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Statistical modelling of cryptocurrencies

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Abstract There has been tremendous interest invested by researchers and academics in Bitcoin since it's introduction to the financial market. However, in recent years there has been an advancement of the cryptocurrency market where other cryptocurrencies such as Ethereum, Litecoin and Ripple have grown relatively quickly and could potentially challenge the dominant placement of Bitcoin. These cryprocurrencies have been utilized globally as a virtual currency for multiple transactions. The returns of cryptocurrencies are known to be volatile and have been observed to fluctuate quite a bit in recent times. This study assesses and differentiates the performance of generalized autoregressive score (GAS) models integrated with a few heavy-tailed distributions in Value-at-Risk (VaR) estimation of the four most popular cryptocurrencies' returns, i.e. Bitcoin returns, Ethereum returns, Litecoin returns and Ripple returns. This paper proposed VaR models for Bitcoin, Ethereum, Litecoin and Ripple returns, i.e. GAS models combined with the generalized hyperbolic distribution (GHD), the variance gamma (VG) distribution, the normal inverse Gaussian (NIG) distribution and the generalized lambda distribution (GLD). The Kupiec likelihood ratio test was adopted to evaluate the proposed models' adequacy and Backtesting VaR was used to select the superior set of models.

Keywords Bitcoin, cryptocurrency, Ethereum, generalized autoregressive score (GAS), generalized hyperbolic distribution (GHD), generalized lambda distribution (GLD), Kupiec likelihood ratio test, Litecoin, normal inverse Gaussian (NIG), Ripple, risk management, Value-at-Risk (VaR), variance gamma (VG)

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1. Introduction

Since Bitcoin's inception in 2008, the cryptocurrency family has been growing both in quantities and functions as Bitcoin's steady growth in market capitalization and price appreciation has influenced the emergence of numerous alternative cryptocurrencies [65]. It has obtained a crucial position in the international financial landscape and the fascination by the media, regulators, government institutions and the general public has been on the rise in recent years. The cryptocurrencies' investing value has also gained enormous support from worldwide investors as a new tool of investment as opposed to the conventional stocks, bonds, EFTs and Futures [76].

Ethereum was first introduced in August 2015 and is a peer-to-peer (P2P) virtual currency, like Bitcoin. However, it contrasts from Bitcoin as Ethereum's cryptocurrency token, Ether possesses no maximum supply. Ethereum has become the second most popular cryptocurrency as it's protocol includes online contractual agreement applications, known as smart contracts, with a very low possibility of downtime, fraud, interference of a third party and censorship. In October 2011, a former Google employee introduced Litecoin. Litecoin has a very similar protocol to that of Bitcoin and has an 84 million unit supply, capped. It was designed to provide a better performing mining process and to allow faster transactions. Although it was designed to be a rival to Bitcoin, Litecoin did not gained the same attention from the public and social media. In the last couple of years, a new set of coins (which include Ripple), alternative to Bitcoin has emerged. Despite Ripple's early invention, it has recently gained value after

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proving itself as a major worldwide settlement network that facilitates transactions between banks. It has been largely used by many banks and organizations such as UniCredit, UBS, and Santander. It boasts a transaction confirmation time of 5 seconds, compared to Bitcoin's slow 10 minutes [76].

Cryptocurrencies have zero exposure to common stock markets as there is no central bank controlling the issuing of it, thus factors like inflation and interest rates have no impact to the cryptocurrency market. The prices of cryptocurrencies is primarily influenced by behavioural factors. Modelling the volatility of cryptocurrencies plays an essential role in risk management and prediction of market losses. Volatility is highly exhibited in cryptocurrencies and tends to behave in a very similar manner to that of other financial time series [53]. There has been numerous studies performed on cryptocurrencies, to date. However, since cryptocurrencies have only been established in 2008, there is still a wide scope for research and analysis to be done to investigate the statistical characteristics exhibited by them.

[60] analysed the capabilities of the normal inverse Gaussian (NIG) distribution, a subset of the generalized hyperbolic (GH) distribution to fit to the returns of Bitcoin. Daily Bitcoin price data from July 2015 to May 2018 were used in the study. The returns of Bitcoin were investigated in comparison to seven major exchange rates. Using a few statistical tests, it was found that the NIG distribution managed to adequately model the Bitcoin returns for all seven exchange rates. Out-of sample VaR and CVaR were also obtained with both the NIG and GH distributions. Even though the results showed minimal differences, the NIG had the higher cumulative density for the expected shortfall. Thus, this paper suggests that the use of the NIG as better was justifiable as the model has fewer parameters to adjust and was able to appropriately capture the highly volatile behaviour of Bitcoin as well as produce excellent out-of-sample VaR and CVaR results.

[20] modelled seven of the most popular cryptocurrencies (excluding Ethereum) using eight statistical distributions. The daily data of each cryptocurrency from June 2014 to February 2018 were utilized in the study. All eight distributions were fitted to the cryptocurrencies' returns. The fits were evaluated using VaR and expected shortfall measures of risk of which the generalized hyperbolic distribution outperformed all the other distributions used in the study.

The flexibility of GH in fitting finacial data is what has made the distribution so appealing in financial studies. [34] fit a GH distribution to DAX index data from 1989 to 1992 and conclude that the GH distrubution produces better approximations for modelling financial returns when compared to the NIG and hyperbolic distribution. However, it was concluded by [8] that the NIG is the better candidate when modelling financial returns. [61] first proposed the application of the GH distribution and it's subsets to the returns of cryptocurrencies. The study statistically analyses seven cryptocurrencies using data from June 2014 to October 2016. It was concluded that the returns of the new financial assets exhibit consistent behaviour with that of financial data and thus the GH distribution and it's family are highly suited to fit to cryptocurrencies.

[71] used heavy-tailed GARCH models and generalized autoregressive score (GAS) models with underlying distributions to model and forecast the risks and returns of Bitcoin. Daily data for the period July 2010 and April 2018 were utilized in the study. Goodness-of-fit tests were applied to evaluate the adequacy of each model and backtesting procedures were conducted to assess the VaR model specifications. It was found that the GAS models with the underlying heavy-tailed distributions were able to outperform the normally distributed GARCH and GAS models. The best conditional and unconditional coverage for the 1% forecasts were produced by the GAS-AST model.

[48] proposed the accelerating generalized autoregressive score (aGAS) technique applied to the Gaussian-Cauchy mixture model. This led to the introduction of a novel time-varying mixture (TVM)-aGAS model. The model was then used to forecast VaR to Bitcoin, Ethereum and Ripple. Daily data from the inception of each cryptocurrency up until June 2015 was used in this paper. The proposed GAS models' results was then compared to that of the classical AR moving average ARMA-GARCH model with various innovations. The out-of-sample VaR forecast GAS results surpassed what had been achieved by the ARMA-GARCH model.

[22] modeled Bitcoin, Ethereum, Litecoin and Ripple using GAS models to investigate the co-dependency nature and portfolio Value-at-Risk of these cryptocurrencies. The time period of the analysis was between January 2016 and December 2021. Multivariate GAS models as well as multivariate DCC (dynamic conditional correlation)-GARCH models were considered in this study. The out-of-sample predictive performance of each model that

was fitted to the respective cryptocurrencies was compared and it was concluded that the GAS model highly outperformed the traditional DCC-GARCH model.

Table 1. Summary of related research on modelling and analysing risk in cryptocurrencies.

Authors	**Data**	**Models**	**Robust models**
Núñez *et al*. (2018)	Bitcoin (Daily data for the period 7 July 2015-16 May 2018)	Normal inverse Gaussian Generalized Hyperbolic	Normal inverse Gaussian
Chan *et al*. (2017)	Bitcoin Dash Litecoin MaidSafeCoin Monero Dogecoin Ripple (Daily data for the period 23 June 2014-28 February 2017)	Student's-*t* Laplace Skew *t* Generalized hyperbolic Normal inverse Gaussian Generalized-*t* Skew Student's-*t* Asymmetric Student's-*t*	Generalized Hyperbolic
Joerg (2017)	Bitcoin Dash Litecoin MaidSafeCoin Monero Dogecoin Ripple (Daily data for the period June 2014- October 2016)	Normal Student's t Skewed generalized Student's t Hyperbolic Generalized Hyperbolic Asymmetric normal inverse Gaussian Asymmetric variance Gamma	Student's t
Troster *et al*. (2018)	Bitcoin (Daily data for the period 19 July 2010-16 April 2018)	GAS-N, GAS-tS GAS-SSTD, GAS-AST GAS-AST1 GARCH, APARCH GJRGARCH, TGARCH CGARCH, NGARCH HGARCH, EGARCH	GAS-AST
Jiang et al. (2022)	Bitcoin Ethereum Ripple (Daily data for the period)	GAS ARMA-GARCH	GAS
Cheng (2023)	Bitcoin Ethereum Litecoin Ripple (Daily data for the period) January 2016-December 2023)	Multivariate GAS(1,1) DCC-GARCH	GAS(1,1)

To the extent of our knowledge, there has been limited research performed on using Generalized hyperbolic distributions and the Generalized lambda distribution as underlying distributions for the GAS model. In spite of the use of GAS models for various cryptocurrencies in [71], [48] and [22], the work done in this paper contributes a comparative VaR analysis on the GAS models with these different underlying heavy-tailed distributions.

