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Solution of a Model for Pricing Options with Hedging Strategy Through Nonlinear Filters

Luis Sánchez [1,2,3,4,*], Freddy Sánchez P. [5], Freddy Sánchez A. [5] and Norma Bargary [1,2,3]

[1] *Faculty of Science & Engineering, Department of Mathematics & Statistics, University of Limerick, Ireland*
[2] *Insight Centre for Data Analytics, Ireland*
[3] *Confirm Centre for Smart Manufacturing, Ireland*
[4] *Departamento de Matemática y Estadística, Instituto de Ciencias Básicas, Universidad Técnica de Manabí, Ecuador*
[5] *Escuela de Administración Comercial y Contaduría Pública, Facultad de Ciencias Económicas y Sociales, Universidad de Carabobo, Venezuela*

Abstract A methodology is presented to estimate the solution states for a non-linear price problem, a model for pricing options with a hedging strategy in the Föllmer-Schweizer sense is defined. The problem is to determine the price of a contingent claim, that is a contract, that pays of an amount at time t in a incomplete market, that is not possible to replicate a payoff by a controlled portfolio of the basic securities. Two algorithms are presented to estimate the solution of the presented problem, the nested sequential Monte Carlo (NSMC) and space-time particle filter (STPF) are defined from sequences of probability distributions. The methodology is validated to use real data from option Asian, the states in real-time are estimated, that is proposed on the basis of the price model. The efficiency of the forecasts of the model is compared. Finally, one goodness-of-fit measure to validate the performance of the model are used, obtaining insignificant estimation error.

Keywords Non-linear price problems, Stochastic Differential Equation, Hedging strategy, Nested sequential Monte Carlo, Space-time particle filter.

AMS 2010 subject classifications 62L10, 62L12, 62L20

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1. Introduction

When a premium pays by a borrower to a creditor for the use of money for a period, that is defined by the interest rate, that is expressed like a percentage per year, interest rates is comparable, that is often known as the price of money. That should refer to interest rates, such as taxes (there are several). That are when the borrowers pay to lenders, the borrowers pay in debt securities and deposits, and the prices of debt and deposits are derived from cash. When the discounted cash flows for the interest rates are paid then the flows pay for the bonds in the future. There are many different interest rates like; time deposit interest rate, base interest rate, report rate, call deposit interest rate, bank interest rate, corporate bond interest rate, government bond interest rate, tradable certificate of deposit interest rate, company commercial paper interest rate, treasury bill interest rate, floating interest rate, effective interest rate, among others. The interest rate models are applied in the economic, financial, energy, productive fields, among others.

When institutional investors and bank are interested in evaluating callable bonds, that is the reason that the interest models are applied in many fields. A mathematical model describes the future evolution of interest rates by the

*Correspondence to: Luis Sánchez (Email:uccursos@gmail.com). Faculty of Science & Engineering, Department of Mathematics & Statistics, University of Limerick, Ireland.

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future evolution of the short rate, that is defined as a short-rate model, that is defined in the context of interest rate derivatives. Many models of interest rates in the discrete and continuo field are proposed, the time series models (in discrete developments) and stochastic differential equations (in continuous cases) are used, [34], [35] and [36], [48] and the [14], and subsequently, [6], [12] and the exponential model of Vasicek, among others. A great breakthrough took place that the CIR model was adopted by Heston [21], who derived a closed-form pricing formula for European options. However, a lot of empirical evidence has shown that the Heston model is not perfect either, [2], [40], [7], [18], another stochastic models are proposed by [20], [19], [10], [9], [51], [50], [23], [25], among others.

There are problems with those models, the models are not calibrated to the yield curve. In fact, the models could not fit an arbitrary prescribed yield curve, the coefficients of the stochastic differential equation (SDE) are assumed constant then the parameters as level of interest rates are fixed, that is the parameters as functions of time are fixed. Additionally, those models do not use strategies to reduce the risk of loss of an existing position. The price of the interest rate are defined as functions of state variables, that is as spot interest rate, spot forward rate, long-term interest rate, among others. Additionally, the consistencies are required, that is the observed initial term structures of interest rates and/or volatilities of interest rates. For this reason, a mathematical model is needed when it concerns to fit the yield curve and to handle the interest rates as a variable, then, a reliable models for interest rate are needed, [31] and [32]. Motivated by the above discussions, a new model by replacing the constant parameters in the stochastic process modeling is needed. The newly proposed model needs to assume parameters of a SDE like random variables, mainly, theses parameters can be concerning to the interest rates. Also, a hedging strategy is needed to potentially help reduce the risk of loss of an existing position, hedging done by companies can help provide greater certainty of future costs.

