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A New Version of the Inverse Weibull Model with Properties, Applications and Different Methods of Estimation

Mohamed Ibrahim[1,*], S. I. Ansari[2], Abdullah H. Al-Nefaie [1], and Haitham M. Yousof[3]

[1]*Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia*
[2]*Department of Business Administration, Azad institute of Engineering and Technology, Lucknow, India*
[3]*Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Benha 13518, Egypt*

Abstract A new extension of the inverse Weibull model is introduced and studied. Some of its statistical properties are derived. Different estimation methods are used for estimating the unknown parameter. We assessed the performance of all methods via simulation study. Two real data applications are used for comparing competitive estimation methods. The importance of the new model is demonstrated via two real data applications. The new model is much better than other competitive models in modeling the two real data sets.
Zero Truncated Poisson Distribution; Inverse Weibull Distribution; Maximum Likelihood; Bootstrapping Estimation; Kolmogorov Estimation; Modeling.

Mathematics Subject Classification: 60E05; 62N01; 62G05; 62N02; 62N05; 62E10; 62P30

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1. Introduction and physical motivation

A probability distribution is a mathematical function used in probability theory and statistics that estimates the likelihood that various possible outcomes of an experiment will occur. In many practical fields, including engineering, medicine, and finance, among others, right or left skewness, bi-modality, or multi-modality are features of data sets that can be modelled using statistical distributions. Because of their straightforward shapes and identifiability characteristics, well-known distributions including normal, Weibull, gamma, and Lindley are frequently utilised. However, during the past ten or so years, much research has concentrated on the more flexible and complicated Generalized or simply G families of continuous distributions, with the aim of improving their modelling capabilities by including one or more shape parameters. The inverse Weibull distribution (IW) has the ability to model failure hazard rates which are common in reliability analysis and biological studies. De Gusmo et al. [36] defined and studied a new three-parameter IW distribution with monotonically decreasing and upside down failure rate. They provide some mathematical properties of the new IW distribution and proposed a location-scale regression model for modeling lifetime real data. Due to de Gusmo et al. [36], a random variable (r.v.) Y is said to have the IW distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by (for $y \geq 0$)

$$h(y) = h_c(y; a, b) = cba^b y^{-b-1}\exp\left[-c\left(a\frac{1}{y}\right)^b\right],$$

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*Correspondence to: Mohamed Ibrahim (Email:miahmed@kfu.edu.sa). Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia.

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and

$$H(y) = H_c(y; a, b) = \mathbf{\Delta}_{\underline{\mathbf{V}}}(y) = \exp\left[-c\left(a\frac{1}{y}\right)^b\right]|_{\underline{\mathbf{V}}=a,b,c},$$

respectively, where $a > 0$ is a scale parameter, c and $b > 0$ are a shape parameters, respectively. We can easily prove that $h_c(y; a, b)$ is a density function by substituting $u = -c\left(a\frac{1}{y}\right)^b$. The standard IW distribution is a special case of $h_c(y; a, b)$ when $c = 1$. For $c = 1$, we get the standard IW distribution. The IW distribution can be simulated by using the nonlinear equation $y_u = a\left\{\frac{1}{c}\left[-\log(u)\right]\right\}^{-\frac{1}{b}}$ where u has the uniform $U(0, 1)$ distribution. For $b = 2$, we get the generalized Inverse Rayleigh distribution (GIR). For $a = 1$, we have the generalized Inverse Exponential (GIEx) distribution. For $c = 1$ and $b = 2$, we get the standard IR distribution. For $c = a = 1$ we get the standard IEx model. For more details about the IW model see Gusmao et al. (2011), Harlow [52], Zaharim et al. [112], Krishna et.al. [70], Barreto-Souza et al. [30], Afify et al. [11], Korkmaz et al. [68], Ul-Haq et al. [97], Yousof et al. ([99], [104], [109] and [106]), Chakraborty et al. [32], Elbiely and Yousof [39], Jahanshahi et al. [62] and Elsayed and Yousof [46]. Recentely, Yousof et al. [110] expanded the IW model by defining a new G family of distributions called extended odd inverse Weibull distribution (EOIW) and studied its properties, applications and then presented a regression mode based on the new family. Salah et al. [83] defined and studied a new IW model called the odd-Burr inverse Weibull (OBIW) model. In their study Salah et al. [83] presented some new bivariate type extensions using Farlie-Gumbel-Morgenstern copula, modified Farlie-Gumbel-Morgenstern copula, Clayton copula, and Renyi's entropy copula. Yousof et al. [108] defined and studied the two-parameter Xgamma inverse Weibull (XgIW) distribution with some Characterization results, different copulas and different classical estimation methods. Al-Babtain et al. [9] presented a new three parameter inverse Weibull model called the generalized odd generalized exponential inverse Weibull (GOGEIW) model with simple type copula, mathematical properties and some applications to breaking stress of carbon fibres and strengths data sets. Bhatti et al. [29] defined and studied the modified Burr XII Inverse Weibull (BXIIIW) distribution. Goual et al. [49] studied the Lomax inverse Weibull (LxIW) model and its properties, applications and presented a modified chi-squared goodness-of-fit test for censored validation.

In this paper we propose and study a new extension of the IW distribution using the zero truncated Poisson (ZTP) distribution. Suppose that a system has N subsystems functioning independently at a given time where N has ZTP distribution with parameter $\lambda = 1$. It is the conditional probability distribution of a Poisson-distributed r.v., given that the value of the r.v. is not zero. The probability mass function (PMF) of N is given by

$$\text{PMF}(N = n)|_{(n=1,2,\ldots)} = \frac{1}{n!}\frac{\exp(-1)}{1 - \exp(-1)}. \tag{1}$$

Note that for ZTP r.v., the expected value $\mathbf{E}(N|\lambda)$ and variance $Var(N|\lambda)$ are, respectively, given by

$$\mathbf{E}(N|\lambda = 1) = \frac{1}{1 - \exp(-1)},$$

and

$$Var(N|\lambda) = \frac{1}{1 - \exp(-1)}\left(2 - \frac{1}{1 - \exp(-1)}\right).$$

Suppose that the failure time of each subsystem has the Burr X inverse Weibull ("BXIW(θ, c, a, b)" for short) defined by the cumulative distribution function (CDF) given by

$$G(y) = \left\{1 - \exp\left[-\mathbf{O}^2_{\underline{\mathbf{V}}}(y)\right]\right\}^\theta, \tag{2}$$

where

$$\mathbf{O}_{\underline{\mathbf{V}}}(y) = \frac{\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)}{1 - \mathbf{\Delta}_{\underline{\mathbf{V}}}(y)}.$$

Let X_i denote the failure time of the ith subsystem and let

$$Y = \min\{X_1, X_2, \cdots, X_N\}. \tag{3}$$

Then the conditional CDF of Y given N is

$$F\left(y|N\right) = 1 - \Pr\left(Y > y \mid N\right) = 1 - \left[1 - G(y)\right]^N. \tag{4}$$

Therefore using (4), the unconditional CDF of the PBXIW model can be expressed as

$$F_{\underline{\Theta}}\left(y\right) = \frac{1}{1-\exp\left(-1\right)}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^2\left(y\right)\right]\right\}^{\theta}\right)\right], \tag{5}$$

with the corresponding probability density function (PDF) as

$$f_{\underline{\Theta}}\left(y\right) = \frac{2\theta bc}{1-\exp\left(-1\right)}\frac{y^{-(b+1)}\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}\left(y\right)\right]^{-3}}{\exp\left[2\left(a\frac{1}{y}\right)^b + \mathbf{O}_{\underline{\mathbf{V}}}^2\left(y\right)\right]}\frac{\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^2\left(y\right)\right]\right\}^{\theta-1}}{\exp\left(\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^2\left(y\right)\right]\right\}^{\theta}\right)}. \tag{6}$$

The hazard rate function (HRF) of the new model can be calculated via $f(y)/\left[1 - F(y)\right]$. For $\theta = 1$, the PBXIW model will reduce to Poisson Rayleigh IW (PRIW) model. For $c = 1$, the PBXIW model will reduce to the four parameter PBXIW. For $b = 2$, the PBXIW model will reduce to PBXIR model. For $a = 1$, the PBXIW model will reduce to PBXIEx model. For $\theta = 1, b = 2$ the PBXIW model will reduce to PRIR model. For $\theta = a = 1$, the PBXIW model will reduce to PRIEx model. For $\theta = c = 1, b = 2$ the PBXIW model will reduce to the two parameter PRIR model. For $\theta = c = a = 1$, the PBXIW model will reduce to the two parameter PRIEx model. For $c = 1, b = 2$ the PBXIW model will reduce to the three parameter PBXIR model. For $c = a = 1$, the PBXIW model will reduce to the three parameter PBXEx model. The PDF of the new model can be right skewed and unimodal with symmetric and asymmetric shapes (see Figure 1) also it can be left skewed (see Table 1). The HRF of the new model can be decreasing-constant-increasing (U-shape or bathtub shape), increasing-constant-increasing, increasing, upside down-bathtub, monomaniacal decreasing, upside down-increasing, monomaniacal increasing and upside down (see Figure 2).