This paper is segmented into five sections including the Introduction. The next section describes the models used in this analysis. A discussion on the results achieved can be found in section 3 and 4. Section 5 concludes the article.

2. Econometric models

2.1. GAS models

A stochastic time series model is predominantly represented by the time-variation of its parameters, that are accountable for capturing the dynamic behaviour that the time series process exhibits (in both univariate and multivariate cases). A time series model can either be characterized as an observation-driven model or as a parameter-driven model [26]. Parameter-driven models have been found perplexing to estimate and/or are unable to adequately capture the shape of the conditional distribution of the data. Thus, the observation-driven approach has become well accepted in many applied scientific fields as this approach removes the complexities of likelihood evaluation. Generalized autoregressive score (GAS) models, proposed by [28], are a set of observation-driven time series models. The GAS approach is different from other observation-driven approaches in that the models' driving mechanism is the scaled score of the likelihood function. The GAS specification presents a framework for initiating time-varying parameters in a range of nonlinear models. It encompasses popular observation-driven models such as the GARCH models, autoregressive conditional density models (ACD) introduced by [35], autoregressive conditional multinomial models (ACM) of [64], and Poisson models for count data.

Define $Y^t = \{y_1, y_2, \cdots, y_t\}$, $F^t = \{f_0, f_1, \cdots, f_t\}$ and $X^t = \{x_1, x_2, \cdots, x_t\}$ where x_t is a vector of independent variables.

Let y_t be an $N \times 1$ random vector at time t with conditional distribution:

$$y_t \sim p(y_t | f_t, F_t; \theta), \tag{1}$$

where $F_t = \{Y^{t-1}, F^{t-1}, X^t\}$ for $t = 1, 2, \cdots, n$ and θ is a vector of static parameters.

The time-varying parameter f_t is assumed to be updated by the following autoregressive equation:

$$f_{t+1} = \omega + \sum_{i=1}^{p} A_i s_{t-i+1} + \sum_{j=1}^{q} \beta_j f_{t-j+1}, \tag{2}$$

where ω is a vector of constants. A_i and B_j are matrices of coefficients that have suitable dimensions for i and j and are functions of θ. Furthermore, s_t is an appropriate function of past data, i.e. $s_t = s_t(y_t, f_t, F_t; \theta)$. The distinguishing characteristic of a GAS model is the choice of the innovation s_t as the local score, ∇_t. We specify the innovation as follows:

$$s_t = S_t \cdot \nabla_t, \qquad \nabla_t = \frac{\Delta \ln p(y_t | f_t, F_t; \theta)}{\Delta f_t}, \qquad S_t = S(t, f_t, F_t; \theta), \tag{3}$$

where $S(\cdot)$ is a matrix function.

These equations are defined to be the generalized autoregressive score function with orders p and q, i.e. GAS (p, q).

Updating the time-varying parameter, f_t, with the use of the score function is known for improving the model's fit in terms of the likelihood at time t, with the current position of f_t being realized. The score, ∇_t, is dependent on the complete density of the observations y_t, which differentiates the GAS model framework from many other observation-driven models found in past literature. The structure is highly flexible in a sense that different choices for the scaling matrix S_t can produce many available observation-driven models introduced in literature, as S_t influences how the score is used to update f_t.

In many occurrences, a form of scaling that is dependent on the variance, is considered. This scaling matrix is defined as,

$$S_t = [I_{t|t-1}]^{-\alpha}, \qquad I_{t|t-1} = E_{t-1}[\nabla_t \nabla_t'], \tag{4}$$

where E_{t-1} denotes an expectation with respect to the conditional density and the additional parameter α is fixed and takes on values in the set $\{0, \frac{1}{2}, 1\}$. When $\alpha = 0$, S_t is an identity matrix and there is no scaling. If $\alpha = 1$ of $\alpha = \frac{1}{2}$ then the conditional score is multiplied by the inverse or the square root of its covariance matrix I_t.

For $S_t = [I_{t|t-1}]^{-1}$, the GAS $(1,1)$ model results in the standard GARCH $(1,1)$ model. If $f_t = \sigma^2$, (2.12) reduces to,

$$f_{t+1} = \omega + A_1(y_t^2 - f_t) + B_1 f_t, \tag{5}$$

where $y_t = \sigma_t \epsilon_t$ is the basic model with the Gaussian innovation ϵ_t of zero mean and unit variance. This is equivalent to,

$$f_{t+1} = \alpha_0 + \alpha_1 y_t^2 + \beta_1 f_t, \tag{6}$$

where $\alpha_0 = \omega, \alpha_1 = A_1$ and $\beta_1 = B_1$ are unknown coefficients and need certain restrictions to achieve stationarity.

2.1.1. ***Parameterizations:*** The GAS model framework automatically adapts to the different choice of parameterization of the observation density, as the parameter vector θ_t has a linear specification and is consequently unbounded.

To illustrate how the GAS dynamics easily accommodates various parameterizations, consider $\tilde{f}_t = h(f_t)$ for some continuous and invertible mapping $h(\cdot)$. Let $\dot{h}_t = \frac{\partial h(f_t)}{\partial f_t'}$ which is deterministic, taking into consideration the information set $\mathcal{F}_t$.

For unsophisticated densities,

$$\begin{aligned} \tilde{\mathcal{J}}'_{t|t-1}\tilde{\mathcal{J}}'_{t|t-1} &= (E_{t-1}[(\dot{h}_t^{-1})'\nabla_t\nabla_t'\dot{h}_t^{-1}])^{-1} \\ &= \dot{h}_t\mathcal{I}_{t|t-1}^{-1}\dot{h}_t{}' \\ &= \dot{h}_t\tilde{\mathcal{J}}'_{t|t-1}\tilde{\mathcal{J}}'_{t|t-1}\dot{h}_t{}', \end{aligned} \tag{7}$$

which implies that the information matrix equals the expected outer product of scores as well as the expected second derivative of the log density. The tildes in (2.17) represents the derivatives that are taken with respect to $\tilde{f}_t$ rather than f_t thus $\tilde{\nabla}_t$ is given by:

$$\tilde{\nabla}_t = \frac{\partial \ln p\,(y_t\,|f_t\,F_t;\,\theta)}{\partial \tilde{f}_t} = (\dot{h}_t')^{-1}\nabla_t. \tag{8}$$

The updating step, unique to the GAS model, for $\tilde{f}_t$ with square root information scaling is as follows:

$$\tilde{s}_t = \tilde{\mathcal{J}'}_{t|t-1}\tilde{\nabla}_t = \tilde{\mathcal{J}'}_{t|t-1}(\dot{h'_t})^{-1}\tilde{\mathcal{J}'}^{-1}_{t|t-1} \cdot s_t, \tag{9}$$

considering $s_t = J_{t|t-1} \cdot \nabla_t$.

For the univariate case,

$$\tilde{\mathcal{J}'}_{t|t-1}(\dot{h'_t})^{-1}\tilde{\mathcal{J}'}^{-1}_{t|t-1} = 1.$$

For the multivariate case,

$$(\tilde{\mathcal{J}'}_{t|t-1}(\dot{h'_t})^{-1}\tilde{\mathcal{J}'}^{-1}_{t|t-1})(\tilde{\mathcal{J}'}_{t|t-1}(\dot{h'_t})^{-1}\tilde{\mathcal{J}'}^{-1}_{t|t-1}) = \tilde{\mathcal{J}'}_{t|t-1}(\dot{h'_t})^{-1}I_{t|t-1}(\dot{h'_t})^{-1}\tilde{\mathcal{J}'}^{-1}_{t|t-1}.$$

Thus, the resultant updating step is an orthogonal linear transformation of the original step under the parameterization. Therefore, the choice of parameterization only has a slight impact on the form of the updating step if the scaling matrix $J_{t|t-1}$ is used. However, various implications can arise based on the choice of scaling.

