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Gaussian quantum systems and Kahler geometrical structure

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Abstract In this article, we study the phase-space distribution of the quantum state as a framework to describe the different properties of quantum systems in continuous-variable systems. The natural approach to quantum systems is given the Gaussian Wigner representation, to unify the description of bosonic and fermionic quantum states, we study the structure of the Kahler space geometry as the geometry generated by three forms under the agreement conditions depended on the nature of the state bosonic or fermionic. Multimode light is studied, and we established that the Fock space vacuum corresponds to a certain homogeneous Gaussian state.

Keywords Wigner function, Fock space, Kahler space, photon, boson, fermion, Gaussian state, Maxwell equation.

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1. Introduction (some definitions and notations)

The central limit theorem establishes that a sum of numbers of the independent and identically distributed random variables, which variances are finite, will approach toward a normal distribution as the number of variables will grow. This statement has many different variations with slightly different conditions on random variables, colloquially speaking, the central limit theorem maintains that the properties of the normalized sums have a tendency to the normalization [1, 3, 25, 26]. From a mathematical perspective, this theorem highlights the impotence of Gaussian (or normal) distributions, from a physical viewpoint, the gaussian states play a central role in the theory of Bose gases and the formalism of the theory of optical coherency. The central limit theorem warrants the Gaussian theory a prominent place in the quantum information theory of the continuous variables [29-35].

The general form of the Gaussian probability density function is $u(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)$, where m is its mean, mode, and median, σ is a standard deviation. Thus, the Gaussian states are completely defined by their mean-field and covariance matrix. Let ξ^a be a vector in the phase space with the symmetric bilinear form g_{ab}, the Wigner function for the bosonic Gaussian states is

$$W(\xi) = \sqrt{\det \frac{g}{\pi}} \exp\left(-\frac{1}{2} g_{ab}\xi^a\xi^b\right).$$

Now, to clarify our considerations, let us introduce some notations and definitions. We will assume that a set of n identical particles is described by a quantum state vector $|\psi\rangle$ in a reflexive Banach space $\mathbf{B}$. The joint state of n particles can be determined by the classical tensor product $|\psi_1\rangle, ..., |\psi_n\rangle$, where $|\psi_i\rangle$ is a state vector for the i-th particle.

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Definition 1. *A linear operator $A^* : \mathbf{B}^* \to \mathbf{B}^*$ on a reflexive Banach space is said to be adjoint to the linear operator $A : \mathbf{B} \to \mathbf{B}$ if $\langle y^*, Ax \rangle = \langle A^* y^*, x \rangle$ holds for all $x \in \mathbf{B}$ and all $y^* \in \mathbf{B}^*$.*

Definition 2. *A linear operator $A : \mathbf{B} \to \mathbf{B}$ on a reflexive Banach space is said to be strictly Hermitian if the following equality $\langle A^* y^*, Ax \rangle = \langle y^*, x \rangle$ holds for all $x \in \mathbf{B}$ and all $y^* \in \mathbf{B}^*$.*

The permutation $\sigma \in S_n$ is defined by strictly Hermitian operator P_σ according to the following formula

$$P_\sigma |\psi_1\rangle \otimes ... \otimes P_\sigma |\psi_n\rangle = \left|\psi_{\sigma(1)}\right\rangle \otimes ... \otimes \left|\psi_{\sigma(n)}\right\rangle. \tag{1}$$

This permutation guarantees the invariance of the observable physics of the identical particle with the same internal attributes.

The span $span\left\{u_1, ..., u_n\right\}$ of a set of vectors $u_1, ..., u_n$ is the set of all linear combinations of these vectors

$$span\left\{u_1, ..., u_n\right\} = \left\{\alpha_1 u_1 + ... + \alpha_n u_n \quad : \quad \alpha_1, ..., \alpha_n \in K\right\}, \tag{2}$$

where K is a field over which the vector space is considered.

Assuming that the particles are identical, postulating the invariance under the permutation gives us that the state vector is either fully symmetric (Bosons) or fully antisymmetric (Fermions) relative to these permutations, and a single particle is symmetric. So, the natural condition to demand is

$$P_\sigma \left|\Psi^{(n)}\right\rangle = \left|\Psi^{(n)}\right\rangle \tag{3}$$

for Bosons or

$$P_\sigma \left|\Psi^{(n)}\right\rangle = -\left|\Psi^{(n)}\right\rangle \tag{4}$$

for Fermions.

