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Inference Based on New Pareto-Type Records With Applications to Precipitation and Covid-19 Data

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Abstract We consider estimation and prediction of future records based on observed records from the new Pareto type distribution proposed recently by Bourguignon et al. (2016), ¡°M. Bourguignon, H. Saulo, R. N. Fernandez, A new Pareto-type distribution with applications in reliability and income data, Physica A, 457 (2016), 166-175¡±. We first obtain the maximum likelihood and Bayesian estimators of the model parameters. We then derive several point predictors for a future record on the basis of the first n observed records. Two real data sets on precipitation and Covid 19 are analysed and a Monte Carlo simulation study has been performed to evaluate the statistical performance of point predictors presented in this paper.

Keywords Records, Pareto distribution, Estimation, Prediction, Simulation

AMS 2010 subject classifications 62G30, 62F10, 62M20

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1. Introduction

The Pareto distribution is a skewed and heavy-tailed model which was originated by Pareto [26] to model income distribution as a result of basic economic mechanisms. This distribution is currently widely used in different fields including insurance, economics, reliability, hydrology and engineering.

The two-parameter new Pareto-type (PT) distribution proposed by Bourguignon et al. [12] has the distribution function

$$F(x;\theta,\sigma) = \frac{x^\theta - \sigma^\theta}{x^\theta + \sigma^\theta}, \quad x \geq \sigma,\ \theta > 0, \sigma > 0, \tag{1}$$

and density function

$$f(x;\theta,\sigma) = \frac{2\theta\,\sigma^\theta\,x^{\theta-1}}{(x^\theta + \sigma^\theta)^2}, \quad x \geq \sigma,\ \theta > 0, \sigma > 0, \tag{2}$$

where θ and σ are shape and scale parameters, respectively. The density function of the PT distribution is decreasing and therefore, like the Pareto distribution, this distribution can be used as a model for income distributions. Depending on the values of its parameters, the PT hazard rate function is decreasing or upside down bathtub shaped. See Bourguignon et al. [12] for more information on PT distribution. In recent years, this distribution has been studied by several authors. Different estimation methods for PT distribution were investigated by Saadati et al. [22]. Saadati et al. [23] considered prediction methods for

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future failure times based on type-II right censored samples. Recently, Saadati et al. [24] investigated the inferential methods for the PT distribution under progressive type-II censoring.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with a distribution function $F(x)$ and probability density function $f(x)$. Then, an observation X_j is called an upper record value of this sequence if $X_j > \max\{X_1, \ldots, X_{j-1}\}, j > 1$. More specifically, if we define the sequence $\{U(n), n \geq 1\}$ as

$$U(1) = 1, \quad U(n) = \min\left\{j : j > U(n-1),\ X_j > X_{U(n-1)}\right\}$$

for $n \geq 2$, then $\{U(n), n \geq 1\}$ is called the sequence of upper record times and the sequence $\left\{X_{U(n)}, n \geq 1\right\}$ is a sequence of upper record values. Record values arise naturally in many real-life applications, including data related to weather, sports, economics, and reliability. For more details, see Arnold et al. [6]. For some recent works on statistical inference using record data, see for example Asgharzadeh et al. [8, 9], Aldallal [4], Chaturvedi and Malhotra [13], Bagheri et al. [10], Basiri et al. [11], Sing et al. [25], Qazi et al. [19], Raqab et al. [20] and Aly et al. [5].

The PT distribution is a heavy tailed distribution since its moments beyond a certain power do not exist (Klugman et al. [16]). It follows that large values are likely to occur frequently. Therefore, extreme values from this distribution deserve special attention, as they are very important to studying the nature of the phenomena being modeled by this distribution. In such a setting, records, especially upper record values arise naturally and have obvious importance. An example of such a situation is given by Hayek et al. [14] where they utilized the theory of record-breaking data to study the evolution of temperature and precipitation during 2003-2019 in Lebanon. They were concerned with the prediction of the intensity of the upcoming ¡°highest¡± temperature and precipitation and to compute the timing probabilities for the future record. There are several other natural occurrences of records are in economics, actuarial science, sports and reliability studies (Arnold et al. [6]) where the prediction of future records is essential for taking remedial and corrective actions.

Although the statistical inference for PT distribution based on complete and censored data has been discussed, to the best of our knowledge, the inference based on record data has not previously been studied in the literature. In this paper, we will discuss the maximum likelihood and Bayesian estimation for the PT model based on record data. We will also consider predicting future records based on the first n observed records.

The paper is organized as follows. We developed the maximum likelihood and Bayesian estimators of the parameters in Section 2. Classical and Bayesian point predictors of future records are obtained in Section 3 and were applied to two real data sets on precipitation and Covid 19 in Section 4. In Section 5 we investigated the performance of the estimators and predictors through a simulation study.

2. Estimation of the parameters

Suppose we observe the first n upper record values $X_{U(1)} = x_1, X_{U(2)} = x_2, \ldots, X_{U(n)} = x_n$ from the $PT(\theta, \sigma)$ distribution. For notational simplicity, we will use X_i instead of $X_{U(i)}$. The likelihood function is given (see Arnold et al., [6]) by

$$L(\theta, \sigma \mid \mathbf{x}) = f\left(x_n; \theta, \sigma\right) \prod_{i=1}^{n-1} \frac{f\left(x_i; \theta, \sigma\right)}{1 - F\left(x_i; \theta, \sigma\right)},$$

where $f\left(x_i; \theta, \sigma\right)$ and $F\left(x_i; \theta, \sigma\right)$ are, respectively, the density and distribution functions of the $PT(\theta, \sigma)$ distribution. The likelihood function is obtained as

$$L(\theta, \sigma \mid \mathbf{x}) = \frac{2\theta^n \sigma^\theta}{(x_n^\theta + \sigma^\theta)^2} \prod_{i=1}^{n} x_i^{\theta-1} \prod_{i=1}^{n-1} (x_i^\theta + \sigma^\theta)^{-1}. \tag{3}$$

2.1. Maximum likelihood estimates

Taking the logarithm of (3), we obtain the log-likelihood function

$$\ln L(\theta,\sigma \mid \mathbf{x}) = \ln 2 + n\ln\theta + \theta\ln\sigma - 2\ln(x_n^\theta+\sigma^\theta) + (\theta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n-1}\ln(x_i^\theta+\sigma^\theta). \tag{4}$$