2.1.2. ***Parameter estimation methods- Maximum likelihood estimation:*** Observation-driven models are commonly simple to estimate due to their dynamics being defined directly in terms of observables. Thus, parameter estimation by maximum likelihood (ML) becomes a relatively uncomplicated way of estimating parameters for these types of models. This advantageous trait applies to GAS models as well. The likelihood may be evaluated by prediction-error decomposition, as the model is stated in terms of conditional densities that are only dependent on a finite number of past observations. Computational issues are highly reduced with observation-driven models due to the likelihood function being available in the closed form, which allows evaluation to be straightforward. The maximization problem is described as follows for an observed time series $y_1, \cdots, y_n$ along with implementing the standard prediction error decomposition,

$$\hat{\theta} = \arg\max_{\theta} \sum_{t=1}^{n} \ell_t, \tag{10}$$

where $\ell_t = \ln p(y_t|f_t F_t;\theta)$. Estimating the log-likelihood function of the GAS model only needs the implementation of the GAS updating equation (2.12) and the evaluation of ℓ_t for a specific value θ^* of θ. However, even when the ML estimator is consistent and asymptotically normal, the computational maximization of the likelihood function may present some challenges due to the non-linearities induced by the link function and the way the observations may enter the scaled score. Therefore, it is essential that good starting values be selected to make it possible to formulate proper recursions for computing the gradient of the likelihood regarding the static parameter vector θ. The gradient for the GAS $(1,1)$ model can be computed by using the chain rule, as follows,

$$\frac{\partial \ell_t}{\partial \theta'} = \frac{\partial \ln p_t}{\partial \theta'} + \frac{\partial \ln p_t}{\partial f'_t} \cdot \frac{\partial f_t}{\partial \theta'}, \tag{11}$$

where $p_t = p\ (y_t|\ f_t\ F_t;\ \theta)$ and

$$\frac{\partial f_t}{\partial \theta'} = \frac{\partial \omega}{\partial \theta'} + A_1\,\frac{\partial s_{t-1}}{\partial \theta'} + B_1\,\frac{\partial f_{t-1}}{\partial \theta'} + (s'_{t-1} \otimes I)\,\frac{\partial \vec{A}_1}{\partial \theta'} + (f'_{t-1} \otimes I)\,\frac{\partial \vec{B}_1}{\partial \theta'}, \tag{12}$$

$$\frac{\partial s_{t-1}}{\partial \theta'} = S_{t-1} \cdot \frac{\partial \nabla_{t-1}}{\partial \theta'} + \frac{\partial \vec{S_{t-1}}}{\partial \theta'}, \tag{13}$$

where $\vec{S}$ denotes the vector with the stacked columns of matrix S and $\otimes$ is the Kronecker matrix product. The gradient recursions can then be determined.

The MLE process is used to estimate the parameters of the GAS models fitted to both cryptocurrencies data. The methods implemented for the GAS models in the statistical software platform **R** are available within the **GAS** package.

2.2. Heavy-tailed distributions

2.2.1. ***Class of Generalized hyperbolic distributions*** The family of GH distributions, first introduced by [9], encompasses a number of relevant distributions useful in finance. These distributions include the hyperbolic, the normal, the inverse Gaussian and the variance-gamma among others [33].

The random varaible X follows a generalized inverse Gaussian (GIG) distribution if it's probability density function is given as,

$$h(x;\lambda,\chi,\psi) = \frac{\chi^{-\lambda}\left(\sqrt{\chi\psi}\right)^{\lambda}}{2K_{\lambda}\left(\sqrt{\chi\psi}\right)} x^{\lambda-1}\exp\left(-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right) \tag{14}$$

for $x > 0$, $\chi\psi > 0$ and K_λ is a modified Bessel function of the third kind with index λ.

Thus the multivariate generalized hyperbolic distribution is dependent on three real parameters: χ, ψ, λ, two parameter vectors: the location parameter (μ) and the skewness parameter (γ) in $\mathbb{R}^d$, and $d \times d$ positive matrix Σ, i.e. $X \sim GH_d(\lambda, \chi, \psi, \mu, \gamma, \Sigma)$ [47].

Adjusting the specified parameters in (14) gives rise to the following distributions:

- **Generalized hyperbolic distribution**

  If $\lambda = 1$, the multivariate generalized hyperbolic distribution is achieved with univariate margins that are one dimensional hyperbolic distributions. This one dimensional distribution is widely used to model univariate financial data.

- **Normal inverse Gaussian distribution**

  If $\lambda = -\frac{1}{2}$, the normal inverse Gaussian distribution is produced. The NIG distribution is also highly favoured to model univariate financial returns.

- **Variance Gamma distribution**

  If $\lambda > 0$ and $\chi = 0$, a limiting case referred to as the variance gamma distribution is generated.

The methods applied for the GH distributions in the statistical software platform **R** are available within the package **ghyp**.

2.2.2. ***Generalized lambda distribution*** The generalized lambda distribution (GLD), initially introduced by [63], is one of the most essential generalized classes of distributions. It is a four-parameter (location, scale, kurtosis and skewness) modification of Tukey's lambda distribution [72] that can be manipulated to produce common statistical distributions, such as Gaussian, lognormal, uniform and Weibull, as special cases. The one-parameter lambda distribution provided the foundation for the GLD, which was generalized to generate random variates for simulation studies of Monte Carlo. Since then, the flexibility of the GLD has proved to be a versatile option in multiple fields of research, specifically in modelling a wide range of financial data. However, the parameter estimation process proves to be a downfall of the inclusive distribution, as acquiring appropriate parameters becomes a challenging issue [49].

The traditional parameterizations of the GLD are known as the RS [63] parameterization and the FMKL [39] parameterization.

The FMKL GLD, the four parameter generalization of (2.24) is given by,

$$Q(u) = \lambda_1 + \frac{\frac{u^{\lambda_3 - 1}}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}}{\lambda_2}, \tag{15}$$

where λ_1 is the location parameter, λ_2 is the scale parameter, and, λ_3 and λ_4 are defined as the shape parameters, i.e., λ_3 is a skewness parameter and λ_4 is a kurtosis parameter.

The methods applied for the GLD models in the statistical software platform **R** are available within the package **GLDEX**.

2.3. Combining the GAS model with the GHD and the GLD

The step-by-step method used to create the GAS-GHD, GAS-VG, GAS-NIG and GAS-GLD models are described as follows:

- The first step is to fit a Gaussian GAS model to the Bitcoin, Ethereum, Litecoin and Ripple returns data.
- The next step is to extract the standardized residuals from the fitted Gaussian GAS models.
- The generalized hyperbolic distribution, variance gamma, normal inverse Gaussian and generalized lambda distribution is then fitted to the extracted sets of standardized residuals.
- The last step is to use the Anderson-Darling test to assess the goodness-of-fit of the model.

3. Empirical results

This section describes the results that were generated by the application of the proposed models (in earlier chapters) to the cryptocurrencies' datasets.

3.1. Data source and description

The data used in this analysis is comprised of the daily Bitcoin, Ethereum, Litecoin and Ripple closing prices in United States dollars (USD). The time periods for each cryptocurrency differs from each other due to data availability constraints. However this paper focuses on the capabilities and comparisons of the GAS models with various underlying heavy-tailed distributions and thus the time period differences among the cryptocurrencies do not inflict any comparability issues.

2095 daily observations were used for the Bitcoin closing prices for the period 28/11/2014 to 22/08/2020, 1646 daily observations for the Ethereum closing prices for the period 09/05/2016 to 09/11/2020, 1295 daily observations for the Litecoin closing prices for the period 25/06/2017 to 09/01/2021, 1431 daily observations for the Ripple closing prices for the period 17/01/2017 to 17/12/2020. The datasets were acquired from:
`https://www.cryptodatadownload.com/data/northamerican/`.

Exploratory data analysis was used to understand the characteristics exhibited by the cryptocurrencies' data.
The time series plots of the closing prices (USD) for Bitcoin, Ethereum, Litecoin and Ripple are shown in Figure 1. The plots illustrate the general trend exhibited by the cryptocurrencies for the specified periods above. It is observed that Bitcoin and Ethereum prices fluctuated at relatively small amounts until early 2017. It appeared that after this period, all four cryptocurrencies' prices exhibited a certain momentum and increased until it peaked in 2018. Thereafter, an unsteady trend was exhibited by all four cryptocurrencies, with Bitcoin and Litecoin peaking slightly again in 2019. This volatile behaviour illustrated by the closing prices of Bitcoin, Ethereum, Litecoin and