The first quantization is a description of a n-particles system. We consider the Boson case. The Banach space $\mathbf{B}_s^{(n)}$ that describes n-Bosons system is a subspace of the Banach space $\mathbf{B}^{\otimes n}$, which consists of all linear combinations of vectors such that $P_\sigma \left|\Psi^{(n)}\right\rangle = \left|\Psi^{(n)}\right\rangle$, and can be written as

$$|\psi_1\rangle \vee ... \vee |\psi_n\rangle = \sum_{\sigma \in S_n} \left|\psi_{\sigma(1)}\right\rangle \otimes ... \otimes \left|\psi_{\sigma(n)}\right\rangle, \tag{5}$$

so

$$\mathbf{B}_s^{(n)} = clos\left[span\left\{|\psi_1\rangle \vee ... \vee |\psi_n\rangle \quad : \quad |\psi_i\rangle \in \mathbf{B}\right\}\right], \tag{6}$$

where closure is understood in the topology generated by the norm of the Banach space.

Let us denote n-particles Boson Banach system by $\mathbf{B}_s^{(n)}$, the direct sum of such systems is

$$\Gamma(\mathbf{B}) = \mathbf{B}_s^{(0)} \oplus \mathbf{B}_s^{(1)} \oplus \mathbf{B}_s^{(2)} \oplus ..... \tag{7}$$

The component $\mathbf{B}_s^{(0)}$ describes the vacuum state with the single state $|0\rangle$.

Pure separable states of Bosons (Fermions) can be described by the following formula

$$|\Psi\rangle = \Psi^{(0)} \oplus \Psi^{(1)} \oplus \Psi^{(2)} \oplus ..., \tag{8}$$

where $\Psi^{(i)} \in \mathbf{B}_s^{(i)}$ are vectors from i-th Banach space. Now, to define the state $|\Psi\rangle$, the formula (8) must be completed by the normalization requirement

$$\|\Psi\| = 1. \tag{9}$$

Creation and annihilation operators will be denoted as $\hat{a}^\dagger$ and $\hat{a}$, the operator $\hat{a}^\dagger$ creates and $\hat{a}$ deletes particles. The creation operator $\hat{a}^\dagger$ can be defined by

$$\hat{a}^\dagger(\varphi)|\Psi\rangle = 0 \oplus \left(\Psi^{(0)}|\varphi\rangle\right) \oplus \left(|\varphi\rangle \vee \Psi^{(1)}\right) \oplus \left(|\varphi\rangle \vee \Psi^{(2)}\right) \oplus ..., \tag{10}$$

correctly defined all $\varphi \in \mathbf{B}$. The annihilation operator $\hat{a}$ can be defined as the conjugation of the operator $\hat{a}^{\dagger}$ with the condition $\hat{a}(\varphi)|0\rangle = 0$.

Creation and annihilation operators are the generators of the algebra of observables, which provides a unique representation of the algebra. The canonical commutation relation on the Fock space is given by $\left[\hat{a}(\psi),\ \hat{a}^{\dagger}(\varphi)\right] = \langle\psi|\varphi\rangle$, which holds all vectors $\varphi \in \mathbf{B}$ and $\psi \in \mathbf{B}^{*}$ in the single-particle Banach space $\mathbf{B}$.

A basis in the Fock space can be constructed as follows. Let set V be a basis in single-particle Banach space $\mathbf{B}$ then the basis in the Fock space consist of all possible Fock states, which can be formed by generating particles in vectors of V. Particles in the vacuum can be created by the creation operator $\hat{a}^{\dagger}$ as

$$|\psi_1\rangle \vee ... \vee |\psi_n\rangle = \hat{a}^{\dagger}(\psi_1)\,\hat{a}^{\dagger}(\psi_2)\,...\hat{a}^{\dagger}(\psi_n)\ |0\rangle,$$

which generates a certain Fock space, the whole Fock space can be obtained as a direct sum of all such Fock spaces. by definition

$$\hat{a}(\varphi)|\Psi\rangle = 0 \oplus \left(\Psi^{(0)}|\varphi\rangle\right) \oplus \left(|\varphi\rangle \vee \Psi^{(1)}\right) \oplus \left(|\varphi\rangle \vee \Psi^{(2)}\right) \oplus ..., \tag{11}$$

correctly defined all $\varphi \in \mathbf{B}$.

2. The classical model of multimode light and its generalization

The light propagates as a wave, which is regulated by Maxwell equations. A vector field $u_1(r,\ t)$ is called a mode of the electromagnetic field. The Maxwell equations yield the following equations

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u_1(r,\ t) = 0, \tag{12}$$

$$\nabla \cdot u_1(r,\ t) = 0, \tag{13}$$

$$\frac{1}{V}\int_V d^3r\,|u_1(r,\ t)|^2 = 1, \tag{14}$$

where V is a volume containing whole considering physical system.

