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On Sensitivity for Portfolio Optimisation Based on a High-dimensional Jump-diffusion Merton Model

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Abstract The problem of singularity of the variance-covariance matrix and its impact on the sensitivity of Markowitz portfolio optimization has been extensively studied in the literature when the underlying model does not include jump terms. In this paper, we first use a jump-diffusion multivariate Merton model to evaluate sensitivity of portfolio optimization and apply principal component analysis (PCA) for dimensionality reduction as a solution to singularity of the variance-covariance matrix. Finally, we provide a numerical study based on the adjusted daily closing price of $S\&P$ 500 stocks to explore the impact of the dimension of the reduced space and jump terms on the sensitivity of the portfolio optimization. Empirical experiments confirm that for models without jump terms, the sensitivity analysis may not reflect the correct assessment of the impact of dimensionality reduction on the portfolio optimization.

Keywords Portfolio optimisation, Error-maximisation, Portfolio sensitivity, Jump-diffusion model

AMS 2010 subject classifications 91G10, 91G70

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1. Introduction

In 1952, Markowitz solved the problem of stock portfolio optimization based on expected returns (average) and standard deviation (variance) using parametric quadratic programming (PQP) [11]. Michalowski et al. [14] and Zhang et al. [21] generalized the Markowitz model and measured risk using absolute and downside deviations instead of variance. Ledoit et al.[9] introduced a nonlinear shrinkage estimator in the Markowitz portfolio selection to solve the problem of estimating the covariance matrix of returns. Goto et al. [7] proposed a “sparse” estimator of the inverse covariance matrix by shrinking the size of trades and reducing the number of stocks in each hedge trade. The portfolio obtained using this estimator compares favorably with those obtained by other methods (equal weighting, small covariance matrix, industry operating model, non-negative constraints) and also has a significantly reduced out-of-sample risk and a much better certainty equivalent returns after transaction costs.

One of the basic assumptions of the Markowitz model is investor risk-aversion (see Borodovsky et al. [3]). An insurance risk manager looking for a risk-averse portfolio prefers to manage portfolios with large numbers of stocks to avoid financial crises. However, when the number of shares increases, some become linearly dependent. As a result, the non-singularity assumption of the variance-covariance matrix

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is violated, and the standard portfolio optimization methods are not applicable. As a solution, common linear dimensionality reduction methods such as principal component analysis are suggested.

The Mean-Variance (MV) portfolio optimizer also has a poor empirical performance due to relatively large errors in estimating the means and variances ([6]). Several approaches have been proposed in the literature to address this problem. For example, Best and Grauer [2] studied the variation of an optimal portfolio set with respect to a non-random perturbation in the input parameters, and suggested an upper bound for the sensitivity of such a portfolio. Chopra et al. [4] investigated the relative impact of estimation errors in means, variance, and covariances and showed that it is important to distinguish between errors in variances and covariances. Moreover, he relative impact of errors in means, variance, and covariances also depends on the investor's risk tolerance.

It is well known that the Markowitz mean–variance portfolio optimization problem is a quadratic programming problem whose first-order conditions require the solution of a linear system and that the optimal portfolio weights are sensitive to parameter estimates, particularly the mean return vector. This has generally been attributed to the interaction of estimation error and optimization, but Hurley et al. [8] provided examples to show that the linear system produced by the first-order condition of the Markowitz mean variance portfolio optimization problem is ill-conditioned, and it is in fact this property that causes the optimal weight sensitivity.

An empirical study by Paskaramoorthy et al. [15] shows that the analytical bounds by Best and Grauer [2] significantly overestimate MV portfolio sensitivity. Moreover, they showed that the condition number of the covariance matrix does not necessarily epitomize the sensitivity of the MV optimizer. Another restrictive assumption of the Markowitz model is the Gaussian distribution assumption on assets prices ([17]). Merton [13] showed that under this assumption, trading must occur continuously in time, and the path of stock price dynamics must be continuous with probability one. Since this assumption may not hold in practice, models including jump terms are suggested for modeling stock prices ([19],[10], and [18]). To study these jump terms we first introduce the portfolio optimization problem based on a jump-diffusion Merton model. In contrast to Paskaramoorthy et al. [15] we consider the effect of jump terms in the sensitivity of the MV optimizer. An empirical study shows that when the model includes jump terms, the ratio of the exact amount to the upper bound for sensitivity of the portfolio optimization depends very much on the dimension of the reduced space,but is almost constant when the jump terms are excluded in which case we obtain results similar to those obtained by Paskaramoorthy et al. [15].

In this paper, we first introduce the portfolio optimization problem based on a jump-diffusion Merton model. We then study sensitivity of the portfolio optimization with respect to perturbations to the model parameters. Finally, using an empirical study we explore the influence of the dimension of the reduced space and jump terms on the ratio of the exact amount to the upper bound for the sensitivity of the portfolio optimization. The remainder of the paper is organized as follows. In Section 2, using the multivariate Merton model with jump terms, we discuss the MV portfolio problem with full investment constraints and obtain the upper bound for the sensitivity of the portfolio optimization measured by adding perturbation to the model parameters. In Section 3, we provide numerical results based on the adjusted closing daily price of $S\&P$ 500 stocks from January of 2010 to March of 2021. We also calculate the bounds of the sensitivity of the portfolio performance obtained based on a multivariate jump-diffusion Merton model based on $S\&P$ 500 stocks data. For the reader's convenience, notations used in this paper are displayed in Table 1. Moreover, we provide the proof of the results in Appendix A, B, C.

2. Portfolio Choice under Changes to the Mean under Merton model

Modern portfolio optimization theory was introduced by Markowitz [11], where he considered a portfolio of assets with jointly normally distributed returns, and defined an optimal portfolio as one that has the minimum variance and meets a targeted return. This problem is equivalent to maximizing the return of a portfolio while controlling the variance. Markowitz [12] and Sharpe [16] proposed the MV portfolio

Table 1. Notations used throughout this paper

Symbol	*Definition*
m	Number of risky assets
τ	Investment horizon
$P_j(t)$	Price of the j-th risky asset at time $t,\ j=1,2,\dots,m$
$\mathfrak{m}_j$	Drift of geometric Brownian motion
σ_j	Volatility of the asset $P_j,\ j=1,2,\dots,m$
$N(t)$	Poisson processes with intensity rates ξ
$N_j(t)$	Poisson processes with intensity rates ξ_j
ξ	intensity rate of $N(t)$
$\xi_{Z_{j,1}}$	intensity rates of $N_j(t)$
$\boldsymbol{\xi}_{Z_1}$	$(\xi_{Z_{1,1}},\dots,\xi_{Z_{m,1}})^\top$
$Z_{k,j,0}$	The jump magnitude of the k-th common jump for all assets $P_j\ in\,(0,t],\ k=1,2,\dots,N(t)$
$\mu_{Z_{j,0}}$	$\mathbb{E}(Z_{k,j,0}),\ j=1,2,\dots,m,$
$\sigma^2_{Z_{j,0}}$	$Var(Z_{k,j,0}),\ j=1,2,\dots,m,$
$\boldsymbol{\mu}_{Z_0}$	$(\mu_{Z_{1,0}},\dots,\mu_{Z_{m,0}})^\top$
$Z_{k,j,1}$	The jump magnitude of the k-th individual jump of asset $P_j,\ in\,(0,t],\ k=1,2,\dots,N_j(t)$
$\mu_{Z_{j,1}}$	$\mathbb{E}(Z_{1,j,1}),\ j=1,2,\dots,m,$
$\sigma^2_{Z_{j,1}}$	$Var(Z_{k,j,1}),\ j=1,2,\dots,m,$
$\boldsymbol{\mu}_{Z_1}$	$(\mu_{Z_{1,1}},\dots,\mu_{Z_{m,1}})^\top$
$\mathfrak{h}_{j,0}$	$E[e^{Z_{\iota(t),j,0}}]-1,\ j=1,2,\dots,m,$
$\mathfrak{h}_{j,1}$	$E[e^{Z_{\iota(t),j,1}}]-1,\ j=1,2,\dots,m,$
$\boldsymbol{\mu}$	$(r+\mathfrak{m}_1-\xi\mathfrak{h}_{1,0}-\xi\mathfrak{h}_{1,1},\dots,r+\mathfrak{m}_m-\xi\mathfrak{h}_{m,0}-\xi_m\mathfrak{h}_{m,1})^\top$
$\mathbf{1}$	Vector of ones with length m
$\mathbf{x}$	$(x_1,\dots,x_m)^\top$ fraction of assets the optimal MV efficient portfolio
B'_j	m-dimensional Brownian motions $j=1,2,\dots,m$
r	Interest rate of risk-free asset
$\rho_{i,j}\,t$	$Cov(B'_i(t+s),B'_j(t))$ for $i,j=1,2,\dots,m$ and $\forall s,t\geq 0$
$\boldsymbol{\Sigma}_{m\times m}$	$\Big(\sigma_i\,\sigma_j\,\rho_{i,j}+\xi\,\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}+(\xi\,\sigma^2_{Z_{i,0}}+\xi_i\,\sigma^2_{Z_{i,1}}+\mu^2_{Z_{i,1}}\,\xi_i)I_{\{i=j\}}\Big)_{i,j}$ $i,j=1,2,\dots,m$