On differentiating (4) with respect to θ and σ and equating partial derivatives to zero, we obtain the estimating equations

$$\frac{\partial \ln L(\theta,\sigma \mid \mathbf{x})}{\partial\theta} = \frac{n}{\theta} + \ln\sigma - 2\frac{x_n^\theta\,\ln x_n + \sigma^\theta\,\ln\sigma}{x_n^\theta+\sigma^\theta} + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n-1}\frac{x_i^\theta\,\ln x_i + \sigma^\theta\,\ln\sigma}{x_i^\theta+\sigma^\theta} = 0. \tag{5}$$

$$\frac{\partial \ln L(\theta,\sigma \mid \mathbf{x})}{\partial\sigma} = \frac{\theta}{\sigma} - \frac{2\theta\,\sigma^{\theta-1}}{x_n^\theta+\sigma^\theta} - \sum_{i=1}^{n-1}\frac{\theta\,\sigma^{\theta-1}}{x_i^\theta+\sigma^\theta} = 0. \tag{6}$$

Equations (5) and (6) can be solved numerically to obtain $\widehat{\theta}$ and $\widehat{\sigma}$, the maximum likelihood estimates (MLEs) of θ and σ.

2.2. Bayes estimation

To obtain the Bayesian estimates of the unknown parameters θ and σ, we need to specify a joint prior for (θ,σ). Here, we consider the joint prior for (θ,σ) as

$$\pi(\theta,\sigma) \propto \theta^\gamma\sigma^{\theta b-1}c^{-\theta}, \quad \theta>0,\ 0<\sigma<d, \tag{7}$$

where the hyper-parameters γ, b, c, d are positive known constants and $d^b<c$. This joint prior distribution is a known prior distribution for Bayesian inference in the two-parameter Pareto distribution, see Lwin [18] and Arnold and Press [7]. We can rewrite this prior as

$$\pi(\theta,\sigma) = \pi(\theta)\pi(\sigma \mid \theta),$$

where $\pi(\theta)$ is the density function of a gamma distribution with parameters γ and $\log c - b\log d$, and $\pi(\sigma \mid \theta)$ is the density function of a power function distribution given as

$$\pi(\sigma \mid \theta) \propto b\,\theta\,\sigma^{b\theta-1}d^{-b\theta}, \quad 0<\sigma<d.$$

When $\gamma=-1, \mathrm{c}=1,\ \mathrm{b}=0$ and $d\to\infty$, it is reduced to the non-informative prior

$$\pi(\theta,\sigma) \propto \frac{1}{\theta\sigma}, \quad \theta>0, \sigma>0.$$

By combining the likelihood function and the joint prior density in (3) and (7), the joint posterior density function of θ and σ is

$$\pi(\theta,\sigma \mid \underline{x}) = \frac{1}{K(\underline{x})}2\theta^{n+\gamma}\,\sigma^{\theta(1+b)-1}\,(x_n^\theta+\sigma^\theta)^{-1}\prod_{i=1}^{n}x_i^{\theta-1}\,(x_i^\theta+\sigma^\theta)^{-1}, \qquad \theta>0,\ 0<\sigma<D, \tag{8}$$

where $D=\min(d,x_1)$ and

$$K(\underline{x}) = \int_0^\infty\int_0^D 2\theta^{n+\gamma}\,\sigma^{\theta(1+b)-1}\,(x_n^\theta+\sigma^\theta)^{-1}\prod_{i=1}^{n}x_i^{\theta-1}\,(x_i^\theta+\sigma^\theta)^{-1}\,d\sigma\,d\theta.$$

Under the squared error (SE) loss function, the Bayes estimate (BE) of any function of θ and σ, say $w(\theta,\sigma)$, is the posterior mean $\widehat{w}_{BE}(\theta,\sigma) = E[w(\theta,\sigma)|\underline{x}]$. By (8), we obtain

$$\widehat{w}_{BE}(\theta,\sigma) = \frac{1}{K(\underline{x})}\Big(\int_0^\infty \int_0^D w(\theta,\sigma) 2\theta^{n+\gamma}\,\sigma^{\theta(1+b)-1}\,(x_n^\theta+\sigma^\theta)^{-1}\prod_{i=1}^n x_i^{\theta-1}\,(x_i^\theta+\sigma^\theta)^{-1}\,d\sigma\,d\theta\Big). \tag{9}$$

Since the BEs of $\widehat{\theta}_{BE}$ and $\widehat{\sigma}_{BE}$ cannot be obtained in explicit forms, the Metropolis-Hastings (MH) algorithm (see for example, Robert and Casella [21]) can be used to calculate the BEs.

2.2.1. MH algorithm. Here the MH algorithm is used for generating random samples from the posterior distribution (8) assuming univariate normal candidate distributions for both parameters θ and σ. The steps are given in the following algorithm.

Algorithm 1

Step 1: Set the initial values $(\theta_0,\,\sigma_0)$ and set $k=1$;
Step 2: Based on the MH algorithm, we generate (θ_k,σ_k) from $\pi(\theta_{k-1},\sigma_{k-1}|\underline{x})$ using the bivariate normal distribution as proposal distribution;
Step 3: Set $k=k+1$;
Step 4: Repeat Steps 2 and 3, N times to get the MCMC samples $(\theta_1,\sigma_1),\ldots,(\theta_N,\sigma_N)$;
Step 5: Based on the above MCMC samples, the approximate Bayes estimates of θ and σ can be computed as

$$\widehat{\theta}_{BE} = \frac{1}{N}\sum_{k=1}^N \theta_k \quad \text{and} \quad \widehat{\sigma}_{BE} = \frac{1}{N}\sum_{k=1}^N \sigma_k,$$

respectively.

Based on the sample-based estimates of θ and σ generated using the previous algorithm, the 100γ-th simulated percentiles of θ and σ are easily obtained and then the simulated credible intervals (CIs) of θ and σ can be constructed. Based on N iterations and the corresponding values of $\theta_i's$, we can construct a $100(1-\gamma)\%(0<\gamma<1)$ Bayesian CI as $\left(\theta_{[\frac{\gamma}{2}N]},\theta_{[(1-\frac{\gamma}{2})N]}\right)$ where $\theta_{[\frac{\gamma}{2}N]}$ and $\theta_{[(1-\frac{\gamma}{2})N]}$ are the $\left[\frac{\gamma}{2}N\right]$-th smallest integer and the $\left[\left(1-\frac{\gamma}{2}\right)N\right]-$ th smallest integer of $\{\theta_t, t=1,2,\ldots,N\}$, respectively. The same method can be used to compute the Bayesian CI for σ.

