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Some Properties of Total Time on Test and Excess Wealth in Bivariate Cases

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Abstract Most of the introduced transformations have many applications in reliability. For example, the total time on test (*TTT*) and excess wealth (*EW*) transforms are useful concepts in various fields. This paper presents bivariate *TTT* and *EW* transforms. Also, the bivariate location independent riskier (*LIR*) transform has been considered. In addition, we present the conditions for establishing the TTT transform ordering in the bivariate mode and its relationship with *EW* order and some stochastic orders. Also, we establish that the bivariate *TTT* transform order as well as the presentation of the new better than used in bivariate *TTT* transform class. Finally, we describe the relationship between *TTT* and *EW* transforms with aging classes in the bivariate mode.

Keywords Total Time on Test Transform, Excess Wealth Transform, Location Independent Riskier, Copula, Bivariate Aging Classes, NBUT

AMS 2010 subject classifications 62H05, 62N05, 62P99.

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1. Introduction

The total time on test (*TTT*) transform has attracted the attention of researchers in several fields of applications, such as reliability engineering, economics, analysis of censored data, maintenance schedule, model identity, stochastic orders, and so on. This concept and its plot were first introduced by Barlow and Proschan (1975). We refer to Barlow and Campo (1975), Barlow (1979), and Klefsjo and Westberg (1996) for further study on the application of the *TTT* plot. In the context of reliability analysis, Nair et al. (2008) investigated the properties of the *TTT* transform order and its applications via quantiles. The *TTT* plot is also a useful tool for analyzing lifetime data. Bergman and Klefsjö (2014) have used this chart to examine the behavior of the failure rate function. Especially, the *TTT* plots were used by Rao and Prasad (2001) to estimate maintenance intervals for failure data with the increasing failure rate. Recently, Gamiz et al. (2020) also used the *TTT* plot as a graphical tool for aging trends recognition.

One of the practical and useful concepts in the study of bivariate distributions is the copula. First presented by Sklar (1959), copulas represent a helpful approach to demonstrating multivariate peculiarities by unraveling the joint dependence structure from the marginal behavior. This approach is especially valid for applications where the adaptability of copulas seems desirable over the immediate fitting of multivariate distributions, which might be challenging to characterize and manage Nelsen (2006).

In this article, we express the three transforms, *TTT*, excess wealth (*EW*), and location independent riskier (*LIR*), for the bivariate state when we have nonnegative variables and then examine their relationship with each other. Also, we specifically examine the properties of *TTT* transform and express its relation with the concepts of dependence and some concepts of reliability, such as the aging and new better than used in bivariate total time on test transform (*NBUT*).

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2. Preliminaries

Let n units be placed on life test at age 0 and let $N(u)$ be the number of units alive (on test) at age u (time u). Then the *TTT* at time x shows by $T_n(x) = \int_0^x N(u)du$. Barlow and Proschan (1975) called $\{T_n(x); x \geq 0\}$ the total time on a test process. Since $\frac{N(u)}{n} = \bar{F}(u)$, where $\bar{F}_n = 1 - F_n$ is the empirical survival function, it is clear that $\frac{1}{n}T_n(x) \to \int_0^x \bar{F}(u)du$ almost surely as $n \to \infty$.

Let X_1 and X_2 be two random variables with the corresponding cumulative distribution functions F_1 and F_2 and probability density functions f_1 and f_2, respectively. In addition, for $i = 1, 2$, let $\bar{F}_i = 1 - F_i$ be the survival (tail) function, let $F_i^{-1}(u_i) = \inf\{x_i : F_i(x_i) \geq u_i\}, u_i \in (0, 1)$, be the quantile function, and let $F_i^{-1}(0)$ and $F_i^{-1}(1)$ be the lower bound and upper bound of the support of the random variable X_i, say S_{F_i}.

Definition 2.1. For the nonnegative continuous random variable X, the TTT transform is defined as

$$T_X(u) = \int_0^{F^{-1}(u)} \bar{F}(x)dx.$$

Given the fact that $0 < \mu = E(X) < \infty$, the scaled *TTT* transform $\varphi(u)$ of X is shown as $\varphi_F(u) = \frac{T_X(u)}{\mu}$. Barlow and Campo (1975) was the first that introduced the scaled *TTT* transform, which is also free scale. The scaled *TTT* transform has proved to be an extremely useful tool in a variety of reliability applications, including model recognition, characterizing different aging properties, and analyzing various maintenance and burn-in problems.

Definition 2.2. Let X and Y be two random variables with distribution functions F and G that have finite means. Then X is said to be smaller than Y in TTT transform order, denoted by $X \leq_{ttt} Y$, if

$$T_X(u) \leq T_Y(u) \quad \text{for all } u \in (0, 1).$$

Definition 2.3. In the bivariate mode, consider the two-variable function

$$C : [0, 1]^2 \longrightarrow [0, 1].$$

Then $C(u, v)$ is a two-dimensional copula if

- $C(u, 0) = C(0, v) = 0$ for every $u, v \in [0, 1]$,
- $C(u, 1) = u$ and $C(1, v) = v$ for every $u, v \in [0, 1]$,
- $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$ for every $u_1, v_1, u_2, v_2 \in [0, 1]$, such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Sklar's theorem shows that if (X_1, X_2) is a random vector with joint distribution F and marginal distribution functions F_1 and F_2, then

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

The most important copula families are Frechet, Gumbel, Normal, FGM, Clayton, and so on. As mentioned, one of the most important copulations that shows a positive dependence is the Clayton copula, which is also used in this article and defined as follows.

Definition 2.4.

a) If $C(\cdot)$ has the form $C_{\mathrm{C}}(u, v) = [u^{-\theta} + v^{-\theta} - 1]^{-\dfrac{1}{\theta}}$, then we say it Clayton copula with parameter $\theta \geq -1, \theta \neq 0$. When $\theta \to 0$, it is reduced to an independent copula.

b) If $C(\cdot)$ has the form $C_{\mathrm{C}}(u, v) = uv \exp\{-\theta \ln(u) \ln(v)\}$, then we say it Clayton copula with parameter $\theta \geq 1$.

There are various criteria for measuring the relationship between random variables. Some of these criteria do not depend on marginal distributions and are called nonparametric correlation criteria. One of the most famous of these criteria is the Kendall's tau, which is denoted by τ and is defined as follows.

