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The Topp-Leone Odd Burr X-G Family of Distributions: Properties and Applications

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Abstract This paper proposes a new generalized family of distributions called the Topp-Leone odd Burr X-G (TLOBX-G) distribution and its special model, Topp-Leone odd Burr X-Weibull (TLOBX-W) is studied in detail. Structural properties are derived, including the hazard rate function, quantile function, density expansion, moments, Rényi entropy, and order statistics. The maximum likelihood technique is used to estimate the parameters of the new family of distributions and a simulation study was carried out to assess the accuracy and consistency of these estimators. Finally, the applicability, usefulness, and flexibility of TLOBX-W distribution are illustrated using two real-life datasets.

Keywords Topp-Leone distribution, odd Burr X-G distribution, Weibull distribution, Maximum Likelihood estimation

AMS 2010 subject classifications 60E05, 62E15, 62F30

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1. Introduction

In many situations, existing traditional distributions do not provide an adequate fit to real data found in areas such as engineering, finance, biology, health, and economics. With that, researchers have been challenged with developing flexible lifetime distributions from classic ones that can fit heavy-tailed and skewed lifetime data. New classes of distributions have been developed by the addition of shape parameters to control the tail weights together with the skewness and kurtosis, hence providing flexibility in modeling skewed and heavy-tailed data. Some of the recent generators that have been proposed include the Marshall-Olkin-G by [1], Gamma-G generator by [2], beta-G by [3], transformer (T-X) by [4], Topp-Leone-G (TL-G) by [5], odd Burr X-G (OBX-G) by [6], Topp-Leone Gompertz-G by [8] and Topp-Leone-Harris-G by [7], to mention just a few.
Using the distribution function of the one-parameter Topp-Leone random variable, [5] developed TL-G family of distributions with probability density function (pdf) and cumulative distribution function (cdf)

$$\mathrm{f}(x;\alpha,\xi) = 2\alpha g(x;\xi)\bar{G}(x;\xi)\left[1-\bar{G}^2(x;\xi)\right]^{\alpha-1} \quad x,\alpha>0 \tag{1}$$

and

$$\mathrm{F}(x;\alpha,\xi) = \left[1-\bar{G}^2(x;\xi)\right]^{\alpha} \quad x,\alpha>0, \tag{2}$$

respectively. TL-G family of distributions has been extended to other family of distributions which include Topp-Leone odd Exponential Half Logistic-G (TLOEHL-G) family of distributions [9], Topp-Leone odd Burr III-G (TLOBIII-G) family of distributions [10], Topp-Leone Kumaraswamy-G (TLK-G) family of distributions [11]

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and Topp-Leone odd Lindley-G (TLOL-G) family of distributions [12]. Special cases of the TL-G distributions studied include Topp-Leone-Exponential (TL-E) distribution [5], Topp-Leone Weibull (TL-W) distribution [13], Topp-Leone Power Lindley (TL-PL) distribution [14] and Topp-Leone Lomax (TL-L) distribution [15].
[6] used the distribution function of the Burr type X random variable to develop the odd Burr X-G (OBX-G) family of distributions with pdf and cdf

$$\mathrm{f}(x;\theta,\xi)=\frac{2\theta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^{3}(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta-1}, \tag{3}$$

and

$$\mathrm{F}(x;\theta,\xi)=\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta} \quad x,\theta>0. \tag{4}$$

Studies which worked around the Burr Type X-G family of distributions include Gamma Burr type X (GBX) distribution [16], Exponentiated Burr X (EBX) distribution [17], Beta Burr X (BBX) distribution [18], Exponentiated Generalized Burr X (EGBX) distribution [19] and Type I Half-Logistic Burr X (TIHLBX) distribution [20].
In this paper, our aim is to develop the Topp-Leone odd Burr X-G (TLOBX-G) family of distributions, a novel and versatile class that combines the TL-G and OBX-G families. The primary motivation behind this unique combination is their collective capacity to effectively model heavy-tailed and asymmetric data, which are frequently encountered in real-world datasets. By exploiting the complementary strengths of TL-G and OBX-G distributions, we endeavor to provide a comprehensive solution for accommodating diverse probability density function (pdf) shapes, including left-skewed, right-skewed, almost symmetric, and reversed-J shapes. Moreover, we seek to construct distributions with hazard rate functions (hrf) that exhibit a wide range of shapes, encompassing increasing, decreasing, bathtub, and upside-down bathtub configurations. Additionally, the incorporation of heavy-tailed distributions within the TLOBX-G family is expected to yield superior fitting capabilities to real data, surpassing those offered by existing distributions.
The rest of the paper is organized as follows; TLOBX-G family is introduced in Section 2 followed by its structural properties which include the quantile function, density expansion, moments, Rényi entropy and order statistics in Section 3. Section 4 provides some special cases of the TLOBX-G family of distributions, while Section 5 presents parameter estimation carried out using maximum likelihood method. Section 6 presents results for the simulation study and Section 7 gives applications. Concluding remarks are given in Section 8.