Recently, many researchers have been keen to derive new probability distributions, but they have taken care of some applied aspects in practical fields such as insurance and actuarial science, and we mention them, for example, see Mohamed et al. ([78], [79] and [81]). While others were concerned with discretizing the continuous probability distributions (continuous G families of probability distributions) and applying the new discrete distributions (discrete G families of probability distributions) to different count and zero-inflated data, for more details see Aboraya et al. [2], Yousof et al. [107], Ibrahim et al. [56], Eliwa et al. [44] and Chesneau t al. [34]. By examining the statistical literature in the field of statistical hypothesis tests, we find that there are many practical applications for commonly used tests and new ones using many of the new probability distributions. For example, many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test and modified Nikulin-Rao-Robson goodness-of-fit test are considered by Ibrahim et al. [61], Goual et al. ([48], [49]), Mansour et al. ([72], [73], [74], [75], [76], [77]), Yadav et al. [98], Goual and Yousof [47], and Ibrahim et al. [54] among others. However, the Bagdonavičius-Nikulin goodness-of-fit test and the modified Bagdonavičius-Nikulin goodness-of-fit test are considered by Aidi et al [7], Ibrahim et al. [55], Yousof et al. ([100], [105] and [102]).

Recent developments in statistical modeling have introduced innovative distributions and methodologies that align well with the potential applications of the PBXIW distribution. Ahmed et al. [4] developed a framework for amputated life-testing using extended Dagum percentiles, emphasizing the role of advanced distributions in life-testing and group inspection plans. Similarly, the PBXIW distribution can be tailored to model failure times under group inspection schemes, optimizing sample sizes and analyzing termination time ratios in reliability studies. Khan et al. [**?**] proposed a heavy-tailed Lomax model for extreme-value applications, incorporating peaks-over-random-threshold Value-at-Risk (VaR) and mean-of-order-p analysis. These approaches highlight the importance

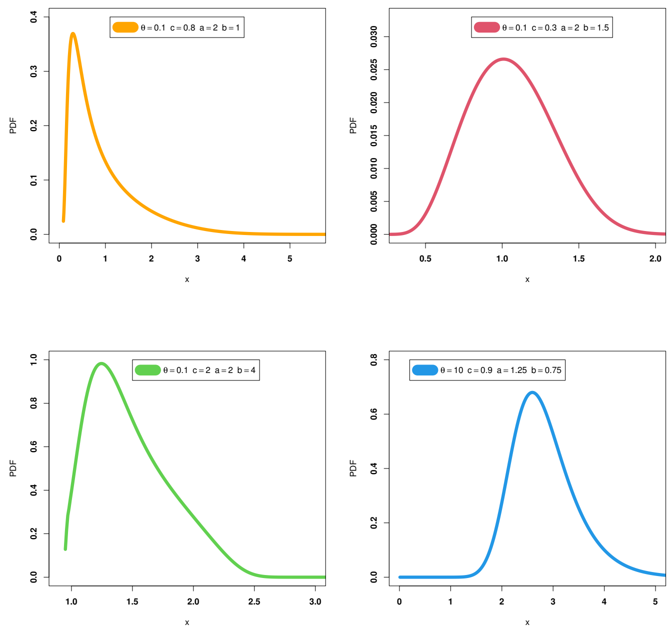

Figure 1. Plots of the new PDF for selected values of the parameter.

of heavy-tailed distributions in risk modeling, which directly connects to PBXIW's ability to capture extreme values and provide effective tools for financial risk management and insurance claims analysis. The work of Abiad et al. [1] on copula-based reliability applications using a new Fisk probability model underscores the importance of modeling dependencies between variables in reliability analysis. PBXIW could be extended to incorporate copula-based frameworks, enabling the modeling of dependent reliability systems and joint risk scenarios in engineering and applied sciences. Alizadeh et al. [18] introduced a weighted Lindley model for analyzing extreme historical insurance claims. This focus on extreme-value modeling in insurance aligns with PBXIW's potential to handle tail risk in claims distributions. PBXIW could also address challenges in modeling and predicting extreme insurance claims, providing robust actuarial solutions. Finally, Das et al. [35] applied the Laplace distribution for economic peaks and Value-at-Risk analysis in real estate markets. Similar to their work, the PBXIW distribution could serve as a flexible model for economic and financial applications, particularly in assessing risk associated with economic extremes, such as housing price volatility.

In this paper, the PBXIW model is studied through some aspects of mathematical theory and practical application. After showing how the new distribution was derived, we dealt with some mathematical and statistical aspects of the new distribution, such as its distinctive statistical properties. Three-dimensional skewness and kurtosis plots are presented to show the wide flexibility of the PBXIW model. The flexibility of the new distribution is influenced by the degrees of the skew coefficient, kurtosis coefficient, failure rate function, and variety in the PDF and failure rate functions. Furthermore, the usefulness and efficiency of the probability distribution in statistical modelling are significant in this context. We looked at the novel PDF and found that it was highly flexible in these and other ways. This inspired us to conduct a detailed analysis of this probability distribution. Different estimation methods of are used for estimating the unknown parameters including the maximum likelihood estimation method, the Cramér–von Mises estimation method, the bootstrapping estimation method, the Kolmogorov estimation method and the Anderson Darling method(left-tail of the second-order). We assessed the performance of all

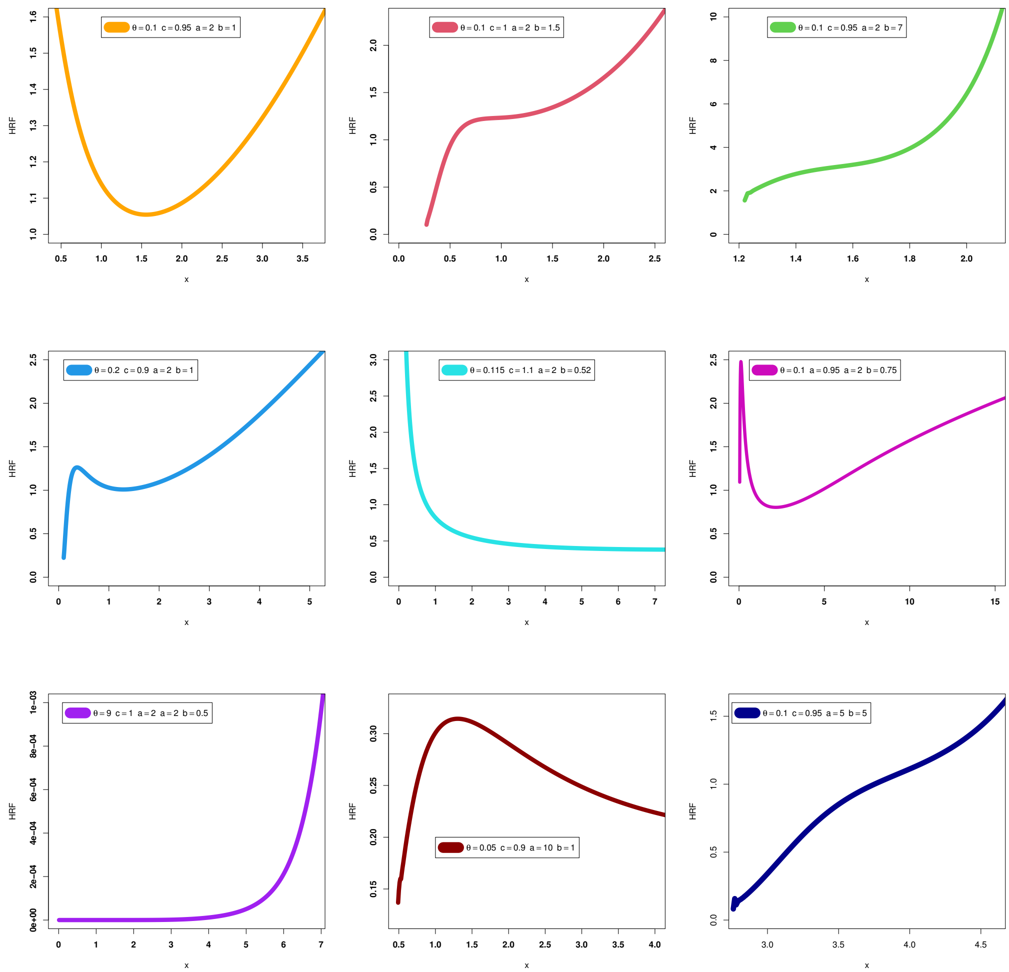

Figure 2. Plots of the new HRF for selected values of the parameter.

methods via simulation study. Two real data applications are used for comparing competitive estimation methods. We are motivated to introduce the PBXIW model for the following reasons:

1. Generating new PDF that can be ”asymmetric and right skewed with a hefty tail”, ”symmetric”. Because the PDF for every new model is so flexible, we may use it to analyze a variety of environmental data sets.

