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Statistical Analysis of Covid-19 Data using the Odd Log Logistic Kumaraswamy Distribution

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Abstract This paper presents a statistical analysis of Covid-19 data using a new distribution. Some mathematical properties of the proposed distribution such as survival and hazard functions, quantile function, median, ordinary and incomplete moments, moment generating function, probability weighted moment, distribution of order statistic, and Renyi entropy are derived. Five estimators are examined for unknown model parameters. The performances of the estimators are compared using an extensive simulation study based on the bias and mean square error criterion. Two Covid-19 data sets representing the percentage of daily recoveries of Covid-19 patients are used to illustrate the utility of the new distribution. The results show that the new model is a superior alternative for some current models with bounded support.

Keywords Odd Log Logistic-G Distribution, Kumaraswamy Distribution, Moments, Quantiles

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1. Introduction

Lifetime distributions are parametric models used for the prediction of the length of life of a device. It also measures the survival time as well as the rate at which failure occurs in a system. Many lifetime distributions have been studied and applied to areas such as biological sciences, engineering, finance, actuarial sciences, etc. Modeling data in the range of (0,1), such as mortality and recovery rates, scores of ability tests, or measurement sciences, has become more important in recent years. Therefore, there is a need to introduce new distributions which are defined on (0,1). Therefore, the Kumaraswamy distribution, introduced by Kumaraswamy [16], is selected as the baseline distribution for this study. The Kumaraswamy distribution, although in existence for decades, received less attention until the work of Jones [15], who addressed the relevance of the distribution over the widely explored beta distribution. A major advantage of the Kumaraswamy distribution over the beta distribution is the explicit expression of the cumulative distribution function and the quantile function for the Kumaraswamy distribution which is not the case for the beta distribution due to the beta function. The cumulative distribution function (cdf) of the Kumaraswamy distribution is given by

$$F(x) = 1 - (1 - x^a)^b, \qquad 0 < x < 1, \quad a, b > 0, \tag{1}$$

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and the corresponding probability density function (pdf) is obtained as

$$f(x) = a\,b\,x^{a-1}\left(1-x^{a}\right)^{b-1}, \qquad 0<x<1, \quad a,b>0. \tag{2}$$

It is inarguably, that the addition of extra parameter(s) to an existing lifetime model increases its flexibility in data analysis. Thus, this has aroused the interest of many researchers in developing several methods of generalizing existing models. Some of these methods include the exponentiated Weibull family by Mudholkar and Srivastava [20], the Marshall-Olkin extended family by Marshall and Olkin [17], the transmuted-G family by Shaw and Buckley [24], the Kumaraswamy-G family by Cordeiro and de Castro [6], the Beta-G family by Eugene et al. [7], the $T-X$ family by Alzaatreh et al. [1], the Weibull-G family by Bourguignon et al. [4], the $T-R\{Y\}$ family by Alzaatreh et al. [2], the Kumaraswamy-Marshall -Olkin family by Alizadeh et al. [3], the Topp-Leone Kumaraswamy-G family by Ibrahim et al. [14], etc.

Gleaton and Lynch [10] introduced the generalized log-logistic (GLL) family of distributions. Cordeiro et al. [5] referred to the GLL family of distributions as the odd log logistic-G (OLL-G) family of distributions and then proposed the beta odd log logistic-G family of distributions. The cdf of the OLL-G family of distributions is defined as

$$G(x,\alpha,\xi) = \frac{F(x,\xi)^{\alpha}}{F(x,\xi)^{\alpha}+\bar{F}(x,\xi)^{\alpha}}, \tag{3}$$

with pdf obtained as

$$g(x,\alpha,\xi) = \frac{\alpha f(x,\xi)\left[F(x,\xi)\right]^{\alpha-1}\left[\bar{F}(x,\xi)\right]^{\alpha-1}}{\left[F(x,\xi)^{\alpha}+\bar{F}(x,\xi)^{\alpha}\right]^{2}}, \tag{4}$$

where $\bar{F}(x,\xi)=1-F(x,\xi)$, $f(x,\xi)=\frac{dF(x,\xi)}{dx}$ is the density function and ξ is the vector of the parameter(s) of the baseline distribution.

The main idea of this paper is to develop a heavy-tailed distribution that accommodates left-skewed, right-skewed, exponentially decreasing (reversed-J), symmetric shapes, exhibits a bathtub-shaped hazard rate property and also provides consistently better fits than existing non-nested lifetime models in data analysis. The remaining Sections of this paper are organized as follows: Section 2 defines the odd log-logistic Kumaraswamy (OLLK) distribution. Section 3 provides some mathematical properties of the OLLK distribution. Section 4 discusses five estimation methods and a Monte Carlo simulation study is conducted to investigate the performances of the estimators. The applicability of the OLLK distribution alongside some existing non-nested models is illustrated in Section 5. Finally, in Section 6, concluding remarks are presented.

2. The Odd Log Logistic Kumaraswamy Distribution

We define the cdf and the pdf of the OLLK distribution by inserting (1) and (2) into (3) and (4) as follows:

$$G(x) = \frac{\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha}}{\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha}+\left(1-x^{a}\right)^{b\alpha}}, \qquad \alpha,a,b>0, \quad 0<x<1, \tag{5}$$

and

$$g(x) = \frac{ab\alpha x^{a-1}\left(1-x^{a}\right)^{b\alpha-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}}{\left[\left(1-\left(1-x^{a}\right)^{b}\right)^{\alpha}+\left(1-x^{a}\right)^{b\alpha}\right]^{2}}, \qquad \alpha,a,b>0, \quad 0<x<1. \tag{6}$$

It is easily seen that (6) reduce to the density function of the Kumaraswamy distribution when $\alpha=1$.

The survival, hazard rate function (hrf) and quantile functions of the OLLK distribution are defined as follows:

$$S(x) \;=\; \frac{(1-x^a)^{b\alpha}}{\left[1 \;-\; (1-x^a)^b\right]^{\alpha} + (1-x^a)^{b\alpha}}, \tag{7}$$

$$h(x) \;=\; \frac{ab\alpha x^{a-1}\left[1 \;-\; (1-x^a)^b\right]^{\alpha-1}}{(1-x^a)\left[\left(1 \;-\; (1-x^a)^b\right)^{\alpha} + (1-x^a)^{b\alpha}\right]} \tag{8}$$

and

$$Q_X\left(u\right) \;=\; \left[1-\left\{\frac{(1-u)^{1/\alpha}}{(1-u)^{1/\alpha}+u^{1/\alpha}}\right\}^{1/b}\right]^{1/a}, \quad 0<u<1. \tag{9}$$

The quantile function allows us to generate random samples from a known probability distribution for simulation study.

Figures 1 and 2 present the plots of the pdf and the hrf of the OLLK distribution for some selected values of the parameters.

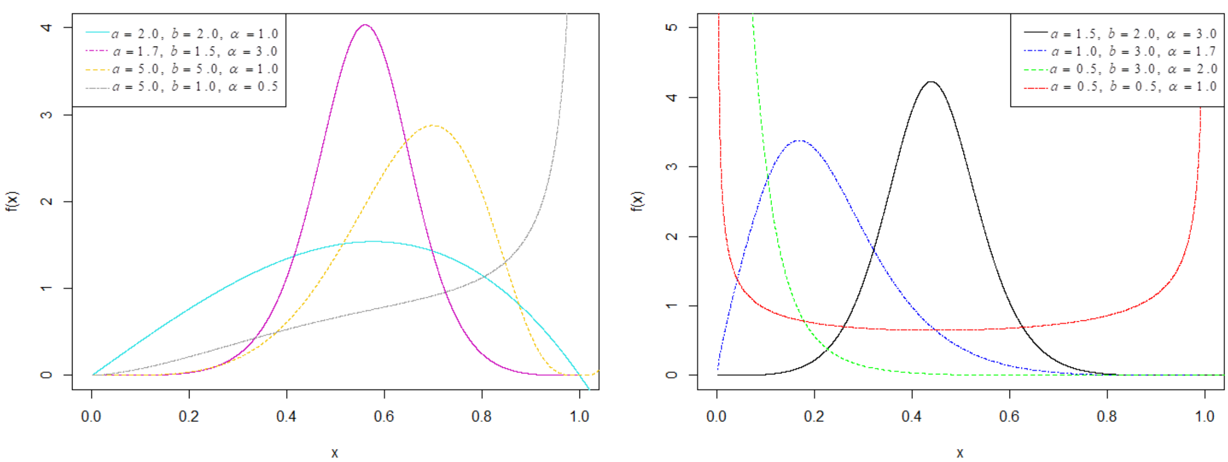

Figure 1. The pdf plots of the OLLK distribution for some values of the parameters.

