STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, July 2024, pp 943–961.
Published online in International Academic Press (www.IAPress.org)

On Truncated Versions of the Xgamma Distribution: Various Estimation Methods and Statistical Modeling

Subhradev Sen[1], Morad Alizadeh [2],Mohamed Aboraya[3], M. Masoom Ali[4], Haitham M. Yousof[5,*]and Mohamed Ibrahim[6,7]

[1] *Alliance School of Applied Mathematics, Alliance University, India*
[2] *Department of Statistics, Faculty of intelligen systems engineering and data science, Persian Gulf University, Bushehr, 75169, Iran*
[3] *Department of Applied, Mathematical and Actuarial Statistics, Faculty of Commerce, Damietta University, Damietta, Egypt*
[4] *Department of Mathematical Sciences, Ball State University, Muncie, Indiana 47306 USA*
[5] *Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Benha 13518, Egypt*
[6] *Quantitative Methods Department, School of Business, King Faisal University, Saudi Arabia*
[7] *Department of Applied, Mathematical and Actuarial Statistics, Damietta University, Damietta, Egypt*

Abstract In this article, we introduced the truncated versions (lower, upper and double) of xgamma distribution (Sen et al. 2016). In particular, different structural and distributional properties such as moments, popular entropy measures, order statistics and survival characteristics of the upper truncated xgamma distribution are discussed in detail. We briefly describe different estimation methods, namely the maximum likelihood, ordinary least squares, weighted least square and L-Moments. Monte Carlo simulation experiments are performed for comparing the performances of the proposed methods of estimation for both small and large samples under the lower, upper and double versions. Two applications are provided, the first one comparing estimation methods and the other for illustrating the applicability of the new model.

Mathematics Subject Classification: 62E10, 60K10, 60N05

DOI: 10.19139/soic-2310-5070-1660

1. Introduction and motivation

In manufacturing industries, final products often pass through a screening inspection before being sent to the customers. It is then a normal practice that if a product's performance falls within certain tolerance limits, it is being judged as conforming and thereby sent to the customer. A product is usually rejected and therefore scrapped or reworked if it fails to conform. In this case, the actual distribution to the customer is truncated. As a standard practice, truncated distributions are applied, in particular, for such industrial applications and settings (see for more details Cho and Govindaluri [4], Kapur and Cho [22], [23], Phillips and Cho [31], [32], Khasawneh et al. [24], [25] and references therein).
As an another example, truncated distributions are well applied in multi-stage production process in which inspection is done at each stage and only conforming items are sent to the next stage. Very useful application can also be found in accelerated life testing situations. In practice, the concept of a truncated distribution plays an important role in analyzing a variety of production processes, process optimization and quality improvement techniques.

*Correspondence to: Haitham M. Yousof (Email:haitham.yousof@fcom.bu.edu.eg). Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Benha 13518, Egypt

ISSN 2310-5070 (online) ISSN 2311-004X (print)

To model intensity statistics in the study of atomic heterogeneity, truncated distributions are utilized (see Mukhopadhyay et al. [29] for more insight). In another situation, measurements match well with a truncated distribution with much better fit over smaller file or request sizes for high-performance Ethernet (see for more details Field et al. [10]).
Truncated versions of several standard continuous probability distributions have been studied by different authors and applied successfully to numerous real life situations (see for example, Hegde and Dahiya [14], Mittal and Dahiya [28], Nadarajah [30], Zaninetti and Ferraro [43] and, Zhang and Xie [44]). Recently, Sen et al. [35] introduced and studied a one-parameter lifetime distribution, named xgamma distribution, with probability density function (PDF) as

$$f(x) = \frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x}, \quad x > 0, \theta > 0.$$

It is denoted by $X \sim xgamma(\theta)$. The corresponding cumulative distribution function (CDF) is given by

$$F(x) = 1 - \frac{\left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x}, \quad x > 0, \theta > 0.$$

The xgamma distribution is applied successfully to time-to-event data set and its different properties are studied. Our aim in this investigation is to introduce and study the truncated version(s) of xgamma distribution for understanding situations for which better modeling is possible. In literature there are many works focused on the xgamma distribution sush as Sen et al. [34] (for the quasi xgamma-Poisson distribution), Yousof et al. [40] (for the two-parameter xgamma-Fréchet distribution with some characterizations results, properties, copulas and different classical estimation methods), Cordeiro et al. [5] (for the xgamma family with censored regression modelling and some applications) and Ibrahim et al. [20] (for a new three-parameter version of the xgamma-Fréchet distribution with applicationss and different methods of estimation).
A truncated distribution in statistics is a conditional distribution that is produced when the domain of another probability distribution is constrained. In situations where the capacity to record, or even to know about, occurrences is restricted to values that lie above or below a particular threshold or within a set range, truncated distributions are created. The dates of birth of students at a school, for instance, would normally be truncated in comparison to those of all students in the neighbourhood since the school only admits students who fall within a certain age range on a particular day. Generally, there are many practical cases in which the researcher has to truncate the data in order to complete the study. This truncation of the data necessarily requires a truncated probability distribution that is suitable for modeling this data. This truncating also results in changes in the mathematical properties of the probability distribution and other results different from the original version before the truncation.
In this paper we are motivates to present the truncated xgamma distribution for the following reasons:

1. The statistical literature in general is still poor on this aspect of probability distributions. This scarcity of truncated distributions has motivated us to present this work, hoping that it will gain a great deal of study and analysis in the aspect of mathematical modeling and statistical analysis.
2. Truncation are distinguished with presenting three versions of the truncated distribution, and these three versions are suitable in the case of lower truncation, upper truncation and double truncation.
3. Providing new truncated distributions with high flexibility, by introducing new density functions and hazard rate functions that have high flexibility.
4. Presenting truncated distributions with skew coefficient and kurtosis coefficient that have a wide flexibility. These two coefficients are among the most important factors that show the elasticity of any probability distribution.
5. In the aspect of statistical inference, we presente a comprehensive simulation study using many classical methods such as the maximum likelihood method, the ordinary least squares method, L-Moments method and weighted least squares method. By examining the statistical literature, there is a great scarcity in this aspect specifically. Therefore, we have compared the four classical methods and highlighted the best method

using the truncation approach. The comparison between the classical methods was also made within the framework of applications on actual data. We have presented an application on truncated actual data for this purpose specifically.

The rest of the article is organized as follows: In section 2, we introduce the truncated versions of xgamma distribution. Sections 3 and 4 deal with basic structural properties and entropy measures respectively. Order statistics and survival characteristics are studied in Sections 5 and 6 respectively. The unknown parameters are estimated by the various method in Section 7. In Section 8, Simulation studies for comparing estimation methods are presented. In Section 9, a real life data set is analyzed for comparing classical methods and for comparing different models. Finally, concluding remarks are offered in Section 10.

2. The truncated xgamma distribution

If X is a continuous random variable then for truncated version of X, we have the following definition:

Definition 1
An absolutely continuous random variable, X, is said to follow a double truncated distribution (DTD) over the interval $[\alpha, \beta]$ if it has the CDF as

$$G(x) = \frac{F(x) - F(\alpha)}{F(\beta) - F(\alpha)}, \alpha \leq x \leq \beta,$$

where $F(.)$ denotes the CDF of the baseline distribution, α and β, points of truncation.
The corresponding PDF is

$$g(x) = \frac{f(x)}{F(\beta) - F(\alpha)}, \alpha \leq x \leq \beta,$$

where $f(.)$ is the PDF of baseline distribution.

We recognize the following three cases:

(i) When $\alpha \to 0$ and $\beta \to \infty$, we have the baseline lifetime distribution with support $(0, \infty)$.
(ii) When $\alpha \to 0$, we have upper truncated distribution (UTD) of the baseline distribution.
(iii) When $\beta \to \infty$, we have lower truncated distribution (LTD) of the baseline distribution.