In the real world, this phenomenon can be modeled by Markov process, that is a continuous time process with continuous trajectories, but not differentiable, that is, the states is evolved over time, [31] and [32]. However, the states of the dynamic systems are partially observed, only approximate measurements of the underlying system are obtained, the samples incomplete, errors and a non-linear behavior are observed. The objectives of the diffusion processes are the prediction of hidden states (price). The Bayesian techniques are indicated, that is to study methods for dynamic systems. The state space formulation introduces the filter algorithms, the a posteriori probability density of the hidden states of the system is calculated. The space state models are Markov processes, that describes the probabilistic dependence between two stochastic processes, that is based on the states and observations equations. There is an extensive literature, see: [29], [15], among others. The structures of that models are widely used, in many applications such as: [12], [21], [37], [11], [13], [33], [44], [16], [47], [46], [45], [22], [16], [1], [24], [27], [26], [42], [43], among others.

A model in recent years is used, the Langevin-type equation for a stochastic (real) quantity x_t (price):

$$d\mathbf{x}_t = \mathcal{M}(\mathbf{x}_t, t)dt + \sigma(\mathbf{x}_t, t)dL_t \tag{1}$$

where $\mathcal{M} : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}^n$ is a non-linear function of the state equation, $\sigma : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}^{n\times s}$ is a function of valued matrices, L_t is a Lévy stable motion with independent components, i.e. each component of L forms an independent scalar α-stable Lévy motion, that is defined as follows for $\alpha \in (0, 2]$. More precisely, the $\hat{I}$to stochastic calculus corresponds to consider dL_t similarly to dB_t (when $\alpha = 2$, that is the Lévy motion L_t coincides with a scaled Brownian motion $\sqrt{2}B_t$), a forward increment in time (that is $dL(t, dt) = L(t + dt) - L(t)$). The value of x at time t by events prior to the application of the stochastic for dL_t is determined, that is from time t to $t + dt$.
The integral form (1) is given, by:

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int \mathcal{M}(\mathbf{x}_t, t)dt + \int \sigma(\mathbf{x}_t, t)dL, \tag{2}$$

where the last term corresponds to a stochastic integration of a stochastic process. The integration of a stochastic process $\sigma(x_t, t)$ with respect to the Lévy motion L is rather straightforward in the case of step processes:

$$\sigma(\mathbf{x}_t, t) = \sigma_n, \qquad for \quad t \in (t_n, t_{n+1}), \qquad n = 0, 1, \ldots, N-1;$$

$$\int \sigma(t) dL = \sum_{n=0}^{N-1} \sigma_n (L(t_{n+1}) - L(t_n)) \tag{3}$$

Since the equation (1) in continuous time is defined, that is the equation of states; the behavior of the system is imperfectly observed through an experimental process $\mathbf{y}_t$, that is related to the process $\mathbf{x}_t$, the observation equation (12) is modeled. That is using Bayesian algorithms, the a posteriori marginal distribution $p(\mathbf{x}_t|\mathbf{y}_{1:t})$ in time t in two steps is estimated, prediction and update, where $\mathbf{x}_{0:t} = (\mathbf{x}_0, \ldots, \mathbf{x}_t)$ are the unknown solutions states, and $\mathbf{y}_{1:t} = (\mathbf{y}_1, \ldots, \mathbf{y}_t)$ are the observed solutions. Then, a posteriori marginal density $p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1})$ is known at a time t, the updated density $p(\mathbf{x}_t|\mathbf{y}_{1:t-1})$ of the state in time t are obtained, that is using the equation of Chapman-Kolmogorov transition:

$$p(\mathbf{x}_t|\mathbf{y}_{1:t-1}) = \int p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1}) p(\mathbf{x}_t|\mathbf{x}_{t-1}) d\mathbf{x}_{t-1} \tag{4}$$

When a new observation $\mathbf{y}_t$ is obtained, the posterior density is updated to use an equation of observation and the Bayes Theorem as:

$$p(\mathbf{x}_t|\mathbf{y}_{1:k}) = \frac{p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1}) p(\mathbf{y}_t|\mathbf{x}_t)}{\int p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1}) p(\mathbf{y}_t|\mathbf{x}_t) d\mathbf{x}_t} \tag{5}$$

The joint probability distribution of $P(\mathbf{x}_t, t, \theta)$ is interested; in particular, the estimation of the joint a posteriori distribution of the states $P(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})$, or the marginal distribution of a state $P(\mathbf{x}_t|\mathbf{y}_{1:t})$ are desired, where $\mathbf{x}_t$ represents a solution of the equation in (1).

