \pi(y)\Pi(y)^{\alpha-1}}{1-[2\Pi(y]^\alpha}. \tag{7}$$

With same way $\left[1-(1-\Pi(y;\Theta))^2\right]^\alpha + [1-\Pi(y;\Theta)]^{2\beta} \to 1$ as $y \to \infty$. For $y \to \infty$,

$$1-F(y) \sim [1-\Pi(y;\Theta)]^{2\beta}, \quad f(y) \sim 2\beta\, \pi(y)\, [1-\Pi(y;\Theta)]^{2\beta-1}, \quad h(y) \sim \frac{2\beta\, \pi(y)}{1-\Pi(y)}. \tag{8}$$

2.3. Linear combination

In this section we obtain a linear combination for cdf of NWTL-Π. By taking $B = [1-\Pi(y)]^2$

$$\begin{aligned}
F(y) &= \frac{(1-B)^\alpha}{(1-B)^\alpha + A^\beta} = \frac{(1-B)^\alpha}{1+(1-B)^\alpha - [1-B^\beta]} \\
&= (1-B)^\alpha \sum_{j_1=0}^{\infty} (-1)^{j_1} \left\{(1-B)^\alpha - \left[1-B^\beta\right]\right\}^{j_1} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1} (-1)^{2\,j_1-j_2} \binom{j_1}{j_2} (1-B)^{\alpha(j_2+1)} (1-B^\beta)^{j_1-j_2} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_1-j_2} (-1)^{2\,j_1-j_2+j_3} \binom{j_1}{j_2}\binom{j_1-j_2}{j_3} (1-B)^{\alpha(j_2+1)} B^{\beta\, j_3} \\
&= \sum_{j_1,j_4=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_1-j_2} (-1)^{2\,j_1-j_2+j_3+j_4} \binom{j_1}{j_2}\binom{j_1-j_2}{j_3}\binom{\beta\, j_3}{j_4} (1-B)^{\alpha(j_2+1)+j_4} \\
&= \sum_{j_1,j_4,l=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_1-j_2} (-1)^{2\,j_1-j_2+j_3+j_4+l} \binom{j_1}{j_2}\binom{j_1-j_2}{j_3}\binom{\beta\, j_3}{j_4}\binom{2\alpha(j_2+1)+2\,j_4}{l} \Pi(y)^l.
\end{aligned} \tag{9}$$

Then

$$F(y) = \sum_{l=0}^{\infty} w_l \Pi(y)^l, \tag{10}$$

where

$$w_l = \sum_{j_1,j_4=0}^{\infty}\sum_{j_2=0}^{i_1}\sum_{j_3=0}^{j_1-j_2} (-1)^{2\,j_1-j_2+j_3+j_4+l} \binom{j_1}{j_2}\binom{j_1-j_2}{j_3}\binom{\beta\, j_3}{j_4}\binom{2\alpha(j_2+1)+2\,j_4}{l}. \tag{11}$$

The pdf of Y is given by

$$f(y) = \frac{dF(y)}{d\,y} = \sum_{l=0}^{\infty} w_{l+1}(l+1)\pi(y)\,\Pi(y)^l. \tag{12}$$

Equations (10) and (12) are useful equations in this section. We can obtain some structural properties such as moments, incomplete moments, moment generating function and order statistics based on the proprties of Exponentiated-Π distributions.

2.4. Order statistics

Order statistics are used for some area of statistical inference. Suppose that $Y_1, \dots, Y_m$ show the random sample from any NWTL-Π distribution with size m. The r-th order statistic is denoted by $Y_{r:m}$. The cdf of $Y_{r:m}$ is given by

$$F_{r:m}(y) = \sum_{j=r}^{m} \binom{m}{j} F(y)^j \left[1 - F(y)\right]^{m-j} = \sum_{j=r}^{m} \sum_{k=0}^{j} (-1)^k \binom{m}{j} \binom{j}{k} [1 - F(y)]^{m-j+k}$$
$$= \sum_{j=r}^{m} \sum_{k=0}^{j} (-1)^k \binom{m}{j} \binom{j}{k} \frac{B^{\beta\, k}}{[B^\beta + (1-B)^\alpha]^{m-j+k}}. \tag{13}$$