Applying the above definition and taking the baseline distribution as xgamma(θ) we have the following definition for truncated version of xgamma distribution:

Definition 2
A continuous random variable, X, is said to follow a double truncated xgamma (DTXG) distribution with support $[\alpha, \beta]$ if its PDF is of the form:

$$g(x; \alpha, \beta, \theta) = \frac{\theta^2}{(1+\theta)} \frac{\left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x}}{[F(\beta) - F(\alpha)]}, \quad \alpha \leq x \leq \beta, \alpha > 0, \beta > 0, \theta > 0,$$

where $F(.)$ is the CDF of xgamma distribution before. Also, we denote it by $X \sim DTXG(\alpha, \beta, \theta)$.

Hereafter, we concentrate mainly in studying the upper truncated version of xgamma distribution with the following definition:

Definition 3
A continuous random variable, X, is said to follow an upper truncated xgamma (UTXG) distribution if its PDF is

given by:

$$g(x;\beta,\theta) = K(\beta,\theta)\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}, \quad 0 \le x \le \beta, \beta > 0, \theta > 0, \tag{1}$$

where $K(\beta,\theta) = \frac{\theta^2}{(1+\theta)(1-e^{-\theta\beta})-\theta\beta\left(1+\frac{\theta\beta}{2}\right)e^{-\theta\beta}}$, a function of β and θ.
We denote it by $X \sim UTXG(\beta,\theta)$.

The corresponding CDF is given by

$$G(x;\beta,\theta) = \frac{F(x)}{F(\beta)}, \quad 0 \le x \le \beta.$$

The characteristic function (CF) of X is derived as

$$\phi_X(t) = K(\beta,\theta)\left[\frac{1}{(\theta - it)}\gamma\{1,(\theta - it)\beta\} + \frac{\theta}{2(\theta - it)^3}\gamma\{3,(\theta - it)\beta\}\right], t \in \mathbb{R}, i = \sqrt{-1}.$$

Figure 1 gives plots of the baseline XG PDF for some parameter values. Figure 2 provides plots of the UTXG PDF for some parameter values. Figure 3 displays plots of the LTXG PDF for some parameter values. Figure 4 shows plots of the DTXG PDF for some parameter values. Due to Figure 1, the density of baseline XG model ($\alpha \to 0$ and $\beta \to \infty$) can be unimodal with right tail and bimodal with right tail. Based Figure 2, the density of baseline UTXG model ($\alpha \to 0$) can be unimodal with right tail, left skewed with no beak and bimodal with right tail. Based to Figure 3, the density of baseline LTXG model ($\beta \to \infty$) can be bimodal with right tail and unimodal with right tail. Based to Figure 4, the density of baseline DTXG model can be right skewed with no peak and unimodal with right tail. Generally, the truncation approach converted the XG model to a more flexible model which contains various useful density shapes for statistical modeling and reliability testing. The degree of the skew coefficient, kurtosis coefficient, failure rate function, and diversity in the density and failure rate functions all play a role in how flexible the probability distribution is. Additionally, the probability distribution's usability and effectiveness in statistical modelling are crucial in this regard. After examining the novel probability distribution, we discovered that it was quite adaptable in the many applied areas. This is what motivated us to investigate this probability distribution thoroughly.

3. Moments and associated measures

The r^{th} order raw moment of $X \sim UTXG(\beta,\theta)$ is obtained as

$$\begin{aligned} \mu_r^{'} = E(X^r) &= \int_0^\beta x^r g(x;\beta,\theta)dx \\ &= \frac{K(\beta,\theta)}{\theta^{r+1}}\left[\gamma(r+1,\theta\beta) + \frac{1}{2\theta}\gamma(r+3,\theta\beta)\right]; r = 1,2,\ldots, \end{aligned} \tag{2}$$

where $\gamma(a,x) = \int_0^x z^{a-1}e^{-z}dz$ is lower incomplete gamma function.
In particular, we have

$$E(X) = \frac{K(\beta,\theta)}{\theta^2}\left[\gamma(2,\theta\beta) + \frac{1}{2\theta}\gamma(4,\theta\beta)\right] = \mu\,(say)$$

and

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{K(\beta,\theta)}{\theta^3}\left[\{\gamma(3,\theta\beta) + \frac{1}{2\theta}\gamma(5,\theta\beta)\} - \frac{K(\beta,\theta)}{\theta}\{\gamma(2,\theta\beta) + \frac{1}{2\theta}\gamma(4,\theta\beta)\}^2\right] \\ &= \sigma^2\,(say) \end{aligned} \tag{3}$$

Table 1. Mean and variance values for selected values of θ and β.

		θ							
$\beta\downarrow$		0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
5	μ	2.6285	2.3682	2.0293	1.6935	1.3996	1.1606	0.9735	0.8288
	σ^2	2.1392	2.0934	1.9335	1.6620	1.3447	1.0473	0.8035	0.6175
10	μ	5.2821	3.9136	2.7518	1.9870	1.5097	1.1998	0.9870	0.8333
	σ^2	7.8938	6.8105	4.6456	2.8830	1.8275	1.2252	0.8661	0.6389
15	μ	7.3244	4.4884	2.8494	1.9997	1.5111	1.2000	0.9870	0.8333
	σ^2	12.3538	10.6627	5.4461	2.9961	1.8408	1.2267	0.8662	0.6389
20	μ	8.7030	4.6336	2.8567	1.9999	1.5111	1.2000	0.9870	0.8333
	σ^2	17.0923	12.3157	5.5432	2.9999	1.8410	1.2267	0.8663	0.6389

Table 1 shows the values of μ and σ^2 for selected values of β and θ.
The moment generating function (MGF) of X is derived as

$$M_X(t) = K(\beta,\theta)\left[\frac{1}{(\theta-t)}\gamma\{1,(\theta-t)\beta\} + \frac{\theta}{2(\theta-t)^3}\gamma\{3,(\theta-t)\beta\}\right], t\in\mathbb{R}.$$

The cumulant generating function (CGF) of X is derived as

$$K_X(t) = \ln K(\beta,\theta) + \ln\left[\frac{1}{(\theta-t)}\gamma\{1,(\theta-t)\beta\} + \frac{\theta}{2(\theta-t)^3}\gamma\{3,(\theta-t)\beta\}\right], t\in\mathbb{R}.$$

The following theorem shows that $UTXG(\beta,\theta)$ is unimodal.

Theorem 1
For $0<\theta\le\frac{1}{2}$, $g(x)$ given in (1) attains a maximum at $x=\frac{1+\sqrt{1-2\theta}}{\theta}$ and for $\theta>\frac{1}{2}$, $g(x)$ decreases in x.

Proof
The proof is straightforward by taking the first derivative of $g(x)$ with respect to x. □

Hence we have for $X\sim UTXG(\beta,\theta)$,

$$Mode(X) = \begin{cases} \frac{1+\sqrt{1-2\theta}}{\theta}\ , & \text{if } 0<\theta\le 1/2. \\ 0\ , & \text{otherwise.} \end{cases} \quad (4)$$

4. Entropy measures

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Rényi entropy. If a non-negative continuous random variable, X, has the probability density function $f(x)$, then Rényi entropy is defined as

$$H_R(\delta) = \frac{1}{1-\delta}\ln\int_0^\infty f^\delta(x)dx \text{ for } \delta>0\,(\neq 1).$$

When $X\sim UTXG(\beta,\theta)$, one can derive

$$\int_0^\beta g^\delta(x;\beta,\theta)dx = [K(\beta,\theta)]^\delta\sum_{j=0}^{\delta}\binom{\delta}{j}\frac{1}{2^j\delta^{2j+1}\theta^{j+1}}\gamma(2j+1,\delta\theta\beta)$$

to obtain Rényi entropy as

$$H_R(\delta) = \frac{\delta}{1-\delta}K(\beta,\theta) + \frac{1}{1-\delta}\ln\left[\sum_{j=0}^{\delta}\binom{\delta}{j}\frac{1}{2^j\delta^{2j+1}\theta^{j+1}}\gamma(2j+1,\delta\theta\beta)\right].$$