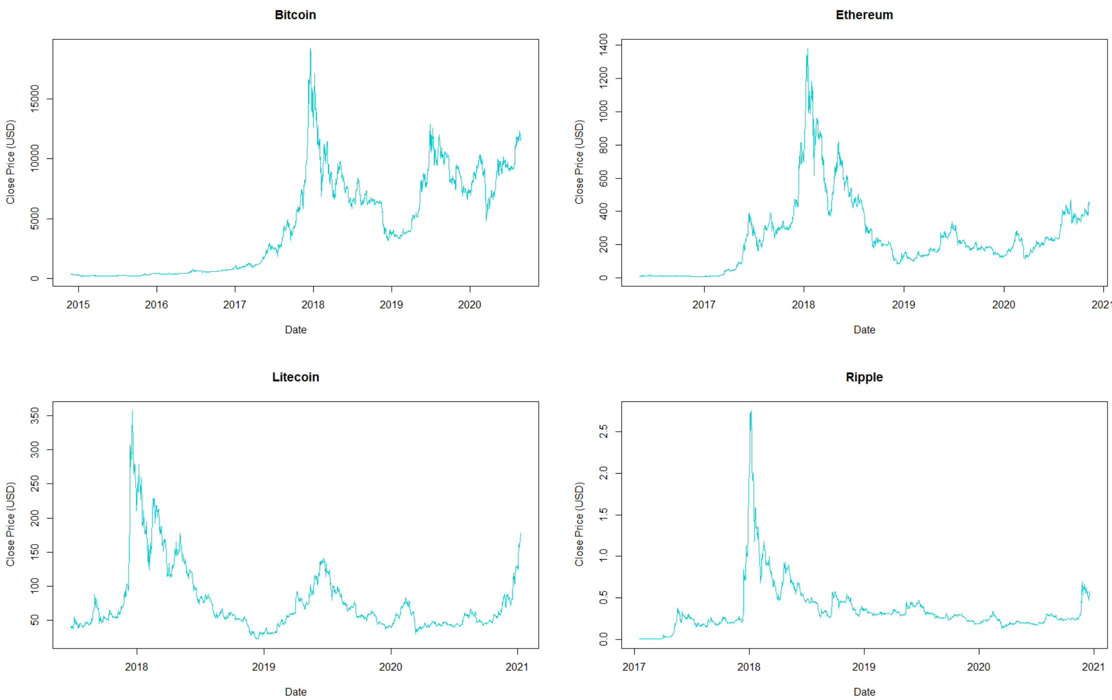

Figure 1. Time series plots of (a) daily Bitcoin prices (USD) for the period 28/11/2014-22/08/2020, (b) daily Ethereum prices (USD) for the period 09/05/2016-9/11/2020, (c) daily Litecoin prices (USD) for the period 25/06/2017-09/01/2021, (d) daily Ripple prices (USD) for the period 17/01/2017-17/12/2020.

Ripple is typical of non-stationarity, which implies non-constant means and high variability.

Returns on investment is of high interest to investors and thus the closing prices of the cryptocurrencies' were transformed to log returns,

$$r_t = \ln \frac{p_t}{p_{t-1}},$$

where r_t is the log return, p_t is the closing price of the cryptocurrency at time t, and p_{t-1} is the closing price of the cryptocurrency at time $t-1$.

The time series plots of the log returns for Bitcoin, Ethereum, Litecoin and Ripple are found in Figure 2.
All four plots in Figure 2 depict a pattern of fluctuations around zero. This indicates that the cryptocurrencies' returns data is now stationary in mean, however the time-varying variance is still observed as there are trends of high and low periods. This behaviour suggests volatility clustering. Leverage effects are also eminent as as negative shocks to returns increase volatility with a larger impact than that of positive shocks.

The descriptive statistics of the log returns of the four daily cryptocurrencies' prices are reported in Table 2. The mean of Bitcoin, Ethereum, Litecoin and Ripple are all slightly greater than zero which implies that the four sets of returns are, to some degree, increasing. Bitcoin and Ethereum appear to be negatively skewed with Bitcoin being more negatively skewed than Ethereum. This indicates that the left tail of both Bitcoin and Ethereum are larger than that of the right. Litecoin and Ripple are found to be positively skewed which in contrast to Bitcoin and Ethereum have larger right tails than that of the left. The negative skewness of Bitcoin and Ethereum suggests

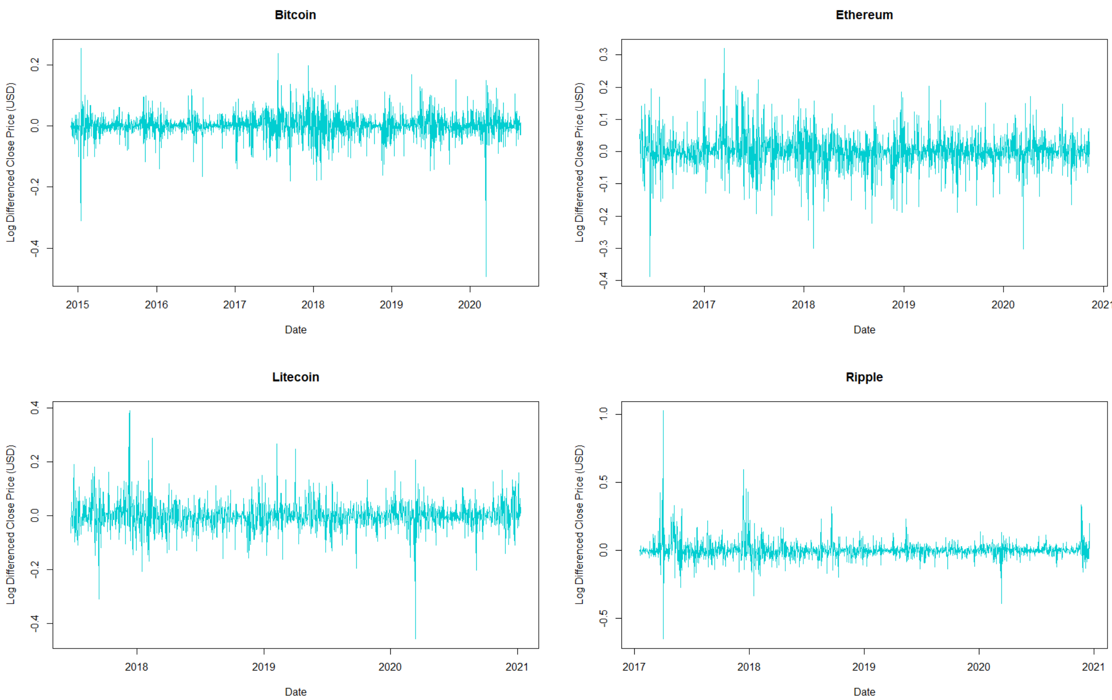

Figure 2. Time series plots of (a) daily Bitcoin log returns for the period 28/11/2014-22/08/2020, (b) daily Ethereum log returns for the period 09/05/2016-9/11/2020, (c) daily Litecoin log returns for the period 25/06/2017-09/01/2021, (d) daily Ripple log returns for the period 17/01/2017-17/12/2020.

Table 2. Descriptive statistics of the log returns of the daily Bitcoin, Ethereum, Litecoin and Ripple prices.

Statistics	Bitcoin	Ethereum	Litecoin	Ripple
Minimum	-0.4940	-0.3878	-0.4574	-0.6530
Maximum	0.2538	0.3194	0.3904	1.0280
Mean	0.0016	0.0024	0.0011	0.0031
STDEV	0.0404	0.0580	0.0581	0.0752
Skewness	-1.0475	-0.1814	0.2808	2.5888
Kurtosis	15.8044	4.2271	7.8965	35.8246

that the returns' losses are greater than the profits for these cryptocurrencies where as the positive skewness of Litecoin and Ripple imply the converse, i.e. the profits are greater that the losses of the returns. All four of the cryptocurrencies have a postive excess kurtosis indicating that the daily returns are heavy-tailed which are consistent with traits of financial data. Figure 3 and 4 presents the density plots and the Q-Q plots, respectively, of the log returns of the cryptocurrecies. Both sets of plots show notable deviations from the normal plot and line. The plots also show that the tails for all four cryptocurrencies are heavier than the tails of a normal distribution. As observed in the density plots, the distributions of the returns substantially differ from that of the normal distribution.

ADF, PP and KPSS tests were done on the daily Bitcoin, Ethereum, Litecoin and Ripple returns to assess the stationarity for all three cases found in Table 3. The resultant p-values for the ADF and PP tests for all four cryptocurrencies are less than 0.05 suggesting the null hypothesis of non-stationarity is rejected at a 5% level of

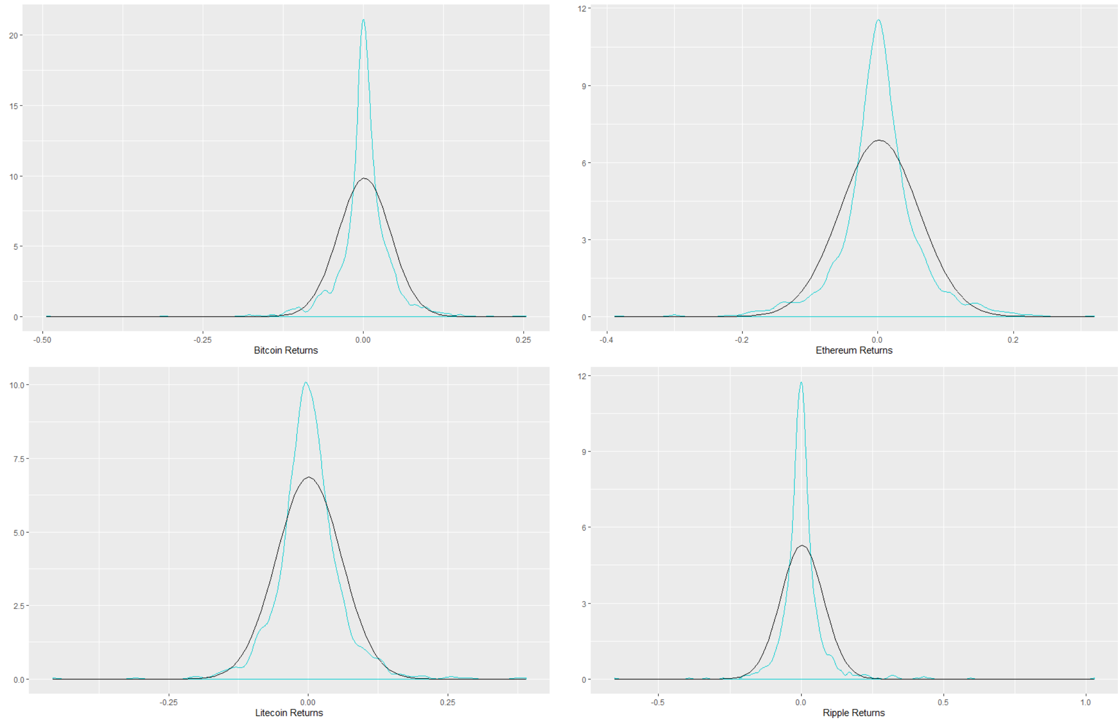

Figure 3. Density plots of the daily (a) Bitcoin, (b) Ethereum, (c) Litecoin and (d) Ripple log returns.

significance. The resultant p-values of the KPSS test are greater than 0.05 supporting the ADF and PP tests of stationarity by failing to reject the null hypothesis of stationarity.