But

$$\frac{B^{\beta\, k}}{[B^\beta + (1-B)^\alpha]^{m-j+k}} = \frac{A^{\beta\, k}}{\{1 + (1-B)^\alpha - [1 - B^\beta]\}^{m-j+k}}$$
$$= B^{\beta\, k} \sum_{j_1=0}^{\infty} (-1)^{j_1} \binom{-m+j-k}{j_1} \left\{(1-B)^\alpha - \left[1 - B^\beta\right]\right\}^{j_1}$$
$$= B^{\beta\, k} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{j_1} (-1)^{2\, j_1 - j_2} \binom{-m+j-k}{j_1} \binom{j_1}{j_2} (1-B)^{\alpha\, j_2} (1 - B^\beta)^{j_1 - j_2}$$
$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_1-j_2} (-1)^{2\, j_1 - j_2 + j_3} \binom{-m+j-k}{j_1} \binom{j_1}{j_2} \binom{j_1 - j_2}{j_3} (1-B)^{\alpha\, j_2} B^{\beta\,(k+j_3)}$$
$$= \sum_{j_1, j_4=0}^{\infty} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_1-j_2} (-1)^{2\, j_1 - j_2 + j_3 + j_4} \binom{-m+j-k}{i_1} \binom{j_1}{j_2} \binom{j_1 - j_2}{j_3} \binom{\beta(k+j_3)}{j_4} (1-B)^{\alpha\, j_2 + j_4}$$
$$= \sum_{j_1, j_4, l=0}^{\infty} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_1-j_2} (-1)^{2\, j_1 - j_2 + j_3 + j_4 + l} \binom{-m+j-k}{j_1} \binom{j_1}{j_2} \binom{j_1 - j_2}{j_3} \binom{\beta(k+j_3)}{j_4} \binom{2\alpha\, i_2 + 2\, i_4}{l} \Pi(y)^l. \tag{14}$$

Finally the cdf of $Y_{r:m}$ is obtained

$$F_{r:m}(Y) = \sum_{l=0}^{\infty} v_l \Pi(y)^l, \tag{15}$$

where

$$v_l = \sum_{j=r}^{n} \sum_{k=0}^{j} \sum_{j_1, j_4=0}^{\infty} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_1-j_2} (-1)^{k + 2\, j_1 - j_2 + j_3 + j_4 + l} \binom{m}{j} \binom{j}{k} \binom{-m+j-k}{i_1}$$
$$\times \binom{j_1}{j_2} \binom{j_1 - j_2}{j_3} \binom{\beta(k+j_3)}{j_4} \binom{2\alpha\, j_2 + 2\, j_4.}{l} \tag{16}$$

The pdf of $Y_{r:m}$ is easily obtained as

$$f_{r:m}(y) = \sum_{l=0}^{\infty} v_{l+1} (l+1) \pi(y) \Pi(y)^l. \tag{17}$$

Eqaution (15) and (15) are important formulas in this part. In fact we can obtain some general properties of rr-th orfer statistics such as moments, incomplete moments and moment generating function of r-th order statistics.

3. Uniform case

In this part, we derive some properties of (3) for uniform case with more focus. By giving $\Pi(y) = y$ and $\pi(y) = 1$ in equations (3) and 4, we obtain a new weighted Topp-Leone distribution (NWTL). The pdf and hrf of NWTL are given by

$$f(y) = \frac{2(1-y)^{2\beta-1}(2\,y-y^2)^{\alpha-1}(\alpha+(\beta-\alpha)(2y-y^2))}{\left[(2y-y^2)^{\alpha}+(1-y)^{2\beta}\right]^2}, \tag{18}$$

$$\psi(y) = \frac{2(2\,y-y^2)^{\alpha-1}(\alpha+(\beta-\alpha)(2y-y^2))}{\left[(2y-y^2)^{\alpha}+(1-y)^{2\beta}\right](1-y)}. \tag{19}$$

Figures 1 and 2 show the pdf and hrf of NWTL for some arbitary subset of parameters. The pdf of NWTL is suitable for modelling right-skew, almost symmetric and left- skew real data. The hazard rate function of NWTL can be increasing, bathtub and upside down-bathtub shape.

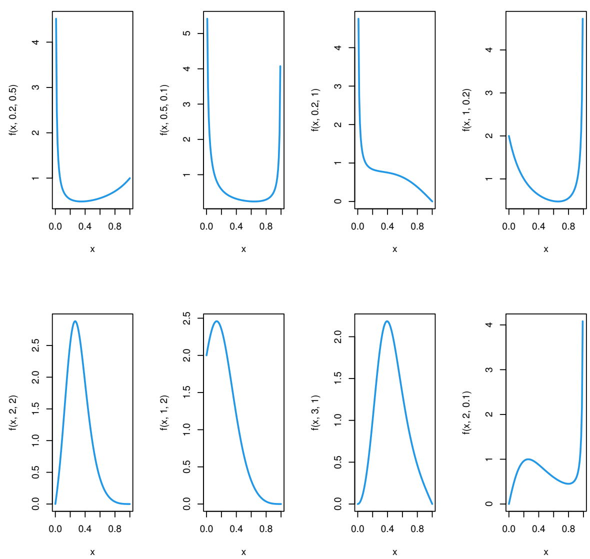

Figure 1. Plots of pdf for some arbitary subset of parameters.