In physics, the Tsallis entropy (see [42]) is a generalization of the standard Boltzmann-Gibbs entropy. For an absolutely continuous non-negative random variable X with PDF $f(x)$, Tsallis entropy (also called q-entropy) is defined as

$$S_q(X) = \frac{1}{q-1}\ln\left[1 - \int_0^\infty f^q(x)dx\right] \text{ for } q > 0\,(\neq 1)$$

When $X \sim UTXG(\beta,\theta)$, Tsallis entropy can be derived as

$$S_q(X) = \frac{1}{q-1}\ln\left[1 - [K(\beta,\theta)]^q\sum_{j=0}^{q}\binom{q}{j}\frac{1}{2^jq^{2j+1}\theta^{j+1}}\gamma(2j+1,q\theta\beta)\right].$$

Shannon measure of entropy is defined as

$$H(g) = E[-\ln g(x)] = -\int_0^\infty \ln[g(x)]g(x)dx.$$

For $X \sim UTXG(\beta,\theta)$, Shannon entropy is obtained as

$$\begin{aligned} H(g) =& \frac{K(\beta,\theta)}{\theta}\left[\gamma(2,\theta\beta) + \frac{1}{2\theta}\gamma(4,\theta\beta)\right] - \ln K(\beta,\theta) \\ &- K(\beta,\theta)\sum_{j=0}^{\infty}(-1)^{j+1}\frac{1}{j2^j\theta^{j+1}}\left[\gamma(2j+1,\theta\beta) + \frac{1}{2\theta}\gamma(2j+3,\theta\beta)\right]. \end{aligned} \quad (5)$$

5. Order statistics

The distributions of order statistics play important role in obtaining system reliabilities (be it biological or mechanical) when the components are connected in series or parallel configurations.
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $X \sim UTXG(\beta,\theta)$.
Denote $X_{(j)}$ as the j^{th} order statistic. Then $X_{(1)}$ and $X_{(n)}$ denote the smallest and largest order statistics for a sample of size n drawn from $UTXG(\beta,\theta)$, respectively. The PDF of $X_{(1)}$ is derived as

$$f_{X_{(1)}}(x;\beta,\theta) = \frac{n\theta^2}{(1+\theta)\{F(\beta)\}^n}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}\{F(\beta)-F(x)\}^{n-1}, \quad 0 \le x \le \beta.$$

The PDF of $X_{(n)}$ is obtained as

$$f_{X_{(n)}}(x;\beta,\theta) = \frac{n\theta^2}{(1+\theta)\{F(\beta)\}^n}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}\{F(x)\}^{n-1}, \quad 0 \le x \le \beta.$$

6. Survival characteristics

In this section we study the important survival properties of a random variable X following $UTXG(\beta,\theta)$.
The survival/reliability function of X is given by

$$S(x;\beta,\theta) = \frac{F(\beta)-F(x)}{F(\beta)}, \quad 0 \le x \le \beta. \quad (6)$$

The failure rate function is obtained as

$$h(x) = \frac{\theta^2}{(1+\theta)} \frac{\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{[F(\beta)-F(x)]}, \quad 0 \le x \le \beta. \tag{7}$$

Now, we investigate aging property of the failure rate function in (7). The distribution in (1) is increasing failure rate (IFR) or decreasing failure rate (DFR) depending on a particular range of X. We have the following theorem:

Theorem 2
The distribution, $UTXG(\beta, \theta)$, is IFR (DFR) if $x > (<)\sqrt{\frac{2}{\theta}}$ for all $\theta > 0$.

Proof
If a continuous non-negative random variable X has PDF $f(x)$, we define

$$\eta(x) = -\frac{f(x)}{f'(x)},$$

where $f^{'}(x)$ is the first derivative of $f(x)$ with respect to x.
Then, $f(x)$ is IFR (DFR) according as $\eta(x)$ is increasing (decreasing) in x (see [9] for the characterization). When $X \sim UTXG(\beta, \theta)$, let us consider

$$\eta_{UTXG}(x) = -\frac{g(x;\beta,\theta)}{g'(x;\beta,\theta)}$$

and, when $X \sim xgamma(\theta)$, let us take

$$\eta_{XG}(x) = -\frac{f(x;\theta)}{f'(x;\theta)}.$$

Then we have,

$$\eta_{UTXG}(x) = \eta_{XG}(x) = \theta - \frac{\theta x}{\left(1+\frac{\theta}{2}x^2\right)},$$

which gives after taking first derivative with respect to x, $\eta^{'}_{XG}(x) = \frac{\theta\left(\frac{\theta}{2}x^2\right)}{\left(1+\frac{\theta}{2}x^2\right)^2}$ and is positive (negative) if $x > (<)\sqrt{\frac{2}{\theta}}$ for all $\theta > 0$. Hence the proof. □

The reverse hazard rate (RHR) function of X is given by

$$r(x) = \frac{\theta^2}{(1+\theta)} \frac{\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{F(x)}, \quad 0 \le x \le \beta.$$

7. Estimation of parameters

Consider the following classical estimation methods:
1-The maximum likelihood (ML) method.
2-the ordinary least squares (OLS) method.
3-The L-Moments method.
4-The whighted least squares (WLS) method.

These methods and many others have recently been used in statistical modeling and evaluation using applications on actual data and using simulation studies. For more information on these and other methods, see Ibrahim et al. ([21] and [18]) for different methods of estimation under a new extension of Lindley distribution and a new Burr type XII distribution, Yadav et al. [37] for different methods of estimation under a new Topp-Leone-Lomax distribution, Mansour et al. [26] for different methods of estimation under a new Burr X Weibull distribution, Mansour et al. [27] for different methods of estimation under a new two parameter Burr XII distribution, Ibrahim et al. ([16] and [15]) for different methods of estimation under discrete analogue of the Weibull G family and a exponential generalized log-logistic model, Ibrahim et al. [17] for different methods of estimation under the Burr XII exponentiated exponential distribution, Aboraya et al. [2] for different methods of estimation under a novel family of discrete distributions, Elgohari et al. [6] for different methods of estimation under a novel version of the Lomax distribution, Salah et al. [33] for different methods of estimation under a novel version of Fréchet distribution, Yousof et al. [38] for different methods of estimation under a new reciprocal Rayleigh extension, Yousof et al. [41] for different methods of estimation under a novel Chen extension, Yousof et al. [39] for different methods of estimation under a novel Nadarajah Haghighi extension, Shehata et al. [36] for different methods of estimation under a novel Nadarajah Haghighi extension. Also, see Hamedani et al. [13], Aboraya et al. [1], Eliwa et al. [7], Hamed et al. [12], Ali et al. [3], Ibrahim et al. [19] and El-Morshedy et al. [8] for more details.

All these methods are discussed in the statistical literature with more details. In this work, we may ignore many of its derivation details for avoiding the replication. However, we propose maximum likelihood of estimation (MLE) for the unknown parameters of double truncated xgamma (DTXG), lower truncated xgamma (LTXG) and upper truncated xgamma (UTXG) distributions with more details since the application section will depend on its results. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a particular realization on that.

The ML method

MLEs for DTXG

In case when X follows a $DTXG(\alpha, \beta, \theta)$, let us denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ as a particular realization on a sample of size n drawn from it.