Figure 1 shows that the pdf plots of the OLLK distribution accommodate exponentially decreasing or increasing, negatively-skewed, positively-skewed and symmetric shapes.

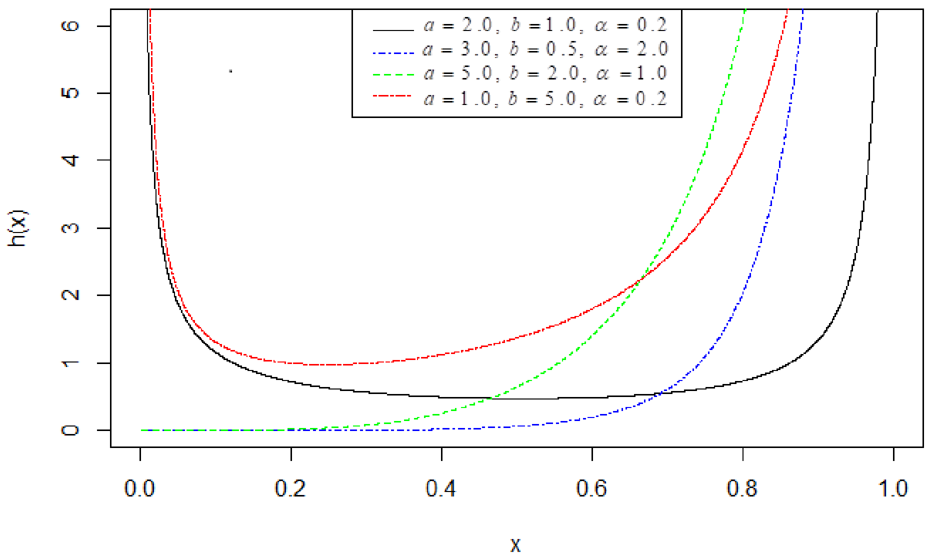

Figure 2. Hazard function of the OLLK distribution for selected values of the parameters.

Figure 2 suggests that the hrf of the OLLK distribution exhibits an increasing and bathtub shaped hazard property.

3. Mathematical Properties

3.1. Linear Representation

The usefulness of obtaining a mixture representation for the density function and the cumulative distribution function of a new model is to allow easy derivation of some other mathematical properties of the model such as the moments, moment generating function and the distribution of order statistic. To obtain the mixture representation for the cumulative distribution function and the density function of the OLLK distribution, we consider the following useful expansion

$$\left(1 - (1-x^a)^b\right)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j (1-x^a)^{bj},$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\alpha}{j} \binom{j}{k} (-1)^{j+k} \left[1-(1-x^a)^b\right]^k,$$

$$= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \binom{\alpha}{j} \binom{j}{k} (-1)^{j+k} \left[1-(1-x^a)^b\right]^k$$

$$= \sum_{k=0}^{\infty} a_k \left[1-(1-x^a)^b\right]^k,$$

where $a_k = \sum_{k=0}^{j} \binom{\alpha}{j} \binom{j}{k} (-1)^{j+k}$.

Also,

$$\left(1 - (1-x^a)^b\right)^\alpha + (1-x^a)^{b\alpha} = \sum_{k=0}^{\infty} a_k \left[1-(1-x^a)^b\right]^k + \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \left[1-(1-x^a)^b\right]^k,$$

$$= \sum_{k=0}^{\infty} b_k \left[1-(1-x^a)^b\right]^k,$$

where $b_k = a_k + \binom{\alpha}{k} (-1)^k$.

Combining these expressions, we obtain the mixture representation for the cumulative distribution function of the OLLK distribution as

$$G(x) = \frac{\sum_{k=0}^{\infty} a_k \left[1-(1-x^a)^b\right]^k}{\sum_{k=0}^{\infty} b_k \left[1-(1-x^a)^b\right]^k} = \sum_{k=0}^{\infty} c_k \left[1-(1-x^a)^b\right]^k, \tag{10}$$

where $c_0 = \frac{a_0}{b_0}$, and for $k \geq 1$, we have $c_k = b_0^{-1}\left[a_k - b_0^{-1}\sum_{r=0}^{k-1} b_r c_{k-r}\right]$,

(See Gradshteyn and Ryzhik, [11], pg.17).

(10) can be rewritten as

$$G(x) = \sum_{k=0}^{\infty} c_k H_k(x) = \sum_{k=0}^{\infty} c_k \left[H(x)\right]^k, \tag{11}$$

where $H_k(x)$ is the cumulative distribution function of the Kumaraswamy distribution with power parameter k. From (11), we deduce the mixture representation of the OLLK distribution as

$$g(x) = \sum_{k=0}^{\infty} c_{k+1} h_{k+1}(x), \tag{12}$$

where $h_{k+1}(x)$ is the density function of the Kumaraswamy distribution with power parameter (k+1).

3.2. The rth ordinary Moments, incomplete Moments and Moment Generating Function

$$E[X^r] = \sum_{k=0}^{\infty} c_{k+1} E\left[Y_{k+1}^r\right], \tag{13}$$

where $E\left[Y_{k+1}^r\right]$ is the moment of the Kumaraswamy distribution with density function of power parameter k+1. That is,

$$E\left[Y_{k+1}^r\right] = ab(k+1)\int_0^1 y^{r+a-1}(1-y^a)^{b-1}\left[1-(1-y^a)^b\right]^k dy, \tag{14}$$

since,

$$\left[1-(1-y^a)^b\right]^k = \sum_{m=0}^{\infty} \binom{k}{m} (-1)^m (1-y^a)^{bm},$$

then the change of variable $y^a = x$, transforms (14) as

$$E\left[Y_{k+1}^{r}\right] = b\left(k+1\right)\sum_{m=0}^{\infty}\binom{k}{m}\left(-1\right)^{m}\int_{0}^{1}x^{r/a}\left(1-x\right)^{b(m+1)-1}dx, \tag{15}$$

but, $B\left(a,b\right) = \int_{0}^{1}x^{a-1}\left(1-x\right)^{b-1}dx,$
so that (15) now becomes

$$\begin{aligned} E\left[Y_{k+1}^{r}\right] &= b\left(k+1\right)\sum_{m=0}^{\infty}\binom{k}{m}\left(-1\right)^{m}B\left(1+{}^{r}/_{a},b\left(m+1\right)\right), \\ &= b\sum_{m=0}^{\infty}\left(-1\right)^{m}\frac{\left(k+1\right)\Gamma\left(k+1\right)}{m!\Gamma\left(k+1-m\right)}B\left(1+{}^{r}/_{a},b\left(m+1\right)\right). \end{aligned} \tag{16}$$

Substituting (16) into (13), we define the r^{th} ordinary moments of the OLLK distribution as

$$E\left[X^{r}\right] = b\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}c_{k+1}\left(-1\right)^{m}\frac{\left(k+1\right)\Gamma\left(k+1\right)}{m!\Gamma\left(k+1-m\right)}B\left(1+{}^{r}/_{a},b\left(m+1\right)\right). \tag{17}$$

The corresponding r^{th} incomplete moments of the OLLK distribution can be obtained from (17) as

$$\varphi_{r}\left(t\right) = b\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}c_{k+1}\left(-1\right)^{m}\frac{\left(k+1\right)\Gamma\left(k+1\right)}{m!\Gamma\left(k+1-m\right)}B_{t}\left(1+{}^{r}/_{a},b\left(m+1\right)\right). \tag{18}$$

where $B_{t}\left(\alpha,\beta\right)=\int_{0}^{t}y^{\alpha-1}\left(1-y\right)^{\beta-1}dy$ is the incomplete beta function.

The mean of the OLLK distribution is obtained by setting r = 1 in (18). Other related measures such as the variance $\left(\sigma^{2}\right)$, skewness $\left(S_{k}\right)$ and kurtosis $\left(K_{s}\right)$ can further be derived from (18) as

$$\sigma^{2} = \mu_{2}^{'}-\left(\mu_{1}^{'}\right)^{2},$$

$$S_{k} = \frac{\mu_{3}^{'}-3\mu_{2}^{'}\mu_{1}^{'}+2\left(\mu_{1}^{'}\right)^{3}}{\left(\mu_{2}^{'}-\left(\mu_{1}^{'}\right)^{2}\right)^{\frac{3}{2}}},$$

$$K_{s} = \frac{\mu_{4}^{'}-4\mu_{3}^{'}\mu_{1}^{'}+6\mu_{3}^{'}\left(\mu_{1}^{'}\right)^{2}-3\left(\mu_{1}^{'}\right)^{4}}{\left(\mu_{2}^{'}-\left(\mu_{1}^{'}\right)^{2}\right)^{2}}.$$

where $\mu_{1}^{'}$, $\mu_{2}^{'}$, $\mu_{3}^{'}$ and $\mu_{4}^{'}$ are the first four r^{th} ordinary moments of the OLLK distribution.