3.1. Asymptotic

Let $Y \sim NWTL(\alpha, \beta)$, then the asymptotics of hrf, pdf and cdf as $y \to 0$ are given by

$$h(y) \sim \frac{\alpha\, 2^{\alpha}\, y^{\alpha-1}}{1-(2y)^{\alpha}}, \quad f(y) \sim \alpha\, 2^{\alpha}\, y^{\alpha-1}, \quad F(y) \sim (2y)^{\alpha}.$$

Also the asymptotics of hrf, pdf and cdf as $y \to 0$ are given by

$$h(y) \sim \frac{2\beta}{1-y}, \quad f(x) \sim 2\beta\,(1-y)^{2\beta-1}, \quad 1-F(y) \sim (1-y)^{2\beta}.$$

Table 1 show the mean, variance, skewness and kurtosis of NWTL(α, β, c) for some parameters.

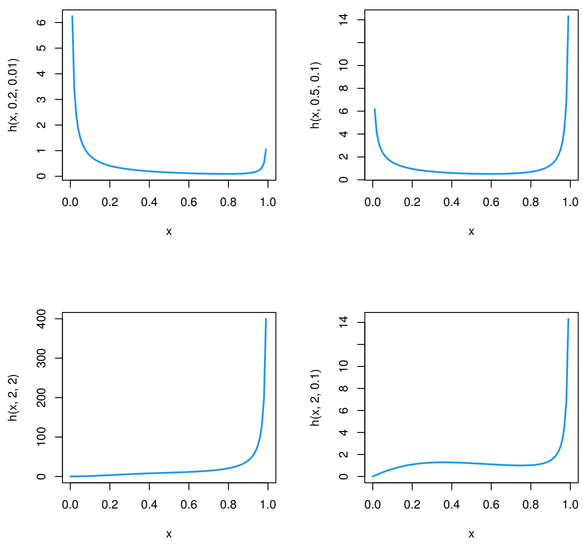

Figure 2. Plots of hrf for some arbitary subset of parameters.

4. Estimation

In this part we study various methods for estimating the parameters of the NWTL(α, β). In the first subsection we study the maximum-likelihood estimation. In the second subsection we study Cramer-von-Mises estimation, Anderson-Darling estimation and right-tailed Anderson–Darling estimation . In the third subsection, we compare all of these methods by numerical simulation tables.

4.1. MLE method

Let $y_1, y_2, \ldots, y_m$ be a any RS of volume m from the NWTL(α, β) distribution. The function the log-likelihood $(l_{\mathfrak{m}}(\alpha, \beta))$ for a vector of parameters can be written as

The log-likelihood function , say $l_{\mathfrak{m}}(\alpha, \beta)$, for the parameters vector of the NWTL(α, β) model is obtained by

$$\begin{aligned} l_{\mathfrak{m}}(\alpha, \beta) &= m \log(2) + (2\beta - 1) \sum_{k=1}^{m} \log(1 - y_k) + (\alpha - 1) \sum_{k=1}^{m} \log(2\, y_k - y_{kl}^2) \\ &+ \sum_{k=1}^{m} \log\left[\alpha + (\beta - \alpha)(2\, y_k - y_k^2)\right] - 2 \sum_{k=1}^{m} \log\left[(2\, y_k - y_k^2)^{\alpha} + (1 - y_k)^{2\beta}\right]. \end{aligned}$$

For maximaizing the $l_{\mathfrak{m}}(\alpha, \beta)$ function with respect to parameters α, β, the score functions are obtained by

$$\frac{\partial l_{\mathfrak{m}}(\alpha, \beta)}{\partial \alpha} = \sum_{k=1}^{m} \log(2\, y_k - y_k^2) + \sum_{k=1}^{m} \frac{(1 - y_k)^2}{\alpha + (\beta - \alpha)(2\, y_k - y_k^2)} - 2 \sum_{k=1}^{m} \frac{(2\, y_k - y_k^2)^{\alpha} \log(2\, y_k - y_k^2)}{(2\, y_k - y_k^2)^{\alpha} + (1 - y_k)^{2\beta}},$$

and

$$\frac{\partial l_{\mathfrak{m}}(\alpha, \beta)}{\partial \beta} = 2 \sum_{k=1}^{m} \log(1 - y_k) + \sum_{k=1}^{m} \frac{2\, y_k - y_k^2}{\alpha + (\beta - \alpha)(2\, y_k - y_k^2)} - 2 \sum_{k=1}^{m} \frac{(1 - y_k)^{2\beta} \log(1 - y_k)}{(2\, y_k - y_k^2)^{\alpha} + (1 - y_k)^{2\beta}}.$$

Table 1. Mean, variance, skewness and Kurtosis of NWTL(α, β, c) for some parameters