The Jarque-Bera and Shapiro-Wilk tests of normality were used to substantiate the deviations observed in Figure 3 and 4. Both tests resulted in p-values less than 0.05 for the Bitcoin, Ethereum, Litecoin and Ripple returns which indicates the rejection of the null hypothesis of normality at a 5% level of significance.

The time-variation of all four cryptocurrencies' returns were investigated using the Cox-Stuart test. The p-values that resulted from the test for the Bitcoin, Ethereum, Litecoin and Ripple returns were all greater than 0.05. This suggests that the null hypothesis of a non-monotonic trend fails to be rejected at a 5% level of significance and all four sets of returns are in fact independent and identically distributed. This implies that the returns' series are uncorrelated, however the Box-Ljung test and the ARCH-LM test performed on the squared returns of the four cryptocurrencies indicated strong ARCH effects.

In order to execute a sign and size bias test, a symmetric GARCH(1,1) model was fitted to the four sets of data.

A summary of the results of the exploratory data analysis is now completed. The analysis has concluded that the daliy Bitcoin, Ethereum, Litecoin and Ripple returns exhibited empirical properties such as volatility clustering and non-linear dependence, heavy-tails, significant serial correlations in the absolute and squared returns, time variation, leverage effects and ARCH effects.

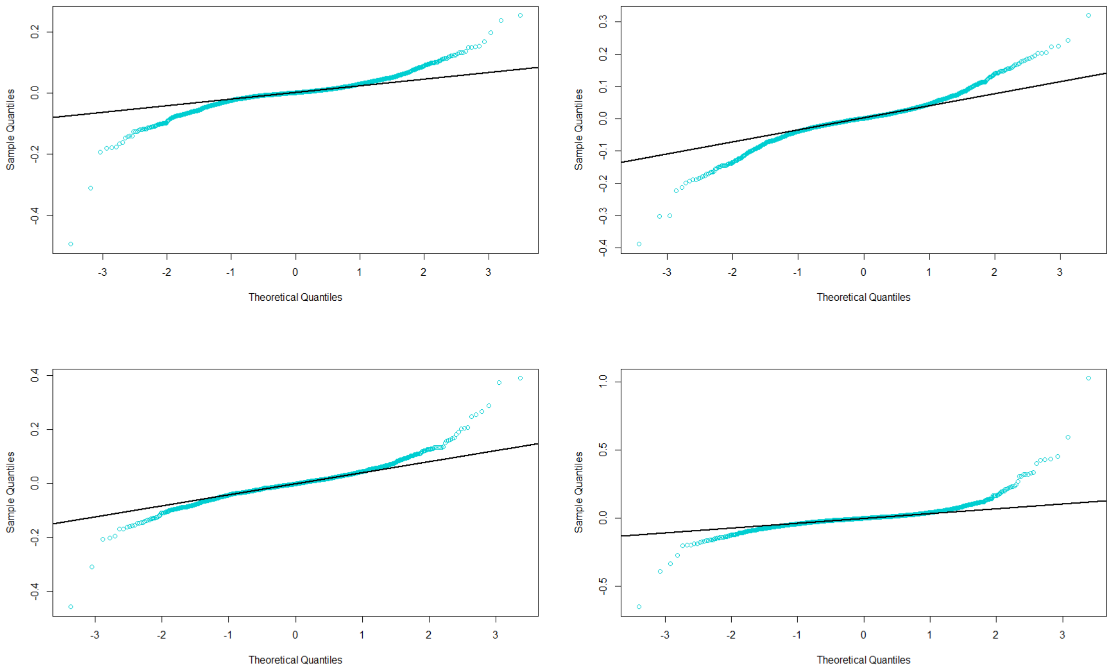

Figure 4. QQ plots of the daily (a) Bitcoin, (b) Ethereum, (c) Litecoin and (d) Ripple log returns.

3.2. *Fitting GAS models with heavy-tailed distributions*

In order to perform a GAS analysis, the first step is to specify an appropriate conditional model. The focal conditional distributions here for the daily Bitcoin, Ethereum, Litecoin and Ripple returns were the Generalized hyperbolic Distribution (GHD), the Variance Gamma distribution (VG), the Normal inverse Gaussian distribution (NIG) and the Generalized Lambda Distribution (GLD). A GAS (1, 1) model was fitted to all four sets of data with different adjustments of the parameters for location, scale, skewness and shape (1 and/or 2) to acquire an adequate fit. These parameters are denoted as μ, ϕ, γ and ν, respectively. The estimates for the parameters of the GAS (1,1) models are reported in Table 4. All parameters of the GAS (1,1) model fitted to the Bitcoin, Ethereum, Litecoin and Ripple returns data were found to be statistically significant at a 10% level of significance.The residuals of the four models were then extracted and analysed.

Table 5 shows the descriptive statistics of the extracted residuals. The mean of the residuals of all four cryptocurrencies appear to be relatively close to zero hence are not significantly differnet from zero. The four sets of residuals appear to have heavy tails as the excess kurtoses are greater than zero. This is observed in the plots in Figure 5, as well. Tests of normality were performed on the residuals and the results are reported in Table 6. The Jarque-Bera and Shapiro-Wilk tests indicate that the residuals are not normally distributed. Figure 5 supports the tests of normality as it appears that residuals' distributions deviates from that of the normal distribution.

The conditional distributions used to model the extracted residuals are obvious candidates as all four models accommodate for the leptokurtic behaviour exhibited by the residuals. A Generalized Hyperbolic distribution and it's two special and limiting cases, the Normal Inverse Gaussian distribution and the Variance Gamma distribution are fitted to the residuals extracted from Gaussian GAS model. A Generalized Lambda distribution with FMKL parameterizations are also fitted to the extracted residuals using the MLE method. The model estimates are reported

Table 3. Formal tests of the log returns of daily Bitcoin prices (BTC/USD), Ethereum prices (ETH/USD), Litecoin prices (LTC/USD) and Ripple prices (XRP/USD).

		Bitcoin		**Ethereum**		**Litecoin**		**Ripple**	
Test		**Test Statistic**	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**
Stationarity	ADF test								
	Case 1: No drift, no trend	-13.0000	0.0100	-11.5000	0.0100	-10.2000	0.0100	-9.5200	0.0100
	Case 2: Drift, no trend	-13.2000	0.0100	-11.5000	0.0100	-10.2000	0.0100	-9.5900	0.0100
	Case 3: Drift and trend	-13.2000	0.0100	-11.6000	0.0100	-10.2000	0.0100	-9.6800	0.0100
	PP test								
	Case 1: No drift, no trend	-2290.0000	0.0100	-1800.0000	0.0100	-1431.0000	0.0100	-1768.0000	0.0100
	Case 2: Drift, no trend	-1795.0000	0.0100	-1792.0000	0.0100	-1430.0000	0.0100	1765.0000	0.0100
	Case 3: Drift and trend	-2282.0000	0.0100	-1242.0000	0.0100	-1430.0000	0.0100	1760.0000	0.0100
	KPSS test								
	Case 1: No drift, no trend	1.5000	0.0671	1.4400	0.0736	0.1240	0.1000	1.4000	0.0774
	Case 2: Drift, no trend	0.3040	0.1000	0.3000	0.1000	0.1350	0.1000	0.3980	0.0781
	Case 3: Drift and trend	0.1320	0.0760	0.1570	0.0411	0.1270	0.0850	0.1680	0.0314
Normality	Jarque-Bera test	22226.0000	< 0.0001	1238.9000	< 0.0001	3393.4000	< 0.0001	78302.0000	< 0.0001
	Shapiro-Wilk	0.8751	< 0.0001	0.9388	< 0.0001	0.9213	< 0.0001	0.7506	< 0.0001
Time variation	Cox-Stuart test	1047.0000	0.2164	822.0000	0.9167	647.0000	0.1569	715.0000	1.0000
Leverage	Sign Bias test	9.1598	0.0272	3.0611	0.3823	7.9619	0.0468	1.5115	0.6796
Autocorrelation	Ljung-Box test	10.5430	0.1598	0.7876	0.3748	1.7099	0.1910	5.4001	0.0611
ARCH effects	Box-Ljung test	10.5430	0.1598	0.7876	0.3748	1.7099	0.1910	5.4001	0.0611
	ARCH-LM test	72.1950	< 0.0001	55.1120	< 0.0001	69.3130	< 0.0001	138.6500	< 0.0001

in Table 7. It appears that all the fitted models are good fits at a 5% level of significance, according to the Anderson-Darling test results found in Table 8.