Taking $u_1(r,\ t)$ as a first element, we can construct an orthogonal mode basis $\{u_m(r,\ t)\}$ with orthogonality condition

$$\frac{1}{V}\int_V d^3r\,(u_m(r,\ t))^{*}\,u_n(r,\ t) = \delta_{mn}. \tag{15}$$

The modes $\{u_m(r,\ t)\}$ fashion a basis for the representation of any solution to the Maxwell equations in the form of a series

$$E^{(+)}(r,\ t) = \sum_m \varepsilon_m u_m(r,\ t), \tag{16}$$

ε_m are the complex amplitudes, which is convenient to present in the form of the sum of the real (amplitude quadrature) and imaginary (phase quadrature) components

$$\varepsilon_m = E_m^{(x)} + iE_m^{(p)}. \tag{17}$$

The space of all solutions to the Maxwell equations constitutes a mode space with the basis $\{u_m(r,\ t)\}$. The series $\sum_m \varepsilon_m u_m(r,\ t)$ has finite numbers of summands since a vector u_m (we will omit arguments, where it is possible, notation independent of any representation) consists of zeros except for one at the m-th position.

In Hilbert spaces, there is a unitary operator U, which defines the unitary transformation from one basis to another basis such that

$$u_m(r,\ t) = \sum_k U_{km} v_k(r,\ t), \tag{18}$$

$$v_m(r,\ t) = \sum_k U_{km}^{\dagger} u_k(r,\ t), \tag{19}$$

the first formula can be rewritten in the form $u_m = \sum_k U_m^k v_k$.
The infinite-dimensional matrix $u_m(r,\ t) = \sum_k U_{km} v_k(r,\ t)$ is such that

$$U_{km} = \frac{1}{V} \int_V d^3 r \left(v_k(r,\ t)\right)^* u_k(r,\ t). \tag{20}$$

The expansion of the electric field of the new basis can be written as

$$E^{(+)}(r,\ t) = \sum_k \tilde{\varepsilon}_k v_k(r,\ t), \tag{21}$$

where $\tilde{\varepsilon}_k = \sum_m U_{km}\varepsilon_m$. Since the unitary transformation U is arbitrary, the mode basis can be chosen in accordance with the optical process, for instance, spatial or frequency Hermite-Gauss modes.

3. Quantum representation of multimode light

Let $\left\{\hat{a}_m^{\dagger}\right\}$ be a set of creation operators and Ube a unitary operator with matrix U_m^k so a new set of operators $\left\{\hat{b}_m^{\dagger}\right\}$ can be written as

$$\hat{b}_m^{\dagger} = \sum_k U_m^k \hat{a}_k^{\dagger} \tag{22}$$

or in the form

$$\hat{a}_k = \sum_k U_m^k \hat{b}_k. \tag{23}$$

Since $U$is a unitary operator, we have

$$\left[\hat{b}_m,\ \hat{b}_k^{\dagger}\right] = \delta_{mk}, \tag{24}$$

and a positive electric field has the following representation

$$\hat{E}^{(+)}(r,\ t) = \sum_k f_k^{(1)} \hat{b}_m u_k(r,\ t), \tag{25}$$

where $\hat{b}_m$ is the one-photon annihilation operator in the mode $u_k(r,\ t)$, such that

$$\left(f_m^{(1)}\right)^2 = \sum_k \left(\varepsilon_k^{(1)}\right)^2 \left|U_m^k\right|^2. \tag{26}$$

Since mode $\tilde{u}_k$ associated with a creation operator $\hat{a}_k^{\dagger}$, the new set of modes relative to the plane wave basis is

$$u_m = \frac{1}{f_m^{(1)}} \sum_k \varepsilon_k^{(1)} U_m^k \tilde{u}_k. \tag{27}$$

Let us assume that a mode basis is established then the general quantum light state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{k_1} ... \sum_{k_n} ...\ C_{k_1 ... k_n ...} |k_1\ :\ u_1\rangle \otimes ... \otimes |k_n\ :\ u_n\rangle \otimes ..., \tag{28}$$

where $|k_n\ :\ u_n\rangle = \frac{\left(\sum_k U_n^k \hat{a}_k^{\dagger}\right)^{k_n} |0\rangle}{\sqrt{k_n!}}$.

Intrinsic properties of the state of the multimode light are those properties that are invariant relatively to the choice of the mode basis. The intrinsic properties are:

1. Structural properties, which are solely determined by the class of the quantum system such as composition, set of the observable, the action (Hamiltonian) of the system.
2. Conditional properties are solely determined by the preparation of the system. For instance, let the particle $|\psi\rangle$ possess a spin $\frac{1}{2}$, then, we can prepare the state with spin projection to z - axis equal to $\frac{\hbar}{2}$, from these assumptions arises no contradictions since there is the value of σ_z.
3. Classical properties.