The estimation of parameters and states through the transition density of the equation (1) in closed form are not evaluated, then techniques to approximate the solution are required. There are a variety of methods in the literature, those include maximum likelihood estimation; Monte Carlo techniques by Markov Chains (MCMC); and Monte Carlo Sequential (MCS) algorithms, such as: the Kalman filter (KF), extended Kalman filter (EKF), particle filter (PF), unscented particle filter (UPF) (see [29], [30], [28], [39], [8], [5], [42], [43], among others).

Indeed, the problem is defined to determine the price of a contingent claim $\mathbf{x} \geq 0$ of maturity t, that is a contract, that pays an amount $\mathbf{x}$ at time t. In a complete market, that is possible to construct a portfolio, that is achieved as final wealth the amount $\mathbf{x}$. Thus, the dynamics of the value of the replicating portfolio $\mathbf{x}$ are given by SDE with generator $\mathcal{M}$, with σ corresponding to the hedging portfolio. Then, the price at time t with the value at time t of the hedging portfolio is naturally associated. However, there are an infinite number of replicating portfolio that is allowed that the price is not well defined, the arbitrage pricing theory imposes restrictions on the integrability of the hedging portfolios. These assumptions are related to a risk- adjusted probability measure, the problem is well defined to use BSDE theory where there is an unique price and an unique hedging portfolio that is integrable ones under the primitive probability, that is by restricting admissible strategies to square. When the market is incomplete, that is not always possible to replicate a payoff by a controlled portfolio of the basic securities. The set super hedging strategies is nonempty that is much milder. Under this assumption, [31], [32] showed that there exists an upper price process $(\mathbf{x}_t)_{t\in[0,T]}$ that is associated with the solutions of nonlinear backward stochastic differential equations, in that paper a general filtration is considered, but here, for expository simplicity, we restrict to a Brownian filtration.

The solutions of option pricing problems from a model is obtained, that is via backward stochastic differential equations (BSDE):

$$\begin{aligned} d\mathbf{x}_t &= \mathcal{M}(\mathbf{x}_t, t)dt + \sigma(\mathbf{x}_t, t)d\mathbf{w}_t \\ d\mathbf{y}_t &= -f(\mathbf{x}_t, t, \mathbf{y}_t, \mathbf{z}_t)dt + \mathbf{z}_t d\mathbf{w}_t \end{aligned} \tag{6}$$

In the context of option pricing is driven by a Brownian motion **w**, **x** is a basket of financial underlying, **y** is the price process of the option, and **z** is a hedging strategy (possibly in the Föllmer-Schweizer sense) is related, see [31]. The driver f is linear in the classical pricing problem of options without early-exercise features, so today's price y_0 reduces to the expectation of the discounted option payoff under an equivalent martingale measure. The driver may become nonlinear, [4] considers different interest rates for borrowing and investing in a bond, and [3] considers the computing utility indifference prices.

Significant contribution of this work is as follows: (i) A mathematical model is based on stochastic differential equations that is defined to model the pricing options with hedging strategy. (ii) This model assume the parameters of a SDE like random variables, mainly, thoses parameters can be concerning to the interest rates, it can consider different interest rates for borrowing and investing in a bond, and considers the computing utility indifference prices. (iii) a Föllmer-Schweizer hedging strategy is adjusted where the price process of a option is related to this hedging strategy, this strategy is modelled by a SDE. (iv) the continuous system is given in (6) that is imperfectly observed, that is related through a process of observation, and the non-linear price problem cannot be evaluated in a closed way, then a methodology for estimating the solution for the non-linear price problem is proposed.

The objective is to introduce a methodology for estimating the solution states for the non-linear price problem. A model for pricing options by a hedging strategy in the Föllmer-Schweizer sense (6) is considered. Additionally, the difficulty is presented in the estimation of the states of the posterior marginal distribution that is jointed in the density transition of equation (1) that cannot be evaluated in a closed way then algorithms are proposed to estimate the states of the posterior marginal distribution, that are used to reconstruct the non-linear dynamic system sensitive to the initial conditions, those algorithms are the nested sequential Monte Carlo and space-time particle filter.

The document is organized as follows. In section 2 a brief description of the stochastic differential equation is given. Section 3 the nested sequential Monte Carlo and space-time particle filter are developed, in Section 4, the results are discussed; and in Section 5, a discussion and conclusions are established.