2. Presenting some new special models with different types of HRFs, such as decreasing-constant-increasing (U-shape or bathtub shape), increasing-constant-increasing, increasing, upside down-bathtub, monomaniacal decreasing, upside down-increasing, monomaniacal increasing and upside down. The distribution’s elasticity increases with the number of different failure rate types. Many practitioners may utilise the new distribution in statistical modelling and mathematical analysis thanks to these forms, which make their work easier. We pay a lot of attention to the issue of monitoring the failure rate function for this specific reason.

3. The degree of skew coefficient and kurtosis coefficient both affect how flexible the new distribution is. Furthermore, the usefulness and efficiency of the probability distribution in statistical modelling are significant in this context. We looked at the novel PDF and found that it was highly flexible in these and other ways. This inspired us to conduct a detailed analysis of this probability distribution.

4. Proposing new continuous models for modeling the "bimodal and left skewed heavy tail data" and "the bimodal and right skewed heavy tail data". As shown in this paper, the new model has shown a remarkable superiority in modeling these types of data, whether "bimodal and left skewed heavy tail data" or "the bimodal and right skewed heavy tail data". As shown in this paper, the new model has shown a remarkable superiority in modeling these types of data, whether "bimodal and left skewed heavy tail data" or "the bimodal and right skewed heavy tail data".The new distribution also showed a remarkable superiority in modeling real data, which contains outliers. It is worth noting that the outliers family of distributions is one of the most popular families used in such cases, and the new distribution can undoubtedly be considered a member of the outliers family.

5. In statistical modeling of the increasing hazard rate count data, the PBXIW model provides adequate results; hence, the BXIW model is recommended for modeling the monotonically increasing hazard rate data. Moreover, the same baseline model is also suitable for modeling the monotonically increasing-constant failure rate data with adequate fitting.

2. Mathematical properties

2.1. Useful expansions

Using the power series, the PDF in (6) can be written as

$$f_{\underline{\Theta}}(y)=\sum_{h=0}^{\infty}\frac{2\theta ba^{b}c\,(-1)^{h}}{h!\,[1-\exp(-1)]}y^{-(b+1)}\frac{\exp\left[-2\left(a\frac{1}{y}\right)^{b}\right]}{\exp\left[\mathbf{O}^{2}_{\underline{\mathbf{V}}}(y)\right]\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)\right]^{3}}\left\{1-\exp\left[-\mathbf{O}^{2}_{\underline{\mathbf{V}}}(y)\right]\right\}^{\theta_{*}-1}. \quad (7)$$

Then, if $\left|\frac{\Upsilon_1}{\Upsilon_2}\right|<1$ and $\Upsilon_3>0$ is a real non-integer, the following power series holds

$$\left(1-\frac{\Upsilon_1}{\Upsilon_2}\right)^{\Upsilon_3-1}=\sum_{\Upsilon_4=0}^{\infty}\frac{(-1)^{\Upsilon_4}\,\Gamma(\Upsilon_3)}{\Upsilon_4!\,\Gamma(\Upsilon_3-\Upsilon_4)}\left(\frac{\Upsilon_1}{\Upsilon_2}\right)^{\Upsilon_4}. \quad (8)$$

Applying (8) to (7) we have

$$f_{\underline{\Theta}}(y)=\frac{2\theta ba^{b}cy^{-(b+1)}}{[1-\exp(-1)]}\sum_{h,i=0}^{\infty}\frac{(-1)^{h+i}\,\Gamma(\theta_*)}{i!\,\Gamma(\theta_*-i)}\exp\left[-2\left(a\frac{1}{y}\right)^{b}\right]\frac{\exp\left[-(i+1)\,\mathbf{O}^{2}_{\underline{\mathbf{V}}}(y)\right]}{\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)\right]^{3}}. \quad (9)$$

Applying the power series to the term $\exp\left[-(i+1)\,\mathbf{O}^{2}_{\underline{\mathbf{V}}}(y)\right]$, equation (9) becomes

$$f_{\underline{\Theta}}(y)=ba^{b}cy^{-(b+1)}\sum_{h,i,j=0}^{\infty}\frac{2\theta\,(-1)^{h+i+j}\,\Gamma(\theta_*)\,(i+1)^{j}\left[\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)\right]^{2j+2}}{i!\,j!\,[1-\exp(-1)]\,\Gamma(\theta_*-i)\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)\right]^{2j+3}}\Big|_{\theta_*=(1+h)\theta}. \quad (10)$$

Consider the series expansion

$$\left(1-\frac{\Upsilon_1}{\Upsilon_2}\right)^{-\Upsilon_3}\Big|_{\left(\left|\frac{\Upsilon_1}{\Upsilon_2}\right|<1,\ c>0\right)}=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\Upsilon_3+\kappa)}{\kappa!\,\Gamma(\Upsilon_3)}\left(\frac{\Upsilon_1}{\Upsilon_2}\right)^{\kappa}. \quad (11)$$

Applying the expansion in (11) to (10) for the term $\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}(y)\right]^{2j+3}$, Equation (10) becomes

$$\begin{aligned}f_{\underline{\Theta}}(y)&=\sum_{h,i,j,\kappa=0}^{\infty}\frac{2\theta c\,(-1)^{h+i+j}\,(i+1)^{j}}{i!\,j!\kappa!\,[2(j+1)+\kappa]\,[1-\exp(-1)]}\frac{\Gamma(\theta_*)\,\Gamma(3+2j+\kappa)}{\Gamma(\theta_*-i)\,\Gamma(2j+3)}\\&\quad\times ba^{b}y^{-(b+1)}\,[2(j+1)+\kappa]\exp\left\{-[2(j+1)+\kappa]\left(a\frac{1}{y}\right)^{b}\right\}.\end{aligned}$$

This can be written as

$$f_{\underline{\Theta}}(y)=\sum_{j,\kappa=0}^{\infty}\varsigma_{j,\kappa}\,h_{c_*}(y;a,b)|_{c_*=c[2(j+1)+\kappa]},\tag{12}$$

where

$$\varsigma_{j,\kappa}=\frac{2\theta c\,(-1)^j\,\Gamma(3+2j+\kappa)}{j!\kappa!\,(1-\exp(-1))\,\Gamma(2j+3)\,[2(j+1)+\kappa]}\sum_{h,i=0}^{\infty}\frac{(-1)^{h+i}\,\Gamma(\theta_*)\,(i+1)^j}{i!\,\Gamma(\theta_*-i)},$$

and $h_{c_*}(y;a,b)$ is the IW PDF with scale parameter $ac_*^{1/b}$ and and shape parameter b. Similarly, the CDF of the PBXIW model can also be expressed as

$$F_{\underline{\Theta}}(y)=\sum_{j,\kappa=0}^{\infty}\varsigma_{j,\kappa}\,H_{c_*}(y;a,b),\tag{13}$$

where $H_{c_*}(y;a,b)$ is the the IW CDF with scale parameter $ac_*^{1/b}$ and shape parameter b.

2.2. Some properties

The quantile function (QF) of Y, where $Y\sim$PBXIW (θ,c,a,b), is obtained by inverting (5) as

$$Q(u)=a\sqrt[b]{\left\{-\ln\left[\left(1+\sqrt{-\ln\left[1-\sqrt[\theta]{(-\ln\{1-u(1-\exp(-1))\})}\right]c^{-1}}\right)\right]\right\}^{-1}}.\tag{14}$$

Simulating the PBXIW r.v. is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the r.v. $y=Q(U)$ follows (6).

The r^{th} ordinary moment of Y, say $\mu'_{r,Y}$, follows from (12) as

$$\mu'_{r,Y}|_{(r<b)}=\mathbf{E}(Y^r)=\sum_{j,\kappa=0}^{\infty}\varsigma_{j,\kappa}\,a^r c_*^{r/b}\,\Gamma(1-r/b).\tag{15}$$

Setting $r=1$ in (15) gives the mean of Y as

$$\mathbf{E}(Y)|_{(1<b)}=\sum_{j,\kappa=0}^{\infty}\varsigma_{j,\kappa}\,ac_*^{1/b}\,\Gamma(1-1/b),$$

where $\Gamma(1+\Upsilon)|_{(\Upsilon\in\mathbb{R}^+)}=\Upsilon!$, and

$$\Gamma(\Upsilon)=\int_0^{\infty}y^{\Upsilon-1}\exp(-t)\,dt.$$

The flexibility of the new distribution is influenced by the degrees of the skew coefficient, kurtosis coefficient. Figure 3 gives some three-dimensional skewness plots. Figure 4 gives some three-dimensional kurtosis plots. Figures 3 and 4 show the wide flexibility of the PBXIW model. In these two figures, we have explored the skew coefficient, kurtosis coefficient numerically, and then we made 3D graphics to illustrate and highlight the elasticity of the new distribution through these two coefficients. From Figure 3 it is clear how important and flexible the new distribution is because the skew coefficient contains various shapes. This diversity in the shape of the skew coefficient is important in the statistical and mathematical modeling of data. Similarly, Figure 4 shows the diversity of the values of the kurtosis coefficient for the new distribution, and this is what gives the distribution great importance in statistical modeling processes. The two figures also show the importance of all the parameters of the model, and that all of these parameters directly affect the skew coefficient and the kurtosis coefficient, and that these parameters have added to the distribution more practical importance and flexibility.