Definition 2.5. Let (X_1, X_2) be a random vector with joint distribution F, marginal distribution functions F_1 and F_2, and copula function C. Then

$$\tau(X_1, X_2) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x_1, x_2)dF(x_1, x_2) - 1$$
$$= 4\int_0^1\int_0^1 C(u, v)dC(u, v) - 1.$$

Definition 2.6. Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random variable with $F(x_1, x_2)$. The distribution $F(x_1, x_2)$ is said to have
a) Bivariate New Better than Used ($BNBU$) if

$$\bar{F}(x_1 + t, x_2 + s) \le \bar{F}(x_1, x_2)\bar{F}(t, s), \qquad x_1, x_2, t, s > 0;$$

b) Bivariate New Better than Used in Average ($BNBUA$) if

$$\int_0^{x_1}\int_0^{x_2} \bar{F}(u + t, v + s)dvdu \le \bar{F}(t, s)\int_0^{x_1}\int_0^{x_2} \bar{F}(u, v)dvdu, \qquad x_1, x_2, t, s > 0;$$

c) Bivariate New Better than Used in Failure Rate ($BNBUFR$) if

$$\bar{F}(x_1 + t, x_2 + s) \le exp[-r(0, 0)\sqrt{t^2 + s^2}]\bar{F}(x_1, x_2), \qquad x_1, x_2, t, s > 0,$$

where $r(\cdot, \cdot) = \frac{f(\cdot,\cdot)}{\bar{F}(\cdot,\cdot)}$ denotes the failure rate of F;
d) Bivariate New Better than Used of Second Order ($BNBU(2)$) if and only if

$$\int_0^{x_1}\int_0^{x_2} \bar{F}(u, v)dvdu \ge \int_0^{x_1}\int_0^{x_2} \frac{\bar{F}(u + t, v + s)}{\bar{F}(t, s)}dvdu, \qquad x_1, x_2, t, s \ge 0;$$

e) Bivariate Harmonic New Better than Used Expectation ($BHNBUE$) if

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty} \bar{F}(u, v)dvdu \le \mu\exp\{-\frac{x_1 + x_2}{\mu}\}, \qquad x_1, x_2 \ge 0,$$

wherein $\mu = E(X_1X_2) = \int_0^{\infty}\int_0^{\infty} \bar{F}(x_1, x_2)dx_1dx_2$.

3. Main Results

The concept of TTT transform is well known due to its applications in different fields of study such as reliability analysis, econometrics, stochastic modeling, tail orderings, and ordering distributions. Bralow and Proschan (1975) extended the *TTT* concept to the multivariate case. They assumed that (X, Y) have joint life distribution $F(x, y)$, that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are n independent observations on (X, Y), and also that $N(u, v)$ is the number of pairs (X_i, Y_i) such that $X_i \ge u$ and $Y_i \ge v$. Then $N(u, v)$ is the number of pairs "on test" at joint times u and v. They called $T_n(x, y) = \int_0^x\int_0^y N(u, v)dudv$, the joint *TTT*. If $\bar{F}_n(x, y) = \frac{N(x,y)}{n}$ is the joint empirical survival probability, then $\frac{1}{n}T_n(x, y) = \int_0^x\int_0^y \bar{F}_n(u, v)dudv$ and

$$\lim_{n\to\infty}\frac{1}{n}T_n(x, y) = \int_0^x\int_0^y \bar{F}(u, v)dudv, \tag{1}$$

where $\bar{F}(u,v) = P(X > u, Y > v)$. It is obvious that

$$\int_0^\infty \int_0^\infty \bar{F}_n(u,v)dudv = \frac{1}{n}\sum_{i=1}^n X_i Y_i.$$

Definition 3.1. Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be n independent observations of nonnegative vector variable (X,Y) and let $(X_{(i)},Y_{(i)})$ be order statistics. In this case, we define the bivariate total time on test statistic in (0,**t**) as follows:

$$\begin{aligned} TTT(t_1,t_2) &= n(X_{(1)},Y_{(1)}) + \cdots + (n-s+1)(X_{(s)} - X_{(s-1)}, Y_{(s)} - Y_{(s-1)}) \\ &\quad + (n-s)(t_1 - X_{(s)}, t_2 - Y_{(s)}), \end{aligned} \tag{2}$$

wherein $X_{(s)} \leq t_1 \leq X_{(s+1)}$ and $Y_{(s)} \leq t_2 \leq Y_{(s+1)}$.

According to the concept presented by Barlow and Proschan (1975), the *TTT* transform in the bivariate mode can be written as follows.

Definition 3.2. Let $\mathbf{X} = (X_1,X_2)$ be a bivariate vector variable that has joint life distribution $F(x_1,x_2)$ and marginal distribution functions F_1 and F_2, respectively. The bivariate *TTT* transform is shown by $T_{X_1,X_2}(p_1,p_2)$ and thus expressed as

$$T_{X_1,X_2}(p_1,p_2) = \int_0^{F_1^{-1}(p_1)} \int_0^{F_2^{-1}(p_2)} \bar{F}(u,v)dvdu, \quad p_1,p_2 \in (0,1), \tag{3}$$

where $F_1^{-1}(p_1)$ and $F_2^{-1}(p_2)$ are quantile functions. It can be easily shown that $T_{X_1,X_2}(p_1,0) = T_{X_1,X_2}(0,p_2) = 0$.

Also, the scale bivariate *TTT* transform is defined as below:

$$\varphi_{X_1,X_2}(p_1,p_2) = \frac{\int_0^{F_1^{-1}(p_1)} \int_0^{F_2^{-1}(p_2)} \bar{F}(u,v)dvdu}{\int_0^\infty \int_0^\infty \bar{F}(u,v)dvdu}, \quad p_1,p_2 \in (0,1). \tag{4}$$

We define a nonparametric estimate for $T_{X_1,X_2}(p_1,p_2), p_1,p_2 \in (0,1)$, as follows:

$$\begin{aligned} \hat{T}_{X_1,X_2}(p_1,p_2) =& n(p_1X_{(1)}, p_2Y_{(1)})I_{[0\leq p_1,p_2 \leq \frac{1}{n}]} \\ &+ \frac{1}{n}[n(X_{(1)},Y_{(1)}) + \cdots + (n-j+1)(X_{(j)} - X_{(j-1)}, Y_{(j)} - Y_{(j-1)})] \\ &+ (n-j)(p_1 - \frac{j}{n}, p_2 - \frac{j}{n})(X_{(j+1)} - X_{(j)}, Y_{(j+1)} - Y_{(j)})I_{[\frac{j}{n}\leq p_i \leq \frac{j+1}{n}]}, \\ & \qquad j = 1,2,\ldots,n-1. \end{aligned} \tag{5}$$

Remark 3.3. The transform introduced in (3) is nondecreasing relative to p_1 and p_2 because

$$\frac{\partial T_{X_1,X_2}(p_1,p_2)}{\partial p_1} = \int_0^{F_2^{-1}(p_2)} \frac{\bar{F}(F_1^{-1}(p_1),v)}{f_1(F_1^{-1}(p_1))}dv \geq 0,$$

$$\frac{\partial T_{X_1,X_2}(p_1,p_2)}{\partial p_2} = \int_0^{F_1^{-1}(p_1)} \frac{\bar{F}(u,F_2^{-1}(p_2))}{f_2(F_2^{-1}(p_2))}du \geq 0,$$

$$\frac{\partial^2 T_{X_1,X_2}(p_1,p_2)}{\partial p_2 \partial p_1} = \frac{\bar{F}(F_1^{-1}(p_1),F_2^{-1}(p_2))}{f_1(F_1^{-1}(p_1))f_2(F_2^{-1}(p_2))} \geq 0.$$

Definition 3.4. Let $\mathbf{X} = (X_1,X_2)$ and $\mathbf{Y} = (Y_1,Y_2)$ be two nonnegative random vectors with finite dual expectation. Then $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the sense of bivariate *TTT* order and denoted by $(X_1,X_2) \leq_{ttt} (Y_1,Y_2)$ if

$$T_{X_1,X_2}(p_1,p_2) \leq T_{Y_1,Y_2}(p_1,p_2), \quad (p_1,p_2) \in (0,1)^2.$$

In the following paragraphs, by examining the bivariate *TTT* plot of the random vector (X_1, X_2) that X_1 and X_2 have Weibull and exponential marginal distribution, respectively, we study the sensitivity analysis of the diagram to the change of parameters.