3. Prediction

Here we investigate the problem of predicting the future records of $Y = X_s\,(s=n+1,n+2,\dots)$ based on the first n upper records $\boldsymbol{X}=(X_1,\ldots,X_n)$ from PT distribution and propose maximum likelihood and conditional predictors.

3.1. Maximum likelihood prediction

The maximum likelihood prediction method was first proposed by Kaminsky and Rodhin [15]. In this method, a generalized likelihood function is used to solve statistical problems involving both fixed unknown parameters and unobserved random variables. Kaminsky and Rodhin [15] used this method to predict the future order statistics and estimate the parameters involved in the model. Let $s>n$, then the predictive likelihood of x_s is given by (see Raqab et al., [20])

$$L^*\left(\theta,\sigma,x_s\right)=f\left(x_s;\theta,\sigma\right)\frac{\Big(\ln\big(1-F(x_n;\theta,\sigma)\big)-\ln\big(1-F(x_s;\theta,\sigma)\big)\Big)^{s-n-1}}{\Gamma(s-n)}\prod_{i=1}^{n}\frac{f\left(x_i;\theta,\sigma\right)}{1-F\left(x_i;\theta,\sigma\right)},\qquad x_s>x_n.$$

For the $PT\left(\theta,\sigma\right)$ distribution we have

$$L^*\left(\theta,\sigma,x_s\right)=\frac{2\theta\sigma^\theta x_s^{\theta-1}}{\left(x_s^\theta+\sigma^\theta\right)^2}\frac{\left[\ln\left(\frac{2\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)-\ln\left(\frac{2\sigma^\theta}{x_s^\theta+\sigma^\theta}\right)\right]^{s-n-1}}{\Gamma(s-n)}\prod_{i=1}^{n}\frac{2\theta\sigma^\theta x_i^{\theta-1}}{\left(x_i^\theta+\sigma^\theta\right)^2}\frac{\left(x_i^\theta+\sigma^\theta\right)}{2\sigma^\theta},\qquad x_s>x_n,$$

which simplifies to

$$L^*\left(\theta,\sigma,x_s\right)=\frac{2\theta^{n+1}\sigma^\theta x_s^{\theta-1}}{\left(x_s^\theta+\sigma^\theta\right)^2}\frac{\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}}{\Gamma(s-n)}\prod_{i=1}^{n}\frac{x_i^{\theta-1}}{\left(x_i^\theta+\sigma^\theta\right)},\qquad x_s>x_n. \tag{10}$$

The predictive log-likelihood function is given by

$$\begin{aligned}l^*\left(\theta,\sigma,x_s\right)=&\,c+(n+1)\ln(\theta)+\theta\ln(\sigma)+(\theta-1)\ln\left(x_s\right)-2\ln\left(x_s^\theta+\sigma^\theta\right)+(\theta-1)\sum_{i=1}^{n}\ln\left(x_i\right)\\&+(s-n-1)\ln\left[\ln\left(x_s^\theta+\sigma^\theta\right)-\ln\left(x_n^\theta+\sigma^\theta\right)\right]-\sum_{i=1}^{n}\ln\left(x_i^\theta+\sigma^\theta\right),\quad x_s>x_n.\end{aligned} \tag{11}$$

Let $\left(\widetilde{\theta},\widetilde{\sigma},\tilde{x}_s\right)$ be the value of θ,σ,x_s at which the predictive log-likelihood function is maximized, then $\tilde{x}_s$ is the maximum likelihood predictor (MLP) of x_s and $(\tilde{\theta},\tilde{\sigma})$ is the predictive maximum likelihood estimator of (θ,σ). The predictive likelihood equations are given by

$$\begin{aligned}\frac{\partial l^*}{\partial\theta}=&\frac{n+1}{\theta}+\ln(\sigma)+\ln\left(x_s\right)-2\frac{x_s^\theta\ln\left(x_s\right)+\sigma^\theta\ln(\sigma)}{x_s^\theta+\sigma^\theta}+\sum_{i=1}^{n}\ln\left(x_i\right)\\&+(s-n-1)\frac{\frac{x_s^\theta\ln(x_s)+\sigma^\theta\ln(\sigma)}{x_s^\theta+\sigma^\theta}-\frac{x_n^\theta\ln(x_n)+\sigma^\theta\ln(\sigma)}{x_n^\theta+\sigma^\theta}}{\ln\left(x_s^\theta+\sigma^\theta\right)-\ln\left(x_n^\theta+\sigma^\theta\right)}-\sum_{i=1}^{n}\frac{x_i^\theta\ln\left(x_n\right)+\sigma^\theta\ln(\sigma)}{x_i^\theta+\sigma^\theta}=0,\\\frac{\partial l^*}{\partial\sigma}=&\frac{\theta}{\sigma}-\frac{2\theta\sigma^{\theta-1}}{x_s^\theta+\sigma^\theta}+(s-n-1)\frac{\frac{\theta\sigma^{\theta-1}}{x_s^\theta+\sigma^\theta}-\frac{\theta\sigma^{\theta-1}}{x_n^\theta+\sigma^\theta}}{\ln\left(x_s^\theta+\sigma^\theta\right)-\ln\left(x_n^\theta+\sigma^\theta\right)}-\sum_{i=1}^{n}\frac{\theta\sigma^{\theta-1}}{x_i^\theta+\sigma^\theta}=0,\\\frac{\partial l^*}{\partial x_s}=&\frac{\theta-1}{x_s}-\frac{2\theta x_s^{\theta-1}}{x_s^\theta+\sigma^\theta}+(s-n-1)\frac{\frac{\theta x_s^{\theta-1}}{x_s^\theta+\sigma^\theta}}{\ln\left(x_s^\theta+\sigma^\theta\right)-\ln\left(x_n^\theta+\sigma^\theta\right)}=0.\end{aligned} \tag{12}$$

Numerical methods are needed to find $\left(\widetilde{\theta},\widetilde{\sigma},\tilde{x}_s\right)$, either by direct maximization of the predictive loglikelihood function given by equation (11) or by simultaneously solving the system of nonlinear equations given by (12).