α	β	mean	variance	skewness	kurtosis
0.1	0.5	0.32183	0.12916	0.57967	1.71545
0.3	0.5	0.35056	0.12029	0.49365	1.70264
0.5	0.5	0.37685	0.11161	0.42517	1.71282
0.7	0.5	0.40010	0.10359	0.37212	1.73628
0.9	0.5	0.42088	0.09640	0.32887	1.76567
1.1	0.5	0.43940	0.09004	0.29359	1.79711
1.3	0.5	0.45591	0.08435	0.26455	1.83011
1.5	0.5	0.47077	0.07927	0.24090	1.86237
0.1	1.5	0.17748	0.04798	1.09624	3.15560
0.3	1.5	0.20316	0.04570	0.96052	2.98548
0.5	1.5	0.22707	0.04308	0.85679	2.89903
0.7	1.5	0.24864	0.04050	0.77785	2.86135
0.9	1.5	0.26795	0.03811	0.71470	2.84986
1.1	1.5	0.28543	0.03584	0.66600	2.85891
1.3	1.5	0.30135	0.03379	0.62495	2.87536
1.5	1.5	0.31591	0.03194	0.59189	2.89476
0.5	0.2	0.47657	0.16161	0.12250	1.30066
0.5	0.4	0.40546	0.12598	0.33968	1.56736
0.5	0.6	0.35228	0.09940	0.49886	1.85752
0.5	0.8	0.31229	0.08000	0.61622	2.13289
0.5	1	0.28115	0.06570	0.70647	2.38407
0.5	1.2	0.25628	0.05489	0.77684	2.60903
0.5	1.4	0.23595	0.04655	0.83280	2.80813
0.5	1.6	0.21884	0.04002	0.87908	2.98505
1.5	0.01	0.59613	0.14220	-0.17595	1.38967
1.5	0.05	0.57507	0.13361	-0.10477	1.42655
1.5	0.10	0.59846	0.11828	-0.14899	1.45310
1.5	0.15	0.56877	0.11316	-0.05832	1.49962
1.5	0.20	0.55018	0.10653	0.00377	1.54734
1.5	0.25	0.53213	0.10034	0.06045	1.60410
1.5	0.3	0.51522	0.09448	0.11250	1.66512
1.5	0.4	0.49909	0.08902	0.16050	1.73052

The root of these equations have not closed form. We can solve these equations numerically by using R software.

4.2. More methods of estimation

In this subsection, we study Cramer-von-Mises estimation, Anderson-Darling estimation and right-tailed Anderson–Darling estimation. Let $w_1, w_2, \cdots w_m$ denote the order statistics of observed vector $y_1, y_2, \cdots y_m$. The Cramer-von-Mises estimation (CME) due to Choi and Bulgren (1968) are obtained by minimizing the following equation with respect to α and β.

$$S_{CME} = (12m)^{-1} + \sum_{j=1}^{m} \left[\frac{(2\,t_j - t_j^2)^{\alpha}}{(2\,t_j - t_j^2)^{\alpha} + (1 - t_j)^{2\beta}} - \frac{2j-1}{2m} \right]^2 . \tag{20}$$

Anderson-Darling estimation estimation (AD) due to Anderson and Darling (1952) are obtained by minimaizing the following equation with respect to α and β. parameters.

$$S_{AD} = -m - \frac{1}{m}\sum_{j=1}^{m}(2j-1)\log\left[\frac{(2\,t_j - t_j^2)^{\alpha}}{(2\,t_j - t_j^2)^{\alpha} + (1-t_j)^{2\beta}}\right]$$
$$-\frac{1}{m}\sum_{j=1}^{m}(2j-1)\log\left[\frac{(1-t_{m+1-j})^{2\beta}}{(2\,t_{m+1-j} - t_{m+1-j}^2)^{\alpha} + (1-t_{m+1-j})^{2\beta}}\right].$$

Right-Tailed Anderson-Darling estimation estimation (RTAD) due to Anderson and Darling (1952) are obtained by minimaizing the following equation with respect to α and β. parameters.

$$S_{RTAD} = \frac{m}{2} - 2\sum_{j=1}^{m}(2j-1)\log\left[\frac{(2\,t_j - t_j^2)^{\alpha}}{(2\,t_j - t_j^2)^{\alpha} + (1-t_j)^{2\beta}}\right]$$
$$-\frac{1}{m}\sum_{j=1}^{m}(2j-1)\log\left[\frac{(1-t_{m+1-j})^{2\beta}}{(2\,t_{m+1-j} - t_{m+1-j}^2)^{\alpha} + (1-t_{m+1-j})^{2\beta}}\right].$$

5. Simulation Study

For $(\alpha, \beta) = (0.8, 0.5), (0.9, 0.3), (2, 1), (3, 1), (1.5, 1.5)$, we performed simulation analysis according to the following algorithm

- Generate N=10^4 samples of the size ζ from (3) for Uniform case.
- Compute the estimates for the N=10^4 samples, say $\left(\hat{\alpha}_{\zeta}, \hat{\beta}_{\zeta}\right)$ for $\zeta = 1, 2, ..., 10^4$.
- Compute the "biases" and "mean squared errors".