4. VaR estimation

In finance, estimating risk measures is of great importance as these estimations are essential for traders to assess the risks that are associated with their portfolios' future values, which allows for the consideration of any potential losses. One of the standard risk measures is VaR. It is estimated at long and short positions. Generally, traders who are selling, i.e. traders at a short position, will incur a loss if the price increases and traders who are buying, i.e. traders who are at a long position, will suffer a loss if the price drops. The VaR estimates for short and long positions are reported in Table 9, which are associated with the right and left quantiles of the returns distribution, respectively. The risk levels 1%, 2.5%, 5%, 95%, 97% and 99% were accounted for.

For the Bitcoin and Ethereum returns, it appears that the GAS-GLD resulted in the lowest VaR estimates for the 1% long position and the 95% and 97.5% short positions and had the highest VaR estimates for the long and short position of 5% and 99%, respectively. For the Litecoin returns, the GAS-GLD produced the smallest VaR estimates at the 95% and 99% short positions and the highest VaR estimates at the long positions 2.5% and 5%. For the Ripple returns, the smallest VaR estimate was at the short position of 95% produced by the GAS-GLD.

Table 4. Estimation of results of the Gaussian GAS(1,1) model for the Bitcoin, Ethereum, Litecoin and Ripple daily returns.

	Parameters	Estimate	Std	p-value
Bitcoin	$\hat{\omega_\mu}$	0.0023	0.0007	0.0009
	$\hat{\omega_\phi}$	-0.5533	0.0300	< 0.0001
	$\hat{a_\phi}$	0.1898	0.0198	< 0.0001
	$\hat{b_\phi}$	0.9138	0.0047	< 0.0001
Ethereum	$\hat{\omega_\mu}$	0.0017	0.0013	0.0967
	$\hat{\omega\phi}$	-0.6802	0.0489	< 0.0001
	$\hat{a_\phi}$	0.1898	0.0208	< 0.0001
	$\hat{b_\phi}$	0.88074	0.0086	< 0.0001
Litecoin	$\hat{\omega_\mu}$	2.2726×10^{-5}	4.1414×10^{-5}	< 0.0001
	$\hat{\omega\phi}$	-8.6745×10^{-1}	9.9656×10^{-2}	< 0.0001
	$\hat{a_\mu}$	1.000×10^{-6}	1.4053×10^{-10}	< 0.0001
	$\hat{a_\phi}$	1.8975×10^{-1}	2.5885×10^{-2}	< 0.0001
	$\hat{b_\mu}$	9.8000×10^{-1}	6.8986×10^{-6}	< 0.0001
	$\hat{b_\phi}$	8.4759×10^{-1}	1.7532×10^{-2}	< 0.0001
Ripple	$\hat{\omega_\mu}$	-0.0021	0.0013	0.0492
	$\hat{\omega_\phi}$	-0.4466	0.0428	< 0.0001
	$\hat{a_\phi}$	0.3793	0.0336	< 0.0001
	$\hat{b_\phi}$	0.9138	0.0080	< 0.0001

Table 5. Descriptive statistics of the residuals extracted from the Gaussian GAS model fitted to the log returns of the daily Bitcoin (BTC/USD), Ethereum (ETH/USD), Litecoin (LTC/USD) and Ripple (XRP/USD) prices.

Statistics	Bitcoin	Ethereum	Litecoin	Ripple
Minimum	-11.6079	-5.8711	-7.0403	-4.8337
Maximum	6.4033	4.8058	5.5628	5.8590
Mean	-0.0220	0.0072	-0.0114	0.0457
STDEV	0.9564	0.9665	0.9716	0.9175
Skewness	-1.0209	-0.2346	0.0801	0.7033
Kurtosis	14.1172	3.6560	5.7186	6.6552

Table 6. Normality tests of the residuals extracted from the Gaussian GAS model fitted to the daily Bitcoin, Ethereum, Litecoin and Ripple log returns

	Bitcoin		**Ethereum**		**Litecoin**		**Ripple**	
	Test Statistic	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**
Jarque-Bera test	17793.0000	< 0.0001	935.3600	< 0.0001	1772.9000	< 0.0001	2767.9000	< 0.0001
Shapiro-Wilk	W = 0.8961	< 0.0001	W = 0.9501	< 0.0001	W = 0.9395	< 0.0001	W = 0.9018	< 0.0001

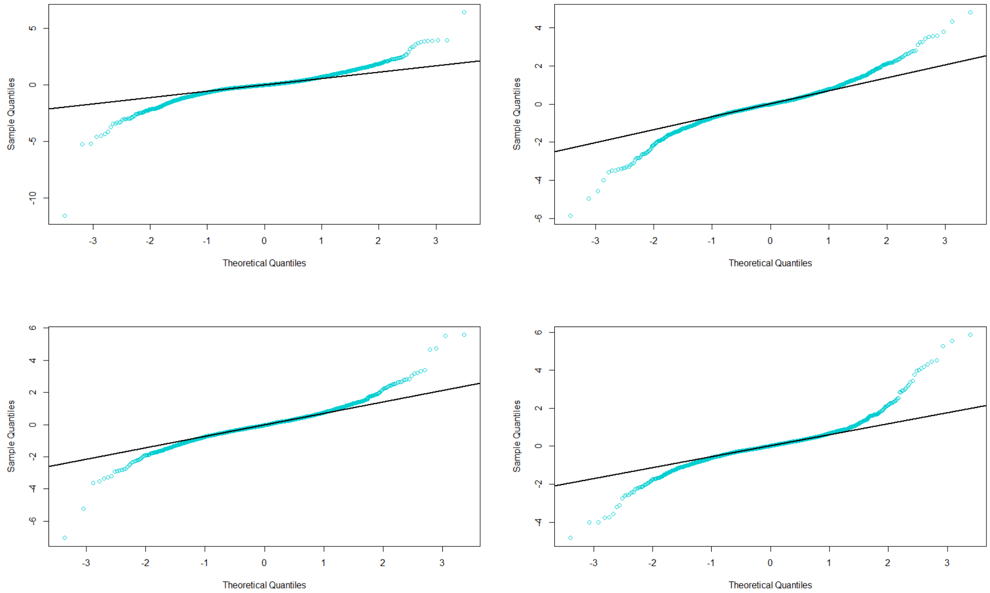

Figure 5. QQ plots of the residuals (a) Bitcoin, (b) Ethereum, (c) Litecoin and (d) Ripple log returns.

Table 7. Models fitted to the extracted Bitcoin, Ethereum, Litecoin and Ripple residuals from the Gaussian GAS model.

Distribution	Parameter estimates	Bitcoin	Ethereum	Litecoin	Ripple
GHD	$\hat{\mu}$	-0.0190	-0.0068	-0.0567	-0.0050
	$\hat{\delta}$	0.1900	0.2717	0.9225	0.6307
	$\hat{\alpha}$	1.0378	1.2206	0.6233	0.5209
	$\hat{\beta}$	-0.0032	0.0150	0.0483	0.0599
	$\hat{\lambda}$	0.3374	0.5379	-0.9301	-0.7345
VG	$\hat{\mu}$	-0.0285	-0.0338	-0.0878	-0.0135
	$\hat{\delta}$	0.0100	0.0100	0.0100	0.0100
	$\hat{\alpha}$	1.3265	1.4049	1.6344	1.5545
	$\hat{\beta}$	0.0001	0.0444	0.0865	0.0791
	$\hat{\lambda}$	0.7374	0.9099	1.1784	0.9019
NIG	$\hat{\mu}$	-0.0025	0.0001	-0.0583	-0.0052
	$\hat{\delta}$	0.5258	0.7021	0.7730	0.5504
	$\hat{\alpha}$	0.5614	0.7316	0.8352	0.6657
	$\hat{\beta}$	-0.0208	0.0444	0.0507	0.0613
	$\hat{\lambda}$	-0.5000	-0.5000	-0.5000	-0.5000
GLD	$\hat{\lambda}_1$	-0.0060	0.0065	-0.0317	0.0150
	$\hat{\lambda}_2$	3.5495	2.7935	2.6397	3.4814
	$\hat{\lambda}_3$	-0.3313	-0.2142	-0.1591	-0.2484
	$\hat{\lambda}_4$	-0.3007	-0.2121	-0.1952	-0.3078