problem, which is given by the following optimization problem for risky assets with constraints,

$$\max_{\mathbf{x}\in\mathbb{R}^m}\big\{s\,\boldsymbol{\mu}^\top\mathbf{x}-\frac{1}{2}\mathbf{x}^\top\boldsymbol{\Sigma}\,\mathbf{x}\mid\mathbf{A}^\top\mathbf{x}\leq\mathbf{b}\big\}, \tag{1}$$

where $\mathbf{x}$ gives portfolio weights, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are expectation and variance-covariance matrix of return rates, s represents the risk tolerance of the investor, $\mathbf{A}$ is the matrix of constraints and $\mathbf{b}$ is the vector of constants. Best and Grauer [2] added a non-random perturbation to input parameters in problem (1) and suggested the following optimization problem

$$\max_{\mathbf{x}\in\mathbb{R}^m}\big\{T(\boldsymbol{\mu}+\kappa\,\mathbf{q})^\top\mathbf{x}-\frac{1}{2}\mathbf{x}^\top\boldsymbol{\Sigma}\mathbf{x}\mid\mathbf{1}^\top\mathbf{x}=1\big\}, \tag{2}$$

where T represents the risk tolerance of the investor, $\boldsymbol{\mu}$ represents the fixed component of the mean returns, $\kappa\,\mathbf{q}$ represents the varying component of the mean, $\mathbf{x}$ represents a vector of portfolio weights, $\boldsymbol{\Sigma}$ is the d variance-covariance matrix, and $\mathbf{1}$ is a vector of ones.

In this paper, we consider the MV portfolio optimisation based on a jump-diffusion multivariate Merton model to account for two different types of jumps: common (caused by any reason that causes a price jump for all stock values, for example changes in investment rate, alterations in economic outlook) and uncommon (caused by new and important information that, for example, may only influence prices of certain stocks or cause an imbalance between their supply and demand).

Here, we consider a jump-diffusion model containing one risk-free asset and m risky assets and assume that (for $j = 1, 2, \ldots, m$), the dynamics of risky assets prices follow

$$\begin{aligned}\frac{dP_j(t)}{P_j(t-)} &= (r + \mathfrak{m}_j - \xi\, \mathfrak{h}_{j,0} - \xi_j\, \mathfrak{h}_{j,1})\, dt + \sigma_j\, dB'_j(t) + (e^{Z_{\iota(t-),j,0}} - 1)\, dN(t) \\ &\quad + (e^{Z_{\iota(t-),j,1}} - 1)\, dN_j(t),\end{aligned} \tag{3}$$

where r is interest rate, $\mathfrak{m}_j$ is the drift of geometric Brownian motion, $\sigma_j > 0$ is the volatility of the asset $P_j\,(j = 1, 2, \ldots, m)$ and B'_j are standard Brownian motions. Moreover, $N(t)$ and $N_j(t)$ are independent Poisson processes with intensity rates ξ and ξ_j, respectively. Discounted changes in assets price are divided into two groups: individual changes corresponding to $N_j(t)$ and common changes corresponding to $N(t)$. As in Vanduffel et al. [19], for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N(t)$, $Z_{k,j,0}$ denotes the jump magnitude of the k-th common jump in $(0, t]$ while for $k = 1, 2, \ldots, N_j(t)$, $Z_{k,j,1}$ denotes the k-th individual jump of asset P_j, in $(0, t]$. Other symbols are introduced in Table 1.

By calculating the expected value of the return of terminal wealth process given in (3), we propose a new optimization problem similar to Markowitz method (Best et al. [2] and Paskaramoorthy et al. [15]) as follows:

$$\max_{\mathbf{x}\in\mathbb{R}^m} \big\{T\big[\boldsymbol{\mu} + \xi\, \boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1} + \kappa\, \mathbf{q}\big]^\top \mathbf{x} - \frac{1}{2}\,(\mathbf{x}^\top\, \boldsymbol{\Sigma}\, \mathbf{x}) \mid \mathbf{1}^\top\, \mathbf{x} = 1\big\}, \tag{4}$$

where $\odot$ is Hadamard product (element-wise multiplication) defined for vectors $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ as $\mathbf{a} \odot \mathbf{b} = (a_1\, b_1, a_2\, b_2, \ldots, a_m\, b_m)$, T is the risk tolerance of the investor, $\kappa\, \mathbf{q}$ indicates the varying components of the mean returns, $\mathbf{x}$ is portfolio weights, and $\boldsymbol{\mu}\, , \xi\, , \boldsymbol{\mu}_{Z_0}\, , \boldsymbol{\xi}_{Z_1}\, , \boldsymbol{\mu}_{Z_1}$, and $\boldsymbol{\Sigma}$ are given in Table 1. A detailed discussion on construction of optimisation problem given in (4) is provided in Appendix A. The objective function of optimization problem in (4) is a quadratic function of $\mathbf{x}$ with a linear constraint, hence using the Lagrange method we can show that the solution is given by

$$\begin{aligned}\mathbf{x}(\kappa\, \mathbf{q}) &= \underbrace{\frac{1}{a_3}\, \boldsymbol{\Sigma}^{-1}\, \mathbf{1} + T\,\Big[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} + \xi\, \boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1}) - \frac{a_1}{a_3}\, \boldsymbol{\Sigma}^{-1}\, \mathbf{1}\Big]}_{\mathbf{h}_0} \\ &\quad + \kappa\, \underbrace{T\,\Big[\boldsymbol{\Sigma}^{-1}\, \mathbf{q} - \frac{a_2}{a_3}\, \boldsymbol{\Sigma}^{-1}\mathbf{1}\Big]}_{\mathbf{h}_1},\end{aligned} \tag{5}$$

where $a_1 = \mathbf{1}^\top\, \boldsymbol{\Sigma}^{-1}\,(\boldsymbol{\mu} + \xi\, \boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1})$, $a_2 = \mathbf{1}^\top\, \boldsymbol{\Sigma}^{-1}\, \mathbf{q}$, $a_3 = \mathbf{1}^\top\, \boldsymbol{\Sigma}^{-1}\, \mathbf{1}$. Note that $\mathbf{h}_0$ and $\mathbf{h}_1$ in (5) represent the fixed and varying components of $\mathbf{x}(\kappa\, \mathbf{q})$, respectively. For reader convenience, a detailed discussion on the solution to the optimisation problem given in (5) is provided in Appendix B. The first term in $\mathbf{h}_0$ is a global minimum variance portfolio and the second term is a zero-cost portfolio, in case jump terms are included in the model. Additionally, $\mathbf{h}_1$ is a zero-cost portfolio showing variation of the optimal portfolio due to changes in the mean vector.

In the following, we provide an upper bound for the L^2 norm of the fixed and varying components in (5). We provide the proof of the Lemmas given in the paper in Appendix C.

Lemma 2.1
Upper bounds for the fixed and varying components of (5) are given by

$$||\mathbf{h}_0|| \leq \frac{\lambda_{\max}}{\sqrt{m}\, \lambda_{\min}} + T\, \frac{||\boldsymbol{\mu} + \xi\, \boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1}||}{\lambda_{\min}}\,(1 + \frac{\lambda_{\max}}{\lambda_{\min}}) \tag{6}$$

and

$$||\mathbf{h}_1|| \leq T\,\frac{||\mathbf{q}||}{\lambda_{\min}}\,(1+\frac{\lambda_{\max}}{\lambda_{\min}})\,, \tag{7}$$

respectively, where $\lambda_{\min}$ and $\lambda_{\max}$ are smallest and largest eigenvalues of the variance-covariance matrix, respectively.