The $bias_{\varepsilon}(\zeta)$ as well as the $MSE_{\varepsilon}(\zeta)$ for $\zeta = 20, 70, \cdots 470$ are computed using the "R-optim function" and the Nelder-Mead method. Tables 2-11 display the results. From tables 2-11, the bias approach zero and the MSEs decreases as m increases. It shows that the consistency of these methods. Also MLE is better than other methods.

Table 2. Estimated Biases for $(\alpha, \beta) = (0.8, 0.5)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.06033	0.05821	0.13956	0.07677	0.06033	0.12300	0.22609	0.13729
70	-0.02905	-0.03279	-.03252	-0.01247	-0.02905	0.04319	0.06142	0.03632
120	-0.03178	-0.03474	-0.03464	-0.02182	-0.03178	0.03738	0.05331	0.02932
170	-0.02851	-0.03117	-0.03347	-0.01940	-0.02851	0.02938	0.03822	0.02342
220	-0.02517	-0.02758	-0.02948	-0.01707	-0.02517	0.03099	0.03980	0.02345
270	-0.02610	-0.02831	-0.02988	-0.01862	-0.02610	0.02528	0.03350	0.01743
320	-0.02531	-0.02789	-0.03081	-0.01747	-0.02531	0.02625	0.03351	0.01754
370	-0.02493	-0.02721	-0.02918	-0.01799	-0.02493	0.02317	0.03101	0.01287
420	-0.02323	-0.02508	-0.02709	-0.01616	-0.02323	0.02002	0.02557	0.01286
470	-0.02160	-0.02325	-0.02473	-0.01632	-0.02160	0.02302	0.02882	0.01597

6. Applications

In this part, we give and analyse three real-life data applications by comparing the fits of the proposed NWTL model with some well-known models. Here we consider beta distribution, Kumaraswamy distribution due to

Table 3. Estimated MSE for $(\alpha, \beta) = (0.8, 0.5)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.04197	0.04515	0.26712	0.04486	0.57440	0.16990	0.36207	0.14831
70	0.00381	0.00416	0.00530	0.00376	0.04611	0.02346	0.03319	0.02043
120	0.00261	0.00296	0.00366	0.00227	0.03316	0.01489	0.02333	0.01036
170	0.00180	0.00199	0.00248	0.00158	0.01613	0.00885	0.01237	0.00686
220	0.00140	0.00157	0.00191	0.00123	0.01231	0.00699	0.00958	0.00571
270	0.00126	0.00142	0.00174	0.00107	0.00979	0.00536	0.00738	0.00416
320	0.00123	0.00139	0.00171	0.00100	0.00890	0.00494	0.00684	0.00357
370	0.00110	0 .00125	0.00155	0.00093	0.00906	0.00443	0.00652	0.00261
420	0.00097	0.00109	0.00138	0.00078	0.00714	0.00389	0.00530	0.00261
470	0.00085	0 .00096	0.00117	0.00072	0.00616	0.00338	0.00462	0.00254

Table 4. Estimated Biases for $(\alpha, \beta) = (0.9, 0.3)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.00025	-0.00400	0.01778	0.02347	0.43188	0.12036	0.26551	0.10675
70	-0.02952	-0.03202	-0.03416	-0.01505	0.10127	0.04781	0.07696	0.02788
120	-0.02346	-0.02592	-0.02899	-0.01033	0.09290	0.05046	0.07367	0.02235
170	-0.02035	-0.02206	-0.02482	-0.01012	0.06088	0.03376	0.04812	0.01692
220	-0.01847	-0.02017	-0.02285	-0.00898	0.05686	0.03140	0.04503	0.01298
270	-0.01782	-0.01917	-0.02146	-0.00940	0.05057	0.03004	0.04082	0.01618
320	-0.01586	-0.01687	-0.01880	-0.00864	0.03588	0.02247	0.02894	0.01379
370	-0.01538	-0.01674	-0.01922	-0.00729	0.04591	0.02795	0.03676	0.01273
420	-0.01457	-0.01570	-0.01794	-0.00724	0.03667	0.02214	0.02933	0.01032
470	-0.01393	-0.01498	-0.01715	-0.00699	0.02941	0.01699	0.02307	0.00655

Table 5. Estimated MSE for $(\alpha, \beta) = (0.9, 0.3)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.01119	0 .01177	0.02648	0.01370	1.79565	0.17909	0.51629	0.12738
70	0.00165	.00185	0.00234	0.00117	0.06842	0.03036	0.04860	0.02340
120	0.00094	0.00109	0.00148	0.00061	0.03903	0.01775	0.02840	0.01150
170	0.00070	0.00081	0.00108	0.00045	0.02676	0.01290	0.01993	0.00827
220	0.00056	0.00065	0.00088	0.00031	0.01639	0.00828	0.01230	0.00581
270	0.00054	0.00062	0.00082	0.00032	0.01444	0.00693	0.01073	0.00444
320	0.00042	0.00047	0.00063	0.00026	0.00911	0.00505	0.00704	0.00397
370	0.00038	0.00044	0.00061	0.00020	0.01061	0.00546	0.00797	0.00343
420	0.00031	0.00036	0.00052	0.00017	0.00910	0.00455	0.00685	0.00290
470	0.00031	0.00036	0.00049	0.00018	0.00653	0.00372	0.00507	0.00275