Galton [8] and Moors [19] introduced an alternative measure of skewness and kurtosis, respectively, based on the quantile function of a probability model. The Galton's skewness and Moors' kurtosis are defined as

$$S_{k} = \frac{Q\left(6/8;a,b,\alpha\right)-2Q\left(4/8;a,b,\alpha\right)+Q\left(2/8;a,b,\alpha\right)}{Q\left(6/8;a,b,\alpha\right)-Q\left(2/8;a,b,\alpha\right)},$$

and

$$K_{s} = \frac{Q\left(7/8;a,b,\alpha\right)-Q\left(5/8;a,b,\alpha\right)+Q\left(3/8;a,b,\alpha\right)-Q\left(1/8;a,b,\alpha\right)}{Q\left(6/8;a,b,\alpha\right)-Q\left(2/8;a,b,\alpha\right)}.$$

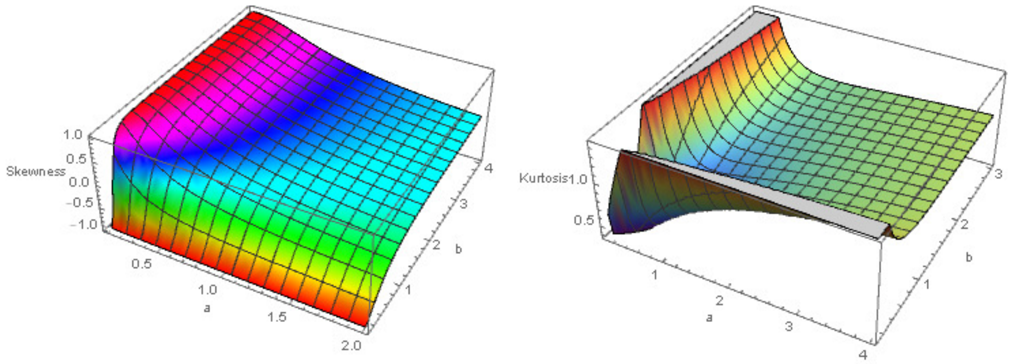

Figure 3. Galton's Skewness and Moors' Kurtosis for OLLK distribution $(a, b,\ 0.5)$

Figure 3 shows the behaviour of the Galton's skewness and Moors' kurtosis for the OLLK distribution when $\alpha = 0.5$.

Table 1. Theoretical moments of the OLLK distribution for selected value of the parameters

a	b	α	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	σ^2	S_k	K_s
1	2	0.5	0.3768	0.2535	0.1955	0.1604	0.1115	0.4281	1.7067
		1	0.3333	0.1667	0.1000	0.0667	0.0556	0.5620	2.4161
		2	0.3074	0.1149	0.0495	0.0239	0.0204	0.5610	3.3442
	4	0.5	0.2543	0.1318	0.0827	0.0572	0.0671	0.8653	2.5922
		1	0.2000	0.0667	0.0286	0.0143	0.0267	1.0498	3.6864
		2	0.1728	0.0381	0.0102	0.0032	0.0082	1.0351	4.4749
3	2	0.5	0.6148	0.4613	0.3768	0.3220	0.0833	-0.3851	1.8648
		1	0.6429	0.4500	0.3333	0.2571	0.0367	-0.4505	2.5611
		2	0.6570	0.4440	0.3074	0.2174	0.0124	-0.3891	3.2724
	4	0.5	0.5248	0.3468	0.2543	0.1976	0.0714	-0.1377	1.8236
		1	0.5341	0.3156	0.2000	0.1335	0.0303	-0.1836	2.4715
		2	0.5392	0.3007	0.1728	0.1019	0.0100	-0.0819	1.6862

It is clearly seen from Table 1 that the OLLK distribution can be right skewed $(S_k > 0)$, left skewed $(S_k < 0)$ and approximately symmetric $(S_k \approx 0)$. Also, at some fixed values of the parameters, the distribution can be leptokurtic $(K_s > 3)$, platykurtic $(K_s < 3)$ as well as mesokurtic $(K_s \approx 3)$.

The moment generating function of a known probability distribution is defined using the Maclaurin expansion of the exponential function as

$$M_X(t) = E\left[e^{t\,x}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[X^n\right]. \tag{19}$$

By inserting (17) in (19), we obtain the moment generating function of the OLLK distribution as

$$M_X(t) = b \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} t^n c_{k+1} (-1)^m \frac{(k+1)\,\Gamma(k+1)}{n!m!\Gamma(k+1-m)} B\left(1 + {}^{n}/_{a}, b(m+1)\right). \tag{20}$$

3.3. Probability Weighted Moments

Greenwood et al. [12] introduced a certain class of moments, called the probability weighted moments (PWMs). The PWMs are generally used to construct the estimators of the parameters and the quantiles of known probability distribution whose cumulative distribution function is invertible. For a random variable X, the $(s,r)^{th}$ PWMs is defined by

$$\rho_{s,r} = E\left[X^r F(x)^s\right] = \int_{-\infty}^{\infty} x^r f(x) F(x)^s \, dx. \tag{21}$$

Combining (5) and (6), we have

$$f(x) F(x)^s = \frac{ab\alpha x^{a-1}(1-x^a)^{b\alpha-1}\left[1 - (1-x^a)^b\right]^{\alpha(s+1)-1}}{\left[\left(1 - (1-x^a)^b\right)^{\alpha} + (1-x^a)^{b\alpha}\right]^{2+s}}, \tag{22}$$

consider the generalized binomial expansion

$$(x+y)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} (-1)^k x^{-s-k} y^k, \tag{23}$$

using (23) in (22), we have

$$f(x) F(x)^s = \alpha ab \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{s+k+1}{k} \binom{\alpha(s+k+1)-1}{m} (-1)^{k+m} x^{a-1} (1-x^a)^{b(m-\alpha(s+k+1))-1},$$

so that (22) now becomes,

$$\begin{aligned} f(x) F(x)^s &= a \sum_{k,m=0}^{\infty} q_{k,m}\, b(m-\alpha(s+k+1))\, x^{a-1}(1-x^a)^{b(m-\alpha(s+k+1))-1}, \\ &= \sum_{k,m=0}^{\infty} q_{k,m}\, h(x,a,b(m-\alpha(s+k+1))), \end{aligned} \tag{24}$$

where

$$q_{k,m} = \binom{s+k+1}{k} \frac{(-1)^{k+m}\alpha\Gamma(\alpha(s+k+1))}{m!\,(m-\alpha(s+k+1))\Gamma(\alpha(s+k+1)-m)},$$

and $h(x,a,b(m-\alpha(s+k+1)))$ represents the pdf of the Kumaraswamy with parameters a and $b(m-\alpha(s+k+1))$.

By setting (24) in (21), we have

$$\begin{aligned} \rho_{s,r} &= \sum_{k,m=0}^{\infty} q_{k,m} \int_0^1 x^r h(x,a,b(m-\alpha(s+k+1)))\, dx, \\ &= a \sum_{k,m=0}^{\infty} q_{k,m}\, b(m-\alpha(s+k+1))\, B\left[r+a,\ b(m-\alpha(s+k+1))\right]. \end{aligned} \tag{25}$$

(25) is readily the $(s,r)^{th}$ PWMs of the OLLK distribution, which is expressed as a linear combination of the Kumaraswamy densities.