In order to identify the role of varying components of the mean returns, i.e. $\kappa\,\mathbf{q}$ in the portfolio optimisation problem given in (4), we first consider the optimization problem without $\kappa\,\mathbf{q}$ and denote the solution to this optimization problem by $\mathbf{x}^\star$. Then, for a fixed risk tolerance T, we investigate the deviation of $\mathbf{x}(\kappa\,\mathbf{q})$ given in (5) from $\mathbf{x}^\star$. We use the L^2 norm as deviance function and provide an upper bound for the deviation between $\mathbf{x}(\kappa\,\mathbf{q})$ and $\mathbf{x}^\star$ as follows.

$$||\mathbf{x}(\kappa\,\mathbf{q})-\mathbf{x}^\star|| \leq \kappa\,||\mathbf{h}_1|| \leq T\,\kappa\,\frac{||\mathbf{q}||}{\lambda_{\min}}\,(1+\frac{\lambda_{\max}}{\lambda_{\min}}).$$

Also the difference between mean and variance of new optimal portfolio (μ_{PF} and σ^2_{PF}) and their initial values ($\mu^\star_{PF}$ and ${\sigma^2_{PF}}^\star$) are given in Equations (8) and (9).

$$|\mu_{PF}-\mu^\star_{PF}| = |(\mathbf{h}_0+\kappa\,\mathbf{h}_1)^\top(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}+\kappa\,\mathbf{q})-\,\mathbf{h}_0^\top(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})|, \tag{8}$$

and

$$|\sigma^2_{PF}-{\sigma^2_{PF}}^\star| = |(\mathbf{h}_0+\kappa\,\mathbf{h}_1)^\top\boldsymbol{\Sigma}\,(\mathbf{h}_0+\kappa\,\mathbf{h}_1)-\mathbf{h}_0^\top\boldsymbol{\Sigma}\,\mathbf{h}_0|. \tag{9}$$

We use (8) and (9) as measures of sensitivity of the portfolio performance and provide upper bounds for these measures in the following lemma.

Lemma 2.2
Upper bound for change in mean and variance of optimal portfolio are given by

$$|\mu_{PF}-\mu^\star_{PF}| \leq \frac{\kappa\,||\mathbf{q}||}{\lambda_{\min}}\left[T\,(1+\frac{\lambda_{\max}}{\lambda_{\min}})(2\,||\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||+\kappa\,||\mathbf{q}||)+\frac{\lambda_{\max}}{\sqrt{m}}\right] \tag{10}$$

and

$$|\sigma^2_{PF}-{\sigma^2_{PF}}^\star| \leq T\,\kappa\,||\mathbf{q}||\,\frac{\lambda_{\max}}{\lambda^2_{\min}}\,(1+\frac{\lambda_{\max}}{\lambda_{\min}})\Big[T\,(1+\frac{\lambda_{\max}}{\lambda_{\min}})\,\big(\kappa\,||\mathbf{q}||+2||\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||\big)+\frac{2\lambda_{\max}}{\sqrt{m}}\Big], \tag{11}$$

respectively.

3. Experimental results

In this section, we use $|\mu_{PF}-\mu^\star_{PF}|$ and $|\sigma^2_{PF}-{\sigma^2_{PF}}^\star|$ to study sensitivity of the portfolio performance for adjusted closing daily stock prices of $S\&P\,500$ from Jan 1^{st}, 2010 to March 1^{st}, 2021. Throughout, we call $|\mu_{PF}-\mu^\star_{PF}|$ and $|\sigma^2_{PF}-{\sigma^2_{PF}}^\star|$ the average and sparsity of sensitivity of the portfolio optimizer, respectively. Our numerical study illustrates that, for a model with jump terms, the dimension of the reduce space has a significant impact on sensitivity of the portfolio optimizer. That is, for the model with jump terms, as dimension of the reduce space increases, sparsity of the sensitivity of portfolio performance significantly increases; however, the sparsity remains similar if we ignore the jump terms.

Example 3.1
For this example we fix the interest rate at $r=0.03$ (see [20] and [5]). Since adjusted closing daily price of stocks are not linearly independent we use principal component analysis (PCA) to obtain the linearly

Table 2. Results for the actual norms $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^\star_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^\star|$.

		n_{eig}					
		5	10	20	35	40	60
	$1/\lambda_{\min}$	$4.2292 \cdot 10^{02}$	$8.5327 \cdot 10^{02}$	$1.4398 \cdot 10^{03}$	$1.9833 \cdot 10^{03}$	$2.2205 \cdot 10^{03}$	$2.9196 \cdot 10^{03}$
	κ	$0.0113 \cdot 10^{00}$	$0.0114 \cdot 10^{00}$	$0.0116 \cdot 10^{00}$	$0.0117 \cdot 10^{00}$	$0.0117 \cdot 10^{00}$	$0.0117 \cdot 10^{00}$
Including jump terms	T	$0.9433 \cdot 10^{00}$	$1.4181 \cdot 10^{00}$	$0.8243 \cdot 10^{00}$	$7.8111 \cdot 10^{01}$	$1.5145 \cdot 10^{00}$	$0.3436 \cdot 10^{00}$
	$\|\|\mathbf{h}_0\|\|$	$0.9646 \cdot 10^{00}$	$1.9474 \cdot 10^{00}$	$2.4252 \cdot 10^{00}$	$2.8372 \cdot 10^{02}$	$5.9281 \cdot 10^{00}$	$1.7237 \cdot 10^{00}$
	$\|\|\mathbf{h}_1\|\|$	$9.1161 \cdot 10^{01}$	$2.5555 \cdot 10^{02}$	$2.7484 \cdot 10^{02}$	$5.0088 \cdot 10^{04}$	$1.1450 \cdot 10^{03}$	$3.2623 \cdot 10^{02}$
	$\|\mu_{PF} - \mu^\star_{PF}\|$	$0.0025 \cdot 10^{00}$	$0.0058 \cdot 10^{02}$	$0.0118 \cdot 10^{00}$	$3.4634 \cdot 10^{00}$	$0.0758 \cdot 10^{00}$	$0.0233 \cdot 10^{00}$
	$\|\sigma^2_{PF} - {\sigma^2_{PF}}^\star\|$	$0.0021 \cdot 10^{00}$	$0.0034 \cdot 10^{00}$	$0.0076 \cdot 10^{00}$	$2.7040 \cdot 10^{02}$	$0.1124 \cdot 10^{00}$	$0.0075 \cdot 10^{00}$
Excluding jump terms	T	$0.0030 \cdot 10^{00}$	$7.6350 \cdot 10^{-4}$	$4.3814 \cdot 10^{-4}$	$8.3110 \cdot 10^{-5}$	$2.1334 \cdot 10^{-4}$	$9.5415 \cdot 10^{-5}$
	$\|\|\mathbf{h}_0\|\|$	$0.5180 \cdot 10^{00}$	$0.3623 \cdot 10^{00}$	$0.3553 \cdot 10^{00}$	$0.1863 \cdot 10^{00}$	$0.3659 \cdot 10^{00}$	$0.2484 \cdot 10^{00}$
	$\|\|\mathbf{h}_1\|\|$	$0.2940 \cdot 10^{00}$	$0.1376 \cdot 10^{00}$	$0.1461 \cdot 10^{00}$	$0.0533 \cdot 10^{00}$	$0.1613 \cdot 10^{00}$	$0.0906 \cdot 10^{00}$
	$\|\mu_{PF} - \mu^\star_{PF}\|$	$0.3014 \cdot 10^{00}$	$0.2951 \cdot 10^{00}$	$0.2080 \cdot 10^{00}$	$0.3031 \cdot 10^{00}$	$0.2095 \cdot 10^{00}$	$0.2418 \cdot 10^{00}$
	$\|\sigma^2_{PF} - {\sigma^2_{PF}}^\star\|$	$1.2076 \cdot 10^{-4}$	$2.6450 \cdot 10^{-5}$	$1.1614 \cdot 10^{-5}$	$5.0235 \cdot 10^{-6}$	$1.1152 \cdot 10^{-5}$	$8.2979 \cdot 10^{-6}$

independent components and denote the the number of PCs by n_{eig}. Then, we estimate parameters of the Merton model for reduced data via PCA using the method of moments. Following Paskaramoorthy et al. [15], we assume $\kappa = ||\boldsymbol{\mu} + \xi\, \boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1}||$ and risk tolerance is assigned to be $T = \frac{1}{|a_1|}$. In the next step, we simulate a random sample of size 100 of random vector $\mathbf{q}$ with dimension n_{eig}, whose marginals have uniform distribution and $||\mathbf{q}|| = 1$. Afterwards, for various value of n_{eig}, we calculate the exact values of $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^\star_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^\star|$ given in (5), (8) and (9), respectively (Table 2). Our findings confirm that the exact value of the average (sparsity) of the sensitivity of the portfolio performance of the multivariate Merton model with jump terms is smaller (larger) than the corresponding value for the Merton model without jump terms. This finding is consistent with the intuition since the Merton model with jump terms is more flexible compared to the Merton model without jump terms, consequently produces less bias and higher variation in estimation.

Minimum, mean, median, mode and maximum values of upper bounds given in (6), (7), (10) and (11) are shown in Table 3. In Table 4, we exclude the jump terms and calculate the minimum, mean, median, mode and maximum values of the upper bounds following Paskaramoorthy et al. [15]. Similar to the results in Table 2, by comparing Tables 3 and 4, we conclude that by adding jump terms, the bounds of the average (sparsity) of the sensitivity of the portfolio performance of multivariate Merton model decreases (increases).