2. Stochastic Differential Equation

The sum of a call and a put option under different rates for borrowing and investing in the money market account is considered. The rate for borrowing is denoted by R and the one for investing by r is denoted. The fair of a straddle in this model is given by y_0, where (y, z) is the solution of the nonlinear BSDE, [4]:

$$dx_t = (bx_t + \sigma x_t h_t)dt + \sigma x_t dw_t$$

$$dy_t = (ry_t + \frac{b-r}{\sigma} z_t - (R-r)(y_t - \frac{z_t}{\sigma}) + z_t h_t)dt + z_t dw_t \tag{7}$$

where, now specify the discrete process h_{t_i}. Let ξ be a square-integrable contingent claim. In the context a strategy (z, π, ϕ), is called a nonadjusted hedging strategy against ξ when:

$$dz_t = rz_t dt + \pi\sigma(dw_t + \theta dt) + d\phi_t \tag{8}$$

where, the process ϕ is a RCLL semimartingale satisfying $\phi_0 = 0$. The process $(-\phi)$ is called the tracking error. In particular, the tracking error measures the spread between the contingent claim ξ and the portfolio value at the terminal time, and ϕ corresponds to the cost of introduced process by [17]. In the context, ϕ is a martingale orthogonal to $\int \sigma_s dw_s = lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\Delta_n} \sigma_{s_{t_{i-1}}}(w_{t_i} - w_{t_{i-1}})$, that is called a Föllmer-Schweizer hedging strategy.

Hence, using the Euler-Maruyama scheme:

$$x_{t+1} = x_t + (bx_t\Delta_t + \sigma x_t h_t)\Delta_t + \sigma x_t \Delta w_t \tag{9}$$

where, $\Delta_t = |t_{i+1} - t_i|$ with $i = 0, \ldots, N-1$, process of Wiener $\Delta w_t = w_t - w_{t-1}$ and x_t is a basket of financial underlings.

$$y_{t+1} = y_t + (ry_t + \frac{b-r}{\sigma} z_t - (R-r)(y_t - \frac{z_t}{\sigma}) + z_t h_t)\Delta_t + z_t \Delta w_t \tag{10}$$

where, y_t is the price process of the option and z is related to a hedging strategy.

$$z_{t+1} = z_t + \rho z_t \Delta_t + \pi\sigma(\Delta w_t + \theta\Delta_t) + \Delta\phi_t \tag{11}$$

where, there exists a predictable and bounded-valued process vector θ, a risk premium is called, π is the portfolio process and the short rate ρ is a predictable and bounded process. Theoretically, the continuous system is given in (7) that is imperfectly observed, that is related through a process of observation $\varphi_k \in \mathbf{R}$ to the process x_k, that is using the statistical model:

$$\mathbf{y}_{t+1} = \mathcal{H}(x_t) + \epsilon_t \tag{12}$$

where $\mathbf{y}_{1:t} = (\mathbf{y}_1, \ldots, \mathbf{y}_t)$ is a vector of measurements, $\epsilon = (\epsilon_1, \ldots, \epsilon_t)$ represent errors, the term of the regression is the realization of a diffusion process $\mathcal{H} : \mathbf{R} \to \mathbf{R}$, where $\mathcal{H}$ is a function, that can be linear or non-linear. The simplest case is linear regression $\mathcal{H}(\mathbf{x}_t) = \mathbf{x}_t$, [38].
So, the model can be written as:

$$\begin{aligned} x_{t+1} &= x_t + (bx_t\Delta_t + \sigma x_t h_t)\Delta_t + \sigma x_t \Delta w_t \\ y_{t+1} &= y_t + (ry_t + \frac{b-r}{\sigma} z_t - (R-r)(y_t - \frac{z_t}{\sigma}) + z_t h_t)\Delta_t + z_t \Delta w_t \\ z_{t+1} &= z_t + \rho z_t \Delta_t + \pi\sigma(\Delta w_t + \theta\Delta_t) + \Delta\phi_t \end{aligned} \tag{13}$$

where $\mathbf{x}_{t+1} = [x_{t+1}, y_{t+1}, z_{t+1}]^T$ at each step of time t, a lineal equation is used for the process of the observations:

$$\mathbf{y}_{t+1} = \mathbf{x}_t + \epsilon_t \tag{14}$$

where, $\mathbf{y}_{t+1} = [x^+_{t+1}, y^+_{t+1}, z^+_{t+1}]^T$ at each step of time t, and $\epsilon_t \sim N(\mathbf{0}, \sigma_\varepsilon \mathbf{I})$, $\mathbf{0}$ is a zeros vector and $\mathbf{I}$ is a identity matrix. Where, $\mathbf{x}_{1:t} = (\mathbf{x}_1, \ldots, \mathbf{x}_t)$ are the unknown solutions states, and $\mathbf{y}_{1:t} = (\mathbf{y}_1, \ldots, \mathbf{y}_t)$ are the observed solutions.