Remark 3.5. Equation (1) can be written as follows for *EW* transform:

$$\lim_{n\to\infty} \frac{1}{n} W_n(x,y) = \int_x^{\infty} \int_y^{\infty} \bar{F}(u,v)dudv, \tag{6}$$

and as a result the bivariate *EW* transform is

$$W_{X_1,X_2}(p_1,p_2) = \int_{F_1^{-1}(p_1)}^{\infty} \int_{F_2^{-1}(p_2)}^{\infty} \bar{F}(u,v)dvdu, \quad p_1,p_2 \in (0,1). \tag{7}$$

Also, the scale bivariate *EW* transform is defined as below:

$$\psi_{X_1,X_2}(p_1,p_2) = \frac{\int_{F_1^{-1}(p_1)}^{\infty} \int_{F_2^{-1}(p_2)}^{\infty} \bar{F}(u,v)dvdu}{\int_0^{\infty} \int_0^{\infty} \bar{F}(u,v)dvdu}, \quad p_1,p_2 \in (0,1). \tag{8}$$

Bivariate *LIR* transform for nonnegative variables can be obtained similarly. For this purpose, assume that (X,Y) is a vector variable consisting of nonnegative variables that have joint life distribution $F(x,y)$, that $(X_1,Y_1),\ldots,(X_n,Y_n)$ are n independent observations on (X,Y), and also that $N^*(u,v)$ is the number of pairs (X_i,Y_i) such that $X_i \le u$ and $Y_i \le v$. Then $LIR_n(x,y) = \int_0^x \int_0^y N^*(u,v)dudv$, is the joint location independent riskier. If $F_n(x,y) = \frac{N^*(x,y)}{n}$ is the joint empirical cumulative distribution, then $\frac{1}{n}LIR_n(x,y) = \int_0^x \int_0^y F_n(u,v)dudv$ and

$$\lim_{n\to\infty} \frac{1}{n} LIR_n(x,y) = \int_0^{x} \int_0^{y} F(u,v)dudv, \tag{9}$$

As a result, the bivariate *LIR* transform is

$$LIR_{X_1,X_2}(p_1,p_2) = \int_0^{F_1^{-1}(p_1)} \int_0^{F_2^{-1}(p_2)} F(u,v)dvdu, \quad p_1,p_2 \in (0,1). \tag{10}$$

Example 3.6. Let $\mathbf{X} = (X_1, X_2)$ be a nonnegative random vector that variables X_1 and X_2 are two components of a system that have Weibull marginal distribution and exponential distribution, respectively, and whose lifetime is positively dependent. We use the Clayton and Gumbel copulas to investigate the lifetime dependence of these two components. The sensitivity analysis of bivariate *TTT* is performed according to changes in the exponential distribution parameter, parameter of copulas, and Weibull distribution shape parameter. To do this, the two parameters are considered constant, and the results are studied based on the changes of the third parameter.

a) As a basic case, we obtain the value of integral (4) for the Weibull distribution with the parameters shape 2 and the scale 1, the standard exponential distribution, and Clayton copula with the parameter $\theta = 1$, which is equal to 58.61538. The value of the integral is called the TTT value, and it is obtained by a calculation for 20 values of p_1 and p_2 of $[0,1]$. Increasing the exponential distribution parameter has no effect on the amount of TTT value and its surface. Indeed with increasing the value of θ, its surface increases slightly in the interval $[-1,-0.6]$, and as a result, the TTT value increases. Then it decreases slightly in the interval $[-0.6, 1.9]$ and finally increases slightly for $\theta > 1.9$. Table 1 shows the values in τ-Kendall and the TTT value in terms of different θ. Finally, increasing the Weibull distribution shape parameter also increases the TTT value. This result is well illustrated in Table 2 and Figure 1. As can be seen, increasing the shape parameter of the Weibull distribution and keeping other parameters constant increase the bivariate *TTT* statistic and the distance of its surface from the origin.

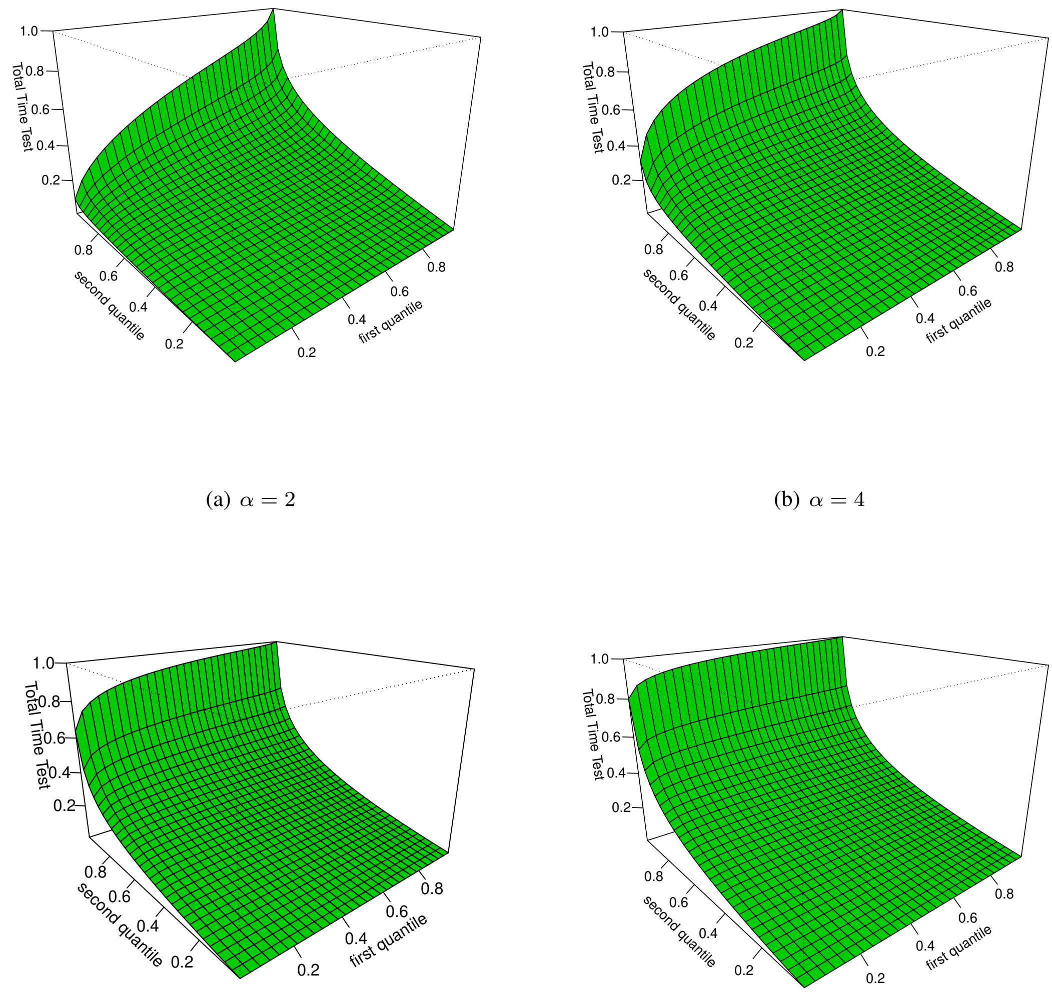
Table 1. TTT value and τ-Kendall for different Clayton copula parameters.