Table 6. Estimated Biases for $(\alpha, \beta) = (2, 1)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.04327	0.03601	0 .08979	0.08161	0.15209	0.07486	0.17462	0.16024
70	0.00037	-0.00098	0.01465	0.01221	0.05830	0.04095	0.07031	0.06186
120	-0.02222	-0.02877	-0.02474	-0.01698	0.06184	0.03180	0.04848	0.04039
170	-0.02500	-0.02933	-0.02653	-0.01916	0.05435	0.03441	0.04578	0.04135
220	-0.02488	-0.02725	-0.02361	-0.01950	0.05830	0.04560	0.05545	0.04870
270	-0.02954	-0.03175	-0.02987	-0.02526	0.04345	0.03288	0.03920	0.03880
320	-0.01975	-0.02377	-0.02141	-0.01882	0.05747	0.04042	0.05072	0.03713
370	-0.02580	-0.02767	-0.02720	-0.02087	0.03370	0.02476	0.02795	0.02933
420	-0.02540	-0.02858	-0.02921	-0.02244	0.04337	0.02981	0.03291	0.03111
470	-0.02742	-0.03082	-0.03018	-0.02581	0 .03883	0.02475	0.03102	0.02222

Kumaraswamy (1980) and Topp-Leone distribution as competitive models. For all examples, the criteria of the Cramér–von Mises (W^*), the criteria of the Anderson-Darling (A^*) and p-value for Kolmogorov-Smirnov-test,

Table 7. Estimated MSE for $(\alpha, \beta) = (2, 1)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.08156	0.08207	0.12530	0.08622	0 .51036	0.31584	0.45147	0.33137
70	0.01544	0.01593	0.02077	0.01492	0.10686	0.07985	0.09757	0.07959
120	0.01028	0.01025	0.01207	0.00923	0.05973	0.04215	0.05309	0.03998
170	0.00622	00.00639	0.00775	0.00549	0.03769	0.02847	0.03531	0.02790
220	0.00459	0.00475	0.00583	0.00406	0.03923	0.02852	0.03390	0.02673
270	0.00457	0.00449	0.00520	0.00410	0.02893	0.02090	0.02507	0.01992
320	0.00301	0.00305	0.00357	0.00275	0.02619	0.01926	0.02293	0.01900
370	0.00293	0.00294	0.00329	0.00257	0.01971	0.01386	0.01619	0.01319
420	0.00279	0.00284	0.00324	0.00254	0.02126	0.01408	0.01701	0.01265
470	0.00244	0.00262	0.00295	0.00238	0.01438	0.01037	0.01220	0.01022

Table 8. Estimated Biases for $(\alpha, \beta) = (3, 1)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.02814	0 .01543	0.06979	0.06045	0.20274	0.09140	0.25092	0.21646
70	0.00354	-0.00312	0.00914	0.01307	0.11674	0.07117	0.11348	0.09702
120	-0.01711	-0.02289	-0.01547	-0.01301	0.11023	0.07391	0.10327	0.08804
170	-0.02462	-0.02914	-0.02389	-0.02133	0.06861	0.04225	0.06426	0.05353
220	-0.02532	-0.02990	-0.02709	-0.02093	0.07947	0.05184	0.06750	0 .05710
270	-0.02379	-0.02780	-0.02600	-0.02163	0.08859	0.06619	0.07771	0.06734
320	-0.02420	-0.02716	-0.02574	-0.02205	0.04832	0.03150	0.03945	0.03722
370	-0.02223	-0.02535	-0.02435	-0.02009	0.06147	0.04353	0.05110	0.04988
420	-0.02384	-0.02633	-0.02682	-0.02099	0.04713	0.03306	0.03529	0.03935
470	-0.02059	-0.02410	-0.02336	-0.02010	0 .07780	0.05839	0.06629	0.05658

Table 9. Estimated MSE for $(\alpha, \beta) = (3, 1)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.07663	0.07020	.10967	.07383	1.01782	.65276	1.06905 .68628	
70	0.01595	0 .01516	0.01906	0.01517	0.21712	0.14435	0.18266	0.14294
120	0.00961	0.00886	0.01020	0.00826	0.15334	0.10453	0.12803	0.09929
170	0.00586	0.00536	0.00592	0.00514	0.09216	0.06480	0.08118	0.06208
220	0.00509	0.00515	0.00587	0.00454	0.07430	0.05128	0.06442	0.04773
270	0.00360	0.00344	0.00376	0.00333	0.05539	0.03895	0.04666	0.03784
320	0.00383	0.00346	0.00374	0.00334	0.04664	0.03169	0.03770	0.02952
370	0.00288	0.00286	0.00330	0.00253	0.03881	0.02851	0.03286	0.02830
420	0.00283	0.00286	0.00330	0.00246	0.03320	0.02651	0.03010	0.02605
470	0.00200	0.00205	0.00234	0.00186	0.03219	0.02305	0.02731	0.02172