This distribution also had the largest VaR estimates at the short positions of 2.5% and 5% and at the long position of 99%. The GAS-VG model produced the lowest VaR estimates for the Bitcoin returns at the 99% short position,

Table 8. AD tests of the models fitted to the extracted residuals of the Gaussian GAS model from the daily Bitcoin, Ethereum, Litecoin and Ripple log returns

	Bitcoin		**Ethereum**		**Litecoin**		**Ripple**	
	Test Statistic	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**	**Test Statistic**	**p-value**
Generalized hyperbolic	0.9224	0.4007	0.2617	0.9639	0.1092	0.9999	0.2067	0.9885
Variance Gamma	1.6448	0.1454	0.5243	0.7222	0.4223	0.8263	1.2707	0.2421
Normal inverse Gaussian	1.4535	0.1880	0.4054	0.8433	0.1127	0.9999	0.2477	0.9716
Generalized Lambda	2.0109	0.0906	0.6314	0.6188	0.1225	0.9998	0.2034	0.9895

Table 9. VaR estimates for the returns at long and short positions

	Distribution	**Long position**			**Short position**		
		1%	**2.5%**	**5%**	**95%**	**97.5%**	**99%**
Bitcoin	GAS-VG	*-2.6455*	*-1.9960*	-1.5116	1.4550	1.9396	**2.5891**
	GAS-GHD	-2.7882	-2.0489	**-1.5131**	*1.4641*	*1.9965*	2.7313
	GAS-NIG	-2.9674	**-2.0608**	-1.4667	1.3863	1.9441	2.7920
	GAS-GLD	**-3.0633**	-2.0350	*-1.4353*	**1.3489**	**1.8908**	*2.7964*
Ethereum	GAS-VG	*-2.6144*	*-1.9946*	-1.5277	*1.5992*	*2.0967*	2.7571
	GAS-GHD	-2.6923	**-2.0266**	**-1.5340**	1.5690	2.0738	**2.7562**
	GAS-NIG	-2.7707	-2.0173	-1.4979	1.5232	2.0518	*2.8194*
	GAS-GLD	**-2.8002**	-1.9961	*-1.4786*	**1.4863**	**2.0002**	2.7974
Litecoin	GAS-VG	*-2.4775*	-1.9256	**-1.5050**	*1.5704*	*2.0376*	**2.6508**
	GAS-GHD	**-2.6201**	-1.9388	-1.4726	1.5073	2.0285	*2.7998*
	GAS-NIG	-2.6167	**-1.9533**	-1.4863	1.5260	2.0443	2.7850
	GAS-GLD	-2.6012	*-1.9233*	*-1.4662*	**1.4907**	**2.0052**	2.7919
Ripple	GAS-VG	*-2.2842*	-1.7353	**-1.3221**	*1.4967*	**1.9545**	**2.5624**
	GAS-GHD	**-2.4843**	-1.7479	-1.2671	1.4418	2.0129	2.9043
	GAS-NIG	-2.4814	**-1.7651**	-1.2833	1.4627	*2.0271*	2.8743
	GAS-GLD	-2.4559	*-1.7124*	*-1.2476*	**1.4135**	1.9790	*2.9297*

[a] Bold values are the lowest VaR estimates and italic values are the highest VaR estimates.

for the Litecoin returns at the long position of 5% and the short position of 99% and for the Ripple returns at the long position of 5% and the short positions of 97.5% and 99%. The GAS-VG model also produced the largest VaR estimates at the long position of 1% for all four cryptocurrencies' returns. For Bitcoin and Ethereum returns, the highest VaR estimates for the long position 2.5% was produced by the GAS-VG. This model had the largest VaR estimates at the short positions 95% and 97.5% for both the Ethereum and Litcoin returns, as well, whereas for the Ripple returns only at the 95% level. For the Bitcoin returns, the GAS-GHD generated the lowest estimate at the 5% long position and the highest estimate at both the 95% and 97.5% short positions. For the Ethereum returns, the lowest VaR estimates were produced by the GAS-GHD model at the 2.5%, 5% and 99% risk levels. For the Litecoin and Ripple returns, the GAS-GHD generated the smallest estimate at the long position of 1% and the highest estimate for just the Litecoin returns at the short position of 99%. The GAS-NIG model had the lowest VaR estimates for the Bitcoin, Litecoin and Ripple returns at the long position of 2.5%. It also produced the highest estimates at the 99% risk level for Ethereum and at the 97.5% risk level for Ripple.

It appears that the models performed very similar for the Bitcoin and Ethereum returns at the 1% and 5% long positions. The models also performed similarly for the Ethereum and Litecoin returns at the short positions of 95% and 97.5%. This similar model performance is also found at the 1%, 2.5% and 95% risk levels for the Litecoin and Ripple returns as well as at the 99% level for Bitcoin and Ripple returns.

4.1. In-sample backtesting

Kupiec likelihood ratio test (1995) was used to perform in-sample backtesting to examine the model adequacy in VaR estimation. The p-values for the in-sample data produced by the Kupiec likelihood ratio test at different levels of risk are found in Table 10. The robustness of the models were based on the highest p-values at each risk level. The p-values less than 0.05 suggests that the null hypothesis of model adequacy is to be rejected at a 5% level of significance at the considered risk levels. However, it appears that all models are adequate at this specified α-quantile except for the GAS-VG model fitted to the Bitcoin returns.

For Bitcoin, the GAS-NIG outperformed the other 3 models at the 1% long position and the 95% short position. At the 2.5% level, the GAS-GHD, GAS-NIG and GAS-GLD all had the highest p-value. The GAS-GHD performed the best at the 5% level. The GAS-GLD appears to have best performed at the 97.5% short postion and the GAS-VG had the highest p-value at the 99% level.

For the Ethereum returns, the GAS-GLD had the highest p-values for all of the levels of the long-positions. All the models appear to have produced the highest p-value at the 2.5% long position. The GAS-NIG model had the largest p-value at the 5% long position and the 95% short position. The GAS-GHD outperformed the other models at the 97.5% level and shared the highest p-value with the GAS-VG model at the 99% level.

With regards to the Litecoin returns, the GAS-VG performed the best at the 2.5% and 5% long positions as well as at the 97.5% short position. The GAS-GHD model had the highest p-values at the 1%, 95% and 97.5% levels. The GAS-NIG shared the best performance at the 1%, 97.5% and 99% risk levels. The GAS-GLD had the highest p-values at the 1%, 5% and 95% levels.

For Ripple, the GAS-Vg model outperformed all the other models at the first two short positions (1% and 2.5%) whereas the GAS-NIG performed the best at the other short position (5%) and at the 97.5% long position. The GAS-GLD was the model robust model at the 95% and 99% risk levels.

4.2. Out-of-sample backtesting

The unconditional coverage (UC) test of Kupiec (1995) was used to apply out-of-sample backtesting which assesses the precision of the predictive capabilities of the VaR model by allocating an estimation period and evaluation period. The out-of-sample periods for Bitcoin, Ethereum, Litecoin and Ripple are 22 August 2020 to 27 August 2021, 9 November 2020 to 27 August 2021, 9 January 2021 to 27 August 2021 and 17 December 2020 to 27 August 2021, respectively. Table 11 presents the Kupiec p-values for both the long and short positions. The null hypothesis of correct model specification should be rejected at a 5% level of significance if the Kupiec p-value is less than 0.05. According to the out-of-sample VaR backtesting results, all of the models were correctly specified to all four cryptocurrencies' returns.

For the Bitcoin returns, the GAS-NIG and GAS-GLD both produce the largest p-values at the 1%, 2.5% and 99% risk levels. The GAS-GLD model outperformed the other models at the 5% risk level. The GAS-VG and GAS-GHD had the highest p-value for the 95% short position and the GAS-GHD had the best model specification at the 97.5% risk level. All models had the same p-values for the 2.5% and 99% long and short positions, respectively.