3.2. The conditional mean and median predictors

Future record statistics satisfy the Markovian (memoryless) property. Therefore, the distribution of the future upper record X_s given the set of the first n upper records $\boldsymbol{X}=(X_1,\ldots,X_n)$ depends only on the current upper record X_n. Hence, the conditional density function of X_s given $X_1,\ldots,X_n$ is the same as the conditional density function of X_s given X_n. The conditional distribution of X_s given X_n is given by (Ahsanullah [2])

$$f\left(x_s;\theta,\sigma \mid x_n\right)=\frac{\Big(\ln\left(1-F(x_n;\theta,\sigma)\right)-\ln\left(1-F(x_s;\theta,\sigma)\right)\Big)^{s-n-1}}{\Gamma(s-n)}\frac{f\left(x_s;\theta,\sigma\right)}{1-F\left(x_n;\theta,\sigma\right)},\qquad x_s>x_n.$$

For the $PT\left(\theta,\sigma\right)$ distribution we obtain

$$\begin{aligned} f\left(x_s;\theta,\sigma \mid x_n\right)&=\frac{\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}}{\Gamma(s-n)}\frac{2\theta\sigma^\theta x_s^{\theta-1}}{\left(x_s^\theta+\sigma^\theta\right)^2}\frac{x_n^\theta+\sigma^\theta}{2\sigma^\theta}\\ &=\frac{\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}}{\Gamma(s-n)}\frac{\theta x_s^{\theta-1}\left(x_n^\theta+\sigma^\theta\right)}{\left(x_s^\theta+\sigma^\theta\right)^2},\qquad x_s>x_n. \end{aligned}\tag{13}$$

If the parameters are known, the best unbiased predictor (BUP) of $Y=X_s(s=n+1,n+2,\dots)$ can be found as the conditional expectation

$$\hat{Y}_{BUP}=E\left(X_s\mid x_n\right)=\int_{x_n}^{\infty}x_s f\left(x_s;\theta,\sigma\mid x_n\right)dx_s.\tag{14}$$

Therefore,

$$\begin{aligned} \hat{Y}_{BUP}&=\int_{x_n}^{\infty}x_s\frac{\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}}{\Gamma(s-n)}\frac{\theta x_s^{\theta-1}\left(x_n^\theta+\sigma^\theta\right)}{\left(x_s^\theta+\sigma^\theta\right)^2}dx_s\\ &=\frac{\left(x_n^\theta+\sigma^\theta\right)}{\Gamma(s-n)}\int_{x_n}^{\infty}\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}\frac{\theta x_s^\theta}{\left(x_s^\theta+\sigma^\theta\right)^2}dx_s. \end{aligned}$$

Applying integration by substitution with $u=\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\rightarrow du=\frac{\theta x_s^{\theta-1}}{x_s^\theta+\sigma^\theta}dx_s$. Noting that $x_s^\theta+\sigma^\theta=\left(x_n^\theta+\sigma^\theta\right)e^u$ and $x_s=\left(\left(x_n^\theta+\sigma^\theta\right)e^u-\sigma^\theta\right)^{\frac{1}{\theta}}$ we obtain

$$E\left(X_s\mid x_n\right)=\frac{1}{\Gamma(s-n)}\int_0^\infty u^{s-n-1}e^{-u}\left(\left(x_n^\theta+\sigma^\theta\right)e^u-\sigma^\theta\right)^{\frac{1}{\theta}}du,\tag{15}$$

which can be found numerically.

The conditional median predictor (CMP) can be obtained by solving

$$\int_{x_n}^{CMP}f\left(x_s;\theta,\sigma\mid x_n\right)dx_s=\frac{1}{2}.\tag{16}$$

Therefore, the CMP should satisfy

$$\frac{\left(x_n^\theta+\sigma^\theta\right)}{\Gamma(s-n)}\int_{x_n}^{CMP}\left[\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\right]^{s-n-1}\frac{\theta x_s^{\theta-1}}{\left(x_s^\theta+\sigma^\theta\right)^2}dx_s=\frac{1}{2}.$$

Applying integration by substitution with $u=\ln\left(\frac{x_s^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)\rightarrow du=\frac{\theta x_s^{\theta-1}}{x_s^\theta+\sigma^\theta}dx_s$. Noting that $x_s^\theta+\sigma^\theta=\left(x_n^\theta+\sigma^\theta\right)e^u$, we obtain

$$\int_{x_n}^{CMP}x_s f\left(x_s;\theta,\sigma\mid x_n\right)dx_s=\frac{1}{\Gamma(s-n)}\int_0^{\ln\left(\frac{CMP^\theta+\sigma^\theta}{x_n^\theta+\sigma^\theta}\right)}u^{s-n-1}e^{-u}du=\frac{1}{2}.$$

Let $M(s-n,1)$ be the median of the $Gamma(s-n,1)$ distribution, we obtain

$$CMP = \left(e^{M(s-n,1)}\left(x_n^\theta + \sigma^\theta\right) - \sigma^\theta\right)^{\frac{1}{\theta}}. \tag{17}$$

A simple, closed form, approximation of CMP may be obtained by using the Wilson-Hilferty transformation (Abramowitz and Stegun [1])

$$M(s-n,1) = (s-n)\left(1 - \frac{1}{9(s-n)}\right)^3. \tag{18}$$

If the parameters are unknown, the best unbiased predictor BUP and the conditional median predictor CMP can be approximated by substituting the MLEs of the unknown parameters.

The best unbiased and the conditional median predictors may be estimated by simulation as follows, consider the random variable Y_C defined by

$$Y_C = \ln\left(1 - F\left(x_n;\theta,\sigma\right)\right) - \ln\left(1 - F\left(X_s;\theta,\sigma\right)\right)$$

It is shown in Aly et al. [5] that $Y_C \sim \text{Gamma}(s-n,1)$. See also Lee et al. [17]. It follows that, for given values of the parameters, the BUP and the CMP of X_s can be estimated using the following algorithm:

Algorithm 2

Step 1: Generate y_C from Gamma $(s-n,1)$ distribution.
Step 2: Calculate x_s by inverting

$$y_C = \ln\left(1 - F\left(x_n;\theta,\sigma\right)\right) - \ln\left(1 - F\left(x_s;\theta,\sigma\right)\right).$$

This leads to

$$y_c = \ln\left(\frac{x_s^\theta + \sigma^\theta}{x_n^\theta + \sigma^\theta}\right) \rightarrow x_s = \left(e^{y_c}\left(x_n^\theta + \sigma^\theta\right) - \sigma^\theta\right)^{\frac{1}{\theta}}$$

Step 3: Repeat steps 1 and 2, N times.
Step 4: Obtain the simulated values of BUP and CMP as the mean and median of the N simulated values of X_s.