The likelihood function is then given by

$$L(\alpha, \beta, \theta|\mathbf{x}) = \prod_{i=1}^{n} \frac{\theta^2 \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta x_i}}{\psi(\alpha, \beta, \theta)},$$

where

$$\psi(\alpha, \beta, \theta) = (1+\theta)\left(e^{-\theta\alpha} - e^{-\theta\beta}\right) + \theta\left(\alpha e^{-\theta\alpha} - \beta e^{-\theta\beta}\right) + \frac{\theta^2}{2}\left(\alpha^2 e^{-\theta\alpha} - \beta^2 e^{-\theta\beta}\right),$$

a function of α, β and θ. It is obvious that the MLEs of α and β, say $\hat{\alpha}$ and $\hat{\beta}$, are

$$X_{(1)} = min\{X_1, X_2, \ldots, X_n\}$$

and

$$X_{(n)} = max\{X_1, X_2, \ldots, X_n\},$$

respectively, i.e., $\hat{\alpha} = X_{(1)}$ = smallest order statistic and $\hat{\beta} = X_{(n)}$ = largest order statistic. Given $\hat{\alpha}$ and $\hat{\beta}$, to obtain the MLE of θ, we have the log-likelihood function as

$$\ln L(\hat{\alpha}, \hat{\beta}, \theta|\mathbf{x}) = 2n \ln\theta + \sum_{i=1}^{n} \ln\left(1 + \frac{\theta}{2} x_i^2\right) - \theta \sum_{i=1}^{n} x_i - n \ln\psi(\hat{\alpha}, \hat{\beta}, \theta),$$

which gives the following log-likelihood equation

$$\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{x_i^2}{2\left(1 + \frac{\theta}{2} x_i^2\right)} - \sum_{i=1}^{n} x_i - n \frac{\psi^{'}(\hat{\alpha}, \hat{\beta}, \theta)}{\psi(\hat{\alpha}, \hat{\beta}, \theta)} = 0,$$

where

$$\psi^{'}(\hat{\alpha}, \hat{\beta}, \theta) = \frac{d}{d\theta} \psi(\hat{\alpha}, \hat{\beta}, \theta).$$

The log-likelihood equation is a non-linear function in θ, we can solve it by any numerical method to obtain $\hat{\theta}$, the MLE of θ.

MLEs for LTXG

In case when X follows LTXG distribution with parameters α and θ, denoting $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ as a particular realization on a sample of size n from it, we have likelihood function as

$$L(\alpha, \theta|\mathbf{x}) = \prod_{i=1}^{n} \frac{\theta^2 \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta(x_i - \alpha)}}{\left(1 + \theta + \theta\alpha + \frac{\theta^2}{2}\alpha^2\right)}.$$

The MLE of α is, obviously

$$\hat{\alpha} = min\{X_1, X_2, \ldots, X_n\} = X_{(1)},$$

the smallest order statistic. Hence for given $\hat{\alpha}$, to obtain MLE of θ, say $\hat{\theta}$, we have the following log-likelihood function

$$\ln L(\hat{\alpha}, \theta|\mathbf{x}) = 2n \ln \theta + \sum_{i=1}^{n} \ln \left(1 + \frac{\theta}{2} x_i^2\right) - \theta \sum_{i=1}^{n} x_i + n\theta\hat{\alpha} - n \ln \left(1 + \theta + \theta\hat{\alpha} + \frac{\theta^2}{2}\hat{\alpha}^2\right).$$

The MLE $\hat{\theta}$ is the solution of the log-likelihood equation

$$\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{x_i^2}{2\left(1 + \frac{\theta}{2} x_i^2\right)} - \sum_{i=1}^{n} x_i + n\hat{\alpha} - \frac{n\left(1 + \hat{\alpha} + \hat{\alpha}^2\theta\right)}{\left(1 + \theta + \theta\hat{\alpha} + \frac{\theta^2}{2}\hat{\alpha}^2\right)} = 0,$$

which can be solved by numerical methods.

MLEs for UTXG

When $X \sim UTXG(\beta, \theta)$, let us take, in a similar fashion as earlier, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to denote a particular realization on a random sample of size n from it. The likelihood function, in this case, is

$$L(\beta, \theta|\mathbf{x}) = \prod_{i=1}^{n} K(\beta, \theta) \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta(x_i - \alpha)}.$$

In this case, the MLE of β is

$$\hat{\beta} = \max\{X_1, X_2, \ldots, X_n\} = X_{(n)},$$

the largest order statistic. Given $\hat{\beta}$, the log-likelihood function is obtained as

$$\begin{aligned} \ln L(\hat{\beta}, \theta|\mathbf{x}) &= 2n \ln \theta + \sum_{i=1}^{n} \ln \left(1 + \frac{\theta}{2} x_i^2\right) - \theta \sum_{i=1}^{n} x_i \\ &\quad - n \ln \left[(1+\theta)\left(1 - e^{-\theta\hat{\beta}}\right) - \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}}{2}\right) e^{-\theta\hat{\beta}}\right] \end{aligned} \tag{8}$$

The MLE of θ, $\hat{\theta}$, is the solution of the log-likelihood equation given by

$$\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{x_i^2}{2\left(1 + \frac{\theta}{2} x_i^2\right)} - \sum_{i=1}^{n} x_i - \frac{n\left[\left(1 - e^{-\theta\hat{\beta}}\right) + \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}^2}{2}\right) e^{-\theta\hat{\beta}}\right]}{\left[(1+\theta)\left(1 - e^{-\theta\hat{\beta}}\right) - \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}}{2}\right) e^{-\theta\hat{\beta}}\right]} = 0,$$

which is again a non-linear equation in θ and is solved using numerical methods.

7.1. The OLS method

Let $F_{\cdot}(x_{[i:n]})$ denote the CDF of TXG model and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ be the n ordered RS. The ordinary least squares estimates (OLSEs) are obtained upon minimizing

$$\mathbf{OLSE}(\cdot) = \sum_{i=1}^{n} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right]^2 .$$

where

$$\tau_{(i,n)} = \frac{i}{n+1}$$

and $F_{\cdot}(x_{[i:n]})$ can be the DTXG(α, β, θ) with $F_{\alpha,\beta,\theta}(x_{[i:n]})$ or UTXG(β, θ) with $F_{\beta,\theta}(x_{[i:n]})$ or the LTXG(α, θ) with $F_{\alpha,\theta}(x_{[i:n]})$ CDFs. The OLSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\alpha}(x_{[i:n]}, \cdot),$$

$$0 = \sum_{i=1}^{n} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\beta}(x_{[i:n]}, \cdot),$$

and

$$0 = \sum_{i=1}^{n} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\theta}(x_{[i:n]}, \cdot),$$

where $\varsigma_{\alpha}(x_{[i:n]}, \cdot)$, $\varsigma_{\beta}(x_{[i:n]}, \cdot)$ and $\varsigma_{\theta}(x_{[i:n]}, \cdot)$ are $\partial F_{\cdot}(x_{[i:n]})/\partial\alpha$, $\partial F_{\cdot}(x_{[i:n]})/\partial\beta$ and $\partial F_{\cdot}(x_{[i:n]})/\partial\theta$ respectively.