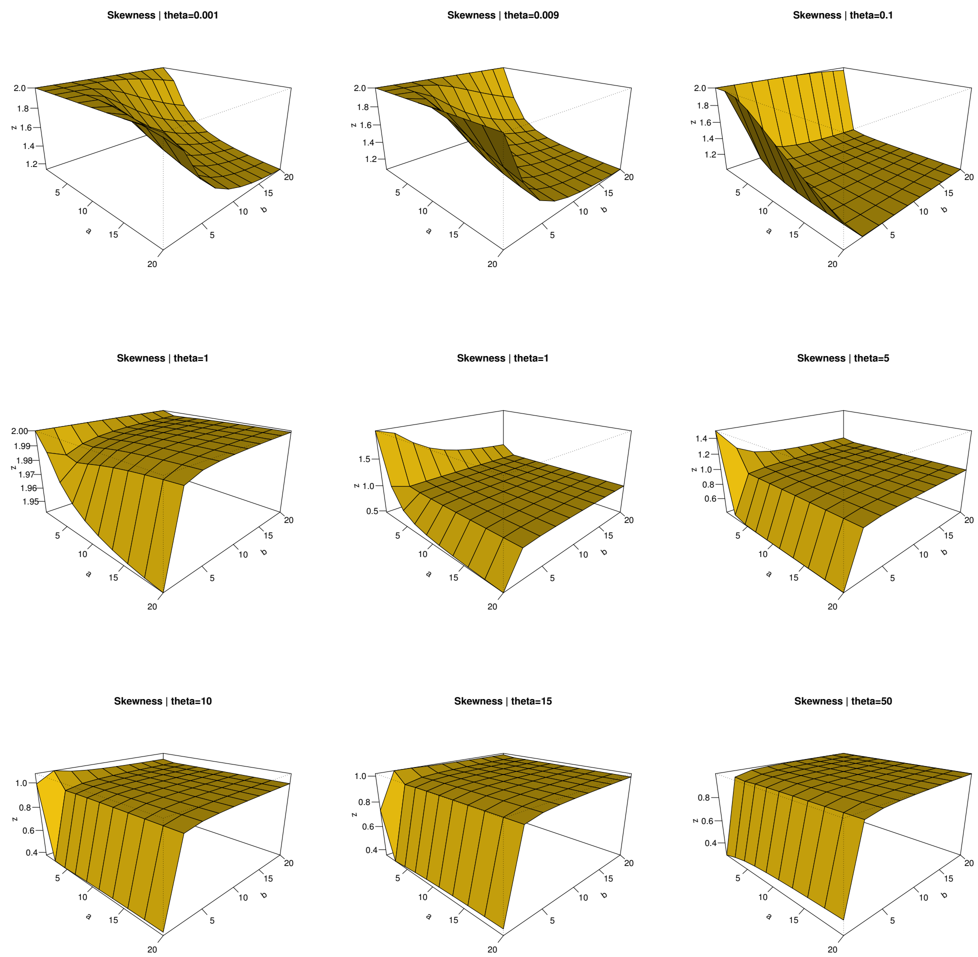

Figure 3. Three-dimensional skewness plots.

The r^{th} incomplete moment of Y is defined by $m_{r,Y}(y) = \int_{-\infty}^{y} y^r\, f(y)dy$. We can write from (12)

$$m_{r,Y}(y)|_{(r<b)} = \gamma\left(1 - r/b, \left(\frac{a}{t}\right)^b\right) \sum_{j,\kappa=0}^{\infty} \varsigma_{j,\kappa}\, a^r c_*^{r/b}. \tag{16}$$

Setting $r = 1$ in (16) gives the 1^{st} incomplete moment of Y as

$$m_{1,Y}(y)|_{(1<b)} = \gamma\left(1 - 1/b, \left(\frac{a}{t}\right)^b\right) \sum_{j,\kappa=0}^{\infty} \varsigma_{j,\kappa}\, a c_*^{1/b},$$

where $\gamma(\xi_1, \xi_2)$ is the incomplete gamma function, where

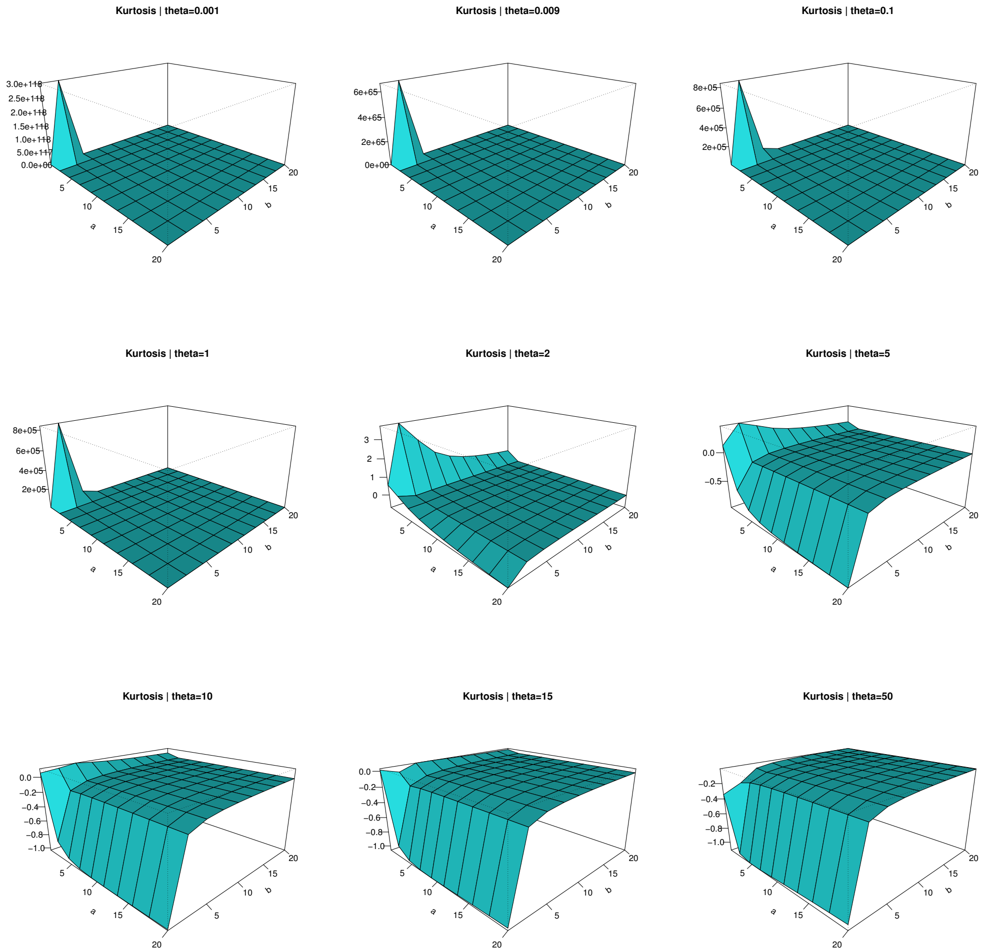

Figure 4. Three-dimensional kurtosis plots.

$$\begin{aligned}\gamma\left(\xi_1,\xi_2\right)|_{(\xi_1\neq 0,-1,-2,\ldots)} &= \int_0^{\xi_2} t^{\xi_1-1}\exp\left(-t\right)dt=\frac{1}{\xi_1}\xi_2^{\xi_1}\left\{{}_1\mathbf{F}_1\left[\xi_1;\xi_1 a+1;-\xi_2\right]\right\}\\ &= \sum_{\kappa=0}^{\infty}\xi_2^{\xi_1+\kappa}\frac{\left(-1\right)^{\kappa}}{\kappa!\left(\xi_1+\kappa\right)}=\Gamma\left(\xi_1\right)-\Gamma\left(\xi_1,\xi_2\right),\end{aligned}$$

the function ${}_1\mathbf{F}_1\left[\cdot,\cdot,\cdot\right]$ is a called the confluent hypergeometric function and

$$\Gamma\left(\xi_1,\xi_2\right)=\int_{\xi_2}^{\infty} t^{\xi_1-1}\exp\left(-t\right)dt,$$

The moment generating function (MGF) of Y, say $M(t)=\mathbf{E}\left(\exp\left(t\,y\right)\right)$, is obtained from (12) as

$$M(t)|_{(r<b)}=\sum_{j,\kappa,r=0}^{\infty}\varsigma_{j,\kappa}\frac{t^r}{r!}a^r c_*^{r/b}\,\Gamma\left(1-r/b\right).$$

3. Estimation

In fact, the statistical literature contains many estimation methods and all of them are of interest and are appreciated by many researchers. In this Section, different estimation methods such as the maximum likelihood estimation method, Cramér–von Mises estimation method, the bootstrapping estimation method, Kolmogorov estimation method and Anderson Darling method (the left-tail of the second order) are used for estimating the unknown parameters. In this work we have neglected many of the methods of appreciation for me for their insignificance but because we must focus our attention only on some of the most famous and efficient methods in order to be able to compare them. But of course there are many ways that can be taken into consideration in future work. For more methods, see Ali et al. [15], Ali et al. [14], Alizadeh et al. [20], Yousof et al. [101], El-Morshedy et al. [45] and Yousof et al. [12].