To consider the accuracy of the upper bounds, based on $n_{eig} = 5, 10, 20, 35$, we calculate the ratio of the exact values of $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^\star_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^\star|$ given in Table 2 and corresponding mean values of upper bounds given in Tables 3 and 4. The ratios (Table 5) illustrate that the ratio of the sparsity of the sensitivity remains the same between the model with or without jump terms; However, the ratio of the mean values of sensitivity decreases are significantly smaller for the model with jump terms. In addition, the ratio is highly dependent on n_{eig}, for the model with jump terms.

4. Conclusion

In this paper, we investigated the sensitivity of the portfolio performance bounds considered in Best and Grauer (1991) and Paskaramoorthy (2021). Our numerical results confirm that the ratio of exact value and the upper bound of the sensitivity of portfolio performance depends on the dimension of the reduced space. In addition, for the Merton Model with jump terms, the ratio of mean sensitivity decreases as dimension of reduced space increases, but it stays the same when the model does not include jump terms. Moreover, when the number of non-zero eigenvalues is small the ratio takes values closer to one. As a future plan, it will be interesting to investigate how portfolio sensitivity is affected by imposing some constraints on jump terms.

Table 3. Results for the upper bounds of $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^{\star}_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}|$ by including jump terms in model.

		n_{eig} 5	10	20	35	40	60
$\|\|\mathbf{h}_0\|\|$	min	$2.3496 \cdot 10^{01}$	$3.5357 \cdot 10^{01}$	$0.0211 \cdot 10^{03}$	$0.0196 \cdot 10^{05}$	$0.0386 \cdot 10^{03}$	$0.0092 \cdot 10^{03}$
	mean	$6.4014 \cdot 10^{01}$	$3.2395 \cdot 10^{02}$	$5.4428 \cdot 10^{02}$	$1.0573 \cdot 10^{05}$	$2.4249 \cdot 10^{03}$	$9.3163 \cdot 10^{02}$
	median	$5.5269 \cdot 10^{01}$	$3.4999 \cdot 10^{02}$	$4.7135 \cdot 10^{02}$	$1.0646 \cdot 10^{05}$	$2.3834 \cdot 10^{03}$	$8.6592 \cdot 10^{02}$
	mod	$2.3496 \cdot 10^{01}$	$3.5357 \cdot 10^{01}$	$2.1112 \cdot 10^{01}$	$1.9599 \cdot 10^{03}$	$3.8608 \cdot 10^{01}$	$9.2309 \cdot 10^{00}$
	max	$1.2202 \cdot 10^{02}$	$7.3402 \cdot 10^{02}$	$1.2251 \cdot 10^{03}$	$2.2006 \cdot 10^{05}$	$5.3596 \cdot 10^{03}$	$2.1156 \cdot 10^{03}$
$\|\|\mathbf{h}_1\|\|$	min	$0.2028 \cdot 10^{04}$	$0.3049 \cdot 10^{04}$	$0.0177 \cdot 10^{05}$	$0.0168 \cdot 10^{07}$	$0.0326 \cdot 10^{05}$	$0.0074 \cdot 10^{05}$
	mean	$5.5797 \cdot 10^{03}$	$2.8240 \cdot 10^{04}$	$4.6685 \cdot 10^{04}$	$9.0617 \cdot 10^{06}$	$2.0712 \cdot 10^{05}$	$7.8834 \cdot 10^{04}$
	median	$4.8100 \cdot 10^{03}$	$3.0502 \cdot 10^{04}$	$4.0394 \cdot 10^{04}$	$9.1246 \cdot 10^{06}$	$2.0354 \cdot 10^{05}$	$7.3220 \cdot 10^{04}$
	mod	$2.0280 \cdot 10^{03}$	$3.0490 \cdot 10^{03}$	$1.7722 \cdot 10^{03}$	$1.6794 \cdot 10^{05}$	$3.2561 \cdot 10^{03}$	$7.3870 \cdot 10^{02}$
	max	$1.0671 \cdot 10^{04}$	$6.4073 \cdot 10^{04}$	$1.0522 \cdot 10^{05}$	$1.8861 \cdot 10^{07}$	$4.5800 \cdot 10^{05}$	$1.7931 \cdot 10^{05}$
$\|\mu_{PF} - \mu^{\star}_{PF}\|$	min	$0.7850 \cdot 10^{00}$	$1.1982 \cdot 10^{00}$	$0.7220 \cdot 10^{00}$	$0.0686 \cdot 10^{03}$	$1.3408 \cdot 10^{00}$	$0.3124 \cdot 10^{00}$
	mean	$2.1525 \cdot 10^{00}$	$1.1058 \cdot 10^{01}$	$1.8881 \cdot 10^{01}$	$3.7004 \cdot 10^{03}$	$8.4922 \cdot 10^{01}$	$3.2707 \cdot 10^{01}$
	median	$1.8566 \cdot 10^{00}$	$1.1944 \cdot 10^{01}$	$1.6341 \cdot 10^{01}$	$3.7261 \cdot 10^{03}$	$8.3462 \cdot 10^{01}$	$3.0385 \cdot 10^{01}$
	mod	$0.7850 \cdot 10^{00}$	$1.1982 \cdot 10^{00}$	$0.7220 \cdot 10^{00}$	$6.8585 \cdot 10^{01}$	$1.3408 \cdot 10^{00}$	$0.3124 \cdot 10^{00}$
	max	$4.1121 \cdot 10^{00}$	$2.5077 \cdot 10^{01}$	$4.2534 \cdot 10^{01}$	$7.7020 \cdot 10^{03}$	$1.8776 \cdot 10^{02}$	$7.4353 \cdot 10^{01}$
$\|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}\|$	min	$0.0977 \cdot 10^{03}$	$0.0224 \cdot 10^{04}$	$0.0008 \cdot 10^{05}$	$0.0007 \cdot 10^{09}$	$0.0003 \cdot 10^{06}$	$0.0001 \cdot 10^{05}$
	mean	$9.9628 \cdot 10^{02}$	$2.8815 \cdot 10^{04}$	$8.1268 \cdot 10^{04}$	$2.9339 \cdot 10^{09}$	$1.5350 \cdot 10^{06}$	$2.2778 \cdot 10^{05}$
	median	$6.0313 \cdot 10^{02}$	$2.2220 \cdot 10^{04}$	$4.0268 \cdot 10^{04}$	$2.0708 \cdot 10^{09}$	$1.0350 \cdot 10^{06}$	$1.3575 \cdot 10^{05}$
	mod	$9.7695 \cdot 10^{01}$	$2.2360 \cdot 10^{02}$	$7.8591 \cdot 10^{01}$	$7.0135 \cdot 10^{05}$	$2.6709 \cdot 10^{02}$	$1.4341 \cdot 10^{01}$
	max	$2.6812 \cdot 10^{03}$	$9.7943 \cdot 10^{04}$	$2.7281 \cdot 10^{05}$	$8.8448 \cdot 10^{09}$	$5.2379 \cdot 10^{06}$	$8.1290 \cdot 10^{05}$

Table 4. Results for the upper bounds of $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^{\star}_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}|$ by excluding jump terms following Paskaramoorthy et al. [15].