7.2. L-Moments method

However, linear combinations of the order statistics can be used to estimate the L-Moments, which are comparable to ordinary moments. They are relatively resistant to the effects of outliers and exist anytime the distribution's mean does, even if certain higher moments do not. Based on the moments of the order statistics, we can derive explicit expressions for the L-moments of x as infinite weighted linear combinations of the means of suitable TXG order statistics. The L-Moments for the population can be obtained from

$$\gamma_r = \frac{1}{r}{}_{m=0}^{r-1} (-1)^m \binom{r-1}{m} \mathbf{E}\left(x_{r-m:m}\right)|_{\,(r\geq 1)}. \tag{9}$$

The first four L-Moments are given by

$$m_1 = \mathbf{E}\left(x_{1:1}\right) = \mu'_1 = \mathbb{L}_1,$$

$$m_2 = \frac{1}{2}\mathbf{E}\left(x_{2:2} - x_{1:2}\right) = \frac{1}{2}\left(\mu'_{2:2} - \mu'_{1:2}\right) = \mathbb{L}_2,$$

and

$$m_3 = \frac{1}{3}\mathbf{E}\left(x_{3:3} - 2x_{2:3} + x_{1:3}\right) = \frac{1}{3}\left(\mu'_{3:3} - 2\mu'_{2:3} + \mu'_{1:3}\right) = \mathbb{L}_3,$$

where $\mathbb{L}_i|_{(i=1.2.3)}$ is the L-Moments for the sample. Then, the L-moments estimators of the parameters α, β and θ can be obtained by solving the following three equations numerically

$$\begin{aligned} m_1\left(\widehat{\alpha}_{(L-Moments)}, \widehat{\beta}_{(L-Moments)}, \widehat{\theta}_{(L-Moments)}\right) &= \mathbb{L}_1, \\ m_2\left(\widehat{\alpha}_{(L-Moments)}, \widehat{\beta}_{(L-Moments)}, \widehat{\theta}_{(L-Moments)}\right) &= \mathbb{L}_2, \end{aligned}$$

and

$$m_3\left(\widehat{\alpha}_{(L-Moments)}, \widehat{\beta}_{(L-Moments)}, \widehat{\theta}_{(L-Moments)}\right) = \mathbb{L}_3.$$

7.3. WLS method

The weighted least squares estimates (WLSEs) are obtained by minimizing the function $\mathbf{WLSE}\,(\cdot)$ with respect to α, β and θ

$$\mathbf{WLSE}\,(\cdot) = \sum_{i=1}^{n} \omega_{(i,n)} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right]^2,$$

where

$$\omega_{(i,n)} = \left[(1+n)^2(2+n)\right] / \left[i(1+n-i)\right].$$

and $F_{\cdot}(x_{[i:n]})$ can be the DTXG(α, β, θ) or UTXG(β, θ) or the LTXG(α, θ) CDFs. The WLSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \omega_{(i,n)} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\alpha}(x_{[i:n]}, \cdot),$$

$$0 = \sum_{i=1}^{n} \omega_{(i,n)} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\beta}(x_{[i:n]}, \cdot),$$

and

$$0 = \sum_{i=1}^{n} \omega_{(i,n)} \left[F_{\cdot}(x_{[i:n]}) - \tau_{(i,n)}\right] \varsigma_{\theta}(x_{[i:n]}, \cdot),$$

where $\varsigma_{\alpha}(x_{[i:n]}, \cdot)$, $\varsigma_{\beta}(x_{[i:n]}, \cdot)$ and $\varsigma_{\theta}(x_{[i:n]}, \cdot)$ are as defined above.

8. Simulation studies for comparing estimation methods

A simulation is an ongoing replica of how a system or process might work in the actual world. Models must be used in simulations; the model reflects the essential traits or behaviors of the chosen system or process, whilst the simulation depicts the model's development through time. Computers are widely applied to run the simulation. There are several applications for simulation, including video games, safety engineering, testing, teaching, and performance tuning or optimization of technology. In order to understand how natural or human systems work, simulation is also employed in scientific modelling, such as in economics. The eventual actual impacts of different circumstances and actions can be demonstrated through simulation. When the real system cannot be utilized, for example, because it is inaccessible, unsafe, or unpleasant to use, it is still being created but has not yet been constructed, or it just doesn't exist, simulation is employed. The acquisition of reliable information from reliable sources about the pertinent selection of key characteristics used to build the model, the use of simplifying approximations and assumptions within the model, and the fidelity and validity of the simulation results are some of the key issues in modelling and simulation. An continuous area of academic study, elaboration, research, and advancement in simulations technology or technique, notably in the work of computer simulation, is procedures and protocols for model verification and validation. In this Section, a numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N=1000 generated data sets from the TXG version where $n = 50, 100, 150$ and 300. The estimates are compared in terms of their bias $\left(\mathrm{BIAS}_{(\cdot)}\right)$ and mean standard error $\left(\mathrm{RMSE}_{(\cdot)}\right)$. Table 2 lists the simulation results for DTXG model. Table 3 lists the simulation results for DTXG model. Table 4 provides the simulation results for LTXG model. From Tables 2, 3 and 4 we note that:

I. The $\mathrm{BIAS}_{(\cdot)}$ tend to zero when n increases which means that all estimators are consistent.

II. The $\mathrm{RMSE}_{(\cdot)}$ tend to zero when n increases which means incidence of consistency property.

Table 2: Simulation results for DTXG.

	n	$BIAS_{(\alpha)}$	$BIAS_{(\beta)}$	$BIAS_{(\theta)}$	$MSE_{(\alpha)}$	$MSE_{(\beta)}$	$MSE_{(\theta)}$
MLE	20	−0.18988	0.33971	−0.00182	5.04622	7.96603	0.00343
LS		−0.20446	−0.46774	−0.00225	19.5937	214.31458	0.0036
L-Moments		−0.01935	0.25724	−0.00209	2.99639	5.88326	0.00354
WLS		0.25051	0.17797	0.00271	17.8097	67.34753	0.00355
MLE	50	−0.13715	0.09408	0.00027	2.16432	3.24087	0.00147
LS		0.02814	0.16385	−0.00016	2.38445	6.74487	0.00154
L-Moments		−0.03911	0.06081	0.00045	1.3282	2.52853	0.0015
WLS		0.0369	−0.10831	0.00003	1.13147	3.00061	0.0015
MLE	150	−0.03291	0.0485	−0.00028	0.65097	1.02794	0.00047
LS		0.06742	0.05911	−0.00082	0.81669	2.11751	0.00047
L-Moments		0.00221	0.03655	0.00004	0.39649	0.77295	0.00045
WLS		−0.04798	0.0231	0.00036	0.39955	0.9581	0.00044
MLE	300	−0.0277	0.01335	0.00009	0.30916	0.48256	0.00022
LS		0.0056	−0.01711	0.00024	0.39732	1.05668	0.00024
L-Moments		−0.01057	0.01284	0.0005	0.21129	0.39241	0.00023
WLS		−0.03965	0.07024	−0.00017	0.22851	0.55418	0.00023

9. Applications

9.1. Application for comparing classical methods

For comparing classical methods, an application to real data set is introduced. We consider the Kolmomogorov-Smirnov (KS) statistic and its p-value. We consider strength data of glass of aircraft window reported by Fullet et al. [11]. The data is presented in Table 5. Table 6 gives the results for DTXG model under the strength data. Table 7 gives the results for UTXG model under the strength data. Table 8 gives the results for LTXG model under the strength data. From Table 6, the OLS method is the best method with KS=0.10860 and p-value=0.85806 then WLS with KS=0.11111 and p-value=0.83870. However, the other methods performed well. From Table 7, the OLS method is the best method with KS=0.19820 and p-value=0.17499. However, the ML and WLS methods performed well. From Table 8, the OLS method is the best method with KS=0.14453 and p-value=0.53648. However, the ML and L-Moment methods performed well. Generally, we can say that the OLS methods is the best methed and can be recommended for comparing the estimation methods.