3.4. Distribution of Order Statistics

Let $(X_1, X_2, ..., X_n)$ be random samples of size *n* from the OLLK distribution. Suppose $X_{r\,:\,n}$ denotes the r^{th} order statistics, then the density function of $X_{r\,:\,n}$ is defined by

$$f_{r\,:\,n}(x) = \frac{1}{B(r, n-r+1)} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j f(x) F(x)^{r+j-1}. \tag{26}$$

Using a similar approach in (22),

$$f(x) F(x)^{r+j-1} = \frac{ab\alpha x^{a-1} (1-x^a)^{b\alpha-1} \left[1 - (1-x^a)^b\right]^{\alpha(r+j)-1}}{\left[\left(1 - (1-x^a)^b\right)^{\alpha} + (1-x^a)^{b\alpha}\right]^{r+j+1}}, \tag{27}$$

it follows from (23) that,

$$f(x) F(x)^{r+j-1} = \alpha ab \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{r+j+k}{k} \binom{\alpha(r+j+k)-1}{m} (-1)^{k+m} x^{a-1} (1-x^a)^{b(m-\alpha(r+j+k))-1}.$$

So that (26) now becomes

$$\begin{aligned} f_{r\,:\,n}(x) &= a \sum_{k,m=0}^{\infty} \psi_{k,m}\, b\,(m-\alpha(r+j+k))\, x^{a-1} (1-x^a)^{b(m-\alpha(r+j+k))-1}, \\ &= \sum_{k,m=0}^{\infty} \psi_{k,m}\, h\,(x, a, b\,(m-\alpha(r+j+k))), \end{aligned} \tag{28}$$

where,

$$\psi_{k,m} = \frac{\alpha}{B(r, n-r+1)} \sum_{j=0}^{n-r} \binom{n-r}{j} \binom{r+j+k}{k} \frac{(-1)^{j+k+m}\, \Gamma(\alpha(r+j+k))}{m!\,(m-\alpha(r+j+k))\, \Gamma(\alpha(r+j+k)-m)},$$

and $h(x, a, b(m-\alpha(r+j+k)))$ denotes pdf of the Kumaraswamy distribution with parameters a and $b(m-\alpha(r+j+k))$. We show that the r^{th} order statistics of the OLLK distribution can be expressed as a linear combination of Kumaraswamy densities.

From (28), we can derive an expression for the s^{th} moment of the r^{th} order statistics of the density of OLLK distribution as

$$\begin{aligned} E\left[X_{r\,:\,n}^s\right] &= \int_0^1 x^s h\,(x, a, b\,(m-\alpha(r+j+k)))\, dx, \\ &= a \sum_{k,m=0}^{\infty} \psi_{k,m}\, b\,(m-\alpha(r+j+k))\; B\left[s+a,\; b\,(m-\alpha(r+j+k))\right]. \end{aligned} \tag{29}$$

3.5. Renyi Entropy

The entropy of a random variable *Y* is the measure of randomness associated with the random variable *Y*. Renyi [23] define the Renyi entropy of a random variable *Y* with pdf $f(y)$, as

$$\tau_R(\xi) = \frac{1}{1-\xi} \log \int_{-\infty}^{\infty} f^{\xi}(y)\, dy, \qquad \xi > 0,\ \xi \neq 0. \tag{30}$$

Inserting the pdf in (6) into (30), the Renyi entropy of the OLLK distribution is defined as

$$\tau_R(\xi) = \frac{1}{1-\xi}\log\int_{-\infty}^{\infty}\left[\frac{ab\alpha x^{a-1}(1-x^a)^{b\alpha-1}\left[1-(1-x^a)^b\right]^{\alpha-1}}{\left[\left(1-(1-x^a)^b\right)^{\alpha}+(1-x^a)^{b\alpha}\right]^2}\right]^{\xi}dx, \quad \xi>0,\ \xi\neq 1. \tag{31}$$

Using the generalized binomial expansion in (23), we have

$$\left[(1-x^a)^{b\alpha}+\left(1-(1-x^a)^b\right)^{\alpha}\right]^{-2\xi} = \sum_{k=0}^{\infty}\binom{2\xi+k-1}{k}(-1)^k\left((1-x^a)^{b\alpha}\right)^{-2\xi-k}\left(1-(1-x^a)^b\right)^{\alpha k},$$

$$\left[1-(1-x^a)^b\right]^{\alpha k+\xi(\alpha-1)} = \sum_{m=0}^{\infty}\binom{\alpha k+\xi(\alpha-1)}{m}(-1)^m(1-x^a)^{bm},$$

so that (31) now becomes,

$$\begin{aligned}\tau_R(\xi) &= \frac{1}{1-\xi}\log\left[a^{\xi}b^{\xi}\alpha^{\xi}\sum_{k,m=0}^{\infty}\psi^*_{k,m}\int_0^1 x^{\xi(a-1)}(1-x^a)^{b(m-\alpha(k+\xi))-\xi}dx\right],\\ &= \frac{1}{1-\xi}\log\left[a^{\xi-1}b^{\xi}\alpha^{\xi}\sum_{k,m=0}^{\infty}\psi^*_{k,m}B\left(\frac{\xi(a-1)+1}{a},b(m-\alpha(k+\xi))-\xi+1\right)\right],\end{aligned} \tag{32}$$

where $\psi^*_{k,m} = \binom{2\xi+k-1}{k}\binom{\alpha k+\xi(\alpha-1)}{m}(-1)^{k+m}$.

4. Point Estimators for Unknown Model Parameters

In this section, some estimators to estimate the unknown parameters of the OLLK distribution are examined. The maximum likelihood, least squares, weighted least squares, Anderson-Darling, and Cramer-von Mises estimators are investigated and performances of these estimators are evaluated via Monte Carlo simulation. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $OLLK(\psi)$distribution,$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represent the associated order statistics and $x_{(i)}$ indicates the observed values of $X_{(i)}$ for $i = 1, 2, \ldots, n$, where $\psi = (a, b, \alpha)$. The likelihood and log-likelihood functions are obtained, respectively, by,

$$L(\psi) = a^n b^n \alpha^n \prod_{i=1}^{n}\frac{{x_i}^{a-1}(1-{x_i}^a)^{b\alpha-1}\left[1-(1-{x_i}^a)^b\right]^{\alpha-1}}{\left[\left(1-(1-{x_i}^a)^b\right)^{\alpha}+(1-{x_i}^a)^{b\alpha}\right]^2}, \tag{33}$$

and

$$\begin{aligned}\ell(\psi) =& n\log(a)+n\log(b)+n\log(\alpha)+(a-1)\sum_{i=1}^{n}\log(x_i)+(b\alpha-1)\sum_{i=1}^{n}\log\{1-{x_i}^a\}\\ &+(\alpha-1)\sum_{i=1}^{n}\log\left\{1-(1-{x_i}^a)^b\right\}-2\sum_{i=1}^{n}\log\left\{\left(1-(1-{x_i}^a)^b\right)^{\alpha}+(1-{x_i}^a)^{b\alpha}\right\}.\end{aligned} \tag{34}$$

Then, the maximum likelihood estimator (MLE) of $\psi$is given by

$$\hat{\psi}_1 = \arg\max_{\psi} \ell\left(\psi\right). \tag{35}$$

The solution to the following non-linear equations is also $\hat{\psi}_1$: $\frac{\partial^2 \ell\left(\psi\right)}{\partial a = 0}$, $\frac{\partial^2 \ell\left(\psi\right)}{\partial b = 0}$ and $\frac{\partial^2 \ell\left(\psi\right)}{\partial \alpha = 0}$, with

$$\begin{aligned}\frac{\partial \ell\left(\psi\right)}{\partial a} =& \frac{n}{a} + \sum_{i=1}^{n} \log\left(x_i\right) - \left(b\alpha - 1\right) \sum_{i=1}^{n} \frac{x^a \log\left(x_i\right)}{1 - {x_i}^a} + \left(\alpha - 1\right) b \sum_{i=1}^{n} \frac{x^a \log\left(x_i\right)\left(1 - {x_i}^a\right)^b}{\left(1 - {x_i}^a\right)\left\{1 \ - \ \left(1 - {x_i}^a\right)^b\right\}} \\ & - 2\sum_{i=1}^{n} \frac{\frac{\alpha b x^a \log(x_i)(1 - {x_i}^a)^b\left(1 \ - \ (1 - {x_i}^a)^b\right)^\alpha}{(1 - {x_i}^a)\left\{1 \ - \ (1 - {x_i}^a)^b\right\}} - \frac{\alpha b x^a \log(x_i)(1 - {x_i}^a)^{b\alpha}}{1 - {x_i}^a}}{\left(1 \ - \ \left(1 - {x_i}^a\right)^b\right)^\alpha + \left(1 - {x_i}^a\right)^{b\alpha}},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell\left(\psi\right)}{\partial b} =& \frac{n}{b} + \alpha \sum_{i=1}^{n} \log\left(1 - {x_i}^a\right) - \left(\alpha - 1\right) \sum_{i=1}^{n} \frac{\left(1 - {x_i}^a\right)^b \log\left(1 - {x_i}^a\right)}{1 \ - \ \left(1 - {x_i}^a\right)^b} \\ & - 2\sum_{i=1}^{n} \frac{\frac{\alpha(1 - {x_i}^a)^b \log(1 - {x_i}^a)\left(1 \ - \ (1 - {x_i}^a)^b\right)^\alpha}{(1 - {x_i}^a)^b - 1} + \alpha \log\left(1 - {x_i}^a\right)\left(1 - {x_i}^a\right)^{b\alpha}}{\left(1 \ - \ \left(1 - {x_i}^a\right)^b\right)^\alpha + \left(1 - {x_i}^a\right)^{b\alpha}},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell\left(\psi\right)}{\partial \alpha} =& \frac{n}{\alpha} + b \sum_{i=1}^{n} \log\left(1 - {x_i}^a\right) + \sum_{i=1}^{n} \log\left\{1 - \left(1 - {x_i}^a\right)^b\right\} \\ & - 2\sum_{i=1}^{n} \frac{\log\left(1 - \left(1 - {x_i}^a\right)^b\right)\left(1 \ - \ \left(1 - {x_i}^a\right)^b\right)^\alpha + b \log\left(1 - {x_i}^a\right)\left(1 - {x_i}^a\right)^{b\alpha}}{\left(1 \ - \ \left(1 - {x_i}^a\right)^b\right)^\alpha + \left(1 - {x_i}^a\right)^{b\alpha}}.\end{aligned}$$