2. The New Model

In this section, the new family of distributions called the Topp-Leone odd Burr X-G (TLOBX-G) family of distributions is provided. Inserting Equations (3) and (4) into Equations (1) and (2), we get TLOBX-G family of distributions with pdf and cdf given by

$$\begin{aligned}\mathrm{f}(x;\alpha,\theta,\xi)=&\frac{4\alpha\theta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^{3}(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta-1}\\&\times\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta}\right]\\&\times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha-1}\end{aligned} \tag{5}$$

and

$$\mathrm{F}(x;\alpha,\theta,\xi)=\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha}, \tag{6}$$

where $x>0$, $\alpha>0$, $\theta>0$, ξ is vector of parameters from the baseline distribution, $\bar{\mathrm{G}}(x;\xi)=1-\mathrm{G}(x;\xi)$, and $\mathrm{G}(x;\xi)$ and $\mathrm{g}(x;\xi)$ are the cdf and pdf of the baseline distribution. The hazard rate function (hrf) of TLOBX-G family of distributions is given as

$$\begin{aligned}\mathrm{h}(x;\alpha,\theta,\xi)=&\frac{4\alpha\theta\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1}\\&\times\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]\\&\times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha-1}\\&\times\left\{1-\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha}\right\}^{-1}.\end{aligned}$$

2.1. Sub-Families

Sub-families of the TLOBX-G family are presented in this subsection.

- When $\alpha=1$, we obtain a reduced TLOBX-G family with cdf $\mathrm{F}(x;\theta,\xi)=1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2$, for $\theta,x>0$ and parameter vector ξ, which is a new extension of OBX-G family.
- When $\theta=1$, we obtain a reduced TLOBX-G family with cdf $\mathrm{F}(x;\alpha,\xi)=\left[1-\exp\left[-2\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right]^{\alpha}$, for $\alpha,x>0$ and parameter vector ξ, which is an new extension of TL-G family.
- When $\alpha=\theta=1$, we obtain a reduced TLOBX-G family with cdf $\mathrm{F}(x;\xi)=1-\exp\left[-2\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]$, with the parameter vector ξ, which is a new family.

3. Some Structural Properties

In this section, quantile function, density expansion, ordinary moments, central moments, moment generating function, incomplete moments, Rényi entropy, order statistics and probability weighted moments are derived.

3.1. Quantile Function

We use the cdf of TLOBX-G distributions to derive the quantile function by solving the non-linear equation

$$u = \left[1 - \left[1 - \left\{1 - \exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^\theta\right]^2\right]^\alpha$$

for $0 < u < 1$. Solving for $\mathrm{G}(x;\xi)$, we get

$$\mathrm{G}(x;\xi) = \frac{\{-\ln[1-(1-(1-u^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}{1+\{-\ln[1-(1-(1-u^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}.$$

Finally, the quantile function is given as

$$\mathrm{Q}(u) = \mathrm{G}^{-1}\left\{\frac{\{-\ln[1-(1-(1-u^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}{1+\{-\ln[1-(1-(1-u^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}\right\}.$$

The quantile function can be used to generate random numbers for the parameters of a specified model. Table 1 presents quantiles of the Topp-Leone odd Burr X-Weibull (TLOBX-W) distribution for various parameter values.