3.3. Bayesian predictor

The predictive probability density function of X_s given X_n is

$$f^*\left(x_s \mid \boldsymbol{x}\right) = \int_0^\infty \int_0^D f\left(x_s;\theta,\sigma \mid x_n\right)\pi(\theta,\sigma \mid \boldsymbol{x})d\sigma d\theta.$$

The Bayes predictor of X_s is given by

$$\begin{aligned} E^*\left(X_s \mid x_n\right) &= \int_{x_n}^\infty x_s f^*\left(x_s \mid x_n\right)dx_s \\ &= \int_{x_n}^\infty x_s\left[\int_0^\infty \int_0^D f\left(x_s;\theta,\sigma \mid x_n\right)\pi(\theta,\sigma \mid x)d\sigma d\theta\right]dx_s \\ &= \int_0^\infty \int_0^D \left[\int_{x_n}^\infty x_s f\left(x_s;\theta,\sigma \mid x_n\right)dx_s\right]\pi(\theta,\sigma \mid x)d\sigma d\theta \\ &= \int_0^\infty \int_0^D \mathcal{J}(\theta,\sigma)\,\pi(\theta,\sigma|x)d\sigma d\theta, \end{aligned} \tag{19}$$

where

$$\mathcal{J}(\theta,\sigma)=\int_{x_n}^{\infty} x_s f\left(x_s;\theta,\sigma \mid x_n\right) dx_s.$$

Due to the complicated form of $f^*\left(x_s \mid x_n\right)$, the Bayes predictor of X_s cannot be computed explicitly. Here, we propose the following algorithm to compute a simulation-based consistent estimate of $f^*\left(x_s \mid x_n\right)$.

Algorithm 3

Step 1: Set the initial values $(\theta_0,\, \sigma_0)$ and set $k=1$;
Step 2: Generate (θ_k,σ_k) from $\pi(\theta_{k-1},\sigma_{k-1}|\underline{x})$ using the MH algorithm as described before;
Step 3: Set $k=k+1$;
Step 4: By repeating Steps 2 and 3 N times, we get the MCMC samples $(\theta_1,\sigma_1),\ldots,(\theta_N,\sigma_N)$;
Step 5: Now, based on the samples (θ_1,σ_1), $\ldots$ (θ_N,σ_N), a simulation-based consistent estimator of $f^*(x_s|\mathrm{x})$ is

$$f^*(x_s|\mathrm{x})=\frac{1}{N}\sum_{k=1}^{N} f(x_s;\theta_k,\sigma_k|x_n); \tag{20}$$

Hence, by using (19) and (20), we can approximate the Bayes predictor of $Y=X_s$ as

$$\begin{aligned}\hat{Y}_{BP} &= \int_{x_n}^{\infty} x_s\, \frac{1}{N}\sum_{k=1}^{N} f(x_s;\theta_k,\sigma_k|x_n)\, dx_s \\ &= \frac{1}{N}\sum_{k=1}^{N}\int_{x_n}^{\infty} x_s\, f(x_s;\theta_k,\sigma_k|x_n)\, dx_s \\ &= \frac{1}{N}\sum_{k=1}^{N}\mathcal{J}(\theta_k,\sigma_k).\end{aligned} \tag{21}$$

4. Real data examples

Here, we present the analysis of monthly total precipitation and COVID-19 record data from PT distribution.

4.1. Precipitation data

Here we illustrate application of the results in Sections 2 and 3 to the monthly total precipitation data during April recorded at New Jersey (Statewide) from 2000 to 2021, see the link [http://climate.rutgers.edu/]. The data are as follows: The Kolmogorov-Smirnov (K-S) test is used to fit the PT distribution

3.33	1.83	3.83	2.86	5.32	4.48	3.64	8.45	2.67	4.40	2.51
5.67	2.87	2.60	4.42	2.67	2.26	3.82	4.17	3.97	3.98	2.35

to the above data. The K-S statistics of the distance between the fitted and the empirical distribution functions (based on the MLEs $\theta=2.235$ and $\sigma=1.83$) is 0.20 and the corresponding p-value is 0.326. Therefore, fitting the PT distribution to the above data is acceptable. From the above data, the upper records until 2021 are :

3.33	3.83	5.32	8.45

For these upper records, we obtained the MLE and Bayes estimates of θ and σ. Bayes estimates and corresponding CIs are obtained using MH algorithm with $N = 10,000$ replicates. In this algorithm, we considered the MLEs $\widehat{\theta}$ and $\widehat{\sigma}$ as the initial values of θ and σ. To calculate the Bayes estimates and corresponding CIs, since we have no prior information, the improper priors on θ and σ, i.e. $c = 1$, $\gamma = -1$, $b = 0$ and $d = \infty$, are used. The results are given in Table 1.

The convergence of samples generated by the MH algorithm can be checked using trace plots. The trace plots of the 10,000 iterations of θ and σ are presented in Figure 1. These plots show that the values are randomly scattered around the average and represent the fine mixing of the chains. Also, the histograms of the 10,000 MH samples of θ and σ in Figure 1, show that choosing normal distributions as proposal distributions is almost reasonable.

We now apply the methods proposed in Section 3 to derive the point predictors of X_s $(s = 5, 6, \ldots,)$. Based on the above four observed records, we predicted the upper records X_5, X_6 and X_7 and presented the results in Table 2.