The MLEs $\hat{a}, \hat{b}$ and $\hat{\alpha}$ have no closed forms. We also determine them with high numerical precision through numerical techniques.

Additionally, the underlying distribution of the estimator $\hat{\psi}_1$, under regular regularity conditions, can be approximately distributed as a normal distribution $N\left(\psi_1, J^*\right)$ for practical applications, where J^* is the observation of the $J = \left(\partial^2 \ell\left(\psi\right) / \partial \psi^2\right)^{-1}\Big|_{\psi = \hat{\psi}}$ from the practical data set.

One of the uses for this result is the asymptotic confidence interval for ψ at the level $(1 - \gamma)100\%$ with $\gamma \in (0, 1)$, which is given by

$$Ci_\gamma = \left[{\psi_1}^* - z_{\gamma/2}\sqrt{J^*}, {\psi_1}^* + z_{\gamma/2}\sqrt{J^*}\right],$$

where $z_{\gamma/2}$ is the $(1 - \gamma)$-quartile of the standard normal distribution. Let us construct the functions that give us the different estimators:

$$Q_{LS}(\psi) = \sum_{i=1}^{n} \left(\frac{\left[1 \ - \ \left(1 - {x_{(i)}}^a\right)^b\right]^\alpha}{\left[1 \ - \ \left(1 - {x_{(i)}}^a\right)^b\right]^\alpha + \left(1 - {x_{(i)}}^a\right)^{b\alpha}} - \frac{i}{n+1} \right)^2, \tag{36}$$

$$Q_{WLS}(\psi) = \sum_{i=1}^{n} \frac{\left(n+2\right)\left(n+1\right)^2}{i\left(n - i + 1\right)} \left(\frac{\left[1 \ - \ \left(1 - {x_{(i)}}^a\right)^b\right]^\alpha}{\left[1 \ - \ \left(1 - {x_{(i)}}^a\right)^b\right]^\alpha + \left(1 - {x_{(i)}}^a\right)^{b\alpha}} - \frac{i}{n+1} \right)^2, \tag{37}$$

$$Q_{AD}\left(\psi\right)=-n-\sum_{i=1}^{n}\frac{2i-1}{n}\left[\log\left\{\frac{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}}{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}+\left(1-x_{(i)}{}^{a}\right)^{b\alpha}}\right\}\right.$$
$$\left.+\log\left\{1-\left\{\frac{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}}{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}+\left(1-x_{(i)}{}^{a}\right)^{b\alpha}}\right\}\right\}\right], \tag{38}$$

and

$$QCvM\left(\psi\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}}{\left[1\ -\ \left(1-x_{(i)}{}^{a}\right)^{b}\right]^{\alpha}+\left(1-x_{(i)}{}^{a}\right)^{b\alpha}}-\frac{2i-1}{2n}\right]^{2}. \tag{39}$$

Then, the least square estimator (LSE), the weighted least square estimator (WLSE), the Anderson-Darling estimator (ADE) and the Cramer-von Mises estimator (CvME) of the ψ are obtained, respectively, by

$$\hat{\psi}_2=\arg\min_{\psi}Q_{LS}\left(\psi\right), \tag{40}$$

$$\hat{\psi}_3=\arg\min_{\psi}Q_{WLS}\left(\psi\right), \tag{41}$$

$$\hat{\psi}_4=\arg\min_{\psi}Q_{AD}\left(\psi\right), \tag{42}$$

and

$$\hat{\psi}_5=\arg\min_{\psi}Q_{CvM}\left(\psi\right). \tag{43}$$

All of the optimization problems in (35), (40), (41), (42), and (43) can be solved using the BFGS methods in the R **optim** function.

In the simulation study, the bias and mean square errors (MSEs) of the five estimators are obtained based on 5000 trials. The quantile function given in (9) is used to generate data from the $OLLK\left(\psi\right)$distribution, where $U\left(0,1\right)$ is the standard uniform distribution. The parameters are selected as $\psi=\left(0.5,0.7,0.9\right)$, $\psi=\left(3,1,0.3\right)$ and $\psi=\left(1,2,0.5\right)$. The sample size $n=25,50,100,250,500,1000$ is considered in the simulation study. The results are reported in Tables 2-4.

Table 2. Average bias and MSEs for $\psi = (0.5, 0.7, 0.9)$

		Bias			MSEs		
	n	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
ψ_1	25	0.7820	1.8373	0.1136	2.3176	17.5880	1.0704
	50	0.4153	0.8082	0.0544	0.8321	3.3289	0.5353
	75	0.2819	0.5138	0.0443	0.4743	1.5407	0.3633
	100	0.2129	0.3779	0.0342	0.3223	0.9193	0.2639
	250	0.1066	0.1693	0.0024	0.1163	0.2384	0.1058
	500	0.0554	0.0846	0.0000	0.0520	0.0926	0.0539
	1000	0.0265	0.0389	0.0007	0.0232	0.0358	0.0266
ψ_2	25	0.0654	0.3033	0.5389	0.4765	1.9484	1.2449
	50	0.0564	0.2674	0.4092	0.3770	1.5866	0.8544
	75	0.0831	0.2706	0.3253	0.3931	1.3741	0.6743
	100	0.0589	0.2115	0.2853	0.2887	0.9780	0.5475
	250	0.0411	0.1166	0.1507	0.1509	0.3256	0.2568
	500	0.0281	0.0738	0.0842	0.0913	0.1743	0.1291
	1000	0.0135	0.0366	0.0495	0.0488	0.0786	0.0674
ψ_3	25	0.0873	0.4426	0.5827	0.6571	6.4575	1.4129
	50	0.0760	0.2821	0.3807	0.4082	1.8177	0.8220
	75	0.0947	0.2745	0.2680	0.3513	1.3854	0.5753
	100	0.0682	0.1967	0.2220	0.2536	0.7523	0.4373
	250	0.0540	0.1154	0.0872	0.1221	0.2493	0.1685
	500	0.0368	0.0715	0.0392	0.0664	0.1219	0.0799
	1000	0.0171	0.0324	0.0205	0.0305	0.0477	0.0380
ψ_4	25	0.1476	0.4917	0.5365	0.7029	4.0503	1.3882
	50	0.1018	0.3132	0.3496	0.4197	1.6590	0.7863
	75	0.0909	0.2496	0.2663	0.3334	1.1017	0.5688
	100	0.0515	0.1610	0.2329	0.2270	0.6054	0.4358
	250	0.0407	0.0952	0.1004	0.1134	0.2239	0.1734
	500	0.0222	0.0515	0.0530	0.0595	0.1056	0.0814
	1000	0.0092	0.0225	0.0289	0.0286	0.0438	0.0388
ψ_5	25	0.2004	0.5741	0.5129	0.6725	3.1638	1.3632
	50	0.1481	0.4389	0.3718	0.4898	2.2418	0.8825
	75	0.1513	0.3932	0.2937	0.4652	1.7853	0.6886
	100	0.1052	0.2842	0.2633	0.3277	1.1155	0.5563
	250	0.0647	0.1505	0.1361	0.1648	0.3666	0.2569
	500	0.0426	0.0929	0.0742	0.0971	0.1881	0.1295
	1000	0.0209	0.0458	0.0440	0.0507	0.0825	0.0675