In this research, a data set $\mathbf{y}_{1:t}$ is defined as the data of asian options, that is also known as average value options; this database contains errors, that come from the measurement instruments themselves and may be due to different causes; These errors cannot be foreseen, since that depends on unknown or stochastic causes. The filtering of the data allows the analysis of the system to be easier and more suitable to observe the important aspects and the tendencies; in this sense, the prices are reconstructed by the graph of the state vector $\mathbf{x}_{0:t}$ in the state space.

3. Methodology

In this paper, two computational algorithms are based on the SMC methods to estimate the solution of the model (13) and (14).

- Nested Sequential Monte Carlo Method (NSMC)
- Space-Time Particle Filter Method (STPF)

3.1. *Nested Sequential Monte Carlo Method*

Inference in complex and high-dimensional statistical models is a very challenging problem, that is ubiquitous in applications. In particular in the sequential Bayesian inference, that involves computing integrals of the form (4) and (5). The Sequential Monte Carlo (SMC) is used to simulate sequentially from the distributions, in particular, that considers the fully adapted SMC sampler, [41]. Specifically, the proposal distribution is given by: $\bar{q}_t(\mathbf{x}_t|\mathbf{x}_{1:t-1}) = P_{qt}(\mathbf{x}_{1:t-1})^{-1}q_t(\mathbf{x}_t|\mathbf{x}_{1:k-1})$, where

$$q_t(\mathbf{x}_t|\mathbf{x}_{1:t-1}) = \frac{q_t(\mathbf{x}_{1:k-1})}{q_{t-1}(\mathbf{x}_{1:t-1})} \tag{15}$$

In addition, the normalising constant $P_{qt}(\mathbf{x}_{1:t-1}) = \int q_{t-1}(\mathbf{x}_{1:t-1})d\mathbf{x}_t$ is further used, that defines the resampling weights, the particles are resampled at time $t-1$, that is according to $P_{qt}(\mathbf{x}_{1:t-1})$ that are propagated before to time t.
The NSMC method is proposed:

- **Step 1.** Set $\hat{P}_{q0} = 1$
- **Step 2.** For $t = 1$ to n
  - Initialise $q^j = Q(q_t(\cdot|\mathbf{x}^j_{i:t-1}), M)$ for $j = 1, \ldots, N$.
  - Set $\hat{P}^j_{qt} = q^t.GetP()$ for $j = 1, \ldots, N$.
  - Compute $\hat{P}_{qt} = \hat{P}_{qt-1}\frac{1}{N}\sum_{j=1}^N \hat{P}^j_{qt}$
  - Draw $m_k^{1:N}$ from a multinomial distribution with probabilities $\frac{\hat{P}^j_{qt}}{\sum_{k=1}^N \hat{P}^k_{qt}}$ for $j = 1, \ldots, N$.
  - Set $L \longleftarrow 0$
  - For $j = 1$ to N
    - Compute $\mathbf{x}^i_t = q^j.Simulate()$ and let $\mathbf{x}^i_{1:t} = (\mathbf{x}^j_{1:t}, \mathbf{x}^i_t)$ for $i = L+1, \ldots, L+m^j_t$
    - Delete q^j
    - Set $L \longleftarrow L + m^j_t$

3.2. *Space-Time Particle Filter Method*

The Space-Time Particle Filter method combines a local filter running d space-step that is using M_d particles with a global filter that is making time-steps to use N particles. This approach has been motivated by the island particle model of [49], where a related method for standard particle filters is developed.
The STPF method is proposed:

- **Step 1.** For $t = 1$ and $i \in 1, \ldots, N$
  - M d-samples for $j = 1$ are generated from

$$q_{1,j}(x_1^{i,l}(j)|x_1^{i,l}(1:j-1), x_0) \tag{16}$$

  - Computes the weights for $l \in 1, \ldots, M_d$

$$G_{1,1}(x_1^{i,l}(x_1^{i,l}(1))) = \frac{\alpha_{1,1}(y_1, x_0, x_1^{i,l}(1))}{q_{1,1}(x_1^{i,l}(1)|x_0)} \tag{17}$$