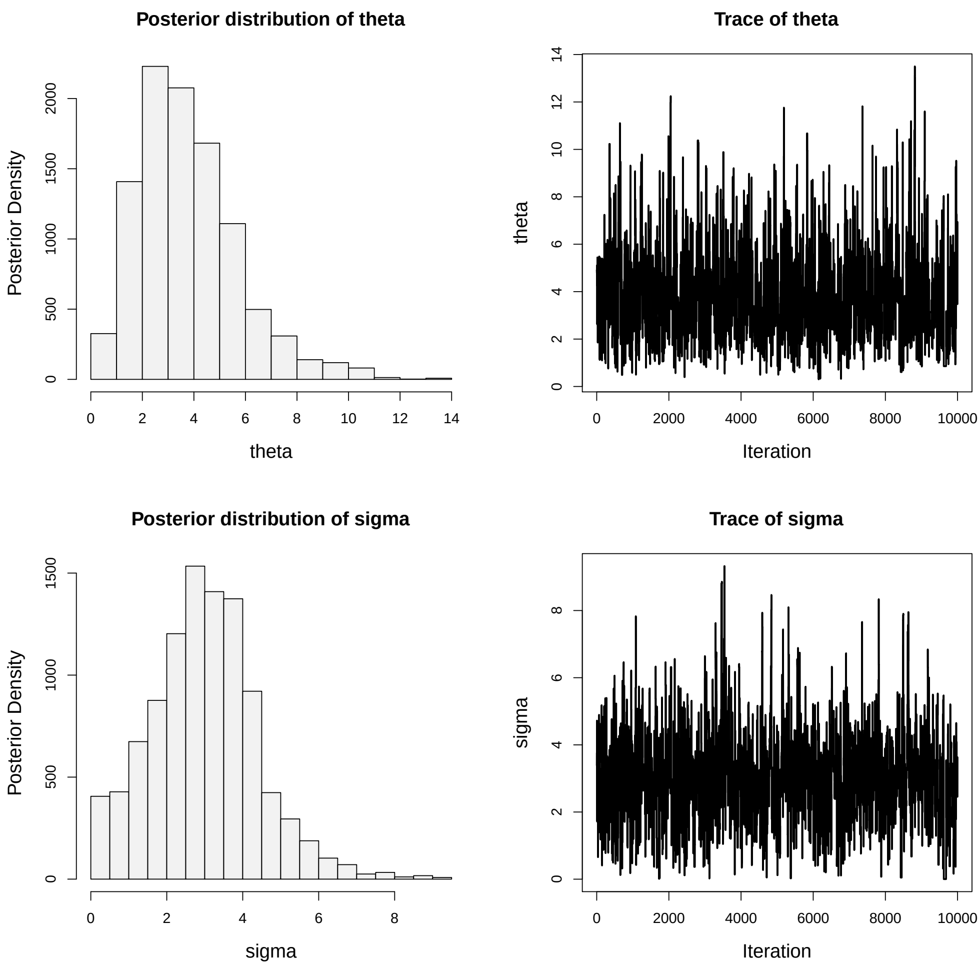

Figure 1. Histogram and trace plots for the precipitation data.

The results show the values of the four predictors developed in this paper. It is clear that the Maximum likelihood predictor is consistently smaller than the other predictors. The BUP and the Bayes estimator are always close to each other. The CMP predictor for the immediately following record is very close to

Table 1. Point estimates and %95 Bayesian CI for the precipitation data

MLE		BEs		Bayesian CI	
$\widehat{\theta}$	$\widehat{\sigma}$	$\widehat{\theta}$	$\widehat{\sigma}$	$\widehat{\theta}$	$\widehat{\sigma}$
4.911	3.032	3.773	2.979	(0.920, 8.727)	(0.365, 6.088)

Table 2. Different predictors for the precipitation data.

	MLP	BUP	CMP	BP
X_5	8.45000	10.6694	9.83133	9.78508
X_6	9.72874	13.3591	11.8237	14.7224
X_7	11.1354	16.7627	14.6206	16.0840

the BUP and the Bayes predictor (an observed feature of the CMP predictor that will be supported by the simulation results in the next section).

4.2. Covid data

The second data is a Covid-19 data belong to Canada of 25 days, from 10 April to 4 May 2020, see the link [https://covid19.who.int/]. See also Almetwally et al. [3]. This data formed of mortality rate and they are as follows:

3.1091	3.3825	3.1444	3.2135	2.4946	3.5146	4.9274	3.3769	6.8686	3.0914
4.9378	3.1091	3.2823	3.8594	4.0480	4.1685	3.6426	3.2110	2.8636	3.2218
2.9078	3.6346	2.7957	4.2781	4.2202					

It can be checked that the PT distribution is fitted well to the data.
The records extracted from the above data set are:

3.1091	3.3825	3.5146	4.9274	6.8686

Based on the above records, the MLEs and Bayes estimates of θ and σ are presented in Table 3. The trace plots and the histograms of the MH sequences of of θ and σ are given in Figure 2. Also, based on the above five observed records, the prediction of the sixth, seventh and eighth upper records X_6, X_7 and X_8 are given in Table 4.

Table 3. Point estimates and %95 Bayesian CI for Covid data.

MLE		BEs		Bayesian CI	
$\widehat{\theta}$	$\widehat{\sigma}$	$\widehat{\theta}$	$\widehat{\sigma}$	$\widehat{\theta}$	$\widehat{\sigma}$
6.946	2.982	5.770	2.619	(1.728, 12.094)	(0.820, 4.166)

Table 4. Different predictors for Covid data.

	MLP	BUP	CMP	BP
X_6	6.86860	8.03149	7.57987	8.70066
X_7	7.64306	9.34186	8.77517	10.3658
X_8	8.47786	10.9479	10.0565	11.5895

The pattern of the results for the predictors in this example follows closely the pattern observed earlier with the precipitation data.