Table 3. Average bias and MSEs for $\psi = (3, 1, 0.3)$

		Bias			MSEs		
	n	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
ψ_1	25	1.6194	0.6572	0.0574	13.8442	1.9784	0.1020
	50	0.8313	0.3332	0.0426	6.0514	0.8115	0.0524
	75	0.5159	0.2149	0.0405	3.9719	0.5145	0.0390
	100	0.4004	0.1598	0.0335	3.0825	0.3793	0.0289
	250	0.1128	0.0506	0.0206	1.2213	0.1415	0.0113
	500	0.0416	0.0194	0.0123	0.6261	0.0695	0.0054
	1000	0.0232	0.0111	0.0062	0.3217	0.0356	0.0026
ψ_2	25	-0.6173	-0.0432	0.3245	5.3625	0.6579	0.4176
	50	-0.2921	0.0499	0.1914	5.2148	0.6647	0.1817
	75	-0.2814	0.0389	0.1484	4.3717	0.5603	0.1145
	100	-0.1842	0.0438	0.1187	4.0815	0.4903	0.0824
	250	-0.1658	0.0100	0.0617	2.1449	0.2534	0.0282
	500	-0.1466	-0.0138	0.0371	1.1399	0.1261	0.0127
	1000	-0.0736	-0.0061	0.0184	0.5983	0.0650	0.0054
ψ_3	25	0.3483	0.3010	0.2487	15.2026	2.2707	0.3214
	50	0.1824	0.1860	0.1366	7.7436	0.9601	0.1179
	75	0.0681	0.1212	0.1069	5.4043	0.6165	0.0785
	100	0.0743	0.1012	0.0861	4.4501	0.5139	0.0570
	250	-0.0145	0.0400	0.0425	1.9447	0.2206	0.0201
	500	-0.0472	0.0074	0.0249	0.9292	0.1028	0.0091
	1000	-0.0186	0.0059	0.0118	0.4679	0.0514	0.0039
ψ_4	25	0.0175	0.1679	0.2761	9.6639	1.2999	0.3508
	50	-0.1127	0.0817	0.1623	5.3651	0.6935	0.1415
	75	-0.1765	0.0413	0.1263	3.9612	0.4890	0.0886
	100	-0.1497	0.0251	0.1013	3.2719	0.3875	0.0619
	250	-0.1694	-0.0121	0.0538	1.5232	0.1746	0.0214
	500	-0.1103	-0.0136	0.0299	0.8523	0.0949	0.0094
	1000	-0.0546	-0.0060	0.0148	0.4499	0.0495	0.0041
ψ_5	25	-0.3011	0.0819	0.3217	5.9659	0.8226	0.4477
	50	-0.0387	0.1417	0.1798	5.8021	0.7720	0.1817
	75	-0.0920	0.1071	0.1393	4.7216	0.6248	0.1140
	100	-0.0161	0.1036	0.1106	4.4662	0.5491	0.0818
	250	-0.0792	0.0395	0.0566	2.2699	0.2719	0.0276
	500	-0.1008	0.0014	0.0343	1.1692	0.1304	0.0124
	1000	-0.0491	0.0020	0.0170	0.6075	0.0662	0.0053

Table 4. Average bias and MSEs for $\psi = (1, 2, 0.5)$

		Bias			MSEs		
	n	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
ψ_1	25	0.6036	3.0507	0.1087	1.7740	44.0300	0.4440
	50	0.2537	1.2305	0.1154	0.6778	9.6036	0.3029
	75	0.1620	0.7615	0.1030	0.4309	4.7026	0.2197
	100	0.1260	0.6060	0.0744	0.3248	3.3748	0.1427
	250	0.0355	0.1902	0.0367	0.1173	0.8623	0.0454
	500	0.0214	0.1009	0.0146	0.0583	0.4084	0.0159
	1000	0.0097	0.0510	0.0061	0.0281	0.1952	0.0061
ψ_2	25	0.1055	1.0089	0.3269	0.9446	11.2901	0.6505
	50	0.0912	0.9852	0.2410	0.7276	10.7536	0.4403
	75	0.0700	0.7402	0.1870	0.5646	7.6914	0.3083
	100	0.0438	0.5843	0.1689	0.4584	5.7007	0.2622
	250	-0.0125	0.1694	0.1008	0.2047	1.6262	0.1187
	500	-0.0017	0.0953	0.0473	0.1088	0.7576	0.0473
	1000	-0.0009	0.0483	0.0188	0.0509	0.3486	0.0148
ψ_3	25	0.2068	2.1566	0.3745	1.5437	60.0749	0.8247
	50	0.0736	0.9404	0.2606	0.7081	17.4782	0.4867
	75	0.0577	0.5967	0.1849	0.4923	6.1996	0.3106
	100	0.0443	0.4664	0.1453	0.3819	4.0783	0.2258
	250	0.0037	0.1452	0.0669	0.1525	1.1087	0.0733
	500	0.0110	0.0895	0.0253	0.0751	0.5089	0.0243
	1000	0.0074	0.0508	0.0081	0.0340	0.2327	0.0069
ψ_4	25	0.2039	1.6810	0.3568	1.2803	25.8724	0.8259
	50	0.0696	0.7796	0.2516	0.6285	8.3372	0.4718
	75	0.0432	0.5112	0.1971	0.4627	4.8276	0.3313
	100	0.0324	0.4157	0.1572	0.3660	3.5433	0.2464
	250	-0.0101	0.1104	0.0799	0.1536	1.0504	0.0877
	500	0.0026	0.0697	0.0319	0.0770	0.5064	0.0292
	1000	0.0030	0.0401	0.0108	0.0347	0.2334	0.0079
ψ_5	25	0.2399	1.4401	0.2564	1.0280	13.5576	0.5321
	50	0.1772	1.2692	0.1897	0.7587	12.2313	0.3633
	75	0.1310	0.9425	0.1532	0.5850	8.8014	0.2630
	100	0.0940	0.7377	0.1391	0.4649	6.2566	0.2286
	250	0.0105	0.2317	0.0878	0.2018	1.6800	0.1076
	500	0.0107	0.1277	0.0408	0.1066	0.7652	0.0429
	1000	0.0049	0.0639	0.0166	0.0503	0.3501	0.0140

From Tables 2-4, it is concluded that the bias and MSEs of all estimators decrease and converge to zero when the sample size increases. Although MLE gives bad results in a and b parameters, MLE shows the best performance in estimating α parameter. It is also observed that the LSE for parameters a and b performed better than other estimators. Additionally, MLE is generally the best method as sample sizes increase.

5. Data Analysis

In this section, the OLLK distribution is applied to two sets of Covid-19 data sets. The data sets represent the percentage of daily total recovery of Covid-19 patients from the total number of confirmed cases in Nigeria and Turkey. The infectious Covid-19 which has spread across nations of the world was confirmed in Nigeria and Turkey on the 28th February and 12 March, 2020, respectively. WHO recorded the first official recoveries from the pandemic virus in Nigeria and Turkey on 18 and 26 March, 2020, whereas, the first death case was recorded on 23 and 17 March, 2020, respectively. The data sets are accessible using the following links; https://covid19.ncdc.gov.ng/ and https://data.humdata.org/dataset/coronavirus-covid-19-cases-and-deaths. The two data sets are calculated as the ratio of daily total number of the recoveries to total number of confirmed cases. The first data set was collated from 18 March to 18 May, 2020, while the second data set was collated from 27 March to 20 April, 2020. The second data set was first reported in Gunduz and Korkmaz [13]. The data sets are presented as follows:

Data set 1

0.12500000, 0.12500000, 0.08333333, 0.04545455, 0.06666667, 0.05000000, 0.04545455, 0.03921569, 0.03076923, 0.04285714, 0.03370787, 0.02702703, 0.06106870, 0.05925926, 0.05172414, 0.10869565, 0.11904762, 0.11682243, 0.14224138, 0.14705882, 0.17322835, 0.15942029, 0.17708333, 0.19016393, 0.22012579, 0.26315789, 0.26530612, 0.26541555, 0.31449631, 0.34389140, 0.32251521, 0.30627306, 0.27113238, 0.28270677, 0.28270677, 0.22565865, 0.20081549, 0.18995434, 0.18781726, 0.18774548, 0.19072550, 0.16644909, 0.17766204, 0.16511387, 0.16175115, 0.14698492, 0.15637217, 0.14882227, 0.16305085, 0.16979332, 0.17044810, 0.17356851, 0.17947483, 0.17685838, 0.19435466, 0.20033424, 0.21524844, 0.22859357, 0.24220183, 0.26187511, 0.26749455, 0.26623482

Data set 2

0.007371007, 0.009456904, 0.011325639, 0.014962594, 0.017958761, 0.021238599, 0.022883926, 0.023134649, 0.032840311, 0.038494218, 0.043882583, 0.046380721, 0.048291739, 0.050659855, 0.051521402, 0.056836698, 0.060502844, 0.064816787, 0.073704904, 0.081767351, 0.095548098, 0.109884654, 0.126966197, 0.138762079, 0.14761486

Figures 4-5 present some graphical plots to describe the nature of the two data sets, respectively.