		n_{eig} 5	10	20	35	40	60
$\|\|\mathbf{h}_0\|\|$	min	$4.6466 \cdot 10^{00}$	$2.0066 \cdot 10^{00}$	$1.3489 \cdot 10^{00}$	$0.8486 \cdot 10^{00}$	$1.0784 \cdot 10^{00}$	$0.8382 \cdot 10^{00}$
	mean	$1.2153 \cdot 10^{01}$	$1.5058 \cdot 10^{01}$	$2.3661 \cdot 10^{01}$	$1.9857 \cdot 10^{01}$	$3.7675 \cdot 10^{01}$	$3.5967 \cdot 10^{01}$
	median	$1.0562 \cdot 10^{01}$	$1.6347 \cdot 10^{01}$	$2.0883 \cdot 10^{01}$	$2.0318 \cdot 10^{01}$	$3.7417 \cdot 10^{01}$	$3.4042 \cdot 10^{01}$
	mod	$4.6466 \cdot 10^{00}$	$2.0066 \cdot 10^{00}$	$1.3489 \cdot 10^{00}$	$0.8486 \cdot 10^{00}$	$1.0784 \cdot 10^{00}$	$0.8382 \cdot 10^{00}$
	max	$2.2841 \cdot 10^{01}$	$3.3178 \cdot 10^{01}$	$5.1705 \cdot 10^{01}$	$3.9348 \cdot 10^{01}$	$8.0802 \cdot 10^{01}$	$7.8326 \cdot 10^{01}$
$\|\|\mathbf{h}_1\|\|$	min	$6.5401 \cdot 10^{00}$	$1.6415 \cdot 10^{00}$	$0.9420 \cdot 10^{00}$	$0.1787 \cdot 10^{00}$	$0.4587 \cdot 10^{00}$	$0.2051 \cdot 10^{00}$
	mean	$1.7994 \cdot 10^{01}$	$1.5204 \cdot 10^{01}$	$2.4815 \cdot 10^{01}$	$9.6417 \cdot 10^{00}$	$2.9176 \cdot 10^{01}$	$2.1893 \cdot 10^{01}$
	median	$1.5511 \cdot 10^{01}$	$1.6422 \cdot 10^{01}$	$2.1471 \cdot 10^{01}$	$9.7086 \cdot 10^{00}$	$2.8672 \cdot 10^{01}$	$2.0334 \cdot 10^{01}$
	mod	$6.5401 \cdot 10^{00}$	$1.6415 \cdot 10^{00}$	$0.9420 \cdot 10^{00}$	$0.1787 \cdot 10^{00}$	$0.4587 \cdot 10^{00}$	$0.2051 \cdot 10^{00}$
	max	$3.4412 \cdot 10^{01}$	$3.4496 \cdot 10^{01}$	$5.5928 \cdot 10^{01}$	$2.0068 \cdot 10^{01}$	$6.4517 \cdot 10^{01}$	$4.9797 \cdot 10^{01}$
$\|\mu_{PF} - \mu^{\star}_{PF}\|$	min	$8.0348 \cdot 10^{00}$	$4.3537 \cdot 10^{00}$	$2.4870 \cdot 10^{00}$	$2.3877 \cdot 10^{00}$	$2.4429 \cdot 10^{00}$	$1.9634 \cdot 10^{00}$
	mean	$2.1711 \cdot 10^{01}$	$3.7200 \cdot 10^{01}$	$5.5477 \cdot 10^{01}$	$8.5864 \cdot 10^{01}$	$1.1993 \cdot 10^{02}$	$1.3504 \cdot 10^{02}$
	median	$1.8770 \cdot 10^{01}$	$4.0253 \cdot 10^{01}$	$4.8349 \cdot 10^{01}$	$8.6995 \cdot 10^{01}$	$1.1831 \cdot 10^{02}$	$1.2631 \cdot 10^{02}$
	mod	$8.0348 \cdot 10^{00}$	$4.3537 \cdot 10^{00}$	$2.4870 \cdot 10^{00}$	$2.3877 \cdot 10^{00}$	$2.4429 \cdot 10^{00}$	$1.9634 \cdot 10^{00}$
	max	$4.1270 \cdot 10^{01}$	$8.3527 \cdot 10^{01}$	$1.2366 \cdot 10^{02}$	$1.7543 \cdot 10^{02}$	$2.6233 \cdot 10^{02}$	$3.0231 \cdot 10^{02}$
$\|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}\|$	min	$3.3371 \cdot 10^{00}$	$0.4840 \cdot 10^{00}$	$0.1002 \cdot 10^{00}$	$0.0359 \cdot 10^{00}$	$0.0288 \cdot 10^{00}$	$0.0152 \cdot 10^{00}$
	mean	$3.2916 \cdot 10^{01}$	$5.3563 \cdot 10^{01}$	$7.0839 \cdot 10^{01}$	$7.6638 \cdot 10^{01}$	$7.8718 \cdot 10^{01}$	$8.9955 \cdot 10^{01}$
	median	$2.0078 \cdot 10^{01}$	$4.1841 \cdot 10^{01}$	$3.6212 \cdot 10^{01}$	$5.5583 \cdot 10^{01}$	$5.4721 \cdot 10^{01}$	$5.5945 \cdot 10^{01}$
	mod	$3.3371 \cdot 10^{00}$	$0.4840 \cdot 10^{00}$	$0.1002 \cdot 10^{00}$	$0.0359 \cdot 10^{00}$	$0.0288 \cdot 10^{00}$	$0.0152 \cdot 10^{00}$
	max	$8.8173 \cdot 10^{01}$	$1.8030 \cdot 10^{02}$	$2.3443 \cdot 10^{02}$	226.6880	$2.6232 \cdot 10^{02}$	$3.1193 \cdot 10^{02}$

Table 5. Results for ratios between the actual norms $||\mathbf{h}_0||$, $||\mathbf{h}_1||$, $|\mu_{PF} - \mu^{\star}_{PF}|$ and $|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}|$ and corresponding upper bounds.

		$\|\|\mathbf{h}_0\|\|$	$\|\|\mathbf{h}_1\|\|$	$\|\mu_{PF} - \mu^{\star}_{PF}\|$	$\|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}\|$
$n_{eig} = 5$	Including jump terms	$0.0151 \cdot 10^{00}$	$0.0163 \cdot 10^{00}$	$0.0012 \cdot 10^{00}$	$0.2108 \cdot 10^{-5}$
	Excluding jump terms	$0.0426 \cdot 10^{00}$	$0.0163 \cdot 10^{00}$	$0.0139 \cdot 10^{00}$	$0.3669 \cdot 10^{-5}$
$n_{eig} = 10$	Including jump terms	$0.0060 \cdot 10^{00}$	$0.0090 \cdot 10^{00}$	$5.2453 \cdot 10^{-4}$	$1.1799 \cdot 10^{-7}$
	Excluding jump terms	$0.0241 \cdot 10^{00}$	$0.0091 \cdot 10^{00}$	$0.0073 \cdot 10^{00}$	$0.4938 \cdot 10^{-6}$
$n_{eig} = 20$	Including jump terms	$0.0045 \cdot 10^{00}$	$0.0059 \cdot 10^{00}$	$6.2497 \cdot 10^{-4}$	$0.0935 \cdot 10^{-6}$
	Excluding jump terms	$0.0150 \cdot 10^{00}$	$0.0059 \cdot 10^{00}$	$0.0037 \cdot 10^{00}$	$1.6395 \cdot 10^{-7}$
$n_{eig} = 35$	Including jump terms	$0.0027 \cdot 10^{00}$	$0.0055 \cdot 10^{00}$	$9.3595 \cdot 10^{-4}$	$0.0922 \cdot 10^{-6}$
	Excluding jump terms	$0.0094 \cdot 10^{00}$	$0.0055 \cdot 10^{00}$	$0.0035 \cdot 10^{00}$	$0.0655 \cdot 10^{-6}$

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A. Appendix

To obtain the optimisation problem (4) which is a MV portfolio optimisation problem based on the multivariate Merton model give in (3), we need to obtain the mean and covariance matrix of the model. Following, Wang [19] the solution to the SDE (3) is given by

$$P_j(t) \quad = \quad e^{(r+\mathfrak{m}_j-\frac{1}{2}\sigma_j^2-\xi\,\mathfrak{h}_{j,0}-\xi_j\,\mathfrak{h}_{j,1})\,t+\sigma_j\,B_j(t)+\sum_{k=1}^{N(t)} Z_{k,j,0}+\sum_{k=1}^{N_j(t)} Z_{k,j,1}} . \tag{12}$$

Therefore, the log return of the j-th $(j = 1, \ldots, m)$ risky asset is given by

$$\begin{aligned} Y_j(t) \quad &= \quad \big(r+\mathfrak{m}_j-\xi\,\mathfrak{h}_{j,0}-\xi_j\,\mathfrak{h}_{j,1}-\frac{1}{2}\,\sigma_j^2\big)+\sigma_j\,\big(B_j(t)-B_j(t-1)\big) \\ &\quad + \sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}+\sum_{k=N_j(t-1)+1}^{N_j(t)} Z_{k,j,1}\,. \end{aligned} \tag{13}$$

Lemma A.1

The expectation of log return of the j-th asset and (i, j)-th $(i, j = 1, 2, \ldots, m)$ element of matrix $\mathbf{\Sigma}$ in Equation (13) are given by

$$\mathbb{E}\big(Y_j(t)\big) \quad = \quad (r+\mathfrak{m}_j-\xi\,\mathfrak{h}_{j,0}-\xi_j\,\mathfrak{h}_{j,1}-\frac{1}{2}\,\sigma_j^2)+\mu_{Z_{j,0}}\,\xi+\mu_{Z_{j,1}}\,\xi_j\,, \tag{14}$$

and

$$Cov\big(Y_i(t),Y_j(t)\big)=\sigma_i\,\sigma_j\,\rho_{i,j}+\xi\,\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}+(\xi\,\sigma^2_{Z_{i,0}}+\xi_i\,\sigma^2_{Z_{i,1}}+\mu^2_{Z_{i,1}}\,\xi_i)I_{\{i=j\}}\,, \tag{15}$$

respectively.