Table 1. Table of quantiles for selected parameter values of the TLOBX-W distribution

	Quantiles for specified values of (δ, θ, α)				
u	(1.5,0.9,131)	(2,20,0.9)	(1.5,0.5,1.5)	(0.1,997,1.8)	(100,2,0.1)
0.1	0.990	0.940	0.228	1.042	0.622
0.2	0.992	0.972	0.312	1.058	0.778
0.3	0.993	0.995	0.378	1.069	0.922
0.4	0.994	1.016	0.435	1.079	1.071
0.5	0.995	1.036	0.489	1.089	1.237
0.6	0.996	1.056	0.542	1.099	1.437
0.7	0.996	1.078	0.598	1.109	1.697
0.8	0.997	1.104	0.660	1.121	2.077
0.9	0.998	1.143	0.740	1.138	2.787

3.2. Series Expansion

This section contain the series expansion of the pdf based on the use of generalized binomial expansion and exponential representation together. Thus, applying the generalized binomial series representation $(1-z)^n = \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}z^i$ which is valid for $|z| < 1$, we can express the TLOBX-G density as

$$\mathrm{f}(x;\alpha,\theta,\xi) = \sum_{p=0}^{\infty}\mathrm{W}_{p+1}\mathrm{h}_{p+1}(x;\xi), \tag{7}$$

(which follows from proofs in Appendix B), where

$$\mathrm{W}_{p+1} = 4\alpha\theta\sum_{i,j,k,m,n=0}^{\infty}\frac{(-1)^{i+j+k+m+n+p}(k+1)^m}{(p+1)m!}\binom{\alpha-1}{i}\binom{2i+1}{j} \times\binom{\theta(j+1)-1}{k}\binom{2m+1}{n}\binom{n-(2m+3)}{p} \tag{8}$$

and $\mathrm{h}_{p+1}(x;\xi) = (p+1)\mathrm{g}(x;\xi)\mathrm{G}^{p+1-1}(x;\xi)$ is the Exponentiated-G (Exp-G) density function with power parameter $(p+1)$. This shows that the structural properties of TLOBX-G family of distributions depends in those of the Exp-G distribution.

3.3. Moments and Generating Functions

Derivation of the moments depends in Equation (7). Thus, the r^{th} ordinary moment of TLOBX-G family of distributions, say μ'_r, is given as

$$\mu'_r = \mathrm{E}(X^r) = \sum_{p=0}^{\infty} \mathrm{w}_{p+1}\mathrm{E}\left(\mathrm{Y}_{p+1}\right) = \sum_{p=0}^{\infty} \mathrm{W}_{p+1}(p+1)\int_0^1 u^{p+1-1}\mathrm{Q}_G^r(u;\xi)du\ , \tag{9}$$

where Y_{p+1} is the Exp-G random variable with power parameter $(p+1)$ and $\mathrm{Q}_G(u;\xi)$ is the quantile function of the baseline distribution with the cdf $\mathrm{G}(x;\xi)$. Following Equation (9), we can derive the n^{th} central moment which is critical in obtaining the skewness and kurtosis as

$$\begin{aligned} M_n = \mathrm{E}(X-\mu'_1)^n &= \sum_{r=0}^{\infty}\binom{n}{r}(-\mu'_1)^{n-r}\mathrm{E}(X^r) \\ &= \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\binom{n}{r}(-\mu'_1)^{n-r}\mathrm{W}_{p+1}\mathrm{E}\left(\mathrm{Y}_{p+1}\right)\ . \end{aligned}$$

The moment generating function of the TLOBX-G distributions, say $\mathrm{M}_X(t)$, can also be derived using Equation (7) as

$$\mathrm{M}_X(t) = \mathrm{E}(e^{tX}) = \sum_{p=0}^{\infty}\mathrm{W}_{p+1}\mathrm{M}_{p+1}(t)\ ,$$

where $\mathrm{M}_{p+1}(t)$ is the moment generating function of the Exp-G random variable Y_{p+1}. Figure 1 and 2 gives the skewness and kurtosis plots of TLOBX-W distribution for different parameter values.