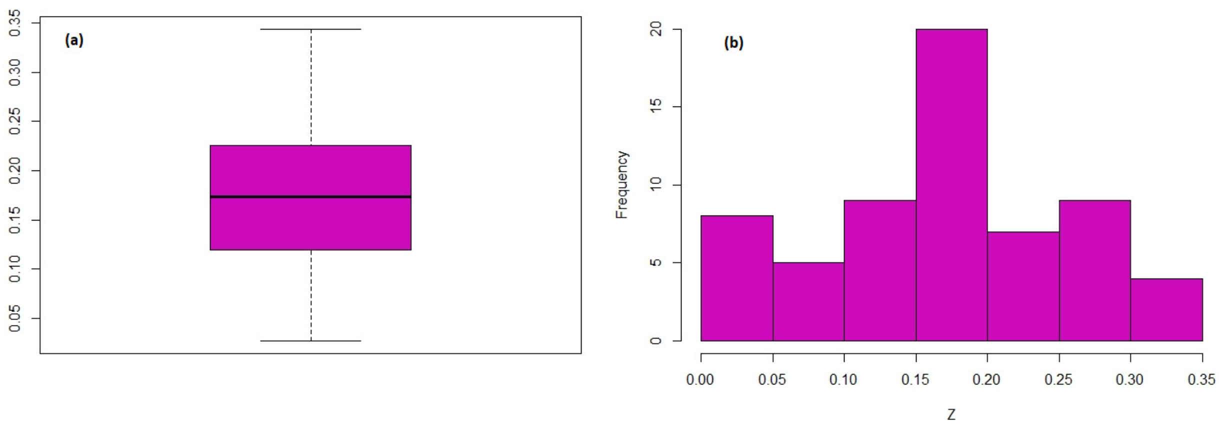

Figure 4. Graphical plots showing the (a) boxplot and (b) histogram of the first data set

In Figure 4(a) and 4(b), it can be said that the data has an approximately symmetrical structure. Further investigation from the boxplot reveals that there is no presence of an outlier in the first data set.

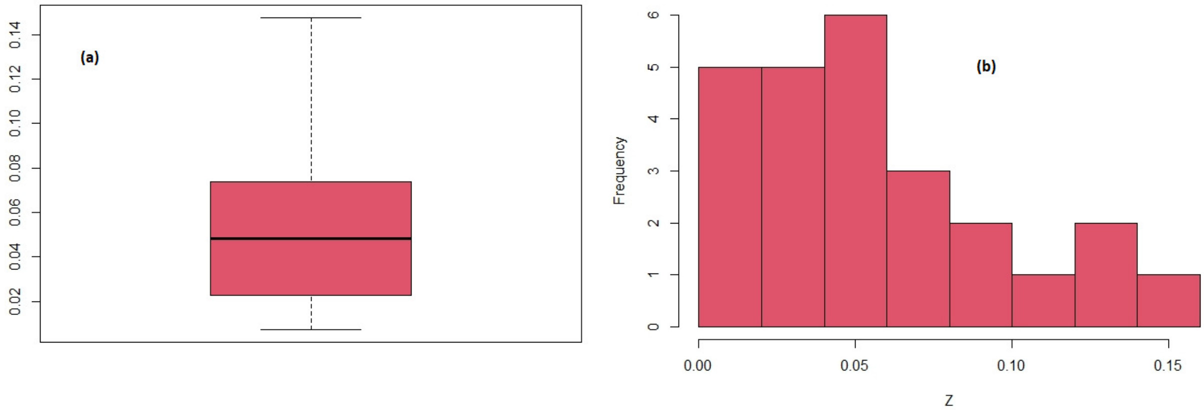

Figure 5. Graphical plots showing the (a) boxplot and (b) histogram of the second data set

One can see clearly from Figure 5 that the second data set is right-skewed and does not contain outliers. Now, we further proceed to compare the fit of the OLLK distribution with the fits obtained from some well-known non-nested models defined on a unit interval whose pdf is defined as

1. Marshall-Olkin Extended Kumaraswamy Distribution (MOEKD) due to George and Thobias [9];

$$f(x, a, b, \alpha) = \frac{\alpha a b x^{a-1}(1-x^a)^{b-1}}{\left[1-\alpha(1-x^a)^b\right]^2},$$

2. Marshall-Olkin Extended Topp-Leone Distribution (MOETLD) due to Opone and Iwerumor [21];

$$f(x, \alpha, \lambda) = \frac{2\alpha\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\left\{1-\left(1-(1-x)^2\right)^{\lambda}\right\}\right]^2},$$

3. Kumaraswamy distribution (KwD) due to Kumaraswamy [16];

$$f(x, a, b) = abx^{a-1}(1-x^a)^{b-1},$$

4. Unit-Gompertz distribution (UGD) due to Mazucheli et al. [18];

$$f(x, a, b) = abx^{-(a+1)}e^{-b\left(x^{-a}-1\right)}.$$

5. Transmuted Marshall-Olkin Extended Topp-Leone Distribution (TMOETLD) due to Opone and Osemwenkhae [22];

$$f(x, \alpha, \beta, \lambda) = \frac{\left[1-(1-x)^2\right]^{\lambda}}{\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\lambda}}\left[1+\beta-\beta\left(\frac{\left[1-(1-x)^2\right]^{\lambda}}{\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\lambda}}\right)\right].$$

The MLE is used in data analysis, in accordance with the results of the simulation experiments and results are reported in Tables 5 and 6. Tables 5 and 6 present the summary statistics of each of the distributions for the two Covid-19 data sets.

Table 5. Summary Statistics for the Covid-19 data set 1

Distributions	Estimates	Log-Lik	AIC	A^*	*p-value*
OLLKD	$a = 3.4464$ $b = 327.47$ $\alpha = 0.5528$	69.944	-133.887	0.8012	0.4797
MOEKD	$a = 1.5586$ $b = 25.0209$ $\alpha = 4.1425$	69.346	-132.693	0.8160	0.4691
MOETLD	$\alpha = 0.0322$ $\lambda = 2.8100$	59.236	-114.472	2.4953	0.0500
KwD	$a = 2.1522$ $b = 34.2331$	68.138	-132.276	1.2284	0.2568
UGD	$a = 0.0757$ $b = 1.1838$	50.473	-96.947	4.1554	0.0074
TMOETLD	$\alpha = 0.0278$ $\beta = -0.1669$ $\lambda = 2.7967$	59.2505	-112.501	2.4908	0.0503

Table 6. Summary Statistics for the Covid-19 data set 2

Distributions	Estimates	Log-Lik	AIC	A^*	*p-value*
OLLKD	$a = 1.5271$ $b = 70.160$ $\alpha = 0.9156$	49.3522	-92.704	0.2109	0.9887
MOEKD	$a = 2.0030$ $b = 86.936$ $\alpha = 0.1796$	49.1729	-92.345	0.3061	0.9329
MOETLD	$\alpha = 0.0062$ $\lambda = 2.0649$	48.1276	-92.255	0.3118	0.9284
KwD	$a = 1.4159$ $b = 50.887$	49.3422	-94.684	0.2126	0.9867
UGD	$a = 0.0186$ $b = 1.1146$	46.5550	-89.110	0.6500	0.6004
TMOETLD	$\alpha = 0.0061$ $\beta = -0.0007$ $\lambda = 2.0647$	48.1278	-90.2552	0.3117	0.9281

The flexibility of the non-nested models in fitting the Covid-19 data sets is examined by considering the fitted value of the log-likelihood, Akaike information criterion (*AIC*) and the Anderson Darling (A^*) test statistic with its corresponding p-value. The model with the best fit for a data set is traceable to the model having the maximized log-likelihood, the least value of *AIC* and A^* test statistics with the highest p-value. Hence, from Tables 5 and 6, we observed that the proposed OLLK distribution having the maximized log-likelihood, the least value of *AIC,* as well as A^* test statistic value outperformed the MOEKD, MOETLD, KwD, UGD and TMOETLD.