Proof

First we consider Equation (14). By Equation (13),

$$\begin{aligned} \mathbb{E}\big(Y_j(t)\big) \quad &= \quad (r+\mathfrak{m}_j-\xi\,\mathfrak{h}_{j,0}-\xi_j\,\mathfrak{h}_{j,1}-\frac{1}{2}\,\sigma_j^2)+\sigma_j\,\mathbb{E}\big(B_j(t)-B_j(t-1)\big) \\ &\quad +\mathbb{E}\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)+\mathbb{E}\big(\sum_{k=N_j(t-1)+1}^{N_j(t)} Z_{k,j,1}\big)\,. \end{aligned} \tag{16}$$

Since Brownian motion has stationary increments, $\mathbb{E}\big(B_j(t) - B_j(t-1)\big) = 0$. The last two expectation terms in (16) can be calculated as follows.

$$\mathbb{E}\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)=\mu_{Z_{j,0}}\,\xi\,, \tag{17}$$

and

$$\mathbb{E}\Big(\sum_{k=N_j(t-1)+1}^{N_j(t)} Z_{k,j,1}\Big) = \mu_{Z_{j,1}}\,\xi_j\,. \tag{18}$$

By substituting the Equations (17) and (18) in (16) the expectation of $Y_j(t)$ is obtained. Of course, for simplicity, we use the vector form for the expectation on Equation (14). Therefore,

$$\mathbb{E}\big(\mathbf{Y}(t)\big) = \boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1} \odot \boldsymbol{\mu}_{Z_1}\,.$$

Now we calculate (i,j)-th $(i,j = 1,2,\ldots,m)$ element of matrix $\boldsymbol{\Sigma}$. From Equation (13),

$$\begin{aligned} Cov\big(Y_i(t),Y_j(t)\big) \quad = \quad & Cov\Big[\big((r+\mathfrak{m}_i - \xi\,\mathfrak{h}_{i,0} - \xi_i\,\mathfrak{h}_{i,1} - \frac{1}{2}\,\sigma_i^2) + \sigma_i\,(B_i(t)-B_i(t-1)) \\ & + \sum_{k=N(t-1)+1}^{N(t)} Z_{k,i,0} + \sum_{k=N_i(t-1)+1}^{N_i(t)} Z_{k,i,1}\big) \\ & ,\big((r+\mathfrak{m}_j - \xi\,\mathfrak{h}_{j,0} - \xi_j\,\mathfrak{h}_{j,1} - \frac{1}{2}\,\sigma_j^2) + \sigma_j\,(B_j(t)-B_j(t-1)) \\ & + \sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0} + \sum_{k=N_j(t-1)+1}^{N_j(t)} Z_{k,j,1}\big)\Big]\,. \end{aligned} \tag{19}$$

First we calculate the covariance related to Brownian motion.

$$\begin{aligned} & Cov\Big[\sigma_i\,\big(B_i(t)-B_i(t-1)\big),\sigma_j\,\big(B_j(t)-B_j(t-1)\big)\Big] = \sigma_i\,\sigma_j\Big\{Cov\big(B_i(t),B_j(t)\big) \\ & -Cov\big(B_i(t),B_j(t-1)\big) - Cov\big(B_i(t-1),B_j(t)\big) \\ & +Cov\big(B_i(t-1),B_j(t-1)\big)\Big\}\,, \end{aligned}$$

and from $Cov(B_i(t),B_j(l)) = \rho_{i,j}\,\min\{t,l\}$ for $t,l \geq 0$, $i,j = 1,2,\ldots,m$ it turns out

$$Cov\Big[\sigma_i\,\big(B_i(t)-B_i(t-1)\big),\sigma_j\,\big(B_j(t)-B_j(t-1)\big)\Big] = \sigma_i\,\sigma_j\,\rho_{i,j}\,. \tag{20}$$

Therefore, the covariance of the common jumps reduces to

$$\begin{aligned} & Cov\Big[\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,i,0}\big),\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)\Big] \\ & = \mathbb{E}\Big\{Cov\Big[\big(\sum_{k_1=N(t-1)+1}^{N(t)} Z_{k_1,i,0}\big),\big(\sum_{k_2=N(t-1)+1}^{N(t)} Z_{k_2,j,0}\big) \mid N(t-1),N(t)\Big]\Big\} \\ & +Cov\Big\{\mathbb{E}\Big[\sum_{k_1=N(t-1)+1}^{N(t)} Z_{k_1,i,0} \mid N(t-1),N(t)\Big],\mathbb{E}\Big[\sum_{k_2=N(t-1)+1}^{N(t)} Z_{k_2,j,0} \mid N(t-1),N(t)\Big]\Big\} \\ & = \mathbb{E}\Big\{\sum_{k_1=N(t-1)+1}^{N(t)}\ \sum_{k_2=N(t-1)+1}^{N(t)} Cov\big(Z_{k_1,i,0},Z_{k_2,j,0}\big)\Big\} \\ & +Cov\Big[\big(\sum_{k_1=N(t-1)+1}^{N(t)} \mu_{Z_{i,0}}\big),\big(\sum_{k_2=N(t-1)+1}^{N(t)} \mu_{Z_{j,0}}\big)\Big]\,. \end{aligned}$$

Since $Z_{k,i,0}\,, i=1,...,m$ are independent for fixed $k=1,\dots,N(t)$, $Cov\big(Z_{k_1,i,0},Z_{k_2,j,0}\big)=0\,$, which implies

$$
\begin{aligned}
&Cov\Big[\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,i,0}\big),\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)\Big]\\
&=\mathbb{E}\Big\{\sum_{k_1=N(t-1)+1}^{N(t)}\sum_{k_2=N(t-1)+1}^{N(t)} Cov\big(Z_{k_1,i,0},Z_{k_2,i,0}\big)\Big\}I_{\{i=j\}}\\
&+\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}\,Cov\Big[\sum_{k_1=N(t-1)+1}^{N(t)}1,\sum_{k_2=N(t-1)+1}^{N(t)}1\Big]\,.
\end{aligned}
$$

Moreover, $Z_{k,i,0}\,, k=1,\dots,N(t)$ are iid for fixed $i=1,\dots,m$, thus

$$
\begin{aligned}
&Cov\Big[\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,i,0}\big),\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)\Big]=\mathbb{E}\Big\{\sum_{k=N(t-1)+1}^{N(t)} Cov\big(Z_{1,i,0},Z_{1,i,0}\big)\Big\}I_{\{i=j\}}\\
&+\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}\,Cov\Big[N(t)-N(t-1),N(t)-N(t-1)\Big]\\
&=Cov\big(Z_{1,i,0},Z_{1,i,0}\big)\,\mathbb{E}\big(N(t)-N(t-1)\big)I_{\{i=j\}}+\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}\,Var\big(N(t)-N(t-1)\big)\,.
\end{aligned}
$$

Since Poisson process has stationary increments,

$$
Cov\Big[\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,i,0}\big),\big(\sum_{k=N(t-1)+1}^{N(t)} Z_{k,j,0}\big)\Big]=\sigma^2_{Z_{i,0}}\xi I_{\{i=j\}}+\mu_{Z_{i,0}}\,\mu_{Z_{j,0}}\,\xi\,. \tag{21}
$$

Finally, we consider covariance of uncommon jumps.

$$
\begin{aligned}
&Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big]\\
&=\mathbb{E}\Big\{Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\\
&\qquad\mid N_i(t-1),N_i(t),N_j(t-1),N_j(t)\Big]\Big\}\\
&+Cov\Big\{\mathbb{E}\Big[\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\mid N_i(t-1),N_i(t),N_j(t-1),N_j(t)\Big]\\
&\qquad,\mathbb{E}\Big[\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\mid N_i(t-1),N_i(t),N_j(t-1),N_j(t)\Big]\Big\}\\
&=\mathbb{E}\Big\{\sum_{k_1=N_i(t-1)+1}^{N_i(t)}\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Cov\big(Z_{k_1,i,1},Z_{k_2,j,1}\big)\Big\}\\
&+Cov\Big[\sum_{k_1=N_i(t-1)+1}^{N_i(t)}\mu_{Z_{i,1}}\,,\sum_{k_2=N_j(t-1)+1}^{N_j(t)}\mu_{Z_{j,1}}\Big]\,.
\end{aligned}
$$

Since $Z_{k,i,1}\,, k=1,\ldots,N_i(t)$ are iid for fixed $i=1,\ldots,m$,

$$
\begin{aligned}
& Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big] \\
& = \mathbb{E}\Big\{\sum_{k=N_i(t-1)+1}^{N_i(t)} Cov\big(Z_{k,i,1},Z_{k,j,1}\big)\Big\} \\
& + Cov\Big[\sum_{k_1=N_i(t-1)+1}^{N_i(t)} \mu_{Z_{i,1}}\,,\sum_{k_2=N_j(t-1)+1}^{N_j(t)} \mu_{Z_{j,1}}\Big]\,.
\end{aligned}
$$