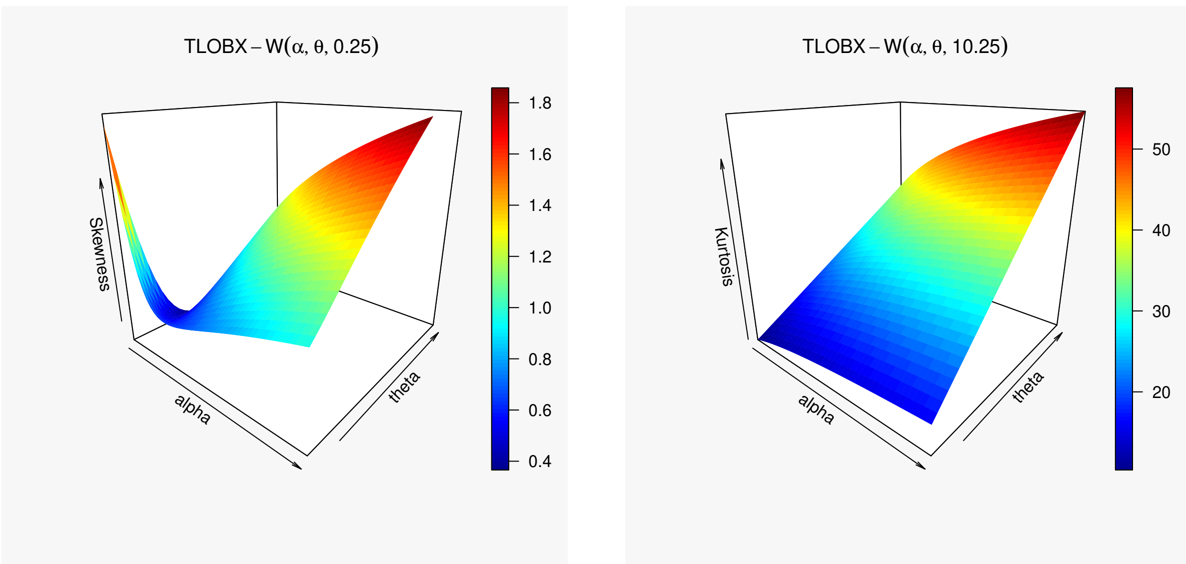


Figure 1. Skewness and kurtosis plots for the TLOBX-W distribution for different parameter values

Holding λ constant in TLOBX-W distribution, large values of α and θ leads to large skewness values, whereas kurtosis gets large with large values of θ as shown in Figure 1. Holding θ in Figure 2, skewness and kurtosis increases as α and λ increases.

3.4. Incomplete Moments

Incomplete moments are needed for the derivation of Bonferroni, Lorenz and Zenga curves. The s^{th} incomplete moments, denoted as $\eta_s(t)$, is given as

$$\eta_s(t) = \int_{-\infty}^{t} x^s \mathrm{f}(x;\alpha,\theta,\xi)dx.$$

Using representation in Equation (7), we get

$$\eta_s(t) = \sum_{p=0}^{\infty} \mathrm{W}_{p+1} \int_{-\infty}^{t} x^s \mathrm{h}_{p+1}(x;\xi)dx, \tag{10}$$

where $\int_{-\infty}^{t} x^s \mathrm{h}_{p+1}(x;\xi)dx$ is the s^{th} incomplete moment of the Exp-G random variable Y_{p+1}. Setting $s=1$ in Equation (10) we get the first incomplete moments of the TLOBX-G family.