θ	τ	TTT value
-1	-1	59.49852
-0.9	-0.818	59.61836
-0.6	-0.429	59.74324
-0.59	-0.418	59.74109
-0.5	-0.333	59.70456
-0.1	-0.053	59.30375
-0.001	-0.0005	59.19368
0.1	0.048	59.09114
0.5	0.2	58.79597
0.9	0.310	58.63971
1	0.375	58.61538
1.5	0.394	58.54916
1.8	0.474	58.53882
1.9	0.487	58.53852
1.91	0.489	58.53856
2	0.5	58.53945
6	0.75	58.81151
200	0.99	59.51554

Table 2. TTT value for different shape parameters with Clayton copula.

α	TTT value
2	58.61538
4	73.02312
10	80.30703
20	87.03152

b) By considering the Gumbel copula with parameter $\theta = 1$, the TTT value is equal to 59.19261 for marginal distributions Weibull(2,1) and Exp(1). As before, increasing the exponential distribution parameter has no effect on the amount of TTT value, but by increasing the value of θ, the TTT value and its surface increase slightly in the interval $[1, 2.1]$. Then it decreases slightly in the interval $[2.1, 164]$ and finally increases slightly for $\theta > 164$. Table 3 shows the values in τ-Kendall and TTT value in terms of different values of the Gumbel copula parameter. Finally, increasing the shape parameter increases the TTT value. This result is well illustrated in Table 4 and Figure 2.



(a) $\alpha = 2$ (b) $\alpha = 4$

(c) $\alpha = 10$ (d) $\alpha = 20$

Figure 1. Bivariate *TTT* plot for different values of shape parameter α with Clayton copula.

3.1. Bivariate TTT transform

The univariate right-spread or *EW* function was introduced by Fernandez-Ponce et al. (1996), and it was extended to a multivariate case by Fernandez-Ponce et al. (2011). In this section, we present a new definition of bivariate *TTT* and *LIR* transforms based on the multivariate case of Fernandez-Ponce et al. (2011). Also, the bivariate *TTT* transform enables us to define a new stochastic order based on the new transform. For this purpose, let $\mathbf{x} = (x_1, x_2)$ be a vector in R^2. Let X be a random vector in R^2 with distribution function $F(\cdot, \cdot)$. The Bivariate u-quantile for X, also called regression representation, was introduced by O'brien (1975) as follows.

Table 3. TTT value and τ-Kendall for different Gumbel copula parameters.

θ	τ	TTT value
1	0	59.19261
1.1	0.091	59.40976
1.6	0.375	59.75881
2	0.5	59.76551
2.1	0.524	59.75849
2.2	0.545	59.75015
3	0.667	59.67657
4	0.75	59.61146
20	0.95	59.49075
100	0.99	59.48428
164	0.994	59.48411
165	0.994	59.48436
170	0.994	59.4857
200	0.995	59.52231
1000	0.999	68.52647

Table 4. TTT value for different shape parameters with Gumbel copula.

α	TTT value
2	59.19261
4	73.37082
10	83.45642
20	87.10735

Let $u_2 = (u_1, u_2)$ be a vector in $[0,1]^2$. Then the bivariate u-quantile for X, denoted by $\hat{\mathbf{x}}(\mathbf{u_2})$, is defined as

$$\hat{x}_1(u_1) = F^{-}_{X_1}(u_1), \quad \hat{x}_2(\mathbf{u_2}) = F^{-}_{X_2|X_1=\hat{x}_1(u_1)},$$

where $F^{-} = \inf\{x : F(x) \geq u\}$ and $\mathbf{u_2} = (u_1, u_2)$. Also the bivariate $\mathbf{x}$-rate vector concept is denoted by $\mathbf{x}^{\star}(\mathbf{x})$, and it is defined as $x_1^{\star}(x_1) = P(X_1 \leq x_1)$ and $x_2^{\star}(x_2) = P(X_2 \leq \mathbf{x}_2 | X_1 = x_1)$, where $\mathbf{x}_2 = (x_1, x_2)$. The right-upper orthant at a point $\mathbf{z}$ is denoted by $C(\mathbf{z})$, and it is defined as $C(\mathbf{z}) = \{\mathbf{x} \in R^2 : \mathbf{z} \leq \mathbf{x}\}$. At the end, the upper-corrected and lower-corrected orthant at a point $\mathbf{z}$, for the random variable X, are shown as $R_{\mathbf{x}}(\mathbf{z})$ and $R'_{\mathbf{x}}(\mathbf{z})$ and defined as

$$R_{\mathbf{x}}(\mathbf{z}) = \{\mathbf{x} \in R^2 : x_1 \geq F^{-}_{X_1}(x_1^{\star}(z_1)), \quad x_2 \geq F^{-}_{X_2|X_1=x_1}(x_2^{\star}(\mathbf{z}_2))\},$$

$$R'_{\mathbf{x}}(\mathbf{z}) = \{\mathbf{x} \in R^2 : x_1 \leq F^{-}_{X_1}(x_1^{\star}(z_1)), \quad x_2 \leq F^{-}_{X_2|X_1=x_1}(x_2^{\star}(\mathbf{z}_2))\}.$$

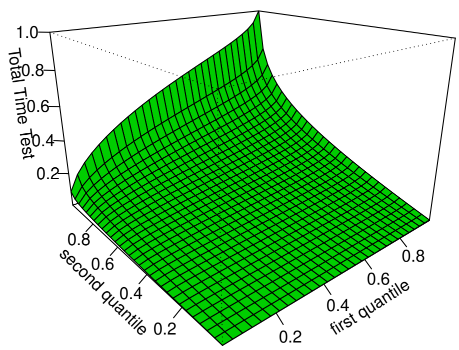
By considering that, the support of a random vector X is characterized as $Supp(\mathbf{X}) = \{\mathbf{x} \in R^2 : P[\mathbf{X} \in B_{\mathbf{x}}(\epsilon)] > 0, \text{ for all } \epsilon > 0\}$, where $B_{\mathbf{x}}(\epsilon)$ is the centered ball at $\mathbf{x}$ with radius ϵ. Then the following definition can be given.