Let η be a mixed state and η_n a minimal span on n modes $u_1, ..., u_n$. The coherency matrix $\left(\Gamma^{(1)}\right)_{mk}$ is

$$\left(\Gamma^{(1)}\right)_{mk} = \left\langle \hat{a}_m^{\dagger},\ \hat{a}_k \right\rangle, \tag{29}$$

and elements of $\left(\Gamma^{(1)}\right)_{mk}$ for $m > n$ and $k > n$ equal to zero, so that matrix $\left(\Gamma^{(1)}\right)_{mk}$ composed of a square $n \times n$ non-zero diagonal matrix. The number n of modes relates to the intrinsic properties of the quantum system and coincides with the rank of the matrix of coherency. The given state coincides with a vacuum for all $k > n$ and $\left\langle \hat{a}_k^{\dagger},\ \hat{a}_k \right\rangle = 0$ for $k > n$.

Let $\left(\tilde{\Gamma}^{(1)}\right)_{mk}$ be a coherency matrix corresponding to the annihilation operators $\hat{b}_k$ of the arbitrary mode basis $\{v_k\}$, so that $\left(\tilde{\Gamma}^{(1)}\right)_{mk} = \left\langle \hat{b}_m^{\dagger},\ \hat{b}_k \right\rangle$. Since the matrix $\left(\tilde{\Gamma}^{(1)}\right)_{mk}$ is Hermitian there is a unitary operator U that transforms $\left(\tilde{\Gamma}^{(1)}\right)_{mk}$ into diagonal form

$$U\left(\tilde{\Gamma}^{(1)}\right)U^{\dagger} = Diag\left[k_1, ..., k_n, 0, 0, ...\right] \tag{30}$$

and the transformation of the creation operators in the vector form $\hat{c}^{\dagger} = U\hat{b}^{\dagger}$. The matrix $U\left(\tilde{\Gamma}^{(1)}\right)U^{\dagger}$ can be presented as

$$\begin{aligned} &U\left(\tilde{\Gamma}^{(1)}\right)U^{\dagger} = Diag\left[k_1, ..., k_n, 0, 0, ...\right] = \\ &= \left\langle U\hat{b}^{\dagger}, \hat{b}^T U^{T*} \right\rangle = \left\langle \hat{c}^{\dagger},\ \hat{c}^T \right\rangle. \end{aligned} \tag{31}$$

So, from the well-known result of linear algebra that a Hermitian matrix can be transformed by a unitary operator to the diagonal form, we have obtained that by the diagonalization of the coherency matrix one can obtain the simplest representation of the given quantum state. The principal eigenvalues correspond with the magnitude of energy of the modes.

4. Exemplar, Gaussian states

The electric field of light is a quantum observable $\hat{E}^{(+)}(r,\ t)$ that can be presented as

$$\hat{E}^{(+)}(r,\ t) = \sum_m \varepsilon_m^{(1)} \frac{\hat{x}_m + i\hat{p}_m}{2} u_m(r,\ t), \tag{32}$$

where $\varepsilon_m^{(1)}$ are electric fields of single-photon; $\hat{x}_m$ and $\hat{p}_m$ are quadrature operators, which must satisfy the Heisenberg inequality $\Delta\hat{x}\ \Delta\hat{p} \geq 1$ and canonical commutation condition $[\hat{x}_m,\ \hat{p}_k] = 2\delta_{mk}$. An observable $\hat{q}(\vec{u})$ can be defined according to the formula

$$\hat{q}(\vec{u}) = \sum_{k=1,...,n} u_{2k-1}\hat{x}_k + u_{2k}\hat{p}_k \tag{33}$$

for any $\vec{u} \in R^{2n}$.

The characteristic function χ for quadrature $\hat{q}(\vec{u})$ is defined as

$$\chi(\lambda) = tr\left[\hat{\eta}\exp\left(i\lambda\hat{q}(\vec{u})\right)\right] = \sum_{k=0,...} \frac{(i\lambda)^k}{k!} tr\left[\hat{\eta}\left(\hat{q}(\vec{u})\right)^k\right] \tag{34}$$

for any $\lambda \in R$. The distribution of the probability can be defined as

$$p(z) = \frac{1}{2\pi} \int_R d\lambda\, \chi(\lambda) \exp(-i\lambda\, z)\,. \tag{35}$$

Let set $\{\vec{u}_1, ..., \vec{u}_n\}$ is such that $[\hat{q}(\vec{u}_m)\,,\ \hat{q}(\vec{u}_k)] = 0$ holds for all m and k, the characteristic function χ defines as

$$\chi(\lambda) = tr\left[\hat{\eta} \exp\left(i\vec{\lambda} \cdot \hat{q}(\vec{u})\right)\right] \tag{36}$$

where $\hat{q}(\vec{u}) = (\hat{q}(\vec{u}_1)\,, ..., \hat{q}(\vec{u}_n))$ and the vector $\vec{\lambda}\ = \lambda_1 \vec{u}_1 + ... + \lambda_n \vec{u}_n$.