  - The $\check{x}_1^{i,l}(1)$ Md-samples are resampled, that are according to their corresponding weights (multinomial resampling).
  - M d-samples from $q_{1,j}$ are generated for $j \in 2, \ldots, d$,

$$G_{1,j}(\check{x}_1^{i,l}(1:j-1), x_1^{i,l}(j)) = \frac{\alpha_{1,j}(y_1, x_0, \check{x}_1^{i,l}(1:j-1), x_1^{i,l}(j))}{q_{1,j}(x_1^{i,l}(j)|x_0, \check{x}_1^{i,l}(1:j-1))} \tag{18}$$

  - The $\check{x}_1^{i,l}(1:d)$ Md-samples are resampled, that are according to the weights.
  - The N particle systems are assigned, weights

$$\mathbf{G}_1(\check{x}_1^{i,M_d}(1:d-1), x_1^{i,M_d}(1:d)) = \prod_{j=1}^{d}\left(\frac{1}{M_d\sum_{l=1}^{M_d} G_{1,j}(\check{x}_1^{i,l}(1:j-1), x_1^{i,l}(j))}\right) \tag{19}$$

- **Step 2.** For $t \geq 2$ and $i \in 1, \ldots, N$
  - M d-samples are generated for $j = 1$ from

$$q_{n,j}(x_n^{i,l}(j)|\breve{x}_n^{i,l}(1:j-1), \breve{x}_{n-1}^{i,l}(1:d)) \tag{20}$$

  - Computes the weights, for $l \in 1, \ldots, M_d$

$$G_{n,j}(\breve{x}_{n-1}^{i,l}(1:d), x_n^{i,l}(1)) = \frac{\alpha_{n,1}(y_n, \breve{x}_{n-1}^{i,l}(1), \breve{x}_n^{i,l}(1:d))}{q_{n,1}(x_n^{i,l}(1)|\breve{x}_{n-1}^{i,l}(1:d))} \tag{21}$$

  - The Md-samples are resampled, that are according to the inclusive weights of the $\breve{x}_{n-1}^{i,l}(1:d)$, which are denoted $\breve{x}_{n-1,j}^{i,l}(1:d)$ at step j.
  - M d-samples from $q_{n,j}$ are generated,

$$G_{n,j}(\breve{x}_{n-1,j-1}^{i,l}(1:d), x_n^{i,l}(1:j-1), \breve{x}_n^{i,l}(j)) = \frac{\alpha_{n,j}(y_n, \breve{x}_{n-1,j-1}^{i,l}(1:d), \breve{x}_n^{i,l}(1:j-1), x_n^{i,l}(j))}{q_{n,j}(x_n^{i,l}(j)|\breve{x}_{n-1,j-1}^{i,l}(1:d), \breve{x}_n^{i,l}(1:j-1))} \tag{22}$$

  - The Md-samples are resampled, that are according to the weights.
  - The N particle systems are assigned, weights

$$\begin{aligned} &\mathbf{G}_n(\breve{x}_{n-1,1:d-1}^{i,1:M_d}(1:d), \breve{x}_n^{i,l:M_d}(1:d-1), x_n^{i,l:M_d}(1:d)) \\ &= \prod_{j=1}^{d}(\frac{1}{M_d}\sum_{l=1}^{M_d} G_{n,j}(\breve{x}_{n-1,j-1}^{i,l}(1:d), \breve{x}_n^{i,l}(1:j-1), x_n^{i,l}(j))) \end{aligned} \tag{23}$$

  - Then resample of the N-particle systems is according to the weights.

4. Empirical Results

In this section, a straddle is numerically evaluated, i.e. that can be the sum of a call and a put option under different rates for borrowing and investment in the money market account. The rate for borrowing by R is denoted, the investment by r is denoted. The fair price of a straddle in this model is given by $\mathbf{x}_0$, where $\mathbf{x} = (x, y, z)$ is written the solution of the nonlinear BSDE (13), the process of observation is $\mathbf{y}_t$, that is related to the process $\mathbf{x}_t$, that is using the following statistical model (14).

We consider the data of asian options, that is also known as average value options, that is a type of option contract whose payoff conditions by the average price of the underlying asset are determined, that is during a predefined period. The payoff conditions make Asian options different from European and American options, whose payoff is defined by the value of the underlying asset at maturity. Asian options offer a series of advantages against regular options. The averaging feature of Asian options makes instruments more efficient at mitigating any volatility. Besides, the price of trend is lower than the European and American options. However, businesses are aiming to protect themselves from foreign currency volatility, they have similar alternatives, like flexible forwards or particularly Dynamic Hedging that is unlike that guarantees an exchange rate throughout the duration of the contract. Then the predictive capability of the algorithms are evaluated with the model, a set of data from Asian Historical Options Data is taken. This example shows how the price a Asian option is considered with two closed form approximations (Nested Sequential Monte Carlo Method and Space-Time Particle Filter Method). The series consists of 51 data, March 2014 until March 2015, see Figure (1). Asian options are securities with payoffs, that are depended on the average value of an underlying asset over a specific period of time. Underlying assets can be stocks, commodities, or financial indices.
The a priori values are to initialize the NSMCM:

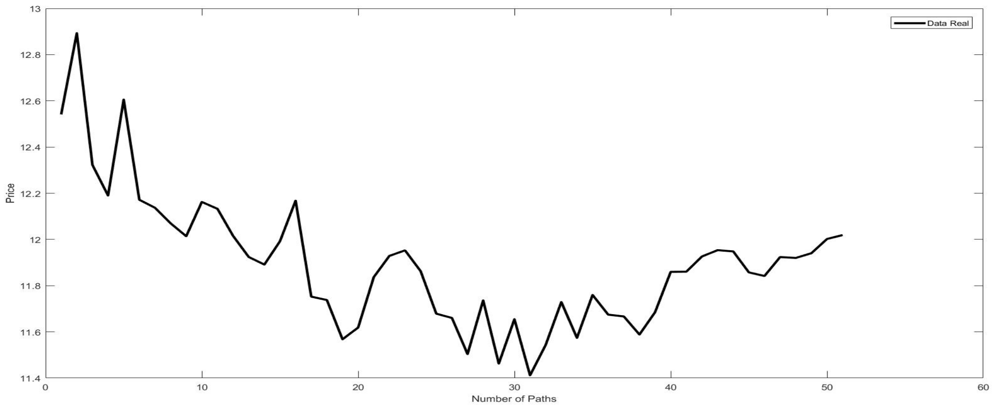

Figure 1. The real price of option asian.

- $T = 51$
- $\Delta = 0.9$
- $\nu = 10$
- $X_0 = 12.5$
- $\sigma = 0.2$
- $\mu = 0.05$
- $r = 0.01$
- $R = 0.06$
- The straddle is supposed to be at the money, i.e. $K = 100$, with maturity $t = 1$ years.

The Figure (2) shows the unobserved states (prices) in red color, and the observed prices in black color, the NSMCM algorithm with the model (13) is implemented, that reconstructs the pattern of unobserved prices with respect to true prices, and in general, that is much similarity between the observed price and the estimated price.
The a priori values are to initialize the STPF algorithm:

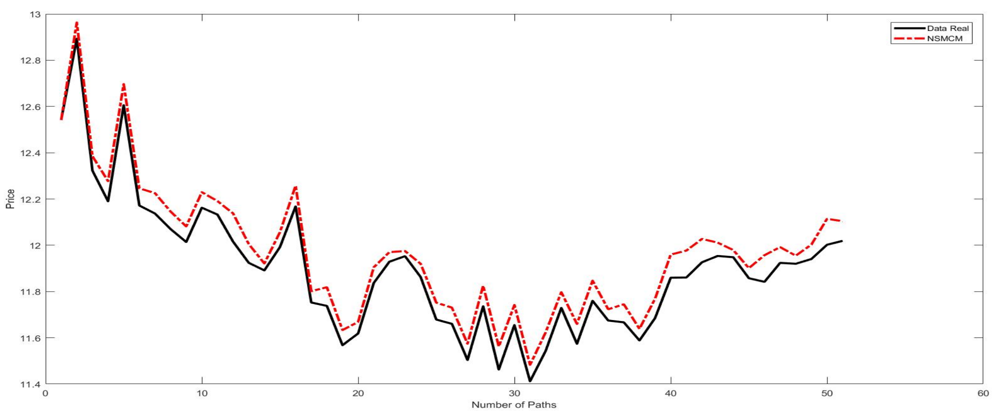

Figure 2. The real data and estimated states by NSMCM algorithm with the model (13).

- $T = 51$
- $\Delta = 0.9$

- $\nu = 10$
- $X_0 = 12.5$
- $\sigma = 0.2$
- $\mu = 0.05$
- $r = 0.01$
- $R = 0.06$
- The straddle is supposed to be at the money, i.e. $K = 100$, with maturity $t = 1$ years.

The Figure (3) shows the unobserved states (prices) in red color, and the observed prices in black color, the STPF algorithm with the model (13) is implemented, that reconstructs the pattern of unobserved prices with respect to observed prices, and in general, that is much similarity between the observed price and the estimated price.

The Figure (4) shows the unobserved states (prices) in red color by STPF algorithm and in blue color by NSMCM algorithm, and the observed prices in black color, the NSMCM and STPF algorithms with the model (13) are implemented, that reconstructs the pattern of unobserved prices with respect to observed prices, and in general, that are much similarity between the observed price and the estimated price.