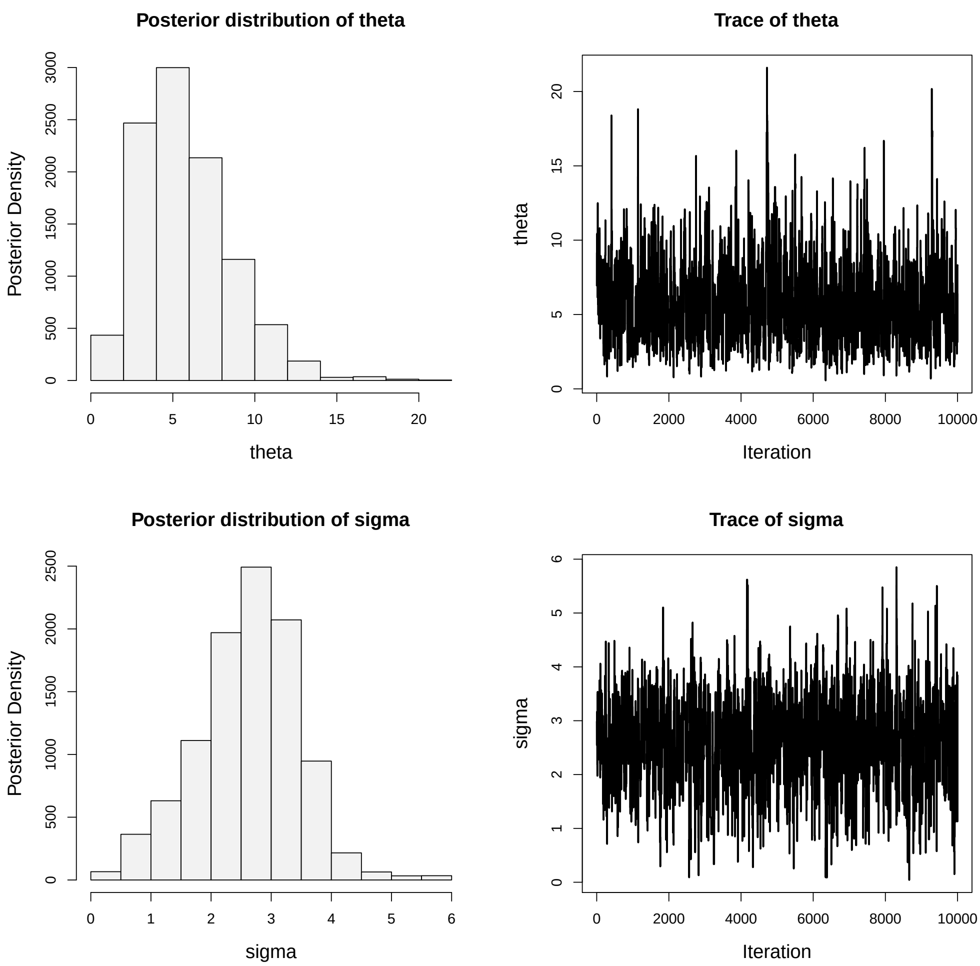

Figure 2. Histogram and trace plots for the Covid data.

5. Simulation study

To compare the performances of the different methods of estimation and prediction presented in the previous sections, a Monte Carlo simulation study is carried out. The performances of the MLEs and Bayes estimates are compared in terms of bias and mean square error (MSE). In this simulation, the values of model parameters are considered to be $(\theta, \sigma) = (2.0, 0.5)$ and $(3.0, 0.5)$ and also, we choose $n = 4, 5, \ldots, 12$. For computing Bayesian estimates and corresponding CIs , we considered the informative and noninformative priors on (θ, σ) as follows:

$$\text{Prior 1: } c = 5,\ \ \gamma = 0.02,\ \ b = 1,\ \ d = 2,$$
$$\text{Prior 2: } c = 1,\ \ \gamma = -1,\ \ b = 0,\ \ d = \infty.$$

Tables 5 and 6 provide the bias and MSEs of MLEs and Bayes estimates of θ and σ over 1000 replications. Also, we compare the performances of the different proposed predictors in term of bias and mean square prediction error (MSPE). We compare the performances of the point predictors MLP, BUP and CMP and Bayesian predictor of $Y = X_s, (s = n + 1, n + 2, \ldots)$ for different chooses n and s. The results are reported in Table 7. All calculations are performed using R software.

From Table 5, the biases and MSEs of the point estimators of θ follow consistently the following pattern. The bias and MSE of the MLE of θ are the highest under almost all situations considered. The MSE of the Bayesian estimator under the informative prior appears to be consistently smaller than the MSE under the non-informative prior. The bias performance of the Bayes estimators is less clear, however, it appears that for values of n less than 9, the bias under the informative prior is less than the bias under the non-informative priors.

For estimating the parameter σ, from Table 6, it appears that the Bayes estimator under the noninformative prior have the worst performance in term of MSE. The Bayes estimator under the informative prior is uniformly better than the MLE in terms of bias and MSE.

From Table 7, the performance of the point predictors appear to follow a clear pattern as follows. The MLP predictor appears to have the largest bias and MSPE under all situations considered. The BUP and the Bayes predictor under the informative prior appear to have the best overall performance with Bayes predictor having often less bias but larger MSPE than the BUP. The CMP appears to be consistently negatively biased and, for larger values of θ, it appears to have the least MSPE among all predictors for predicting the next immediate record. However, for further records, the BUP and the Bayes under informative priors always have smaller MSPEs.

Table 5. Simulated biases and MSEs of the MLE and Bayes estimates of θ.

		$\theta = 2.0$, $\sigma = 0.5$			$\theta = 3.0$, $\sigma = 0.5$		
n		MLE	BEs		MLE	BEs	
			Prior 1	Prior 2		Prior 1	Prior 2
4	Bias	0.9033	-0.3743	0.5246	0.9458	0.5109	-1.2490
	MSE	1.1802	0.1844	0.7446	1.2321	1.0181	1.6203
5	Bias	0.8080	-0.2412	0.5112	0.9413	0.4833	-1.0681
	MSE	0.8359	0.1066	0.6449	1.2031	0.6315	1.2174
6	Bias	0.7602	-0.2289	0.4892	0.9149	0.4675	-0.9188
	MSE	0.6684	0.0729	0.4666	1.1518	0.6131	0.9257
7	Bias	0.7176	0.2261	0.4936	0.9125	-0.6012	-0.7860
	MSE	0.5598	0.0704	0.4001	1.1431	0.4943	0.6864
8	Bias	0.6295	0.1649	0.4317	0.7808	-0.4578	-0.7357
	MSE	0.4154	0.0342	0.3090	0.7745	0.3397	0.5863
9	Bias	0.5099	0.3444	0.3415	0.6889	-0.3560	0.3764
	MSE	0.2663	0.0192	0.2101	0.5862	0.2577	0.4313
10	Bias	0.6879	0.1274	0.3175	0.6629	-0.1968	0.0122
	MSE	0.4822	0.0190	0.1182	0.5187	0.1699	0.1783
11	Bias	0.6358	0.0788	0.2500	0.6547	0.2562	0.1285
	MSE	0.4073	0.0080	0.0739	0.4878	0.1114	0.1741