Table 10. Estimated Biases for $(\alpha, \beta) = (1.5, 1.5)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.04785	0.02942	0.12129	0.08971	0.14154	0.07305	0.16498	0.13032
70	0.01521	0.01079	0.03239	0.02224	0.03985	0.02552	0.04732	0.03603
120	0.01924	0.01571	0.02828	0.02803	0.03338	0.02140	0.03384	0 .02994
170	-0.00575	-0.01059	-0.00720	0.00205	0.03026	0.01939	0.02501	0.02565
220	0.00613	0.00258	0.00950	0.00967	0.02613	0.01800	0.02608	0.01990
270	-0.01800	-0.02176	-0.01803	-0.01394	0.01649	0.00807	0.01349	0.01132
320	-0.01969	-0.02289	-0.01877	-0.01675	0.02381	0.01678	0.02235	0.01704
370	-0.01677	-0.02072	-0.01558	-0.01659	0.02449	0.01664	0.02395	0.01527
420	-0.02200	-0.02811	-0.02704	-0.02074	0.01998	0.00843	0.01313	0.00796
470	-0.02443	-0.02609	-0.02555	-0.02038	0.01033	0.00645	0.00780	0.01028

Akaike information criterion (AIC), Bayesian information criterion (BIC) and the value of maximum likelihood function (l) are considered. Parameters are estimated by maximum likelihood estimation method (MLE).

Table 11. Estimated MSE for $(\alpha, \beta) = (1.5, 1.5)$

n	$RTAD(\hat{\beta})$	$AD(\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$	$RTAD(\hat{\alpha})$	$AD(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$
20	0.18575	0 .17574	0.27234	0.17590	0.33861	0.17795	0.28527	0.18315
70	0.04364	0.04055	0.04722	0.04053	0.07219	0.04931	0.06236	0.04531
120	0.02587	0.02464	0.03069	0.02480	0.04216	0.02752	0.03438	0 .02541
170	0.01662	0.01637	0.01828	0.01584	0.02077	0.01540	0.01836	0.01484
220	0.01109	0.01064	0.01311	0.01008	0.02019	0.01496	0.01750	0.01363
270	0.00804	0.00817	0.01004	0.00721	0.01308	0.00977	0.01153	0.00958
320	0.00726	0.00704	0.00819	0.00660	0.01303	0.00950	0.01119	0.00902
370	0.00701	0.00653	0.00738	0.00633	0.01191	0.00873	0.01073	0.00800
420	0.00543	0.00534	0.00621	0.00508	0.01006	0.00708	0.00845	0.00653
470	0.00483	0.00497	0.00570	0.00467	0.00704	0.00552	0.00644	0.00554

The first data set (data set I) represent the failure time of 20 components due to Murthy et al. (2004). The second data set (data set II), represent 50 observation, the whole diameter was 9 mm and the sheet thickness was 2 mm. These data sets analyzed by Dey et al. (2018). The third data set (data set III), represent the 48 obsevation of rock samples related to a petroleum reservoir. These data analyzed by Cordeiro and Brito (2012). Table 12 show the mean, varinace, skewness and kurtosis for data sets I, II and III.
An overview of the estimated MLE's and fitted information criteria for all data sets using various models can be seen in Tables 13 and 16. The NWTL distribution provide better fit than other competitive models with additional criteria. The histogram of lifetime data set, as well as the fitted pdf plots, are shown in Figures 3.

Table 12. Mean, variance, skewness and kurtosis for data sets I, II and III

	mean	variance	skewness	kurtosis
data set I	0.121	0.007	3.585	15.203
data set II	0.152	0.006	0.0061	2.301
data set III	0.218	0.006	1.169	4.109

Table 13. Estimated parameters with standard errors in parenthesis

model	data set I		data set II		data set III	
NWTL(α, β)	1.434	9.859	1.218	5.086	2.335	4.940
	(0.301)	(2.111)	(0.182)	(0.653)	(0.303)	(0.661)
Beta(α, β)	1.586	21.817	2.400	13.521	5.940	21.201
	(0.244)	(10.177)	(0.451)	(2.770)	(1.181)	(4.346)
Kw(α, β)	3.112	21.823	1.958	31.266	2.714	44.382
	(0.936)	(7.041)	(0.244)	(13.164)	(0.292)	(17.443)
TL(α)	0.624		0.680		0.989	
	(0.139)		(0.096)		(0.142)	