The Table (1) shows the errors by the NSMCM and STPF algorithms for the model, that is considering the option

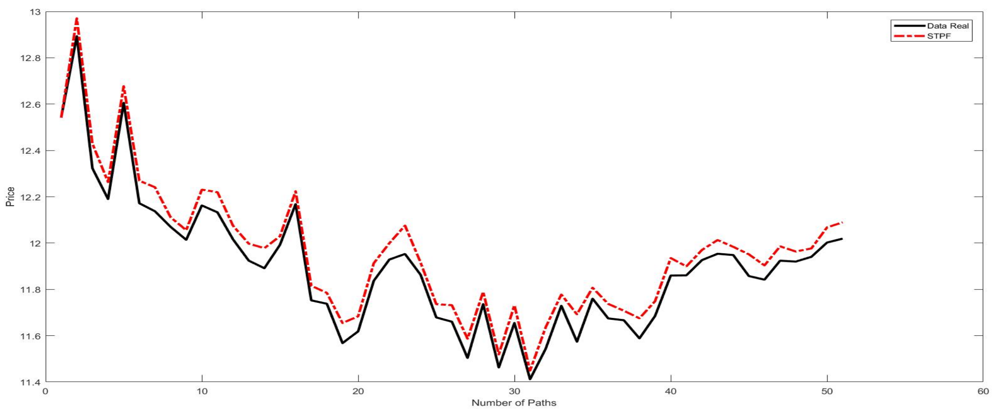

Figure 3. The real data and estimated states by STPF algorithm with the model (13).

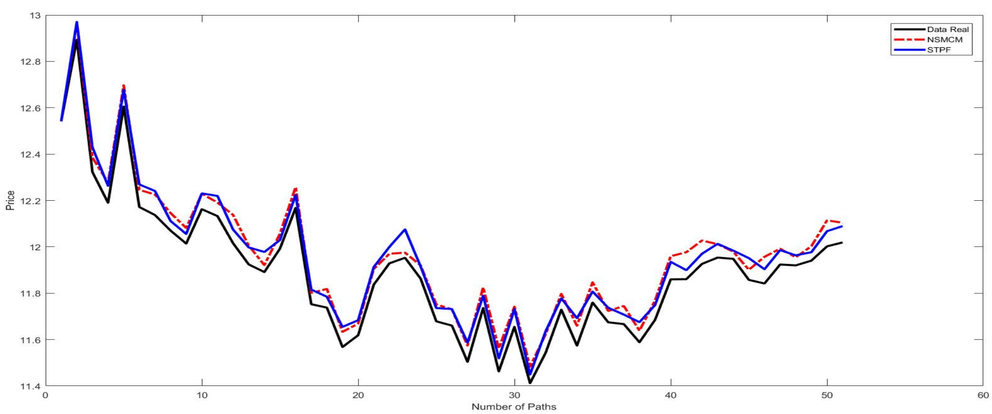

Figure 4. The real data and estimated states by NSMCM and STPF algorithms with the model (13).

asian data, that is observing little variability between the true states and the estimates states.

Table 1. RSME estimated for the model (13).

RSME	NSMCM Algorithm	STPF Algorithm
Model for option asian	0.2873	0.3015

5. Discussions and Conclusions

A methodology for the estimation of the states for non-linear problems are represented, non-linear BSDE is used, a model is defined for pricing options by a hedging strategy in the Föllmer-Schweizer sense. In the context of option pricing by a Brownian motion is driven, the states are a basket of financial underlying, the observations are the price of the option, and there is a variable that a hedging strategy (possibly in the Föllmer-Schweizer sense) is related. Two algorithms are presented to estimate the solution states, the nested sequential Monte Carlo (NSMC) and space-time particle filter (STPF) are defined for sampling from sequences of probability distributions. The proposed methodology is using real data from option asian, the states are estimated in real time that is proposed on the basis of the one price models, under an assumption, the set super hedging strategies is nonempty that is much milder, there exists an upper price process $(\mathbf{x}_t)_{t\in[0,T]}$ that is associated with the solutions of nonlinear backward stochastic differential equations.
The predictive capability of the algorithms are evaluated with the model, a set of data from Asian Historical Options Data is taken. This market is incomplete, then that is not always possible to replicate a payoff by a controlled portfolio of the basic securities. That is shown that there exists an upper price process that is associated with the solutions of nonlinear backward stochastic differential equations.
Finally, one goodness-of-fit measure to validate the performance of the model are used, obtaining insignificant estimation error.

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