Table 3: Simulation results for UTXG.

	n	$BIAS_{(\beta)}$	$BIAS_{(\theta)}$	$MSE_{(\beta)}$	$MSE_{(\theta)}$
MLE	20	0.00594	0.0245	6.04792	0.00378
LS		−0.69331	−0.04098	234.60137	0.04706
L-Moments		0.42751	−0.0029	10.93383	0.00284
WLS		0.62148	0.05271	150.44423	0.00972
MLE	50	−0.08771	0.01758	1.80590	0.00210
LS		0.12593	−0.03441	6.19452	0.00612
L-Moments		0.10946	−0.00015	4.48167	0.00119
WLS		−0.08933	0.03799	2.78863	0.00669
MLE	150	−0.12502	0.00853	0.55195	0.00099
LS		0.01661	−0.01933	1.90316	0.00325
L-Moments		0.05303	−0.00041	1.33857	0.00035
WLS		0.02207	0.02069	0.94507	0.00343
MLE	300	−0.14934	0.0075	0.29553	0.00067
LS		−0.02073	−0.00895	0.97574	0.00156
L-Moments		0.01221	0.00009	0.68479	0.00018
WLS		0.06206	0.01546	0.50154	0.00249

Table 4: Simulation results for LTXG.

	n	$BIAS_{(\alpha)}$	$BIAS_{(\theta)}$	$MSE_{(\alpha)}$	$MSE_{(\theta)}$
MLE	20	0.08377	0.00406	2.96808	0.00115
LS		0.19299	0.00316	1.64559	0.00222
L-Moments		−0.08404	0.0024	8.81319	0.00081
WLS		−0.07499	−0.0028	0.97036	0.00168
MLE	50	−0.013	0.0026	1.27578	0.00046
LS		0.13619	0.00173	0.93617	0.00059
L-Moments		0.15616	0.00091	0.87484	0.00056
WLS		−0.03297	−0.00176	0.43075	0.00054
MLE	150	0.00381	0.00065	0.41299	0.00014
LS		0.06359	0.00031	0.29735	0.00017
L-Moments		0.01413	0.00001	1.21324	0.0001
WLS		−0.049	−0.00037	0.16495	0.00016
MLE	300	0.00288	0.00026	0.19375	0.00007
LS		0.02240	0.00040	0.14642	0.00009
L-Moments		−0.01916	0.00019	0.62064	0.00005
WLS		−0.03706	−0.00036	0.09411	0.00008

Table 5: Data on Glass strength of aircraft window

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.80
26.69	26.770	26.78	27.05	27.67	29.90	31.11	33.20	33.73	33.76
33.890	34.76	35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29
45.381									

Table 6: Results for DTXG model under the strength data.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	KS	p-value
ML	18.83	45.381	0.08618	0.11300	0.82347
OLS	18.83	45.881	0.08799	0.10860	0.85806
WLS	18.83	45.381	0.08696	0.11111	0.83870
L-Moment	18.83	45.381	0.08606	0.11329	0.82109

Table 7: Results for UTXG model under the strength data.

Method	$\widehat{\beta}$	$\widehat{\theta}$	KS	p-value
ML	45.381	0.03487	0.19824	0.17480
OLS	45.881	0.03485	0.19820	0.17499
WLS	45.381	0.02614	0.20032	0.16605
L-Moment	45.381	0.01531	0.22277	0.09221

Table 8: Results for LTXG model under the strength data.

Method	$\widehat{\alpha}$	$\widehat{\theta}$	KS	p-value
ML	18.83	0.13768	0.16553	0.36358
OLS	18.83	0.12406	0.14453	0.53648
WLS	18.83	0.16457	0.28994	0.01090
L-Moment	18.83	0.13756	0.16512	0.36659

9.2. Application for comparing models

In this section, applicability of the truncated versions xgamma distribution is illustrated by a real data analysis. We consider strength data of glass of aircraft window reported by Fullet et al. [11]. We have fitted the data by exponential distribution with rate λ, Weibull distribution with shape α and scale λ, Lindley distribution with parameter θ, xgamma(θ), LTXG(α, θ), UTXG(β, θ) and DTXG(α, β, θ) distributions. Maximum likelihood estimates are obtained for the unknown parameters of each model. As model comparison criteria, we have considered log-likelihood values, KS, Akaike information criteria (AIC) and Bayesian information criteria (BIC). We note that,

$$\begin{aligned} \text{AIC} &= -2\ln(\text{likelihood}) + 2k\,; \\ \text{BIC} &= -2ln(\text{likelihood}) + k\ln(n), \end{aligned}$$

where k is the number of parameters in a distribution, n is the sample size. Lower the value of AIC and/or BIC, better is the model. The result of the data analysis is listed in Table 9. Along with MLEs the standard errors of estimation are shown in parentheses. Based on the ? table, it is clear to us that the new distribution has a high applicability and wide flexibility in statistical modeling processes. This practical ability is demonstrated by the standards values of the new distribution. Where the new distribution showed the best results and were as follows:

1. Regarding the results for DTXG model under the strength data: The DTXG model is the best model with $-$Log-likelihood= 87.89, AIC= 177.78, BIC= 179.21 and KS= 0.11300.

2. Regarding the results for UTXG model under the strength data: The UTXG model is the best model with $-$Log-likelihood$=109.66$, AIC$=221.33$, BIC$=222.76$ and KS$=0.1982$.
3. Regarding the results for LTXG model under the strength data: The LTXG model is the best model with $-$Log-likelihood$=106.32$, AIC$=214.65$, BIC$=216.08$ and KS$=0.1655$.
4. Generally, the DTXG version is the best the LTXG model then UTXG model. Hence, it is recommended to use the DTXG version in the processes of statistical analysis and mathematical and statistical modeling of real data.

Table 9: The MLEs of model parameters and model selection criteria for glass strength data

Distributions	Estimate(Std. Err.)	$-$Log-likelihood	AIC	BIC	KS
Exponential(λ)	$\hat{\lambda}$=0.0324(0.0058)	137.26	276.53	277.96	0.4586
Weibull(α, λ)	$\hat{\alpha}$=4.6349(0.6292)				
	$\hat{\lambda}$=33.673(1.3829)	105.49	214.98	217.84	0.1525
Lindley(θ)	$\hat{\theta}$=0.0629(0.0080)	126.99	255.99	257.42	0.3655
Xgamma(θ)	$\hat{\theta}$=0.0937(0.0098)	122.27	246.55	247.98	0.3245
UTXG(β, θ)	$\hat{\beta}$=45.381				
	$\hat{\theta}$=0.0349(0.0166)	-109.66	221.33	222.76	0.1982
LTXG(α, θ)	$\hat{\alpha}$=18.83				
	$\hat{\theta}$=0.1377(0.0169)	-106.32	214.65	216.08	0.1655
DTXG(α,β, θ)	$\hat{\alpha}$=18.83				
	$\hat{\beta}$=45.381				
	$\hat{\theta}$=0.1480(0.0186)	-87.89	177.78	179.21	0.11300

10. Concluding remarks

In this article, we introduced the truncated versions of xgamma distribution called upper truncated, lower truncated and double truncated xgamma distribution. Particularly, the properties of the upper truncated xgamma distribution such as moments, popular entropy measures, order statistics are discussed. We briefly described different estimation methods, namely the maximum likelihood, ordinary least squares, weighted least squareand L-Moments. The maximum likelihood estimators are constructed for estimating the unknown parameters of the upper truncated xgamma as well as lower truncated and double truncated xgamma distributions in some details. Monte Carlo simulation experiments are performed for comparing the performances of the proposed methods of estimation for both small and large samples under the lower, upper and double versions. Two applications are provided, the first one comparing estimation methods and the other for illustrating the applicability of the new model. In comparing methods, we consider the Kolmomogorov-Smirnov statistic and its p-value. In comparing models, the goodness-of-fits of the exponential, Weibull, Lindley, xgamma and truncated (lower, upper, double) xgamma distributions have been compared through the negative log-likelihood, Akaike information criteria, Bayesian information criteria and Kolmomogorov-Smirnov statistics and found that the double truncated xgamma distribution fits well the data of the window strengths. Finally, it may be concluded that the truncated distributions can be quite effectively used to model the real problems and truncated xgamma distributions are potential models in explaining the uncertainty coming from various fields of applications where truncated data are commonly observed.
Regarding the assessment of the classical estimation methods, some important results can be summarized as follows:

1. Regarding the results for DTXG model under the strength data: The ordinary least squares method is the best method with KS$= 0.10860$ and p-value$= 0.85806$ then the weighted least squares method with KS$= 0.11111$ and p-value$= 0.83870$. However, the other methods performed well.
2. Regarding the results for UTXG model under the strength data: The ordinary least squares method is the best method with KS$= 0.19820$ and p-value$= 0.17499$. However, the maximum likelihood and weighted least squares methods performed well.
3. Regarding the results for LTXG model under the strength data: The ordinary least squares method is the best method with KS$= 0.14453$ and p-value$= 0.53648$. However, the maximum likelihood and L-Moment methods performed well. Generally, we can say that the ordinary least squares methods is the best methed and can be recommended for comparing the estimation methods.