Table 14. Goodness-of-fit statistics for data set I

model	W^*	A^*	AIC	BIC	$p-value$	$-l$
NWTL (α, β)	0.302	1.961	-53.76	-51.77	0.588	28.88
Beta(α, β)	0.370	2.315	-51.76	49.77	0.152	27.88
Kw (α, β)	0.436	2.649	-47.29	-45.30	0.126	25.64
TL (α)	0.339	2.156	-25.48	-24.49	0.0001	13.74

Table 15. Goodness-of-fit statistics for data set II

model	W^*	A^*	AIC	BIC	$p-value$	$-l$
NWTL (α,β)	0.190	1.127	-110.34	-106.52	0.187	57.13
Beta(α,β)	0.276	1.534	-107.86	-104.03	0.039	55.96
Kw (α,β)	0.206	1.171	-111.04	-107.21	0.114	57.21
TL (α)	0.303	1.670	-58.86	-56.95	$1.33e-06$	30.43

Table 16. Goodness-of-fit statistics for data set III

model	W^*	A^*	AIC	BIC	$p-value$	$-l$
NWTL (α,β)	0.131	0.774	-106.49	-102.74	0.44	55.24
Beta(α,β)	0.128	0.778	-107.20	-103.45	0.282	55.60
Kw (α,β)	0.208	1.278	-100.98	-97.24	0.215	52.49
TL (α)	0.119	0.721	-40.33	-38.46	$4.51e-06$	21.16

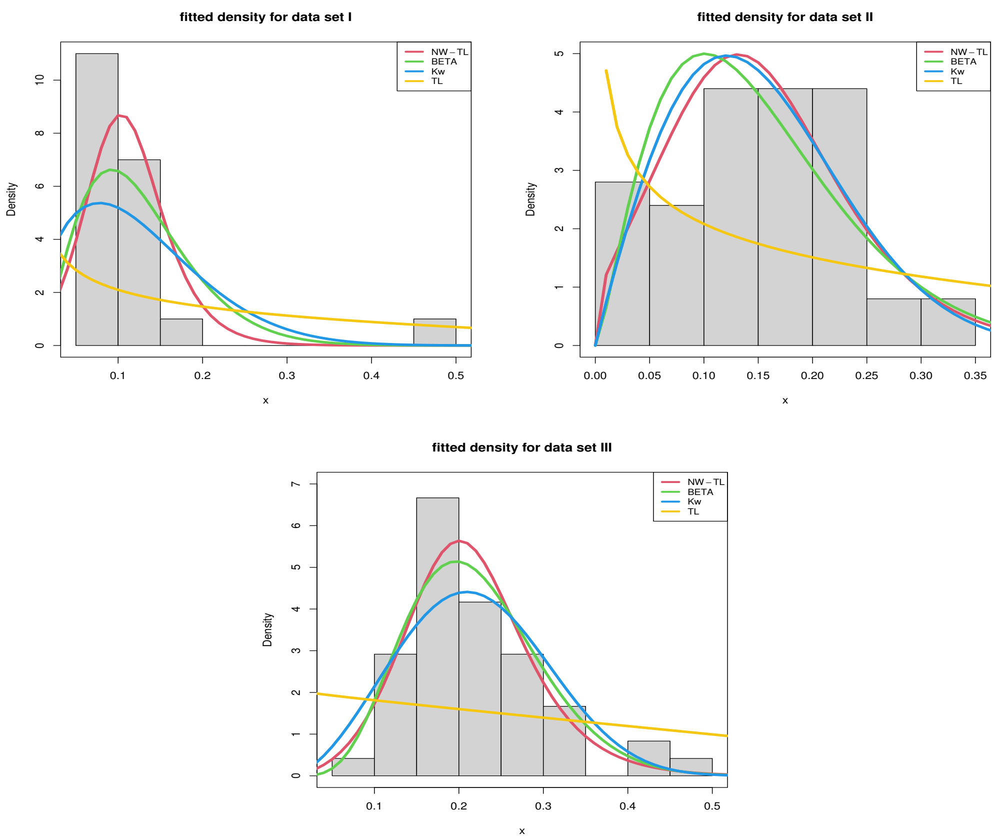

Figure 3. Histogram and fitted pdf for data set I, II and III.

7. Conclusions

In above study, we studied a new weighted Topp-Leone family of distributions called the new weighted TL-II (NWTL-II) class of distributions. Some essential properties of this family, such as the quantile function, the asymptotic, residual entropy index and the order statistics are obtained. Then, we studied Lindley special member. We used some estimation methods for estimate the unknown parameters. The estimated Bias and MSE of parameters for all methods of estimation, converge to zero by increasing the sample size, it represent the consistency of these methods. Then to show the flexibility of the proposed model we fitted to some real data sets and compare with some Lindley extensions. The numerical results show that this model provide better fit.

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