Further evidence of the superiority of OLLK distribution over the competitor models is shown in Figures 6-9. In particular, Figures 6 and 7, show the estimated pdf and cdf fits, and Probability-Probability (P-P) plots of the models for the first data set, whereas, Figures 8 and 9, show the estimated pdf and cdf fits, and Probability-Probability (P-P) plots of the models for the second data set, respectively.

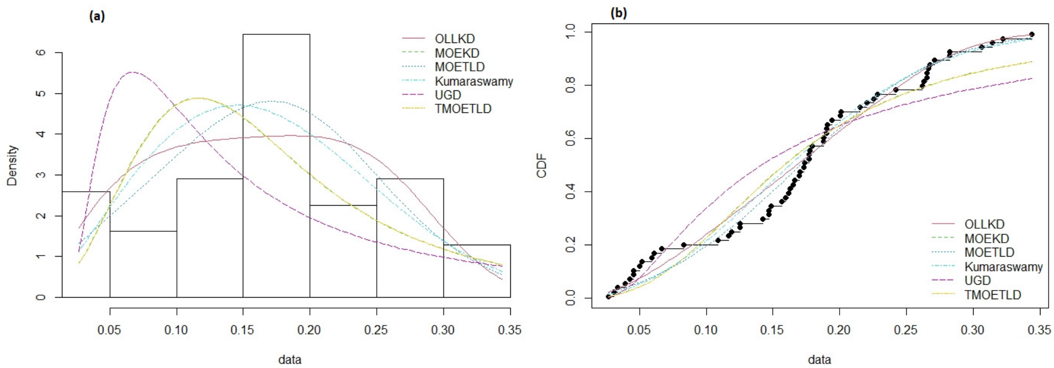

Figure 6. The estimated pdf (a) and cdf (b) fits of the models for the first data set

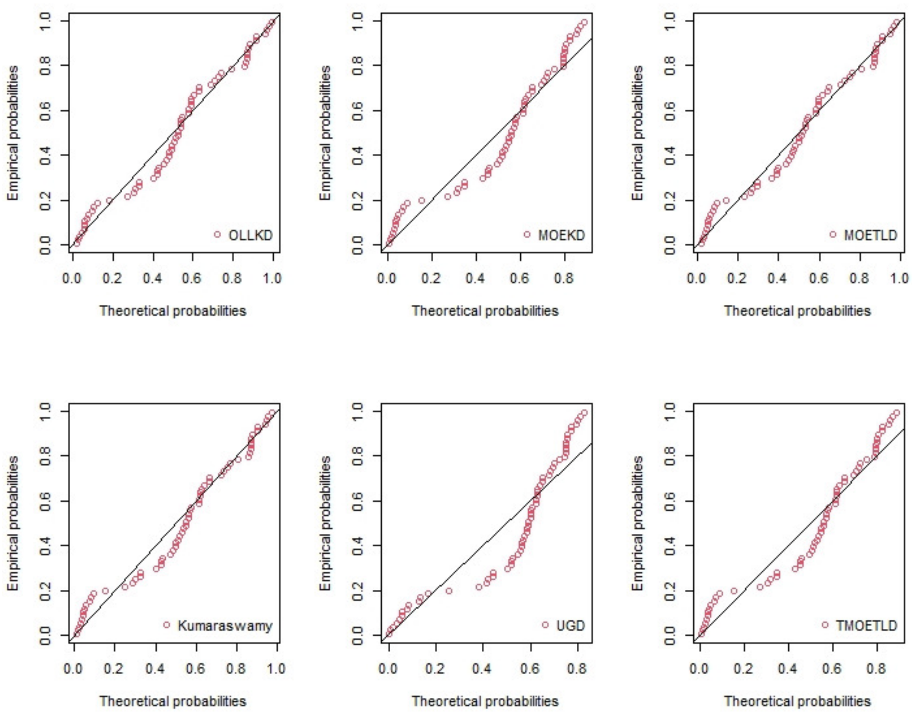

Figure 7. The P-P plots of the models for the first data set

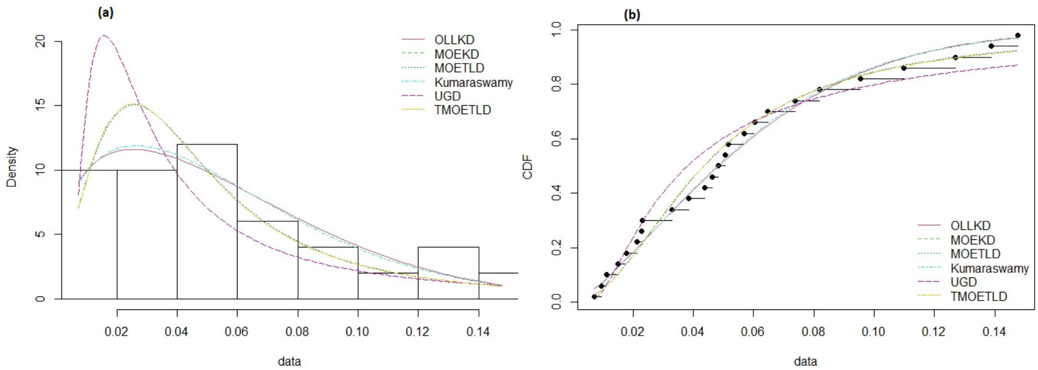

Figure 8. The estimated pdf (a) and cdf (b) fits of the models for the second data set

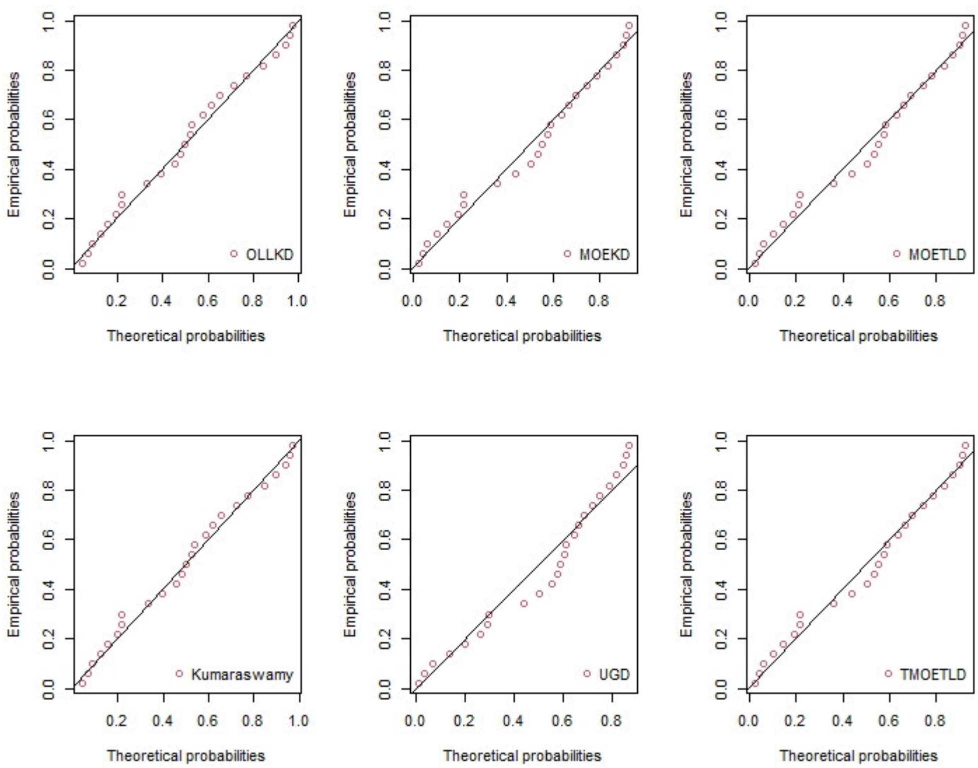

Figure 9. The P-P plots of the models for the second data set

6. Conclusion

In this paper, a new bounded statistical model called the odd log logistic Kumaraswamy distribution is introduced. Some mathematical properties of the proposed OLLK distribution are derived. For the estimation problem of the unknown parameters of the new model, some estimators are examined. A Monte Carlo simulation study is carried out to observe the performance of these estimators. The flexibility of the OLLK distribution and its superiority over some existing bounded models are illustrated using two Covid-19 data sets.

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