Regarding the comparison of the competitive model, some important results can be summarized as follows:

1. Regarding the results for DTXG model under the strength data: The DTXG model is the best model with $-$Log-likelihood$= 87.89$, AIC$= 177.78$, BIC$= 179.21$ and KS$= 0.11300$.
2. Regarding the results for UTXG model under the strength data: The UTXG model is the best model with $-$Log-likelihood$= 109.66$, AIC$= 221.33$, BIC$= 222.76$ and KS$= 0.1982$.
3. Regarding the results for LTXG model under the strength data: The LTXG model is the best model with $-$Log-likelihood$= 106.32$, AIC$= 214.65$, BIC$= 216.08$ and KS$= 0.1655$.
4. Generally, the DTXG version is the best the LTXG model then UTXG model. Hence, it is recommended to use the DTXG version in the processes of statistical analysis and mathematical and statistical modeling of real data.

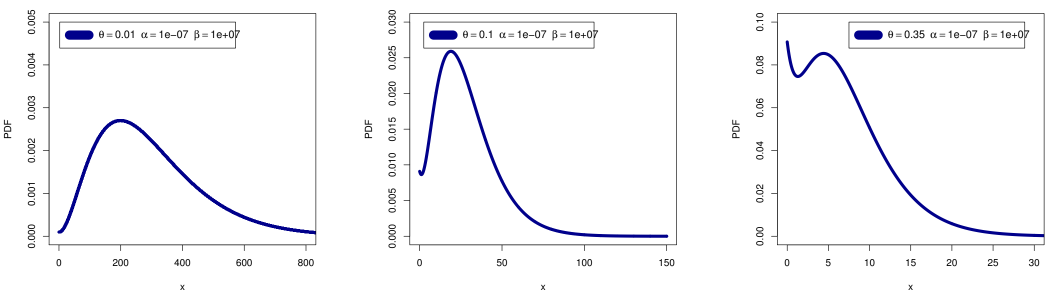

Figure 1. Plots of the baseline XG PDF for some parameter values.

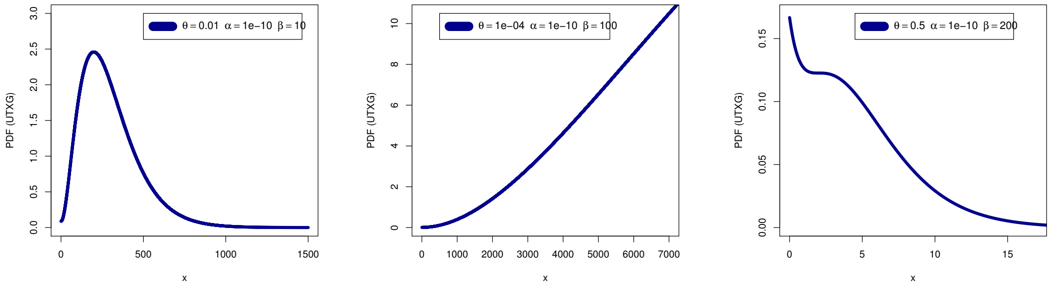

Figure 2. Plots of the baseline UTXG PDF for some parameter values.

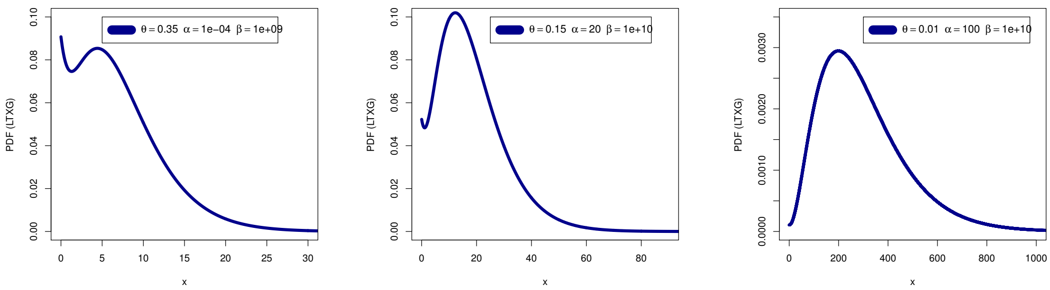

Figure 3. Plots of the baseline LTXG PDF for some parameter values.

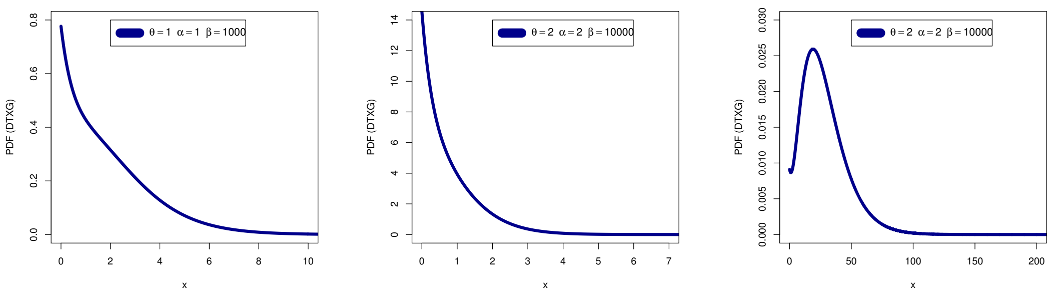

Figure 4. Plots of the baseline DTXG PDF for some parameter values.

Acknowledgment

All of the authors have confirmed their signatures and contributions to the paper with equal portion. Moreover, there is no conflict of interest.