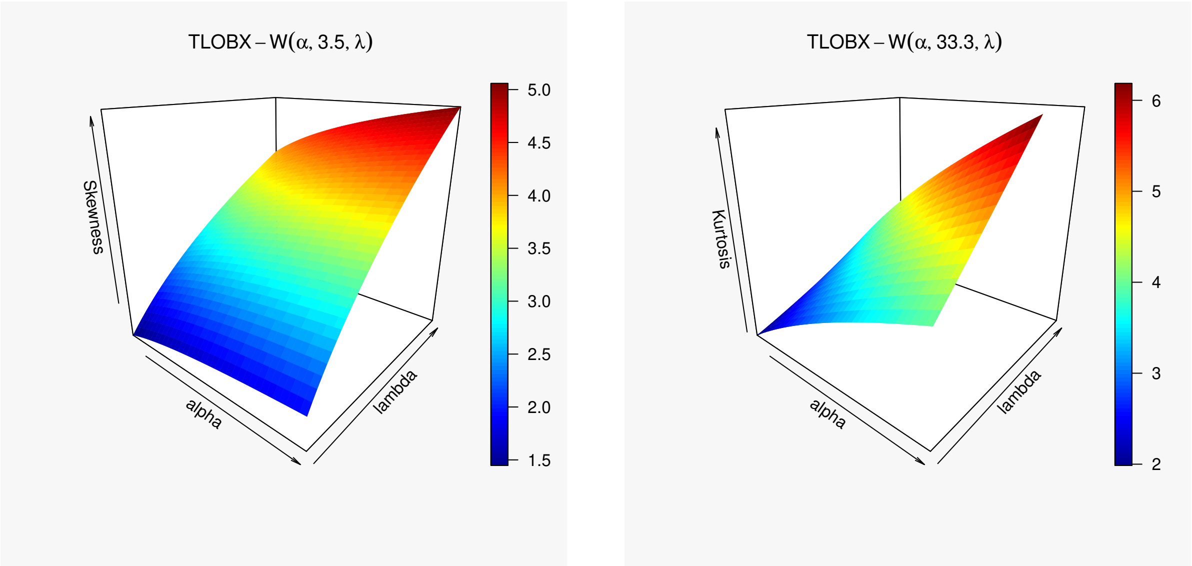


Figure 2. Skewness and kurtosis plots for the TLOBX-W distribution for different parameter values

3.5. Rényi Entropy

The Rényi entropy, which is an extension of the Shannon entropy is defined as

$$\mathrm{I}_R(v) = \frac{1}{1-v}\log\left[\int_0^{\infty} f^v(x;\alpha,\theta,\xi)dx\right],$$

where $v>0,\ v\neq 1$. Expanding $f^v(x;\alpha,\theta,\xi)$ using generalized binomial series and exponential series representation as shown in Appendix C, we get the Rényi entropy for the TLOBX-G family as

$$\mathrm{I}_R(v)=\frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}W_{\frac{p}{v}+1}\exp\left[(1-v)I_{REG}\right]\right],$$

where $I_{REG}=(1-v)^{-1}\log\left[\int_0^{\infty}\left(\left[\frac{p}{v}+1\right]\mathrm{g}(x;\xi)\mathrm{G}^{\frac{p}{v}}(x;\xi)\right)^v dx\right]$ is the Rényi entropy of Exp-G family with power parameter $(\frac{p}{v}+1)$ and

$$W_{\frac{p}{v}+1}=(4\alpha\theta)^v\sum_{i,j,k,m,n=0}^{\infty}\frac{(-1)^{i+j+k+m+n+p}(k+v)^m}{\left[\frac{p}{v}+1\right]^v m!}\times\binom{v(\alpha-1)}{i}\binom{2i+v}{j}\binom{\theta(j+v)-v}{k}\binom{2m+v}{n}\binom{n-(2m+3v)}{p}.$$

3.6. Order Statistics

Let $X_1,X_2,X_3,...,X_n$ be independent and identically distributed TLOBX-G random variables of size n. The pdf of the i^{th} order statistics for the TLOBX-G family is given as

$$\mathrm{f}_{i:n}(x;\alpha,\theta,\xi)=\sum_{q=0}^{\infty}\mathrm{W}_{q+1}\mathrm{h}_{q+1}(x;\xi), \tag{11}$$

(which follows from the derivations in Appendix D), where

$$\mathrm{W}_{q+1}=\frac{4\alpha\theta n!}{(i-1)!(n-i)!}\sum_{h,j,k,m,p=0}^{\infty}\sum_{t=0}^{n-i}\frac{(-1)^{h+j+k+m+p+q+t}(k+1)^m}{(q+1)m!}\binom{n-i}{t}\times\binom{\alpha(t+i)-1}{h}\binom{2h+1}{j}\binom{\theta(j+1)-1}{k}\binom{2m+1}{p}\binom{p-(2m+3)}{q}$$

and $\mathrm{h}_{q+1}(x;\xi)=(q+1)\mathrm{g}(x;\xi)\mathrm{G}^{q+1-1}(x;\xi)$ is the Exp-G distribution of power parameter $(q+1)$.

3.7. Probability Weighted Moments

The $(r,s)^{th}$ probability weighted moments for the TLOBX-G family is given as

$$M_{r,s}=\mathrm{E}[X^r\mathrm{F}^s(X)]=\int_{-\infty}^{\infty}x^r\mathrm{f}(x)\mathrm{F}^s(x)dx,$$

where

$$\mathrm{f}(x)\mathrm{F}^s(x)=\frac{4\alpha\theta\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1}\times\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]\times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha(s+1)-1}.$$