The inverse Fourier transformation

$$W(\vec{z}) = \frac{1}{(2\pi)^{2n}} \int_{R^{2n}} d\vec{\lambda}\, \chi\left(\vec{\lambda}\right) \exp\left(-i\vec{\lambda}^{\mathrm{T}} \cdot\, \vec{z}\right) \tag{37}$$

is called the Wigner function.

The Gaussian quantum state is the state, which the Wigner function has a Gaussian form

$$W(\vec{z}) = \frac{1}{(2\pi)^n \sqrt{Det\, \Gamma}} \exp\left(-\frac{1}{2}\left(\vec{z} - \vec{\xi}\right)^{\mathrm{T}} \Gamma^{-1} \left(\vec{z} - \vec{\xi}\right)\right), \tag{38}$$

where Γ is the covariance matrix and $\vec{\xi}$ is the displacement vector with the property

$$\vec{\xi}^{\mathrm{T}} \cdot \vec{u} = tr\left[\hat{\eta}\hat{q}(\vec{u})\right]. \tag{39}$$

The Gaussian state is invariant relative to the symplectic transformation SL, which means that the Gaussian state remains Gaussian under symplectic transformation.

The value $\frac{1}{Det\, \Gamma}$ is called the purity P of a Gaussian state. The covariance matrix transforms as

$$\tilde{\Gamma} = S\Gamma S^{\mathrm{T}} \tag{40}$$

where $S \in SL$. For a gaussian state to be pure, it is necessary and sufficient that its covariance matrix was a positive symplectic matrix so that $\Gamma = S^{\mathrm{T}} S$, the symplectency of the covariance matrix guarantees the purity of the state.

We assume that symplectic space is R^{2n} equipped with the symplectic form determined by a nonsingular, skew-symmetric matrix in the form

$$\varpi = \underset{i=1,\ldots,n}{\oplus \omega}, \quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{41}$$

so that ϖ is immune to the orthogonal transformations.

Definition 3. The set of all completely positive maps from one Gaussian state to another Gaussian state, which preserves trace, is called a Gaussian channel G.

The Gaussian channel G maps the displacement vector $\vec{\xi}$ and covariance matrix Γ as follows

$$G\ :\ \Gamma \to Z\Gamma Z^{\mathrm{T}} + \Gamma_N \tag{42}$$

$$G\ :\ \vec{\xi} \to Z\vec{\xi} + \vec{\xi}_P, \tag{43}$$

where the matrix Z is a transform and reshaping of the covariance matrix, the matrix Γ_N is Gaussian noise and vector $\vec{\xi}_P$ is additional displacement in phase space. The Gaussian channel G transforms as

$$G\ :\ \exp\left(i\hat{q}\left(\vec{\lambda}\right)\right) \to \exp\left(i\hat{q}\left(Z^{\mathrm{T}}\vec{\lambda}\right) + i\vec{\xi}_P{}^{\mathrm{T}}\vec{\lambda} - \frac{1}{2}\vec{\lambda}\Gamma_N\vec{\lambda}\right) \tag{44}$$

and the mapping of the Wigner function

$$G \,:\, W\left(\vec{z}\right) \to$$
$$\int_{R^{2k}} d\,\vec{x}\,W\left(Z^{-1}\vec{z}-\vec{x}\right)\frac{\exp\left(-\frac{1}{2}\left(\vec{x}-\vec{\xi}_P\right)^{\mathrm{T}}\Gamma_N{}^{-1}\left(\vec{x}-\vec{\xi}_P\right)\right)}{\left(2\pi\right)^k\sqrt{Det\,\Gamma_N}}.$$

The matrices Z and Γ_N must satisfy the following condition

$$\Gamma_N - i\varpi + iZ\varpi Z^{\mathrm{T}} \geq 0, \tag{45}$$

which guarantees $Z\Gamma Z^{\mathrm{T}} + \Gamma_N$ will be the well-defined covariance matrix.