Table 10. In-sample VaR backtesting for the Bitcoin and Ethereum returns

	Distribution	Long position			Short position		
		1%	2.5%	5%	95%	97.5%	99%
Bitcoin	GAS-VG	0.3869	0.0492	0.7420	0.4973	0.1337	**0.1694**
	GAS-GHD	0.6560	**0.1484**	**0.8182**	0.3766	0.0505	0.0608
	GAS-NIG	**0.8352**	**0.1484**	0.5980	**0.5314**	0.1337	0.0608
	GAS-GLD	0.1048	**0.1484**	0.3576	0.2257	**0.2297**	0.0608
Ethereum	GAS-VG	0.0491	**0.9842**	0.5485	0.5485	0.7350	**0.3748**
	GAS-GHD	0.1909	**0.9842**	0.4054	0.6278	**0.9842**	**0.3748**
	GAS-NIG	0.1909	**0.9842**	**0.8873**	**0.9325**	0.7688	0.0848
	GAS-GLD	**0.2796**	**0.9842**	**0.8873**	0.4508	0.4498	0.1507
Litecoin	GAS-VG	0.5746	**0.5439**	**0.9287**	0.2046	**0.9082**	0.7701
	GAS-GHD	**0.7701**	0.4281	0.6762	**0.5441**	**0.9082**	0.5780
	GAS-NIG	**0.7701**	0.3270	0.8687	0.4608	**0.9082**	**0.9866**
	GAS-GLD	**0.7701**	**0.5439**	0.6762	**0.5441**	0.7707	0.5780
Ripple	GAS-VG	**0.8536**	**0.8986**	0.4992	0.7603	0.4798	0.0956
	GAS-GHD	0.5295	0.5180	0.8561	0.9516	0.7059	0.2343
	GAS-NIG	0.5295	0.4105	**0.9516**	0.9516	**0.8332**	0.2343
	GAS-GLD	0.7257	0.8332	0.5887	**0.9517**	0.5874	**0.4859**

[b] Bold values are largest p-values.

All the models fitted to the Ethereum returns produced similar results at the 1%, 97.5% and 99% risk levels. The GAS-VG and GAS-GLD models were the best models at the 2.5% long position. The GAS-NIG model and GAS-GLD model outperformed the other two models at the 5% risk level. The GAS-GHD model produced the largest p-value at the 95% short position.

For Litecoin, the GAS-GHD model outperformed the rest of the models at the 5% risk level. The GAS-GHD, GAS-NIG and GAS-GLD all have the same p-values which appear to better than the p-values of the GAS-VG model for the 1% and 2.5% long positions. All models have the same Kupiec p-values for all of the short positions for the Litecoin returns.

All four models fitted to the Ripple returns had the same p-value for the 1% level and the 97.5% level. The GAS-GLD model, for Ripple, outperformed the other models at the 2.5% risk level and shared the best performance with the GAS-GHD model at the 5% long position. The GAS-NIG had the highest p-value at the 95% short position. All the models except the GAS-VG model produced the same p-value for the 99% short position which was greater than the p-value of the GAS-VG model.

5. Conclusion

Over the past few years, cryptocurrencies have become highly favoured and very popular around the world due to it's fast, digital and secure transactions. Since it does not seem like it is going away any time soon, identifying and understanding the behaviour of these evolving assets play a crucial role to the global financial markets. Researching models that have the capability to forecast the risks affiliated with investment opportunities are of great significance to investors of cryptocurrency.

Table 11. Out-of-sample VaR backtesting for the Bitcoin, Ethereum, Litecoin and Ripple returns

	Distribution	Long position			Short position		
		1%	**2.5%**	**5%**	**95%**	**97.5%**	**99%**
Bitcoin	GAS-VG	0.2702	**0.4340**	0.5420	**0.3002**	0.8052	**0.7053**
	GAS-GHD	0.5191	**0.4340**	0.3887	**0.3002**	**0.9334**	**0.7053**
	GAS-NIG	**0.8770**	**0.4340**	0.7169	0.2087	0.8052	**0.7053**
	GAS-GLD	**0.8770**	**0.4340**	**0.9047**	0.0570	0.8052	**0.7053**
Ethereum	GAS-VG	**0.9626**	**0.6153**	0.6618	0.8712	**0.1767**	**0.1910**
	GAS-GHD	**0.9626**	0.3611	0.6618	**0.9148**	**0.1767**	**0.1910**
	GAS-NIG	**0.9626**	0.3611	**0.8712**	0.5295	**0.1767**	**0.1910**
	GAS-GLD	**0.9626**	**0.6153**	**0.8712**	0.5295	**0.1767**	**0.1910**
Litecoin	GAS-VG	0.1234	0.9249	0.6323	**0.4735**	**0.4288**	**0.3290**
	GAS-GHD	**0.3114**	**0.9250**	**0.8671**	**0.4735**	**0.4288**	**0.3290**
	GAS-NIG	**0.3114**	**0.9250**	0.6323	**0.4735**	**0.4288**	**0.3290**
	GAS-GLD	**0.3114**	**0.9250**	0.6323	**0.4735**	**0.4288**	**0.3290**
Ripple	GAS-VG	**0.7779**	0.5733	0.0739	0.8389	**0.8871**	0.1707
	GAS-GHD	**0.7779**	0.5733	**0.4202**	0.7125	**0.8871**	**0.7779**
	GAS-NIG	**0.7779**	0.5733	0.2621	**0.9314**	**0.8871**	**0.7779**
	GAS-GLD	**0.7779**	**0.8871**	**0.4202**	0.7125	**0.8871**	**0.7779**

[b] Bold values are the largest p-values.

Due to its dynamic and flexible nature, GAS models have become a particularly appropriate choice for modeling and forecasting financial data, including cryptocurrencies. However, it should be noted that the model can become rather complex as the number of parameter increases which can, in turn, lead to estimation difficulties. The capabilities of the GAS models also relies highly on the correct choice of a score function and thus an inappropriate specification may lead to unsatisfactory performance and estimates. These models are also known for it's versatility however they are not adequate for all sets of time series as issues may arise when certain features are exhibited such as regime changes, structural breaks, etc. Nevertheless, GAS models have proven to possess numerous advantageous features which outweigh the drawbacks when it comes to modeling financial data.

In this study, the performance of GAS models with underlying heavy-tailed distributions in estimating the VaR of Bitcoin, Ethereum, Litecoin and Ripple returns were investigated. The adequacy of the models studied here were assessed using in-sample backtesting by utilising the Kupiec test. Table 12 lists the most appropriate models that were found for each of the four cryptocurrencies. At every VaR level, the GAS models with the different underlying distributions appear to be most adequate for at least one of the cryptocurrencies- except at the 95% VaR level where the GAS-VG does not seem appropriate for any of the cryptocurrencies.

The forecasting capabilities of all the models used in this paper were also compared using out-of-sample backtesting. In Table 13, the results of the most robust models are found for each of the VaR levels. All GAS models seem to have exhibited adequate predictive power at different VaR levels of both the long and short positions for all four sets of returns.

The key outcomes of this study are summarised in Table 12 and 13 which clarifies that GAS models coupled with underlying distributions of different variations of the Generalized hyperbolic distributions are highly suitable for VaR estimation of Bitcoin, Ethereum, Litecoin and Ripple.

The findings in this study could prove to be of leading value in the financial industry to risk-managers & analysts, traders, portfolio managers etc as the explorations in this study illustrates how GAS models, when coupled with a suitable heavy-tailed model can obtain adequate risk-management & volatility forecasting outcomes as well as improve trading strategies. This paper will also be of great use to the numerous stakeholders in the cryptocurrency market as the results suggest that GAS models are well equipped to handle extreme volatility. It can also be used to better assess the risks associated and assist with decision making.

The outcomes of this paper may also be useful to academics and researchers who are investigating and modeling the behaviour of other cryptocurrencies and financial data which are observed to possess similar trends such as the cryptocurrencies studied here.

For future research, incorporating GAS models with other heavy-tailed distributions such as the standard statistical distributions built within the **R** GAS package may be beneficial for exploring the modelling and forecasting of Bitcoin, Ethereum, Litecoin and Ripple returns. Also, following a method similar to the one employed in the analysis of this paper, whereby the innovations of the Gaussian GAS model is extracted, may allow more complex models to be fitted as the underlying distributions of the GAS model. Other avenues of research may include using the models explored in this study to analyze and forecast other popular cryptocurrencies such as Tether, Binance Coin, Dogecoin, etc.

Table 12. Most appropriate model selected for Bitcoin, Ethereum, Litecoin and Ripple (In-sample)

VaR level	Bitcoin	Ethereum	Litecoin	Ripple
1%	GAS-NIG	GAS-GLD	GAS-GHD GAS-NIG GAS-GLD	GAS-VG
2.5%	GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GLD	GAS-VG
5%	GAS-GHD	GAS-NIG GAS-GLD	GAS-VG	GAS-NIG
95%	GAS-NIG	GAS-NIG	GAS-GHD GAS-GLD	GAS-GLD
97.5%	GAS-GLD	GAS-GHD	GAS-VG GAS-GHD GAS-NIG	GAS-NIG
99%	GAS-VG	GAS-VG GAS-GHD	GAS-NIG	GAS-GLD

Table 13. Most appropriate model selected for Bitcoin, Ethereum, Litecoin and Ripple (Out-of-sample)

VaR level	Bitcoin	Ethereum	Litecoin	Ripple
1%	GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD
2.5%	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GLD	GAS-GHD GAS-NIG GAS-GLD	GAS-GLD
5%	GAS-GLD	GAS-NIG GAS-GLD	GAS-GHD	GAS-GHD GAS-GLD
95%	GAS-VG GAS-GHD	GAS-GHD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-NIG
97.5%	GAS-NIG	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD
99%	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-VG GAS-GHD GAS-NIG GAS-GLD	GAS-GHD GAS-NIG GAS-GLD

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