3.1. The maximum likelihood estimation (MLE) method

A statistical method known as maximum likelihood estimation (MLE) is used to estimate the parameters of a probability distribution that has been assumed in light of certain observed data. To do this, a likelihood function is maximized to increase the probability of the observed data under the presumptive statistical model. The parameter space position where the likelihood function is maximized is known as the maximum likelihood estimate. Since its justification is understandable and flexible, maximum likelihood is a well-liked method for drawing statistical conclusions. If the likelihood function is differentiable, then maxima can be determined using the derivative test. For instance, the ordinary least squares estimator increases the likelihood of the linear regression model, enabling in some cases to explicitly solve the first-order conditions of the likelihood function. However, it is frequently necessary to employ numerical methods to ascertain the maximum of the probability function. From the perspective of Bayesian inference, MLE is often equivalent to maximum a posteriori (MAP) estimates under a uniform prior distribution of the parameters. When likelihood serves as the goal function in frequentist inference, MLE is a special illustration of an extremum estimator. Consider a random sample from the PBXIW, then the log likelihood function can be expressed as

$$\begin{aligned}
\log \mathbf{L} &= n\log 2 + n\log\theta + n\log b + nb\log a + n\log c \\
&\quad -n\log\left[1-\exp(-1)\right] - (b+1)\sum_{i=1}^{n}\log y_i - 3\log\left[1-\mathbf{\Delta}_{\underline{\mathbf{V}}}(y_i)\right] \\
&\quad +2\sum_{i=1}^{n}\log\mathbf{\Delta}_{\underline{\mathbf{V}}}(y_i) - \sum_{i=1}^{n}\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}^{\theta} \\
&\quad -\sum_{i=1}^{n}\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i) + (\theta-1)\sum_{i=1}^{n}\log\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}.
\end{aligned}$$

The maximum likelihood method and its procedures are available in the literature with details. The components of the score vector are $\mathbf{U}(\underline{\Theta}) = \frac{\partial \ell}{\partial \underline{\Theta}} = \left(\frac{\partial \log \mathbf{L}}{\partial \theta}, \frac{\partial \log \mathbf{L}}{\partial a}, \frac{\partial \log \mathbf{L}}{\partial b}, \frac{\partial \log \mathbf{L}}{\partial c},\right)^{\mathsf{T}}$.

3.2. The Cramér–von Mises estimation (CVME) method

The CVME of the parameter vector $\underline{\Theta}$ are obtained via minimizing the following expression with respect to θ, a, b and c , where

$$\mathbf{CVM}_{(\underline{\Theta})} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n}\left[F_{\underline{\Theta}}(y_i) - c_{(i,n)}^{[1]}\right]^2,$$

and $c_{(i,n)}^{[1]} = \frac{2i-1}{2n}$, then

$$\mathbf{CVM}_{(\underline{\Theta})} = \sum_{i=1}^{n}\left[F_{\underline{\Theta}}(y_i) - c_{(i,n)}^{[1]}\right]^2.$$

Then, CVME of the parameters θ, a, b and c are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left\{\frac{1}{1-\exp(-1)}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}^{\theta}\right)\right]-c_{(i,n)}^{[1]}\right\}\mathcal{D}_{(\theta)}(y_i,\underline{\Theta})=0,$$

$$\sum_{i=1}^{n}\left\{\frac{1}{1-\exp(-1)}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}^{\theta}\right)\right]-c_{(i,n)}^{[1]}\right\}\mathcal{D}_{(a)}(y_i,\underline{\Theta})=0,$$

$$\sum_{i=1}^{n}\left\{\frac{1}{1-\exp(-1)}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}^{\theta}\right)\right]-c_{(i,n)}^{[1]}\right\}\mathcal{D}_{(b)}(y_i,\underline{\Theta})=0,$$

and

$$\sum_{i=1}^{n}\left\{\frac{1}{1-\exp(-1)}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{O}_{\underline{\mathbf{V}}}^{2}(y_i)\right]\right\}^{\theta}\right)\right]-c_{(i,n)}^{[1]}\right\}\mathcal{D}_{(c)}(y_i,\underline{\Theta})=0,$$

where $\mathcal{D}_{(\theta)}(y_i,\underline{\Theta})=\partial F_{\underline{\Theta}}(y_i)/\partial\theta$, $\mathcal{D}_{(a)}(y_i,\underline{\Theta})=\partial F_{\underline{\Theta}}(y_i)/\partial a$, $\mathcal{D}_{(b)}(y_i,\underline{\Theta})=\partial F_{\underline{\Theta}}(y_i)/\partial b$ and $\mathcal{D}_{(c)}(y_i,\underline{\Theta})=\partial F_{\underline{\Theta}}(y_i)/\partial c$ are the first derivatives of the CDF of PBXIW distribution with respect to θ, a, b and c respectively.

3.3. Bootstrapping estimation (Bootst.E) method

The wider category of resampling techniques includes bootstrapping, a form of test or measure that uses random sampling with replacement to replicate the sampling procedure. Bootstrapping provides sample estimates with accuracy ratings for bias, variance, confidence intervals, prediction error, and other factors. This method provides estimate of the sample distribution for almost any statistic using random sampling techniques. One popular choice for an approximation distribution is the empirical distribution function of the observed data. When a set of observations can be assumed to come from a separate population with a same distribution, a few resamples with replacement of the observed dataset can be created (and of equal size to the observed dataset). So, the bootstrapping method is a powerful statistical technique which is useful especially when the sample size is small. Under the normal circumstances, sample sizes of less than 40 cannot be dealt with by assuming a "normal" or a "t " distribution. Bootstraping techniques work quite well with samples that have less than 40 observation. The reason for this is that bootstrapping involves resampling. These kinds of techniques assume nothing about the distribution of our data. Bootstrapping has become more popular as computing resources have become more readily available. This is because for bootstrapping to be practical a computer must be used (see Efron and Tibshirani [37] and Hesterberg [53]).

3.4. KE method

The Kolmogorov estimates (KEs) $\widehat{\theta}, \widehat{a}, \widehat{b}$ and $\widehat{c}$ of θ, a, b and c are obtained by minimizing the function

$$K=\max_{1\leq i\leq n}\left\{\frac{i}{n}-F_{\underline{\Theta}}(y_{i:n}),F_{\underline{\Theta}}(y_{i:n})-c_{(i,n)}^{[2]}\right\},$$

where $c_{(i,n)}^{[2]}=\frac{i-1}{n}$.

3.5. Anderson Darling method-2LD (Left-Tail Second-Order)

The Anderson Darling-2LT estimates (AD2LEs) $\widehat{\theta}_{(\text{AD2LE})}$, $\widehat{a}_{(\text{AD2LE})}$, $\widehat{b}_{(\text{AD2LE})}$and $\widehat{c}_{(\text{AD2LE})}$ of θ, a, b and c are obtained by minimizing

$$\mathbf{AD2LE}(\underline{\Theta})=2\sum_{i=1}^{n}\log\left[F_{\underline{\Theta}}(y_{i:n})\right]+\frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{F_{\underline{\Theta}}(y_{i:n})}.$$

Then, the parameter estimates of $\widehat{\theta}_{(\text{AD2LE})}$, $\widehat{a}_{(\text{AD2LE})}$, $\widehat{b}_{(\text{AD2LE})}$and $\widehat{c}_{(\text{AD2LE})}$ can be obtained by solving the nonlinear equations

$$\partial\left[\mathbf{AD2LE}\left(\underline{\Theta}\right)\right]/\partial\theta=0,\partial\left[\mathbf{AD2LE}\left(\underline{\Theta}\right)\right]/\partial a=0,\partial\left[\mathbf{AD2LE}\left(\underline{\Theta}\right)\right]/\partial b=0$$

and

$$\partial\left[\mathbf{L.T.\ ADE}\left(\underline{\Theta}\right)\right]/\partial c=0.$$

4. Simulation for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N=1000 generated data sets from the OBLx version where $n = 50, 100, 150$ and 300 and

	θ	a	b	c
I	1.2	2.0	1.5	0.6
II	2.0	1.5	0.5	1.5

The estimates are compared in terms of their Average values (AVs) and mean squared errors MSEs($\underline{\Theta}$). From Tables 1, 2, 3 and 4 we note that the MSE($\underline{\Theta}$) tend to zero when n increases which means incidence of consistency property.

Table 1: AVs and the corresponding MSEs (in parentheses) for $n = 50$.