Table 6. Simulated biases and MSEs of the MLE and Bayes estimates of σ.

n		$\theta = 2.0$, $\sigma = 0.5$			$\theta = 3.0$, $\sigma = 0.5$		
		MLE	BEs		MLE	BEs	
			Prior 1	Prior 2		Prior 1	Prior 2
4	Bias	0.5623	0.4746	0.4791	0.3331	0.2544	0.9519
	MSE	0.5135	0.4082	0.4808	0.1635	0.1104	1.0837
5	Bias	0.4995	0.4462	0.4691	0.3217	0.2512	0.8366
	MSE	0.4326	0.4013	0.4598	0.1592	0.1106	0.8801
6	Bias	0.4464	0.3811	0.4221	0.3135	0.2495	0.7714
	MSE	0.2983	0.2873	0.3941	0.1522	0.1103	0.7720
7	Bias	0.3912	0.3216	0.3653	0.3028	0.2465	0.7262
	MSE	0.2353	0.2075	0.2924	0.1492	0.1125	0.7092
8	Bias	0.3714	0.3067	0.3205	0.2763	0.2266	0.6651
	MSE	0.2200	0.1914	0.2296	0.1251	0.0965	0.5967
9	Bias	0.3566	0.2811	0.2850	0.2626	0.2171	0.6350
	MSE	0.1992	0.1706	0.1820	0.1205	0.0964	0.5687
10	Bias	0.3412	0.2736	0.2936	0.2582	0.2162	0.6119
	MSE	0.1883	0.1581	0.1627	0.1143	0.0924	0.5270
11	Bias	0.2769	0.2553	0.2740	0.2340	0.1965	0.5406
	MSE	0.1375	0.1309	0.1378	0.1011	0.0848	0.4477

Table 7. Simulated biases and MSPEs of different predictors.

n	s		$\theta = 2.0$, $\sigma = 0.5$					$\theta = 3.0$, $\sigma = 0.5$				
			MLP	BUP	CMP	BP		MLP	BUP	CMP	BP	
						Prior 1	Prior 2				Prior 1	Prior 2
2	3	Bias	-1.2962	0.4045	-0.7849	0.3901	0.0818	-0.7168	0.3203	-0.3763	0.1581	0.5556
		MSPE	3.1168	1.2749	1.8637	1.3887	1.3927	1.1469	0.7420	0.7088	0.9657	1.1083
	4	Bias	-2.7248	0.2412	-1.7045	-0.1805	-0.6168	-1.4084	0.2236	-0.7866	0.2398	-0.2923
		MSPE	11.134	2.4968	5.9763	2.7233	4.1818	3.9683	1.7455	2.3796	1.9557	2.0092
3	4	Bias	-1.4866	0.5232	-0.6336	0.0425	0.3605	-0.8797	0.3620	-0.3172	0.4204	0.2032
		MSPE	3.7715	1.3541	1.6168	1.3785	1.5458	1.6034	0.8684	0.7832	1.0795	1.1500
	5	Bias	-3.3549	0.1636	-1.8093	-0.4787	-0.7376	-1.5868	0.2558	-0.7002	0.1202	-0.1549
		MSPE	15.681	2.6472	6.5663	3.7590	4.5769	4.6264	1.7958	2.2617	2.0424	2.2127
4	5	Bias	-1.8950	0.5838	-0.7243	0.2371	-0.0134	-1.0903	0.3623	-0.3598	0.0834	0.3647
		MSPE	5.5037	1.5819	1.9239	1.6511	1.6855	2.3037	1.0761	1.0438	1.1127	1.1895
	6	Bias	-4.1898	0.1845	-2.1276	-0.3072	-0.7651	-1.9102	0.2780	-0.7821	0.0936	-0.2662
		MSPE	21.814	2.5759	7.5742	3.6265	4.8932	6.4451	2.1767	2.8685	2.3115	2.4434
5	6	Bias	-2.4499	0.5922	-0.9486	0.1874	0.0314	-1.2555	0.4530	-0.3579	0.1240	0.3432
		MSPE	8.1345	1.5699	2.3595	1.5770	1.6561	2.8176	1.2483	1.1396	1.2898	1.3302
	7	Bias	-5.1719	0.2856	-2.5141	-0.6156	-0.9408	-2.3921	0.2787	-0.9829	-0.1875	-0.4326
		MSPE	31.640	2.3586	9.4019	4.8181	5.5586	9.1496	2.1620	3.4103	3.0171	3.3748
6	7	Bias	-3.2635	0.6256	-1.3019	0.0911	-0.1629	-1.5655	0.4696	-0.4744	0.2752	0.0715
		MSPE	12.769	1.4947	3.0156	1.8914	2.1169	4.1045	1.3305	1.4080	1.3759	1.3906
	8	Bias	-6.5273	0.2757	-3.1699	0.2226	-0.1854	-2.8244	0.2405	-1.1768	-0.2309	-0.4452
		MSPE	47.325	2.4953	13.116	5.1736	5.7660	11.470	2.3055	3.9293	2.8501	3.2632
7	8	Bias	-4.1422	0.7563	-1.6462	0.0410	-0.1693	-1.7427	0.5733	-0.4799	0.1240	0.0506
		MSPE	19.175	1.4503	3.8286	1.9065	2.4937	4.9607	1.4640	1.5174	1.5478	1.5801
	9	Bias	-8.3213	0.3564	-3.9810	-0.8115	-0.9052	-3.3566	0.2499	-1.4048	-0.2790	-0.5381
		MSPE	74.135	2.0830	18.661	5.1848	5.6547	15.354	2.4738	4.8518	3.1293	3.9868

6. Conclusion

In this paper, we considered the estimation of unknown parameters and the prediction of future records from the new Pareto-type distribution based on records. Classical and Bayesian approaches were used for estimation and prediction. Two real data sets and a Monte Carlo simulation study have been conducted to analyze and evaluate different methods. Comparing different estimators, from simulation results, it appears that the Bayes estimator under the informative prior is uniformly better than the MLE and the Bayes estimator under the non informative prior in terms of MSE. Comparing different predictors, the BUP and the Bayes predictor under informative priors are generally better than other predictors for predicting further records.

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