Definition 3.7. Let $\mathbf{X}$ be a nonnegative random vector. Then the dual expectation associated to $\mathbf{X}$, when it exists, is defined as the real value

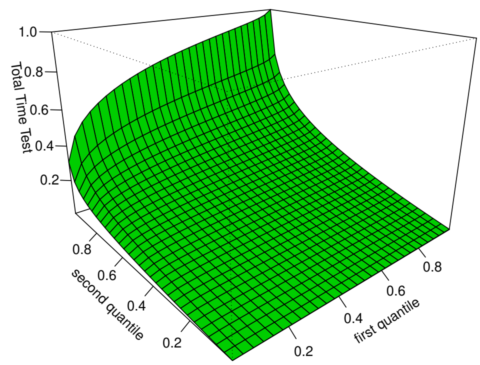
$$\bar{\mu}_{\mathbf{X}} = \int_{Supp(\mathbf{X})} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] d\mathbf{t}.$$

Fernandez-Ponce et al. (2011) showed that for a bivariate random variable $\mathbf{X} = (X_1, X_2)$

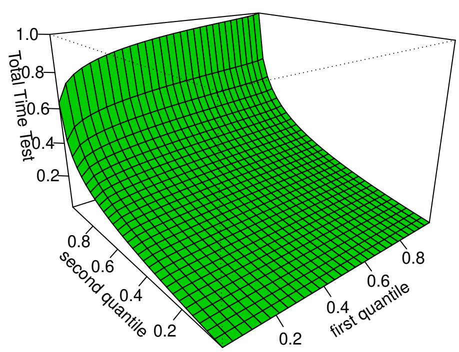
$$\bar{\mu}_{\mathbf{X}} = \nu_{\mathbf{X}} - \int_{F^{-}_{\mathbf{X}_1}(0)}^{\infty} \bar{F}_1(t) F^{-}_{\mathbf{X}_2|\mathbf{X}_1=t}(0) dt,$$



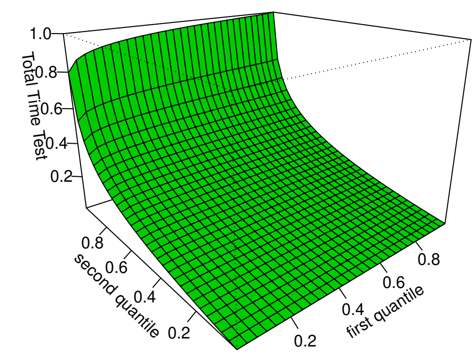
(a) $\alpha = 2$



(b) $\alpha = 4$



(c) $\alpha = 10$



(d) $\alpha = 20$

Figure 2. Bivariate *TTT* plot for different values of shape parameter α with Gumbel copula.

where $\nu_{\mathbf{X}} = \int_{F^{-}_{\mathbf{X}_1}(0)}^{\infty} \bar{F}_1(t) E[\mathbf{X}_2|\mathbf{X}_1 = t]dt$. Especially, if $\mathbf{X}$ is a nonnegative lifetime random variable, then $\bar{\mu}_{\mathbf{X}} \leq \nu_{\mathbf{X}}$. Also, if $Supp(\mathbf{X}) = [0, +\infty)^2$, then $\bar{\mu}_{\mathbf{X}} = \nu_{\mathbf{X}}$ holds. Fernandez-Ponce et al. (2011) defined the bivariate *EW* transform as follows:

$$EW_{\mathbf{X}}(\mathbf{u}) = \int_{R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})]d\mathbf{t}, \quad \mathbf{u} \in [0,1)^2. \tag{11}$$

Now, based on the said content, we denote the lower-corrected orthant at a point $\mathbf{z}$ for the random variable X, by $R^{'}_{\mathbf{x}}(\mathbf{z})$, defined as

$$R^{'}_{\mathbf{x}}(\mathbf{z}) = \{\mathbf{x} \in R^2 : x_1 \leq F^{-}_{X_1}(x^{\star}_1(z_1)), \quad x_2 \leq F^{-}_{X_2|X_1=x_1}(x^{\star}_2(\mathbf{z_2}))\}.$$

Now based on this concept, we define the *TTT* and *LIR* transforms in a bivariate mode and their corresponding stochastic orders in Definitions 3.8 and 3.9.

Definition 3.8. Let $\mathbf{X}$ be a nonnegative random vector with finite dual expectation. The bivariate *TTT* and *LIR* transforms associated to $\mathbf{X}$ is, respectively, defined as

$$T_{\mathbf{X}}(\mathbf{u}) = \int_{R^{'}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})]d\mathbf{t}, \quad \mathbf{u} \in [0,1)^2, \tag{12}$$

$$LIR_{\mathbf{X}}(\mathbf{u}) = \int_{R^{'}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]} P[\mathbf{X} \in R^{'}_{\mathbf{X}}(\mathbf{t})]d\mathbf{t}, \quad \mathbf{u} \in [0,1)^2. \tag{13}$$

If $\mathbf{X}$ is a nonnegative random vector with finite dual expectation, then the following results can be easily obtained:

- $T_{\mathbf{X}}(\mathbf{u})$ is increasing.
- $0 \leq T_{\mathbf{X}}(\mathbf{u}) \leq T_{\mathbf{X}}(\infty) = \bar{\mu}_{\mathbf{X}}$ for all $\mathbf{u}$ in $[0,1)^2$.
- If X_1 and X_2 are independent, then $T_{\mathbf{X}}(\mathbf{u}) = T_{\mathbf{X_1}}(\mathbf{u_1})T_{\mathbf{X_2}}(\mathbf{u_2})$.

Definition 3.9. Let $\mathbf{X}$ and $\mathbf{Y}$ are two nonnegative random vectors with finite dual expectation. Then $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the sense of bivariate *TTT* order, denoted by $\mathbf{X} \leq_{ttt} \mathbf{Y}$, if

$$T_{\mathbf{X}}(\mathbf{u}) \leq T_{\mathbf{Y}}(\mathbf{u}), \quad \mathbf{u} \in (0,1)^2, \tag{14}$$

also $\mathbf{Y}$ is in the sense of bivariate location independent riskier order, denoted by $\mathbf{X} \leq_{lir} \mathbf{Y}$, if

$$LIR_{\mathbf{X}}(\mathbf{u}) \leq LIR_{\mathbf{Y}}(\mathbf{u}), \quad \mathbf{u} \in (0,1)^2. \tag{15}$$

It can be easily written

$$T_{\mathbf{X}}(u_1,u_2) = \int_0^{F^{-}_{X_1}(u_1)} \bar{F}_{X_1}(t_1)T_{X_2|X_1=t_1}(u_2)dt_1.$$

Therefore for $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, we can write $\mathbf{X} \leq_{ttt} \mathbf{Y}$ if

$$X_1 \leq_{ttt} Y_1,$$

$$[X_2|X_1 = x_1] \leq_{ttt} [Y_2|Y_1 = y_1] \qquad whenever\ x_1 \leq y_1.$$

According to the results of section 6.B of the Shaked and Shantikomar (2007), the following results are easily obtained.