REFERENCES

1. Aboraya, M., Ali, M. M., Yousof, H. M. and Ibrahim, M. (2022). A Novel Lomax Extension with Statistical Properties, Copulas, Different Estimation Methods and Applications. Bulletin of the Malaysian Mathematical Sciences Society, (2022) https://doi.org/10.1007/s40840-022-01250-y
2. Aboraya, M., M. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). A new family of discrete distributions with mathematical properties, characterizations, Bayesian and non-Bayesian estimation methods. Mathematics, 8, 1648.
3. Ali, M. M., Ali, I., Yousof, H. M. and Ibrahim, M. (2022). G Families of Probability Distributions: Theory and Practices. CRC Press, Taylor & Francis Group.
4. Cho, B. R. and Govindaluri, M. S. (2002). Optimal screening limits in multi-stage assemblies. International journal of production research, 40(9), 1993-2009.
5. Cordeiro, G. M., Altun, E., Korkmaz, M. C., Pescim, R. R. and Afify, A. Z. and Yousof, H. M. (2020). The xgamma Family: Censored Regression Modelling and Applications. Revstat Statistical Journal, 18(5), 593–612.
6. Elgohari, H., Ibrahim, M. and Yousof, H. M. (2021d). A New Probability Distribution for Modeling Failure and Service Times: Properties, Copulas and Various Estimation Methods. Statistics, Optimization & Information Computing, 8(3), 555-586.
7. Eliwa, M. S., El-Morshedy, M. and Yousof, H. M. (2022). A Discrete Exponential Generalized-G Family of Distributions: Properties with Bayesian and Non-Bayesian Estimators to Model Medical, Engineering and Agriculture Data. Mathematics, 10, 3348. https://doi.org/10.3390/math10183348
8. El-Morshedy, M., Eliwa, M. S., Al-Bossly, A. and Yousof, H. M. (2022). A New Probability Heavy-Tail Model for Stochastic Modeling under Engineering Data. Journal of Mathematics, 2022, 1-20, https://doi.org/10.1155/2022/1910909
9. Gupta, R. (2001). System and Bayesian Reliability: Essays in Honour of Professor R.E Barlow. Series on Quality, Reliability and Engineering Statistics, chap. Nonmonotonic failure rates and mean residual life functions, World Scientfic Press, Singapore, 147-163.
10. Field, T., Harder, U. and Harrison, P. (2004). Network traffic behaviour in switched Ethernet systems. Performance Evaluation, 58(2), 243-260.
11. Fuller Jr, E. R., Freiman, S. W., Quinn, J. B., Quinn, G. D. and Carter, W. C. (1994). Fracture mechanics approach to the design of glass aircraft windows: a case study. In SPIE's 1994 International Symposium on Optics, Imaging and Instrumentation, 419-430.
12. Hamed, M. S., Cordeiro, G. M. and Yousof, H. M. (2022). A New Compound Lomax Model: Properties, Copulas, Modeling and Risk Analysis Utilizing the Negatively Skewed Insurance Claims Data. Pakistan Journal of Statistics and Operation Research, 18(3), 601-631. https://doi.org/10.18187/pjsor.v18i3.3652
13. Hamedani, G. G., Korkmaz, M. C., Butt, N. S. and Yousof, H. M. (2021). The Type I Quasi Lambert Family: Properties, Characterizations and Different Estimation Methods. Pakistan Journal of Statistics and Operation Research, 17(3), 545-558.
14. Hegde, L. M. and Dahiya, R. C. (1989). Estimation of the parameters of a truncated gamma distribution. Communications in Statistics-Theory and Methods, 18(2), 561-577.
15. Ibrahim. M., Aidi, K., Ali, M. M. and Yousof, H. M. (2021). The Exponential Generalized Log-Logistic Model: Bagdonavičius-Nikulin test for Validation and Non-Bayesian Estimation Methods. Communications for Statistical Applications and Methods, 29(1), 681–705.
16. Ibrahim, M., Ali, M. M. and Yousof, H. M. (2021). The discrete analogue of the Weibull G family: properties, different applications, Bayesian and non-Bayesian estimation methods. Annals of Data Science, forthcoming.
17. Ibrahim, M., Altun, E. and Yousof, H. M. (2020). A new distribution for modeling lifetime data with different methods of estimation and censored regression modeling. Statistics, Optimization & Information Computing, 8(2), 610-630.
18. Ibrahim, M., Altun, E., Goual, H., and Yousof, H. M. (2020). Modified goodness-of-fit type test for censored validation under a new Burr type XII distribution with different methods of estimation and regression modeling. Eurasian Bulletin of Mathematics, 3(3), 162-182.
19. Ibrahim, M., Hamedani, G. G., Butt, N. S. and Yousof, H. M. (2022). Expanding the Nadarajah Haghighi Model: Copula, Censored and Uncensored Validation, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research, 18(3), 537-553. https://doi.org/10.18187/pjsor.v18i3.3420
20. Ibrahim. M., Handique, L., Chakraborty, S., Butt, N. S. and M. Yousof, H. (2021). A new three-parameter xgamma Fréchet distribution with different methods of estimation and applications. Pakistan Journal of Statistics and Operation Research, 17(1), 291-308.
21. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H. and Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Communications for Statistical Applications and Methods, 26(5), 473-495.
22. Kapur, K. C. and Cho, B. R. (1994). Economic design and development of specifications. Quality Engineering, 6(3), 401-417.
23. Kapur, K. C. and Cho, B. R. (1996). Economic design of the specification region for multiple quality characteristics. IIE transactions, 28(3), 237-248.
24. Khasawneh, M. T., Bowling, S. R., Kaewkuekool, S. and Cho, B. R. (2004). Tables of a truncated standard normal distribution: A singly truncated case. Quality Engineering, 17(1), 33-50.
25. Khasawneh, M. T., Bowling, S. R., Kaewkuekool, S. and Cho, B. R. (2005). Tables of a truncated standard normal distribution: a doubly truncated case. Quality Engineering, 17(2), 227-241.
26. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M. and Elrazik, E. A. (2020). A New Parametric Life Distribution with Modified Bagdonavičius–Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. Entropy, 22(5), 592.
27. Mansour, M., Yousof, H. M., Shehata, W. A. M. and Ibrahim, M. (2020). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. Journal of Nonlinear Science and Applications, 13(5), 223-238.
28. Mittal, M. M. and Dahiya, R. C. (1989). Estimating the parameters of a truncated Weibull distribution. Communications in Statistics-Theory and Methods, 18(6), 2027-2042.

29. Mukhopadhyay, A., Bhowmick, K. and Mitra, G. B. (2000). Edgeworth series expansion of the truncated Cauchy function and its effectiveness in the study of atomic heterogeneity. Zeitschrift für Kristallographie-Crystalline Materials, 215(12), 718-726.
30. Nadarajah, S. (2008). A truncated inverted beta distribution with application to air pollution data. Stochastic Environmental Research and Risk Assessment, 22(2), 285-289.
31. Phillips, M. D. and Cho, B. R. (1998). Quality improvement for processes with circular and spherical specification regions. Quality Engineering, 11(2), 235-243.
32. Phillips, M. D. and Cho, B. R. (2000). Modelling of optimal specification regions. Applied Mathematical Modelling, 24(5), 327-341.
33. Salah, M. M., El-Morshedy, M., Eliwa, M. S. and Yousof, H. M. (2020). Expanded Fréchet Model: Mathematical Properties, Copula, Different Estimation Methods, Applications and Validation Testing. Mathematics, 8(11), 1949.
34. Sen, S., Korkmaz, M. C. and Yousof, H. M. (2018). The quasi xgamma-Poisson distribution. ISTATISTIK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION, 11(3), 65-76.
35. Sen, S., Maiti, S. S. and Chandra, N. (2016). The xgamma Distribution: Statistical Properties and Application. Journal of Modern Applied Statistical Methods, 15(1), 774-788.
36. Shehata, W. A. M. and Yousof, H. M. (2022). A novel two-parameter Nadarajah-Haghighi extension: properties, copulas, modeling real data and different estimation methods. Statistics, Optimization & Information Computing, 10(3), 725-749.
37. Yadav, A. S., Goual, H., Alotaibi, R. M., Rezk, H., Ali, M. M. and Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry, 12(1), 57.
38. Yousof, H. M., Ali, M. M., Goual, H. and Ibrahim. M. (2021). A new reciprocal Rayleigh extension: properties, copulas, different methods of estimation and modified right censored test for validation, Statistics in Transition New Series, 23(3), 1-23.
39. Yousof, H. M., Ali, M. M., Hamedani, G. G., Aidi, K. & Ibrahim, M. (2022). A new lifetime distribution with properties, characterizations, validation testing, different estimation methods. Statistics, Optimization & Information Computing, 10(2), 519-547.
40. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). The Two-parameter Xgamma Fréchet Distribution: Characterizations, Copulas, Mathematical Properties and Different Classical Estimation Methods. Contributions to Mathematics, 2 (2020), 32-41.
41. Yousof, H. M., Korkmaz, M. Ç., K., Hamedani, G. G and Ibrahim, M. (2022). A novel Chen extension: theory, characterizations and different estimation methods. Eur. J. Stat, 2(2022), 1-20.
42. Tsallis, C. (1988). Possible generalization of boltzmann-gibbs statistics. Journal of statistical physics, 52(1-2), 479-487.
43. Zaninetti, L. and Ferraro, M. (2008). On the truncated Pareto distribution with applications. Open Physics, 6(1), 1-6.
44. Zhang, T. and Xie, M. (2011). On the upper truncated Weibull distribution and its reliability implications. Reliability Engineering & System Safety, 96(1), 194-200.