Parameters	MLE	CVM	Bootstrap	KE	AD2LE
$\theta = 1.2$	1.22326	1.22430	1.20642	1.18581	1.17724
	(0.02515)	(0.03297)	(0.02939)	(0.03272)	(0.02980)
$c = 2$	2.00485	2.00670	2.00611	1.98146	1.96121
	(0.01063)	(0.01229)	(0.01141)	(0.01237)	(0.03050)
$a = 1.5$	1.50827	1.51093	1.51001	1.47941	1.45794
	(0.01675)	(0.01940)	(0.01801)	(0.01905)	(0.04443)
$b = 0.6$	0.60187	0.60074	0.59748	0.60995	0.59066
	(0.02461)	(0.00125)	(0.02935)	(0.00159)	(0.00320)
$\theta = 2$	2.02161	2.04038	1.79441	1.97785	1.97131
	(0.06436)	(0.08956)	(0.09829)	(0.08750)	(0.07593)
$c = 1.5$	1.49878	1.50343	1.43937	1.48966	1.48008
	(0.00386)	(0.00425)	(0.00706)	(0.00426)	(0.00935)
$a = 0.5$	0.49963	0.50066	0.48635	0.49760	0.49533
	(0.00019)	(0.00021)	(0.00036)	(0.00021)	(0.00047)
$b = 1.5$	1.50758	1.50230	1.56312	1.52119	1.48829
	(0.06395)	(0.00669)	(0.01431)	(0.00808)	(0.01556)

Table 2: AVs and the corresponding MSEs (in parentheses) for $n = 100$.

Parameters	MLE	CVM	Bootstrap	KE	AD2LE
$\theta = 1.2$	1.21132	1.21137	1.20031	1.19149	1.18022
	(0.01080)	(0.01516)	(0.00722)	(0.01613)	(0.01486)
$c = 2$	2.00094	2.00225	1.96626	1.99017	1.97332
	(0.00517)	(0.00603)	(0.00614)	(0.00611)	(0.01479)
$a = 1.5$	1.50225	1.50406	1.45911	1.48899	1.46976
	(0.00812)	(0.00946)	(0.00936)	(0.00950)	(0.02198)
$b = 0.6$	0.60170	0.60091	0.61244	0.60490	0.59279
	(0.01068)	(0.00061)	(0.00737)	(0.00072)	(0.00167)
$\theta = 2$	2.01715	2.01065	2.10828	1.97735	1.95809
	(0.03361)	(0.04005)	(0.03948)	(0.04256)	(0.04747)
$c = 1.5$	1.50012	1.49943	1.46454	1.49234	1.47950
	(0.00185)	(0.00206)	(0.00245)	(0.00215)	(0.00601)
$a = 0.5$	0.49998	0.49982	0.49206	0.49825	0.49529
	(0.00009)	(0.00010)	(0.00012)	(0.00011)	(0.00031)
$b = 1.5$	1.50346	1.50420	1.59567	1.51368	1.48755
	(0.03333)	(0.00330)	(0.01431)	(0.00399)	(0.00873)

Table 3: AVs and the corresponding MSEs (in parentheses) for $n = 200$.

Parameters	MLE	CVM	Bootstrap	KE	AD2LE
$\theta = 1.2$	1.20332	1.20197	1.17288	1.19176	1.17684
	(0.00611)	(0.00725)	(0.00601)	(0.00771)	(0.00939)
$c = 2$	2.00020	1.99890	1.96973	1.99253	1.97345
	(0.00265)	(0.00299)	(0.00354)	(0.00308)	(0.00977)
$a = 1.5$	1.50080	1.49925	1.46290	1.49131	1.46887
	(0.00414)	(0.00467)	(0.00538)	(0.00478)	(0.01433)
$b = 0.6$	0.600454	0.60110	0.60746	0.60340	0.59256
	(0.00610)	(0.00030)	(0.00533)	(0.00037)	(0.00098)
$\theta = 2$	2.00978	2.01364	2.02303	1.99593	1.97617
	(0.01696)	(0.02088)	(0.01472)	(0.02201)	(0.02136)
$c = 1.5$	1.50059	1.50179	1.50930	1.49764	1.49011
	(0.00098)	(0.00108)	(0.00106)	(0.00107)	(0.00259)
$a = 0.5$	0.50011	0.50037	0.50204	0.49945	0.49774
	(0.00005)	(0.00005)	(0.00005)	(0.00005)	(0.00013)
$b = 1.5$	1.50114	1.49936	1.48914	1.50469	1.48944
	(0.01687)	(0.00166)	(0.01431)	(0.00190)	(0.00457)

Table 4: AVs and the corresponding MSEs (in parentheses) for $n = 250$.

Parameters	MLE	CVM	Bootstrap	KE	AD2LE
$\theta = 1.2$	1.20132	1.20633	1.24601	1.19828	1.18597
	(0.00430)	(0.00590)	(0.00655)	(0.00620)	(0.00768)
$c = 2$	1.99945	2.00252	1.99351	1.99704	1.98267
	(0.00206)	(0.00243)	(0.00194)	(0.00242)	(0.00767)
$a = 1.5$	1.49974	1.50366	1.49229	1.49680	1.47995
	(0.00322)	(0.00380)	(0.00302)	(0.00378)	(0.01133)
$b = 0.6$	0.60060	0.59976	0.60922	0.60168	0.59316
	(0.00429)	(0.00024)	(0.00452)	(0.00029)	(0.00078)
$\theta = 2$	2.00739	2.01055	1.88229	1.99714	1.97661
	(0.01244)	(0.01638)	(0.02632)	(0.01723)	(0.02134)
$c = 1.5$	1.50031	1.50146	1.46312	1.49821	1.48964
	(0.00074)	(0.00085)	(0.00209)	(0.00085)	(0.00281)
$a = 0.5$	0.50005	0.50030	0.49175	0.49958	0.49763
	(0.00004)	(0.00004)	(0.00011)	(0.00004)	(0.00014)
$b = 1.5$	1.50115	1.49932	1.54121	1.50372	1.48821
	(0.01238)	(0.00131)	(0.01417)	(0.00150)	(0.00406)

5. Real data modeling

5.1. Real data modeling for comparing competitive estimation methods

The 1^{st} data set from Bjerkedal [31] which consists of 72 observations of survival times Guinea pigs injected with different doses of tubercle bacilli. This set of data has received a great deal of study and analysis using many new probability distributions, perhaps because its failure rate is increasing-constant, or perhaps because it contains some extreme observations, and this is what prompted many researchers to study and analyze it (see, for example, Afify et al [10] and Almazah et al. [22]).We consider the Cramér-Von Mises ($W^\star$) and the Anderson-Darling ($A^\star$) statistis. Figure 5 gives probability-probability (P-P) plots for comparing all methods under the failure times data set. From Table 5 (part I), the CVME method is the best method with $W^\star$=0.11519 and $A^\star$=0.65323 then MLE method with $W^\star$=0.12540 and $A^\star$=0.69132. The 2^{nd} data set is obtained from Smith and Naylor [95]. The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set consisting of 63 observations. Also, this set of data has received a huge deal of study and analysis under many new probability distributions, perhaps because its failure rate is monotonically increasing, or perhaps because it contains many extreme observations, and this is what prompted many researchers to study and analyze it (see Ibrahim et al. [60]). Figure 6 gives P-P plots for comparing all methods under the failure times data set. From Table 5 (part II), the Bootst.E method is the best method with $W^\star$=0.56078 and $A^\star$=3.06719 then the KE and AD2LE methods with $W^\star$=0.70103, 0.70039 and $A^\star$=3.83111, 3.83573.

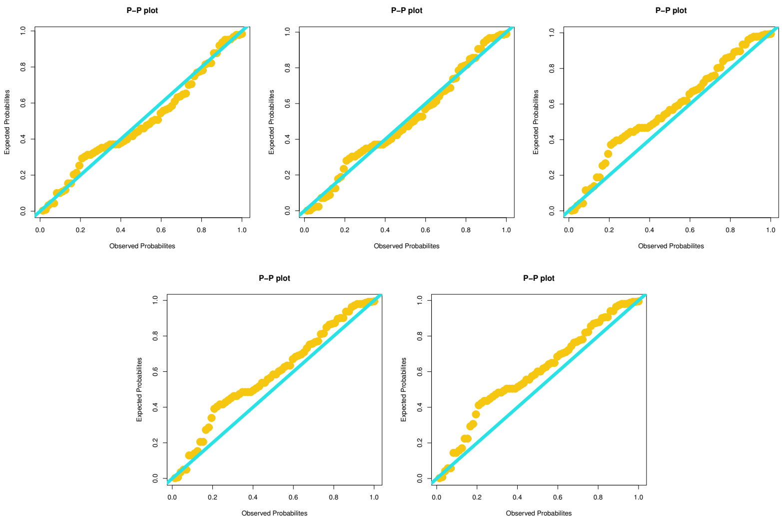

Figure 5. P-P plots for comparing classical methods (MLE, CVME , KE, Bootst.E and AD2LE) under survival times.

Table 5: The values of estimators A★ and W★ under the two real data sets.