Remark 3.10. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors.

- If $\mathbf{X} \leq_{st} \mathbf{Y}$, then $\mathbf{X} \leq_{ttt} \mathbf{Y}$, wherein $\leq_{st}$ is a usual stochastic order.
- If $\mathbf{X} \leq_{sst} \mathbf{Y}$, then $\mathbf{X} \leq_{ttt} \mathbf{Y}$, wherein $\leq_{sst}$ is a strong stochastic order.
- If $X_i \leq_{st} Y_i, i = 1, 2, \ldots, n$, X, and Y have a common copula, then $\mathbf{X} \leq_{ttt} \mathbf{Y}$.
- If $X_i \leq_{ttt} Y_i, i = 1, 2, \ldots, n$, X, and Y have a common copula and $E(X_i) = E(Y_i)$, then $\mathbf{X} \leq_{ew} \mathbf{Y}$.

Also, consider the relationship between stochastic order and *TTT* transform order. Then Theorem 6.B.15 of Shaked and Shantikomar (2007) can be rewritten as follows.

Corollary 3.11. Let $\{Z_i\}_{i=1}^n$ be a sequence of independent random variables. If $Z_1 \leq_{lr} Z_2 \leq_{lr} \leq_{lr} \cdots \leq_{lr} Z_n$, then

$$(Z_1, Z_1 + Z_2, \ldots, \sum_{i=1}^{n} Z_i) \leq_{ttt} (Z_{\pi_1}, Z_{\pi_1} + Z_{\pi_2}, \ldots, \sum_{i=1}^{n} Z_{\pi_i})$$
$$\leq_{ttt} (Z_n, Z_n + Z_{n-1}, \ldots, \sum_{i=1}^{n} Z_i),$$

for every permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of $(1, 2, \ldots, n)$.

In a special case, if $X_1 \leq_{lr} X_2$, then

$$(X_1, X_1 + X_2) \leq_{ttt} (X_2, X_1 + X_2).$$

Fagiuoli et al. (1999) showed that $X \leq_{lir} Y \iff X \leq_{ew} Y$. Accordingly, every result that holds for the *EW* order can be reworded by means of the location independent riskier order.

Example 3.12. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two vector variables with bivariate exponential distribution introduced by Marshall and Olkin (1967), given by

$$\bar{F}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}, \tag{16}$$

wherein $x_1, x_2, \lambda_1, \lambda_2 > 0$ and $\lambda_{12} \geq 0$. Marshall and Olkin (1967) presented the properties of the introduced bivariate distribution and its applications and used the $BVE(\lambda_1, \lambda_2, \lambda_{12})$ to represent it. Now let $\mathbf{X} \sim BVE(\lambda_1, \lambda_2, \lambda_{12})$ and let $\mathbf{Y} \sim BVE(\theta_1, \theta_2, \theta_{12})$. By performing simple calculations, for $i = 1, 2$, we have $\bar{F}_i(x_i) = \exp\{-(\lambda_i + \lambda_{12})x_i\}$ and $F_i^{-1}(p) = -\frac{\ln(1-p)}{\lambda_i + \lambda_{12}}$.

As a result, if $\lambda_i + \lambda_{12} \geq \theta_i + \theta_{12},\ i = 1, 2$, and $\lambda_2 \geq \theta_2$, then it is simply shown that

$$\begin{aligned} &X_1 \leq_{ttt} Y_1, \\ &[X_2 | X_1 = x_1] \leq_{ttt} [Y_2 | Y_1 = y_1] \qquad whenever \quad x_1 \leq y_1, \end{aligned}$$

and consequently, $\mathbf{X} \leq_{ttt} \mathbf{Y}$.

Example 3.13. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two vector variables with bivariate exponential distribution introduced by Gumbel (1960), given by

$$\bar{F}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \delta x_1 x_2\}, \quad x_1, x_2, \lambda_1, \lambda_2, \delta > 0. \tag{17}$$

Lai and Xie (2006) also studied this distribution and examined its aging properties. We use $BGE(\lambda_1, \lambda_2, \delta)$ to represent it. Now let $\mathbf{X} \sim BGE(\lambda_1, \lambda_2, \delta_1)$ and let $\mathbf{Y} \sim BGE(\theta_1, \theta_2, \delta_2)$. By performing simple calculations, for $i = 1, 2$, we have $\bar{F}_i(x_i) = \exp\{-\lambda_i x_i\}$ and $F_i^{-1}(p) = -\frac{\ln(1-p)}{\lambda_i}$.

If $\lambda_1 \geq \theta_1$ and $\lambda_2 + \delta_1 x_1 \geq \theta_2 + \delta_2 x_2$, then

$$\begin{aligned} &X_1 \leq_{ttt} Y_1, \\ &[X_2 | X_1 = x_1] \leq_{ttt} [Y_2 | Y_1 = y_1] \qquad whenever \quad x_1 \leq y_1, \end{aligned}$$

and as a result $\mathbf{X} \leq_{ttt} \mathbf{Y}$.

4. New Better than Used in Total Time on Test Transform

The NBUT concept was introduced by Ahmad et al. (2005). Based on that a random variable X or F is said to be new better than used in *TTT* transform order, denoted by $NBUT$, if

$$X_t \leq_{ttt} X,$$

or equivalently, $X \in NBUT$ if and only if

$$\int_0^{F_{X_t}^{-1}(p)} \bar{F}(x+t)dx \leq \bar{F}(t) \int_0^{F^{-1}(p)} \bar{F}(x)dx, \quad p \in (0,1),$$

where $X_t = [X - t | X > t], t \in \{x : F(x) < 1\}$, named residual life variable, and denotes a random variable whose distribution is the same as the conditional distribution of X_t given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. We have a simple calculation for $p \in (0,1)$,

$$F_{X_t}^{-1}(p) = F^{-1}(F(t)(1-p)+p) - t.$$

In this section, the NBUT concept for the bivariate mode is presented, and its properties are examined. Also the bivariate TTT, EW, and LIR transforms provided for inactivity time variables. Zahedi (1985) defined a residual life variable in bivariate as follows:

$$\mathbf{X_t} = [X_1 - t_1, X_2 - t_2 | X_1 > t_1, X_2 > t_2], \quad t_1, t_2 > 0, \tag{18}$$