Next, tet us consider a mixed state as a statistical ensemble of pure states with a density matrix as follows

$$\hat{\eta} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \tag{46}$$

where $|\Psi_k\rangle$ is a pure state and p_k is a fraction of the ensemble for each $|\Psi_k\rangle$. Let the variance of the pure state $|\Psi_k\rangle$ be $\Delta_k^2\hat{q}\left(\vec{u}\right)$ and $\Delta^2\hat{q}\left(\vec{u}\right)$ be the variance of the mixed state.

The Heisenberg inequality yields the following estimation

$$\begin{aligned}&\Delta^2\hat{q}\left(\vec{u}\right)\Delta^2\hat{q}\left(\Omega\,\vec{u}\right) \geq \\ &\geq \textstyle\sum_k p_k^2\Delta_k^2\hat{q}\left(\vec{u}\right)\ \Delta_k^2\hat{q}\left(\Omega\,\vec{u}\right) + \sum_{k\neq i} p_k p_i \Delta_k^2\hat{q}\left(\vec{u}\right)\ \Delta_i^2\hat{q}\left(\Omega\,\vec{u}\right) \geq 1.\end{aligned} \tag{47}$$

However, Jensen's inequality renders the estimation

$$\Delta^2\hat{q}\left(\vec{u}\right) \geq \sum_{k\neq i} p_k\Delta_k^2\hat{q}\left(\vec{u}\right)\ . \tag{48}$$

The terms $p_k p_i \Delta_k^2\hat{q}\left(\vec{u}\right)\ \Delta_i^2\hat{q}\left(\Omega\,\vec{u}\right)$ in (47) show that the mixed state can only saturate Heisenberg's inequality when the state is pure so only pure Gaussian states saturate Heisenberg's inequality. Thus, the Heisenberg inequality can be saturated if and only if the covariance matrix is symplectic. The covariance matrix is symplectic

5. Kahler space

Now, let us add in our consideration the metric structure of the physical space-time continuum. A Kahler manifold is a Riemannian manifold equipped with a symplectic structure and with a complex structure. The Kahler structure provides the mathematical framework for the unification of the description of bosonic and fermionic states with the Wigner function in the Gaussian form.

Bosons and fermions can be described by a vector $\xi = \{\vec{x},\ \vec{p}\}$ of $2n$-dimensional phase space and an adjoint vector of observables v. The Riemannian structure is presented by the symmetric covariant metric tensor g_{ab}, its contravariant form G^{ab} such that $g_{ac}G^{cb} = \delta_a{}^b$. The symplectic structure is given by a symplectic form Ω^{ab} and its adjoint ω_{ab}. The complex structure is presented by linear form on the phase space as follows $g_{ac}\Omega^{cb} = J_a{}^b$.

The essential difference between the description of bosonic and fermionic states is hidden in the geometric structure of the space of the observables. To describe the bosonic state, the adjoint to phase space is equipped with the symplectic structure Ω^{ab} and the phase space with its dual form ω_{ab} under the condition $\Omega^{ac}\omega_{cb} = \delta^a{}_b$. In order to describe the fermions state, the phase space is metricized by positive form G^{ab} and on adjoint space metric g_{ab}.

For arbitrary Gaussian state $|\psi\rangle$, we can write

$$\left\langle\psi\left|\hat{\xi}^a\hat{\xi}^b\right|\psi\right\rangle - \left\langle\psi\left|\hat{\xi}^a\right|\psi\right\rangle\left\langle\psi\left|\hat{\xi}^b\right|\psi\right\rangle = \frac{1}{2}G^{ab} + \frac{i}{2}\Omega^{ab}.$$

The bosonic system is commutative and the symplectic form is defined independently from a specific state, the canonic commutation relations are

$$\left[\hat{\xi}^a,\, \hat{\xi}^b\right] = i\,\Omega^{ab}.$$

The fermionic system is anti-commutative and the metric does not depend on the state, and the canonic anticommutation relations are

$$\left\{\hat{\xi}^a,\, \hat{\xi}^b\right\} = G^{ab}.$$

Let us consider the classical bosonic state with one degree of freedom, so $\xi = \{x,\, p\}$. The creation and annihilation operators are $a^\dagger = \frac{1}{\sqrt{2}}\,(x - ip)$ and $a = \frac{1}{\sqrt{2}}\,(x + ip)$. The Gaussian state is defined as such that satisfies the equation $a|\psi\rangle = 0$. The Bogolubov transformation is giving

$$\begin{aligned}&\tilde{a} = \alpha a + \beta a^\dagger\\ &\tilde{a}^\dagger = \alpha^* a^\dagger + \beta^* a.\end{aligned}$$