		Estimates				Statistics	
Data	Methods	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$	W*	A*
I	MLE	9.93006	5.68762	0.001870	0.22937	0.12540	0.69132
	CVME	**22.20835**	**3.09761**	**0.00668**	**0.20829**	**0.11519**	**0.65323**
	KE	0.22721	2.35263	0.00126	0.00695	0.15748	0.85344
	Bootst.E	7.71116	6.08833	0.00716	0.26370	0.12996	0.71170
	AD2LE	4.16669	10.60278	0.00509	0.30077	0.15251	0.82587
II	MLE	1.12286	2.55974	0.74915	1.62329	7.56496	45.4009
	CVME	27.27497	17.20453	0.01051	0.73736	0.88524	4.82125
	KE	0.22608	2.31098	0.00126	0.00714	0.70103	3.83111
	Bootst.E	**3.10980**	**46.73873**	**0.07795**	**1.33608**	**0.56078**	**3.06719**
	AD2LE	5.77463	8.51306	0.00478	0.47296	0.70039	3.83573

5.2. Real data modeling for comparing competitive distributions

In this section we provide two applications of the PBXIWdistribution using two real data sets. For the 1^{st} application we shall compare the PBXIWdistribution with related models namely: the odd log logistic IW (OLLIW) the Marshall-Olkin IW (MOIW), Kumaraswamy IW (KIW), beta IW (BIW), Kumaraswamy Marshall-Olkin Inverse exponential (KMOIE), Kumaraswamy Marshall-Olkin Inverse Rayleigh (KMOIR) and IW distributions. For the 2^{nd} application we shall compare the PBXIWdistribution with related models namely: the MOIW, BIW, KMOIR and IW distributions. The total time test (TTT) plot, the quantile-quantile (Q-Q) plot, box plot and nonparametric Kernel density estimation (KDE) plot for the1^{st} real data sets are presented in Figure 7. Based on TTT plot, the empirical HRF of 1^{st} data sets is "upside down then increasing". The KDE plot is "bimodal and right skewed with heavy tail. The TTT, the Q-Q plot, box plot and the KDE plot for the 2^{nd} real data sets is presented in Figure 8. Based on TTT plot, the empirical HRF of 2^{nd} data sets is "increasing". The KDE plot is "bimodal and left skewed with heavy tail. In order to compare the distributions, we consider some criteria like $-2\widehat{\ell}$ (Maximized Log-likelihood), $C^{(1)}$ (Akaike-Information-Criterion), $C^{(4)}$ (the consistent-Akaike-Information-Criterion), $C^{(2)}$ (Bayesian-Information-Criterion) and $C^{(3)}$ (Hannan-Quinn-Information criterion) for the real data

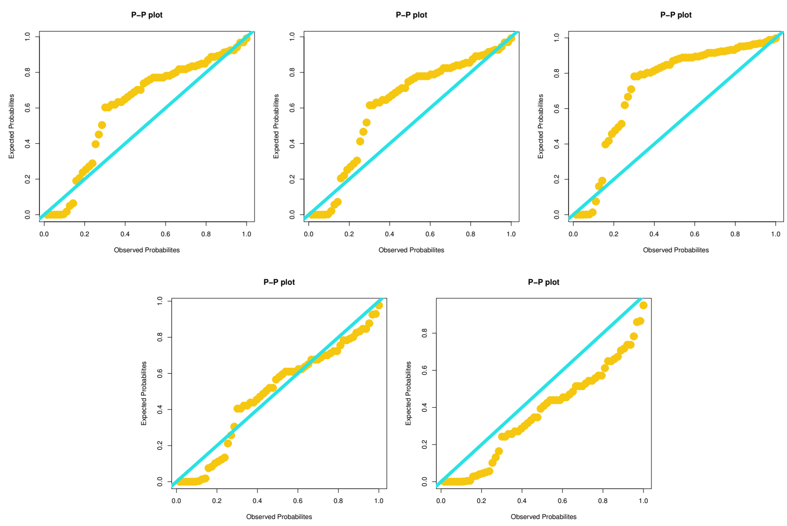

Figure 6. P-P plots for comparing classical methods (MLE, CVME , KE, Bootst.E and AD2LE) under glass fibres data.

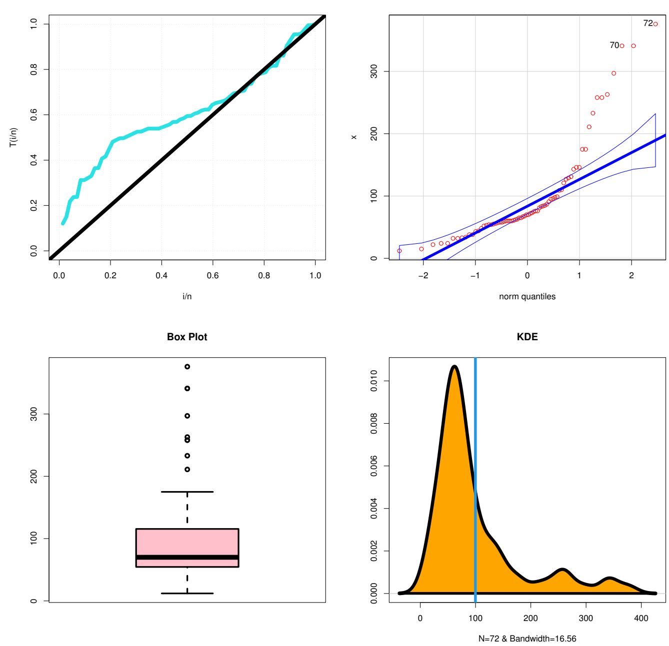

Figure 7. TTT, QQ, box, KDE plots for the 1^{st} real data.

set. Tables 6 and 8 list the MLEs and their corresponding standard errors (SEs). The numerical values of $-2\widehat{\ell}$, $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$ are listed in Tables 7 and 9, respectively. Figure 9 and 10 give the fitted PDF, fitted CDF and fitted HRF and for the two data sets respectively. Tables 7 and 9 compares the PBXIW distribution with other extensions of IW distribution. We note that the PBXIW distribution gives the lowest values for the $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$ statistics among all fitted models. So, the PBXIW distribution could be chosen as the best model.

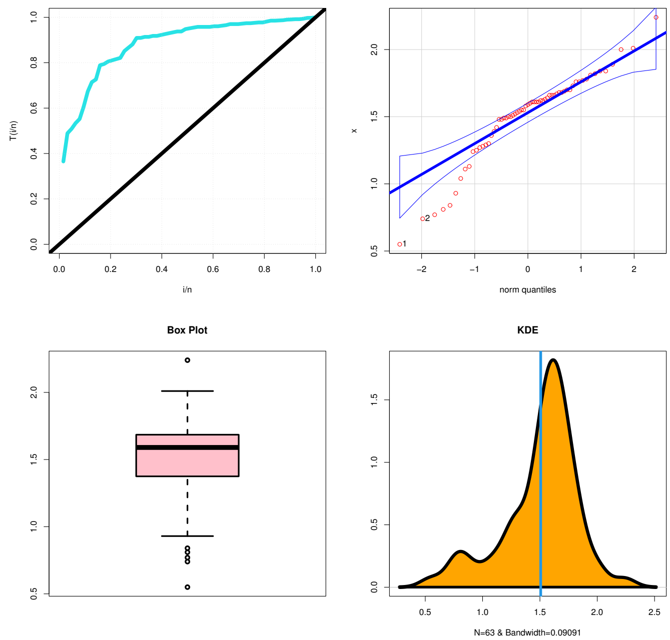

Figure 8. TTT, QQ, box, KDE plots for the 2^{nd} real data.

Table 6: MLEs and their SEs for the 1^{st} data.

Model	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$
PBXIW	9.93006	5.68762	0.001870	0.22937
	(0.4322)	(0.69)	(0.0001)	(0.0205)
OLLGIW	4.7989	1.3108	13.9901	0.38
	(5.1585)	(1.889)	(55.23)	(0.404)
KIW	0.6207	0.7111	45.7326	8.2723
	(0.003)	(0.013)	(0.092)	(0.979)
BIW	0.322	24.5032	19.9786	20.1331
	(0.0012)	(0.087)	(7.246)	(7.26)
KMOIE	8.8727	0.1758	68.1393	2.6258
	(1.174)	(0.000)	(0.020)	(0.512)
KMOIR	9.993	1.6788	58.4697	0.6389
	(1.972)	(0.001)	(0.105)	(0.098)
MOIW	14.9816	1.7855	13.991	
	(4.631)	(0.193)	(2.96)	
IW	1.4148	54.1888		
	(0.003)	(0.111)		

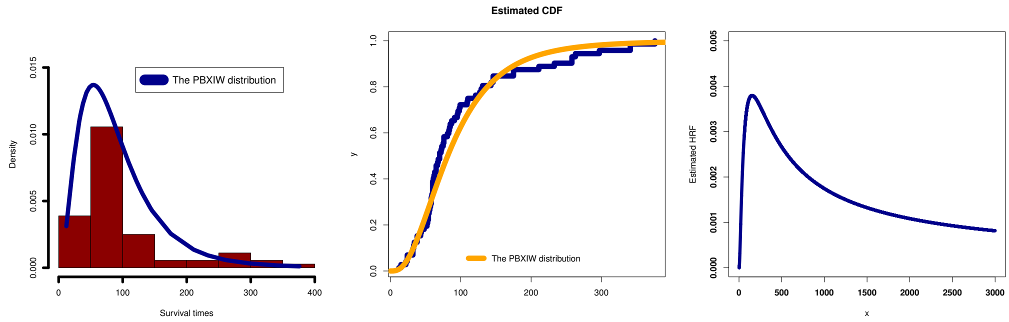

Figure 9. The fitted PDF, CDF and estimated HRF and for the first data set.