The survival function of $\mathbf{X_t}$ is defined as below:

$$\bar{F}_{\mathbf{X_t}}(x_1, x_2) = \frac{\bar{F}(x_1 + t_1, x_2 + t_2)}{\bar{F}(t_1, t_2)}. \tag{19}$$

Now the NBUT concept is defined as follows in the bivariate case:

$$\mathbf{X} \in NBUT \Longleftrightarrow \mathbf{X_t} \leq_{ttt} \mathbf{X}.$$

According to equation (3), this means that

$$\int_0^A \int_0^B \frac{\bar{F}(x_1 + t_1, x_2 + t_2)}{\bar{F}(t_1, t_2)} dx_1 dx_2 \leq T_{X_1, X_2}(p_1, p_2), \tag{20}$$

wherein

$$A = F_1^{-1}(F_1(t_1)(1-p_1)+p_1) - t_1,$$
$$B = F_2^{-1}(F_2(t_2)(1-p_2)+p_2) - t_2.$$

Example 4.1. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two vector variables with bivariate Pareto distribution introduced by Mardia (1970), given by

$$\bar{F}(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2)^{-\lambda}, \quad x_1, x_2, a_1, a_2, \lambda > 0. \tag{21}$$

Lai and Xie (2006) also studied this distribution in Example 9.3, p. 272, and we use the $BPE(a, b, \lambda)$ to represent it. By performing simple calculations, for $i = 1, 2$, we have $\bar{F}_i(x_i) = (1 + a_i x_i)^{\lambda}$, $F_i^{-1}(p) = \frac{(1-p)^{-\frac{1}{\lambda}} - 1}{a_i}$, and

$$T_{X_1, X_2}(p_1, p_2) = \frac{[(1-p_1)^{-\frac{1}{\lambda}} + (1-p_2)^{-\frac{1}{\lambda}} - 1]^{2-\lambda} - (1-p_1)^{\frac{\lambda+2}{\lambda}} - (1-p_2)^{\frac{\lambda+2}{\lambda}} + 1}{a_1 a_2 (1-\lambda)(2-\lambda)},$$

also $\bar{F}_{\mathbf{X_t}}(x_1,x_2) = \frac{[1+a_1(x_1+t_1)+a_2(x_2+t_2)]^{-\lambda}}{(1+a_1t_1+a_2t_2)^{-\lambda}}$. Hence for $t_1,t_2>0$ and $p_1,p_2 \in (0,1)$, according to equation (20), we have

$$\mathbf{X} \in NBUT \Longleftrightarrow \int_0^A \int_0^B \frac{[1+a_1(x_1+t_1)+a_2(x_2+t_2)]^{-\lambda}}{(1+a_1t_1+a_2t_2)^{-\lambda}} dx_2 dx_1 \leq T_{X_1,X_2}(p_1,p_2).$$

Remark 4.2. For any vector variable $\mathbf{X}=(X_1,X_2)$, let

$$\mathbf{X^t} = [t_1 - X_1, t_2 - X_2 | X_1 < t_1, X_2 < t_2], \quad t \in \{x : F(x) < 1\}. \tag{22}$$

When $\mathbf{X}$ is the lifetime of two devices, $\mathbf{X^t}$ can be regarded as the bivariate inactivity time of the devices at time $\mathbf{t}$ and

$$\bar{F}_{\mathbf{X^t}}(x_1,x_2) = \frac{F(x_1+t_1,x_2+t_2)}{F(t_1,t_2)}. \tag{23}$$

Therefore

$$T_{\mathbf{X^t}}(\mathbf{p}) = \int_0^{A'} \int_0^{B'} \frac{F(x_1+t_1,x_2+t_2)}{F(t_1,t_2)} dx_2 dx_1, \tag{24}$$

$$EW_{\mathbf{X^t}}(\mathbf{p}) = \int_{A'}^{\infty} \int_{B'}^{\infty} \frac{F(x_1+t_1,x_2+t_2)}{F(t_1,t_2)} dx_2 dx_1, \tag{25}$$

$$LIR_{\mathbf{X^t}}(\mathbf{p}) = \int_0^{A'} \int_0^{B'} 1 - \frac{F(x_1+t_1,x_2+t_2)}{F(t_1,t_2)} dx_2 dx_1, \tag{26}$$

wherein

$$A' = F_1^{-1}(F_1(t_1)(1-p_1)) - t_1,$$
$$B' = F_2^{-1}(F_2(t_2)(1-p_2)) - t_2.$$

5. Relationship Aging Classes in Bivariate

Rizwan and Hussainy (2016) and Basu et al. (1983) have expressed some aging classes in a bivariate mode and examined their properties. In this section, we describe the relationship between *TTT* transform and aging classes in a bivariate mode.

Proposition 5.1. Let $\mathbf{X}=(X_1,X_2)$ be a bivariate random variable with $F(x_1,x_2)$. The following results can be written, for $F(x_1,x_2)$, as follows:
a) $F(x_1,x_2) \in BNBU, BNBU(2), BNBUA$ if

$$T_{X_1,X_2}(p_1,p_2) \geq \int_0^{F_1^{-1}(p_1)} \int_0^{F_2^{-1}(p_2)} \bar{F}_{\mathbf{X_t}}(u,v) du dv, \quad p_1,p_2 \in (0,1).$$

b) $F(x_1,x_2) \in BNBUFR$ if

$$T_{X_1,X_2}(p_1,p_2) \geq \int_0^{F_1^{-1}(p_1)} \int_0^{F_2^{-1}(p_2)} \frac{\bar{F}(u+t,v+s)}{exp[-r(0,0)\sqrt{t^2+s^2}]} du dv, \quad p_1,p_2 \in (0,1).$$

c) $F(x_1,x_2) \in BHNBUE$ if

$$EW_{X_1,X_2} \leq \mu \exp\{-\frac{F_1^{-1}(p_1)+F_1^{-1}(p_2)}{\mu}\}, \qquad p_1,p_2 \in (0,1),$$

wherein $\mu = \int_0^\infty \int_0^\infty \bar{F}(u,v)dvdu$.

Proof To prove each of the cases, use Definition 2.6. By integrating the sides of the relevant inequalities and also replacing $x_1 = F_1^{-1}(p_1)$ and $x_2 = F_2^{-1}(p_2)$ by performing simple calculations, the desired result is obtained.