But $Z_{k,i,1}\,, i=1,...,m$ are independent for fixed $k=1,\ldots,N_i(t)$. Thus, for $i\neq j$, $Cov(Z_{k,i,1},Z_{k,j,1})=0\,$, which implies

$$
\begin{aligned}
& Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big] \\
& = \mathbb{E}\Big\{\sum_{k=N_i(t-1)+1}^{N_i(t)} Cov\big(Z_{k,i,1},Z_{k,i,1}\big)\Big\} I_{\{i=j\}} \\
& + Cov\Big[\sum_{k_1=N_i(t-1)+1}^{N_i(t)} \mu_{Z_{i,1}}\,,\sum_{k_2=N_j(t-1)+1}^{N_j(t)} \mu_{Z_{j,1}}\Big]\,,
\end{aligned}
$$

and

$$
\begin{aligned}
& Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big] \\
= \;& \mathbb{E}\Big\{\sum_{k=N_i(t-1)+1}^{N_i(t)} Cov\big(Z_{1,i,1},Z_{1,i,1}\big)\Big\} I_{\{i=j\}} \\
& + \mu_{Z_{i,1}}\,\mu_{Z_{j,1}}\, Cov\Big[N_i(t)-N_i(t-1),N_j(t)-N_j(t-1)\Big]\,,
\end{aligned}
$$

thus

$$
\begin{aligned}
& Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big] \\
= \;& Cov\big(Z_{1,i,1},Z_{1,i,1}\big)\mathbb{E}\big\{N_i(t)-N_i(t-1)\big\} I_{\{i=j\}} \\
& + \mu_{Z_{i,1}}\,\mu_{Z_{j,1}}\, Cov\Big[(N_i(t)-N_i(t-1)),(N_j(t)-N_j(t-1))\Big]\,.
\end{aligned}
$$

Since $N_j(t)\,, j=1,2,\ldots,m$ are independent Poisson processes,

$$\begin{aligned}
& Cov\Big[\big(\sum_{k_1=N_i(t-1)+1}^{N_i(t)} Z_{k_1,i,1}\big),\big(\sum_{k_2=N_j(t-1)+1}^{N_j(t)} Z_{k_2,j,1}\big)\Big] \\
= \; & Cov\big(Z_{1,i,1},Z_{1,i,1}\big)\,\mathbb{E}\big\{(N_i(t)-N_i(t-1))\big\}I_{\{i=j\}} \\
& +\mu_{Z_{i,1}}\,\mu_{Z_{i,1}}\,Cov\Big[(N_i(t)-N_i(t-1)),(N_i(t)-N_i(t-1))\Big]I_{\{i=j\}} \\
= \; & \sigma^2_{Z_{i,1}}\,\mathbb{E}\big[N_i(1)\big]I_{\{i=j\}}+\mu_{Z_{i,1}}\,\mu_{Z_{i,1}}\,Var\big[N_i(t)-N_i(t-1)\big]I_{\{i=j\}} \\
= \; & \big(\sigma^2_{Z_{i,1}}\,\xi_i\,+\mu^2_{Z_{i,1}}\,\xi_i\big)I_{\{i=j\}}\,. \qquad (22)
\end{aligned}$$

By plug in (20), (21) and (22) into (19) the result is obtained. □

B. Appendix

To obtain the solution to the optimisation problem (4) we use Lagrange method as follows.

$$L(\mathbf{x},\theta)=T\left[(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})+\kappa\,\mathbf{q}\right]^{\top}\mathbf{x}-\frac{1}{2}\,(\mathbf{x}^{\top}\,\boldsymbol{\Sigma}\,\mathbf{x})+\theta\,(1-\mathbf{1}^{\top}\,\mathbf{x})\,.$$

The gradient of $L(\mathbf{x},\theta)$ is given in the following.

$$\frac{\partial L(\mathbf{x},\theta)}{\partial\mathbf{x}}=T\left[(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})+\kappa\,\mathbf{q}\right]-\boldsymbol{\Sigma}\,\mathbf{x}-\theta\,\mathbf{1}\,. \qquad (23)$$

By simplifying the equation $\frac{\partial L(\mathbf{x},\theta)}{\partial\mathbf{x}}=0$, we reach out to the following equation.

$$T\,\boldsymbol{\Sigma}^{-1}\left[(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})+\kappa\,\mathbf{q}\right]-\theta\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}=\mathbf{x}\,,$$

given $\mathbf{1}^{\top}\,\mathbf{x}=1$. To solve the equation, we multiply both sides by $\mathbf{1}^{\top}$, to obtain

$$T\,\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\left[(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})+\kappa\,\mathbf{q}\right]-\theta\,\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\,\mathbf{1}=1\,.$$

Therefore,

$$\theta=T\,\frac{a_1+\kappa\,a_2}{a_3}-\frac{1}{a_3}\,, \qquad (24)$$

where

$$\begin{aligned}
a_1 &= \mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}\right] \\
a_2 &= \mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\,\mathbf{q} \\
a_3 &= \mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\,\mathbf{1}\,.
\end{aligned}$$

By plug in (24) in the Equation (23), we conclude that

$$T\left[(\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1})+\kappa\,\mathbf{q}\right]-\boldsymbol{\Sigma}\,\mathbf{x}-\left[T\,\frac{a_1+\kappa\,a_2}{a_3}-\frac{1}{a_3}\right]\mathbf{1}=\mathbf{0}\,,$$

therefore

$$\mathbf{x}=T\,\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu}+\xi\,\boldsymbol{\mu}_{Z_0}+\boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}\right]+T\,\kappa\,\boldsymbol{\Sigma}^{-1}\,\mathbf{q}-T\frac{a_1}{a_3}\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}-T\,\kappa\,\frac{a_2}{a_3}\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}+\frac{1}{a_3}\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}\,.$$

Therefore, (5) follows.

C. Appendix

Proof of Lemma 2.1:

Proof
By using triangular inequality and (5),

$$||\mathbf{h}_0|| \quad \leq \quad \frac{1}{|a_3|}\,||\boldsymbol{\Sigma}^{-1}\,\mathbf{1}|| + T\left\{||\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}\right)|| + \left|\frac{a_1}{a_3}\right|\cdot||\boldsymbol{\Sigma}^{-1}\,\mathbf{1}||\right\}. \tag{25}$$

Suppose λ_j and $\mathbf{e}_j$ $(j = 1, 2, \ldots, m)$ are eigenvalues and normalized eigenvectors of matrix $\boldsymbol{\Sigma}$, respectively. According to the definition of $\mathbf{h}_0$, $\mathbf{h}_1$, a_1, a_2 and a_3 in Equation (5), first we find the upper bound for norms of $\boldsymbol{\Sigma}^{-1}\,\mathbf{1}$, $\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}\right]$, a_1, a_2 and a_3^{-1}. By spectral decomposition, $|\boldsymbol{\Sigma}^{-1}| = |\sum_{i=1}^{m}\frac{1}{\lambda_i}\mathbf{e}_i\,\mathbf{e}_i^{\top}|$. Therefore, we have

$$\begin{aligned}
&||\boldsymbol{\Sigma}^{-1}\,\mathbf{1}|| = |\big(\sum_{i=1}^{m}\frac{1}{\lambda_i}\mathbf{e}_i\,\mathbf{e}_i^{\top}\big)\,\mathbf{1}| \leq \frac{1}{\lambda_{\min}}||\mathbf{I}\,\mathbf{1}|| = \frac{1}{\lambda_{\min}}||\mathbf{1}|| = \frac{1}{\lambda_{\min}}\sqrt{m},\\
&||\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}\right]|| \leq \frac{1}{\lambda_{\min}}||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||,\\
&|a_1| \leq \left|\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\right|\cdot||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| \leq \frac{\sqrt{m}}{\lambda_{\min}}\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||,\\
&|a_2| = ||\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\,\mathbf{q}|| \leq ||\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}||\cdot||\mathbf{q}|| \leq \frac{1}{\lambda_{\min}}||\mathbf{1}||\cdot||\mathbf{q}|| = \frac{\sqrt{m}}{\lambda_{\min}}\,||\mathbf{q}||,
\end{aligned} \tag{26}$$

where the last two inequalities are follows by Cauchy-Schwartz inequality. We can obtain a lower bound for a_3 as follows.

$$|a_3| = |\mathbf{1}^{\top}\boldsymbol{\Sigma}^{-1}\,\mathbf{1}| = |\mathbf{1}^{\top}\big(\sum_{i=1}^{m}\frac{1}{\lambda_i}\mathbf{e}_i\,\mathbf{e}_i^{\top}\big)\,\mathbf{1}| \geq \frac{1}{\lambda_{\max}}|\mathbf{1}^{\top}\,\mathbf{I}\,\mathbf{1}| = \frac{m}{\lambda_{\max}},$$

or equivalent,

$$|a_3|^{-1} \leq \frac{\lambda_{\max}}{m}. \tag{27}$$

By plug in the obtained upper bounds in (26) and (27) into (25),

$$||\mathbf{h}_0|| \leq \frac{\lambda_{\max}}{\sqrt{m}\,\lambda_{\min}} + T\,\frac{1}{\lambda_{\min}}\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||.$$