Table 7: $-2\widehat{\ell}$, $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$ for 1^{st} data.

Model	$-2\widehat{\ell}$	$C^{(1)}$	$C^{(2)}$	$C^{(3)}$	$C^{(4)}$
PBXIW	**721.8**	**731.86**	**743.24**	**736.4**	**732.77**
OLLGIW	779.2	787.4	796.5	791	788
Kw-IW	780.5	788.5	797.6	792.1	789.1
BIW	780.6	788.6	797.7	792.3	789.2
KMOIE	782.7	790.7	799.8	794.3	791.3
IW	791.3	795.3	799.9	797.1	795.5
MOIW	790.1	796.1	802.9	798.8	796.5
KMOIR	800.2	808.2	817.3	811.8	808.8

Table 8: MLEs and their SEs for the 2^{nd} data.

Model	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$
PBXIW	1.12286	2.55974	0.74915	1.62329
	(0.0043)	(0.0054)	(0.0541)	(0.0089)
OLLGIW	28.31	0.604	3.068	0.197
	(17.17)	(0.201)	(4.689)	(0.118)
BIW	0.685	1.331	19.591	30.411
	(0.181)	(1.085)	(18.115)	(18.238)
Kw-MOIR	1	2.7498	0.5971	5.7974
	(0.192)	(0.079)	(0.034)	(0.008)
MOIW	0.4816	2.3876	1.5441	
	(0.252)	(0.253)	(0.226)	
IW	2.888	1.264		
	(0.234)	(0.059)		

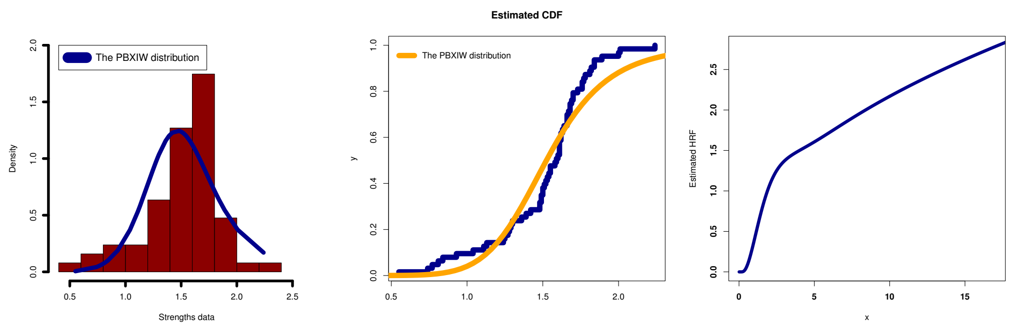

Figure 10. The fitted PDF, CDF and estimated HRF and for the second data set.

Table 9: $-2\widehat{\ell}$, $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$ for 2^{nd} data.

Model	$-2\widehat{\ell}$	$C^{(1)}$	$C^{(2)}$	$C^{(3)}$	$C^{(4)}$
PBXIW	**41.6**	**31.72**	**21.01**	**27.5**	**30.67**
OLLGIW	46.8	54.73	63.32	58.05	55.42
BIW	61.7	69.69	78.29	73.09	70.43
Kw-MOIR	67.3	75.33	83.88	78.73	76.45
IW	93.7	97.70	102.1	99.42	97.91
MOIW	95.7	101.7	108.2	104.4	102.1

6. Concluding remarks and future points

A new extension of the inverse Weibull model is introduced and studied. Some of its statistical properties are derived. The density of the new model can be right skewed and unimodal with symmetric and asymmetric shapes also it can be left skewed. The failure rate function of the new model can be decreasing-constant-increasing (U-shape or bathtub shape), increasing-constant-increasing, increasing, upside down-bathtub, monomaniacal decreasing, upside down-increasing, monomaniacal increasing and upside down. Three-dimensional skewness and kurtosis plots are presented to show the wide flexibility of the PBXIW model. Different estimation methods of are used for estimating the unknown parameters. We assessed the performance of all methods via simulation study. Two real data applications are used for comparing competitive estimation methods. For modeling the survival times Guinea pigs, the the CVME method is the best method with $W^{\star}$=0.11519 and $A^{\star}$=0.65323 then MLE method with $W^{\star}$=0.12540 and $A^{\star}$=0.69132. For modeling the strengths of glass fibres, the Bootst.E method is the best method with $W^{\star}$=0.56078 and $A^{\star}$=3.06719 then the KE and AD2LE methods with $W^{\star}$=0.70103, 0.70039 and $A^{\star}$=3.83111, 3.83573. The importance of the new model is demonstrated via two real data applications. The new model is much better than other competitive models in modeling two real data sets.

We may employ a variety of novel beneficial goodness-of-fit tests, such the Nikulin-Rao-Robson goodness-of-fit test and the Bagdonaviius-Nikulin goodness-of-fit test, for right censored validation as a potential future project.Bayesian analysis can also be considered using various loss functions and making a comparison between the loss function and each other. The new distribution can also be applied in the field of insurance and reinsurance, especially with regard to insurance claims and data on reinsurance returns. It is also possible to create a set of acceptance sampling plans (single, double and multiple), which are of great importance in solving problems of results, examination and statistical quality control (see Ahmed and Yousof [5] and Ahmed et al. [6]). Bayesian and classical inference of the reliability in the multicomponent stress-strength under the new model can also be considered (see Rasekhi et al. [82], Saber and Yousof [84], Saber et al. [86] and Saber et al. [87]). The new distribution qualifies to be used in censored regression modeling processes and their consequences and future predictions (see Korkmaz et al. ([65], [66] and [67]), Ibrahim et al. [57], Hamedani et al. [51] and Altun et al. ([24], [25], [26], [27], [28] and [23])).

Finally, the new distribution is flexible enough and it is expected that many researchers will be motivated to derive more bivariate and multivariate distributions accordingly. Derivation of more bivariate and multivariate distributions helps us and researchers in applied fields to facilitate statistical and mathematical modeling of second variable and multivariate data. In fact, data is available in binary or multivariate form, because natural phenomena are often complex and many variables are intertwined with each other. Recently, many bivariate and multivariate distributions have been presented using the copula methods, and the copulas in general are mathematical functions from which more than one bivariate (or multivariate) versions of the same distribution can be derived. Given the limited space in this work, we will mention some of these works, and for more details see Salah et al. ([88]), Al-babtain et al. [8], Ali et al. ([13] and [16]), Aboraya et al. [3], Elgohari and Yousof ([40], [41] and [42]), Saber et al. [85], Elgohari et al. ([43]), Shehata and Yousof ([90], [91] and [92]), Shehata et al. ([93]), Hamed et al. ([50]), Ibrahim et al. ([59]) and Chesneau et al. [33].

Future research on the PBXIW distribution can take various promising directions inspired by the referenced works. Following Ahmed et al. [6], PBXIW can be employed in acceptance sampling plans to enhance decision-making in quality and risk management. Inspired by Alizadeh et al. [17], statistical threshold risk analysis under extreme stresses could be explored, particularly for insurance and natural disaster modeling. Building on Alizadeh et al. [17] and Khedr et al. [64], PBXIW could be extended for compound risk models and applied to reinsurance revenue and actuarial data. Survival and reliability analysis, as discussed by Loubna et al. [71] and Teghri et al. [96], could benefit from frailty models based on PBXIW for censored and uncensored schemes under heterogeneous data. Financial applications like Value-at-Risk (VaR) and peaks-over-threshold analysis, as highlighted by Korkmaz et al. [69] and Aljadani et al. [21], are additional areas to explore. Drawing from Shrahili et al. [94] and Yousof et al. [103], PBXIW can be applied to model bimodal, asymmetric, and heavy-tailed data, particularly in insurance and automobile claims. Bayesian and non-Bayesian methods for risk and reliability estimation, inspired by Ibrahim et al. [58], offer another potential research avenue. Investigating entropy and information measures, as emphasized by Elbatal et al. [38], can quantify uncertainty in losses or revenues and order-p mean analysis. Extensions or generalizations of PBXIW, incorporating skewness and kurtosis for improved modeling of left-skewed and financial datasets, align with the works of Salem et al. [89] and Rasekhi et al. [80]. Lastly, performance assessment under extreme values, as explored by Yousof et al. [111], can validate PBXIW in insurance and natural hazard data.

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