Example 5.2. Let $\mathbf{X} = (X_1, X_2)$ be a vector variable with bivariate exponential distribution introduced in (16). By doing the desired calculations, we have

$$\begin{aligned}\mu =& \int_0^\infty \int_0^\infty \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)\}dx_2dx_1\\ =&\frac{1}{\lambda_2}(\frac{1}{\lambda_1+\lambda_{12}})I_{[x_1>x_2]} + \frac{1}{\lambda_1}(\frac{1}{\lambda_2+\lambda_{12}})I_{[x_2>x_1]} + \frac{1}{\lambda_1+\lambda_2+\lambda_{12}}I_{[x_1=x_2]},\end{aligned}$$

$$\begin{aligned}T_{X_1,X_2}(p_1,p_2) =& \int_0^{-\frac{\ln(1-p_1)}{\lambda_1+\lambda_{12}}} \int_0^{-\frac{\ln(1-p_2)}{\lambda_2+\lambda_{12}}} \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)\}dx_2dx_1\\ =&(\frac{p_1}{\lambda_1+\lambda_{12}})(\frac{1}{\lambda_2}(1-(1-p_2)^{\frac{\lambda_2}{\lambda_2+\lambda_{12}}}))I_{[x_1>x_2]}\\ &+(\frac{p_2}{\lambda_2+\lambda_{12}})(\frac{1}{\lambda_1}(1-(1-p_1)^{\frac{\lambda_1}{\lambda_1+\lambda_{12}}}))I_{[x_2>x_1]} + T_X(p)I_{[x_1=x_2]},\end{aligned}$$

wherein $T_X(p) = \frac{p}{\lambda_1+\lambda_2+\lambda_{12}}$. Moreover,

$$\begin{aligned}EW_{X_1,X_2}(p_1,p_2) =& \int_{-\frac{\ln(1-p_1)}{\lambda_1+\lambda_{12}}}^\infty \int_{-\frac{\ln(1-p_2)}{\lambda_2+\lambda_{12}}}^\infty \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)\}dx_2dx_1\\ =&(\frac{1-p_1}{\lambda_1+\lambda_{12}})(\frac{1}{\lambda_2}(1-p_2)^{\frac{\lambda_2}{\lambda_2+\lambda_{12}}})I_{[x_1>x_2]}\\ &+(\frac{1-p_2}{\lambda_2+\lambda_{12}})(\frac{1}{\lambda_1}(1-p_1)^{\frac{\lambda_1}{\lambda_1+\lambda_{12}}})I_{[x_2>x_1]} + EW_X(p)I_{[x_1=x_2]},\end{aligned}$$

wherein $EW_X(p) = \frac{1-p}{\lambda_1+\lambda_2+\lambda_{12}}$.

$$\begin{aligned}LIR_{X_1,X_2}(p_1,p_2) =& \int_0^{-\frac{\ln(1-p_1)}{\lambda_1+\lambda_{12}}} \int_0^{-\frac{\ln(1-p_2)}{\lambda_2+\lambda_{12}}} [\exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)\}\\ &+\exp\{-(\lambda_1+\lambda_{12})x_1\} + \exp\{-(\lambda_2+\lambda_{12})x_2\} - 1]dx_2dx_1\\ =&\{(\frac{p_1}{\lambda_1+\lambda_{12}})(\frac{1}{\lambda_2}(1-(1-p_2)^{\frac{\lambda_2}{\lambda_2+\lambda_{12}}}))\\ &-\frac{p_1\ln(1-p_2)-p_2\ln(1-p_1)}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})}\}I_{[x_1>x_2]}\\ &+\{(\frac{p_2}{\lambda_2+\lambda_{12}})(\frac{1}{\lambda_1}(1-(1-p_1)^{\frac{\lambda_1}{\lambda_1+\lambda_{12}}}))\\ &-\frac{p_2\ln(1-p_1)-p_1\ln(1-p_2)}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})}\}I_{[x_2>x_1]}\\ &+\frac{\ln(1-p_1)\ln(1-p_2)}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} + [-\frac{\ln(1-p)}{\lambda_1+\lambda_2+\lambda_{12}} - T_X(p)]I_{[x_1=x_2]}.\end{aligned}$$

Now, we can examine each of the cases of Proposition 5.1.

- For $p_1, p_2 \in (0,1), \quad x_1 > x_2$, we have
$\frac{\lambda_1}{\lambda_2}\frac{\lambda_2+\lambda_{12}}{\lambda_1+\lambda_{12}} \leq (1-p_1)^{\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_{12}}}(1-p_2)^{\lambda_1-\frac{\lambda_2}{\lambda_1+\lambda_{12}}} \Longrightarrow F(x_1,x_2) \in BHNBUE.$
- For $p_1, p_2 \in (0,1), \quad x_1 < x_2$, we have
$\frac{\lambda_2}{\lambda_1}\frac{\lambda_1+\lambda_{12}}{\lambda_2+\lambda_{12}} \leq (1-p_2)^{\frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2+\lambda_{12}}}(1-p_1)^{\lambda_2-\frac{\lambda_1}{\lambda_2+\lambda_{12}}} \Longrightarrow F(x_1,x_2) \in BHNBUE.$
- For $p_1, p_2 \in (0,1), t_1, t_2 > 0$ and $\quad x_1 > x_2$,

  $F(x_1,x_2) \in BNBU, BNBU(2), BNBUA$ if

  $$(\frac{p_1}{\lambda_1+\lambda_{12}})(\frac{1}{\lambda_2}(1-(1-p_2)^{\frac{\lambda_2}{\lambda_2+\lambda_{12}}}))\bar{F}(t_1,t_2)$$
  $$\geq (\frac{p_1\exp\{-(\lambda_1+\lambda_{12})t_1\}}{\lambda_1\lambda_2+\lambda_{12}\lambda_2})[\exp\{-\lambda_2 t_2\} - \exp\{\frac{\lambda_2}{\lambda_2+\lambda_{12}}\ln(1-p_2)\}],$$

  and for $x_1 < x_2$, $F(x_1,x_2) \in BNBU, BNBU(2), BNBUA$ if

  $$(\frac{p_2}{\lambda_2+\lambda_{12}})(\frac{1}{\lambda_1}(1-(1-p_1)^{\frac{\lambda_1}{\lambda_1+\lambda_{12}}}))\bar{F}(t_1,t_2)$$
  $$\geq (\frac{p_2\exp\{-(\lambda_2+\lambda_{12})t_2\}}{\lambda_1\lambda_2+\lambda_{12}\lambda_1})[\exp\{-\lambda_1 t_1\} - \exp\{\frac{\lambda_1}{\lambda_1+\lambda_{12}}\ln(1-p_1)\}].$$

Note that for $x_1 = x_2$, the results will be the same as the univariate case.

6. Conclusions

In this article, we expressed the *TTT*, *EW*, and *LIR* transforms in a bivariate mode and examined its features. Also, considering an example, drawing a diagram of this transform, and assuming different modes of changing the parameters of the marginal distributions and the copula parameter were considered to show the dependence of the two variables. We performed the sensitivity analysis for this diagram. In continuation, based on the right-upper orthant concept, *TTT* and *LIR* transforms were rewritten, and some notes about stochastic orders in the bivariate mode were stated. Then we introduced the class $NBUT$ and all three transforms for the inactivity time variable in a bivariate mode. Finally, the relation of bivariate TTT with some concepts of aging was obtained.

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