Similarly for $\mathbf{h}_1$,

$$||\mathbf{h}_1|| \leq T\left[||\boldsymbol{\Sigma}^{-1}\,\mathbf{q}|| + \left|\frac{a_2}{a_3}\right|\cdot||\boldsymbol{\Sigma}^{-1}\,\mathbf{1}||\right] \leq T\left[\frac{1}{\lambda_{\min}}\,||\mathbf{q}|| + \frac{\sqrt{m}}{\lambda_{\min}}\,||\mathbf{q}||\,\frac{\lambda_{\max}}{m}\,\frac{\sqrt{m}}{\lambda_{\min}}\right]$$

A similar argument can be used to obtain an upper bound for $\mathbf{h}_1$, to complete the proof of the results. □

Proof of Lemma 2.2:

Proof
By using triangular inequality and Equation (8), we have

$$|\mu_{PF} - \mu^{\star}_{PF}| \quad \leq \quad \kappa\,||\mathbf{h}_0||\cdot||\mathbf{q}|| + \kappa\,||\mathbf{h}_1||\left[||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| + \kappa\,||\mathbf{q}||\right].$$

By using (26) and (27),

$$
\begin{aligned}
|\mu_{PF} - \mu^{\star}_{PF}| &\leq \kappa\,||\mathbf{q}||\Big[\frac{\lambda_{\max}}{\sqrt{m}\,\lambda_{\min}} + T\,\frac{1}{\lambda_{\min}}\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}||\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\Big] \\
&\quad + \kappa\,T\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\frac{||\mathbf{q}||}{\lambda_{\min}}\big(||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| + \kappa\,||\mathbf{q}||\big)\,,
\end{aligned}
$$

so

$$
|\mu_{PF} - \mu^{\star}_{PF}| \leq \kappa\,\frac{||\mathbf{q}||}{\lambda_{\min}}\Big[\frac{\lambda_{\max}}{\sqrt{m}} + T\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\big(2\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| + \kappa\,||\mathbf{q}||\big)\Big]\,.
$$

But for calculating upper bound for change in variance, we use triangular inequality and Equation (9) to obtain

$$
\left|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}\right| \leq 2\kappa\left|\mathbf{h}_0^{\top}\boldsymbol{\Sigma}\mathbf{h}_1\right| + \kappa^2\left|\mathbf{h}_1^{\top}\boldsymbol{\Sigma}\mathbf{h}_1\right|,
$$

from spectral decomposition $|\boldsymbol{\Sigma}| = |\sum_{i=1}^{m}\lambda_i\,\mathbf{e}_i{}^{\top}\mathbf{e}_i| \leq \lambda_{\max}\,|\mathbf{I}|$, which result in

$$
\begin{aligned}
\left|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}\right| &\leq 2\kappa\,||\mathbf{h}_0^{\top}\boldsymbol{\Sigma}||\cdot||\mathbf{h}_1|| + \kappa^2\,||\mathbf{h}_1^{\top}\boldsymbol{\Sigma}||\cdot||\mathbf{h}_1|| \\
&\leq 2\kappa\,\lambda_{\max}\,||\mathbf{h}_0||\cdot||\mathbf{h}_1|| + \kappa^2\,\lambda_{\max}\,||\mathbf{h}_1||^2 \\
&= \kappa\,\lambda_{\max}\,||\mathbf{h}_1||\Big[2\,||\mathbf{h}_0|| + \kappa\,||\mathbf{h}_1||\Big]\,. \qquad (28)
\end{aligned}
$$

By plug in (26) and (27) into (28),

$$
\begin{aligned}
|\sigma^2_{PF} - {\sigma^2_{PF}}^{\star}| &\leq \kappa\,\lambda_{\max}\,T\,\frac{1}{\lambda_{\min}}\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\,||\mathbf{q}|| \\
&\quad \times\Big[2\,\frac{\lambda_{\max}}{\sqrt{m}\,\lambda_{\min}} + 2\,T\,\frac{1}{\lambda_{\min}}\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| \\
&\quad + \kappa\,T\,\frac{1}{\lambda_{\min}}\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\,||\mathbf{q}||\Big] \\
&\leq T\,\kappa\,\lambda_{\max}\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big)\,\frac{||\mathbf{q}||}{\lambda^2_{\min}}\Big[\frac{2\,\lambda_{\max}}{\sqrt{m}} + T\,\big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\big) \\
&\quad \times\big(2\,||\boldsymbol{\mu} + \xi\,\boldsymbol{\mu}_{Z_0} + \boldsymbol{\xi}_{Z_1}\odot\boldsymbol{\mu}_{Z_1}|| + \kappa\,||\mathbf{q}||\big)\Big]\,.
\end{aligned}
$$

which completes the proof. □

REFERENCES

1. D. Bauder, T. Bodnar, N. Parolya, and W. Schmid, Bayesian mean–variance analysis: optimal portfolio selection under parameter uncertainty, Quantitative Finance, vol. 21, no. 2, pp. 221–242, 2021.
2. M. J. Best, and R. R. Grauer, On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results, The Review of Financial Studies, vol. 4, no. 2, pp. 315–342, 1991.
3. L. Borodovsky, and M. Lore, Professional's Handbook of Financial Risk Management, Elsevier, 2000.
4. V. K. Chopra, and W. T. Ziemba, The effect of errors in means, variances, and covariances on optimal portfolio choice, In Handbook of the fundamentals of financial decision making: Part I, pp. 365–373, 2013.
5. J. Dhaene, S. Vanduffel, M. J. Goovaerts, R. Kaas, and D. Vyncke, Comonotonic approximations for optimal portfolio selection problems, The Journal of Risk and Insurance, vol. 72, no. 2, pp. 253–300, 2005.
6. V. DeMiguel, L. Garlappi, and R. Uppal, Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?, The Review of Financial Studies, vol. 22, no. 5, pp. 1915–1953, 2009.
7. S. Goto, and Y. Xu, Improving mean variance optimization through sparse hedging restrictions, Journal of Financial and Quantitative Analysis, vol. 50, no. 6, pp. 1415–1441, 2015.

8. W. J. Hurley, and J. Brimberg, A note on the sensitivity of the strategic asset allocation problem, Operations Research Perspectives, vol. 2, pp. 133–136, 2015.
9. O. Ledoit, and M. Wolf, Nonlinear shrinkage of the covariance matrix for portfolio selection: Markowitz meets Goldilocks, The Review of Financial Studies, vol. 30, no. 12, pp. 4349–4388, 2017.
10. S. Lindset, Pricing American exchange options in a jump-diffusion model, Journal of Futures Markets, vol. 27, no. 3, pp. 257–273, 2007.
11. H. Markowitz, Portfolio selection, Journal of Finance, vol. 7, no. 1, pp. 77–91, 1952.
12. H. Markowitz, Portfolio selection: efficient diversification of investments, Wiley, New York, 1959.
13. R. Merton, Theory of rational option pricing, Bell Journal of Economics and Management Science, vol. 4, no. 1, pp. 141–183, 1973.
14. W. Michalowski, and W. Ogryczak, Extending the MAD portfolio optimization model to incorporate downside risk aversion, Naval Research Logistics (NRL), vol. 48, no. 3, pp. 185–200, 2001.
15. A. Paskaramoorthy, and M. Woolway, An Empirical evaluation of sensitivity bounds for mean-variance portfolio optimisation, Finance Research Letters, vol. 44, pp. 102065 ,2022.
16. W. F. Sharpe, Portfolio theory and capital markets, McGraw-Hill, New York, 1970.
17. A. D. Sleire, B. Støve, H. Otneim, G. D. Berentsen, D. Tjøstheim, and S. H. Haugen, Portfolio allocation under asymmetric dependence in asset returns using local Gaussian correlations, Finance Research Letters, vol. 46, pp. 102475, 2021.
18. R. Tompkins, Power options: hedging nonlinear risks, Journal of Risk, vol. 2, no. 2, pp. 29–45, 2000.
19. X. Wang, Pricing power exchange options with correlated jump risk, Finance Research Letters, vol. 19, pp. 90–97, 2016.
20. L. Xu, Ch. Gao, G. Kou, and Q. Liu, Comonotonic approximation to periodic investment problems under stochastic drift, European Journal of Operational Research, vol. 262, no. 1, pp. 251–261, 2017.
21. Y. Zhang, X. Li, and S. Guo, Portfolio selection problems with Markowitz's mean–variance framework: a review of literature, Fuzzy Optimization and Decision Making, vol. 17, no. 2, pp. 125–158, 2018.