The communication relations are $\left[a,\, a^\dagger\right] = \left[\tilde{a},\, \tilde{a}^\dagger\right] = 1$, where α and β such that $|\alpha|^2 - |\beta|^2 = 1$. Thus, the Bogolubov transformation can be presented in the form

$$\begin{aligned}&\alpha = exp\,(i\varphi)\cosh\,(r)\\ &\beta = exp\,(i\nu)\sinh\,(r)\,.\end{aligned}$$

Assume an initial state is $|\psi\rangle$ and state after the Bogolubov transformation is denoted by $\left|\tilde{\psi}\right\rangle$, so the Bogolubov transformation from $\left(a,\, a^\dagger\right)$ to $\left(\tilde{a},\, \tilde{a}^\dagger\right)$ induce linear mapping $X^b{}_a$ on the vector space spanned by ξ^a, such that $X^b{}_a\tilde{\xi}^a = \xi^b$. From the invariancy of the commutation relations, for a symplectic Ω, we deduce the following condition

$$\left(X\,\Omega X^{\mathrm{T}}\right)^{ab} = \Omega^{ab}.$$

Let us denote an operator of correlation as $\tilde{G}^{ab} = \left\langle\tilde{\psi}\left|\left\{\xi^a,\, \xi^b\right\}\right|\tilde{\psi}\right\rangle$ then we have $\tilde{G}^{ab} = X^a{}_c G^{cd}\left(X^{\mathrm{T}}\right)_d{}^b = \left(X\,GX^{\mathrm{T}}\right)^{ab}$, which gives the value of the expectation of the operator ξ^a in the state $\left|\tilde{\psi}\right\rangle$ after transformation.

The lineal Bogolubov transformation can be represented by a symplectic matrix

$$\begin{aligned}&[X]^b{}_a =\\ &\begin{bmatrix}\cos\,(\varphi)\cosh\,(r) + \cos\,(\nu)\sinh\,(r) & \sin\,(\varphi)\sinh\,(r) - \sin\,(\nu)\cosh\,(r)\\ \sin\,(\varphi)\cosh\,(r) + \sin\,(\nu)\sinh\,(r) & \cos\,(\varphi)\cosh\,(r) - \cos\,(\nu)\sinh\,(r)\end{bmatrix}\end{aligned}$$

assuming that the initial state corresponds with $G = 1$, we obtain

$$\left[\tilde{G}\right]^{ab} = \begin{bmatrix}\cosh\,(2r) + \cos\,(\varphi+\nu)\sinh\,(2r) & \sin\,(\varphi+\nu)\sinh\,(r)\\ \sin\,(\varphi+\nu)\sinh\,(r) & \cosh\,(2r) - \cos\,(\varphi+\nu)\sinh\,(2r)\end{bmatrix}.$$

Next, we are going to consider the Gaussian state in the case of two fermions. Similar to the bosons, let the creation operator $a^\dagger{}_i$ creates a fermion in a quantum state i, which is described by ψ_i, and annihilation operator creates the corresponding antiparticle. The fermionic operators are defined as

$$x_i = \frac{1}{\sqrt{2}}\left(a^\dagger{}_i + a_i\right)$$

and

$$p_i = \frac{i}{\sqrt{2}}\left(a^\dagger{}_i + a_i\right).$$

The anti-communication relations are $\{x_i,\, x_k\} = \delta_{ik} = \{p_i,\, p_k\}$ and $\{x_i,\, p_k\} = 0$**.** The matrix G in the basis $\xi = \{x,\, p\}$ is an identity matrix $G = 1$. The Gaussian state $|\psi\rangle$ is given by the anti-symmetric correlation operator as

$$\Omega^{ab} = -i \left\langle \psi \left| \left\{ \xi^a ,\, \xi^b \right\} \right| \psi \right\rangle ,$$

if the state $|\psi\rangle$ is annihilated by a_i, Ω^{ab} is symplectic and we have

$$\Omega^{ab} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} .$$

The pair of different Gaussian states can be defined as $a_i|\psi_i\rangle = 0$ and $\tilde{a}_i\left|\tilde{\psi_i}\right\rangle = 0$. The Bogolubov transformation is a linear mapping $\left\{a_i,\ a^\dagger{}_i\right\}$ into $\left\{\tilde{a}_i,\ \tilde{a}^\dagger{}_i\right\}$ (here the parentheses $\{\quad ,\quad \}$ denotes a set). The requirement for the preservation of the anti-commutation relation, we have

$$G^{ab} = X^a{}_c G^{cd} \left(X^{\mathrm{T}}\right)_d{}^b = \left(X\, G X^{\mathrm{T}}\right)^{ab} ,$$

where $\xi^a = X^a{}_c \tilde{\xi}^c$. Then the transformation of the anti-symmetric correlator is

$$\tilde{\Omega}^{ab} = \left(X\, \Omega X^{\mathrm{T}}\right)^{ab} .$$

In the case of single pair, let us define the linear Bogolubov mapping as

$$\begin{aligned} &\tilde{a} = \alpha a + \beta a^\dagger \\ &\tilde{a}^\dagger = \alpha^* a^\dagger + \beta^* a. \end{aligned}$$

From the preserving anti-communication relation $\left\{a_i,\ a^\dagger{}_i\right\}$, we obtain the following conditions $|\alpha|^2 + |\beta|^2 = 1$and $(\tilde{a})^2 = 0 = \left(\tilde{a}^\dagger{}_i\right)$. These conditions lead to the conclusion that the creation and annihilation operators interchange under Bogolubov transformation in the sense $\tilde{a} = a^\dagger$.

For two pairs of creation and annihilation operators $\left\{a_1,\ a^\dagger{}_1\right\}$ and $\left\{a_2,\ a^\dagger{}_2\right\}$ of fermions, we have

$$\begin{aligned} &\tilde{a}_1 = \alpha a_1 - \beta a^\dagger{}_2 \\ &\tilde{a}^\dagger{}_2 = \beta^* a_1 + \alpha^* a^\dagger{}_2, \end{aligned}$$

which corresponds to the Gaussian states $a_i|\psi_i\rangle = 0$ and $\tilde{a}_i\left|\tilde{\psi_i}\right\rangle = 0$. The linear Bogolubov transformation can be represented in the parametrized form as

$$\begin{aligned} &\alpha = \cos(\nu) \\ &\beta = exp\,(i\varphi) \sin(\nu) . \end{aligned}$$

The mapping $\tilde{\xi}^c$ into ξ^c can be represented by the symplectic matrix

$$[X]^b{}_a = \begin{bmatrix} \cos(\nu) & \sin(\nu)\cos(\varphi) & 0 & \sin(\nu)\sin(\varphi) \\ -\sin(\nu)\cos(\varphi) & \cos(\nu) & -\sin(\nu)\sin(\varphi) & 0 \\ 0 & \sin(\nu)\sin(\varphi) & \cos(\nu) & -\sin(\nu)\cos(\varphi) \\ -\sin(\nu)\sin(\varphi) & 0 & \sin(\nu)\cos(\varphi) & \cos(\nu) \end{bmatrix} .$$

The anticommutation relation for the fermionic quantum systems is given by the formula $G^{ab} = \left\{\hat{\xi}^a,\ \hat{\xi}^b\right\}$, form G^{ab} is the symmetric metric on the adjoint to phase space. This transformation satisfies the condition $G^{ab} = \left(X\, G X^{\mathrm{T}}\right)^{ab}$ since this transformation continuously reaches identity transformation. The creation operator changes on annihilation operator at $\nu = \frac{\pi}{2}$ and annihilation on creation operators so that $\{\tilde{a}_1,\ \tilde{a}_2\} = \left\{-\tilde{a}^\dagger{}_2,\ \tilde{a}^\dagger{}_1\right\}$, when $\nu = \frac{\pi}{2}$ and $\varphi = 0$ from one Gaussian state $|\psi\rangle$ to the different Gaussian state $\left|\tilde{\psi}\right\rangle$.

The pure Gaussian state $|\Gamma\rangle$ (bosonic and fermionic) can be described by the linear complex structure Γ as $\frac{1}{2}\left(\delta^a{}_c + i\Gamma^a{}_c\right)\hat{\xi}^c|\Gamma\rangle$ under the condition of homogeneity of the Gaussian state for fermions.

Thus, the structure of the Kahler space is completely defined by the linear complex structure synchronically with the symplectic correlator Ω^{ab} in the case of bosonic state or by the metric G^{ab} for the fermions, for the bosons, the metric is defined as $G^{ab} = \Gamma^a{}_c \Omega^{cb}$, or for fermions, the correlator is given by $\Omega^{ab} = -\Gamma^a{}_c G^{cb}$. Then, we can calculate the covariance matrix

$$\left\langle \Gamma \left| \left\{ \hat{\xi}^a, \, \hat{\xi}^b \right\} \right| \Gamma \right\rangle = \frac{1}{2} \left(G^{ab} + i\Omega^{ab} \right).$$

The Fock space vacuum corresponds to the homogeneous Gaussian state. Assume Γ and $\tilde{\Gamma}$ are pair of Gaussian states, there is the corresponded Fock space vacuum representation, if and only if the Hilbert-Schmidt norm $\left\| \Gamma - \tilde{\Gamma} \right\|_{HS} < \infty$ is bounded.

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