The expansion of $\mathrm{f}(x)\mathrm{F}^s(x)$ is similar to the expansion of the pdf $\mathrm{f}(x;\alpha,\theta,\xi)$. Thus, using the results of the expansion of $\mathrm{f}(x;\alpha,\theta,\xi)$ obtained in Equation (18) in Appendix B, we can express $\mathrm{f}(x)\mathrm{F}^s(x)$ as

$$\begin{aligned}\mathrm{f}(x)\mathrm{F}^s(x) &= 4\alpha\theta \sum_{i,j,k,m,n,p=0}^{\infty} \frac{(-1)^{i+j+k+m+n+p}(k+1)^m}{m!}\binom{\alpha(s+1)-1}{i}\binom{2i+1}{j} \\ &\quad\times \binom{\theta(j+1)-1}{k}\binom{2m+1}{n}\binom{n-(2m+3)}{p}\mathrm{g}(x;\xi)\mathrm{G}^p(x;\xi) \\ &= \sum_{p=0}^{\infty} \mathrm{W}*_{p+1}\mathrm{h}_{p+1}(x;\xi)\ ,\end{aligned}$$

where $\mathrm{h}_{p+1}(x;\xi)$ has been discussed under Equation (7) and

$$\begin{aligned}\mathrm{W}*_{p+1} &= 4\alpha\theta \sum_{i,j,k,m,n=0}^{\infty} \frac{(-1)^{i+j+k+m+n+p}(k+1)^m}{(p+1)m!}\binom{\alpha(s+1)-1}{i}\binom{2i+1}{j} \\ &\quad\times \binom{\theta(j+1)-1}{k}\binom{2m+1}{n}\binom{n-(2m+3)}{p}\ .\end{aligned}$$

The $(r,s)^{th}$ probability weighted moments then becomes

$$M_{r,s} = \sum_{p=0}^{\infty} \mathrm{W}*_{p+1} \int_{-\infty}^{\infty} x^r \mathrm{h}_{p+1}(x;\xi)dx\ .$$

This shows that probability weighted moments of TLOBX-G family can be obtained from those of Exp-G family.

4. Some Special Cases

This section provides some special cases for the TLOBX-G family, where the baseline distributions are taken to be Weibull distribution, ℓog-ℓogistic distribution and Uniform distribution.

4.1. Topp-Leone odd Burr X-Weibull (TLOBX-W) Distribution

Suppose the baseline distribution is the Weibull distribution with pdf and cdf $\mathrm{g}(x;\lambda) = \lambda x^{\lambda-1}e^{-x^\lambda}$ and $\mathrm{G}(x;\lambda) = 1 - e^{-x^\lambda}$, then the pdf, cdf and hrf of the TLOBX-W distribution are

$$\begin{aligned}\mathrm{f}(x;\alpha,\theta,\lambda) &= \frac{4\alpha\theta\lambda x^{\lambda-1}(1-e^{-x^\lambda})}{e^{-2x^\lambda}}\exp\left[-\left(e^{x^\lambda}-1\right)^2\right]\left\{1-\exp\left[-\left(e^{x^\lambda}-1\right)^2\right]\right\}^{\theta-1} \\ &\quad\times\left[1-\left\{1-\exp\left[-\left(e^{x^\lambda}-1\right)^2\right]\right\}^{\theta}\right] \\ &\quad\times\left[1-\left[1-\left\{1-\exp\left[-\left(e^{x^\lambda}-1\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha-1},\end{aligned} \tag{12}$$

$$\mathrm{F}(x;\alpha,\theta,\lambda) = \left[1-\left[1-\left\{1-\exp\left[-\left(e^{x^\lambda}-1\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha} \tag{13}$$

and

$$\mathrm{h}(x;\alpha,\theta,\lambda)=\frac{4\alpha\theta\lambda x^{\lambda-1}(1-e^{-x^{\lambda}})}{e^{-2x^{\lambda}}}\exp\left[-\left(e^{x^{\lambda}}-1\right)^{2}\right]\left\{1-\exp\left[-\left(e^{x^{\lambda}}-1\right)^{2}\right]\right\}^{\theta-1}$$
$$\times\left[1-\left\{1-\exp\left[-\left(e^{x^{\lambda}}-1\right)^{2}\right]\right\}^{\theta}\right]$$
$$\times\left[1-\left[1-\left\{1-\exp\left[-\left(e^{x^{\lambda}}-1\right)^{2}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha-1}$$
$$\times\left\{1-\left[1-\left[1-\left\{1-\exp\left[-\left(e^{x^{\lambda}}-1\right)^{2}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha}\right\}^{-1},$$

respectively, where $x,\alpha,\theta,\lambda>0$. Figure 3 gives the plots of the pdf and hrf of TLOBX-W distribution for different parameter values.

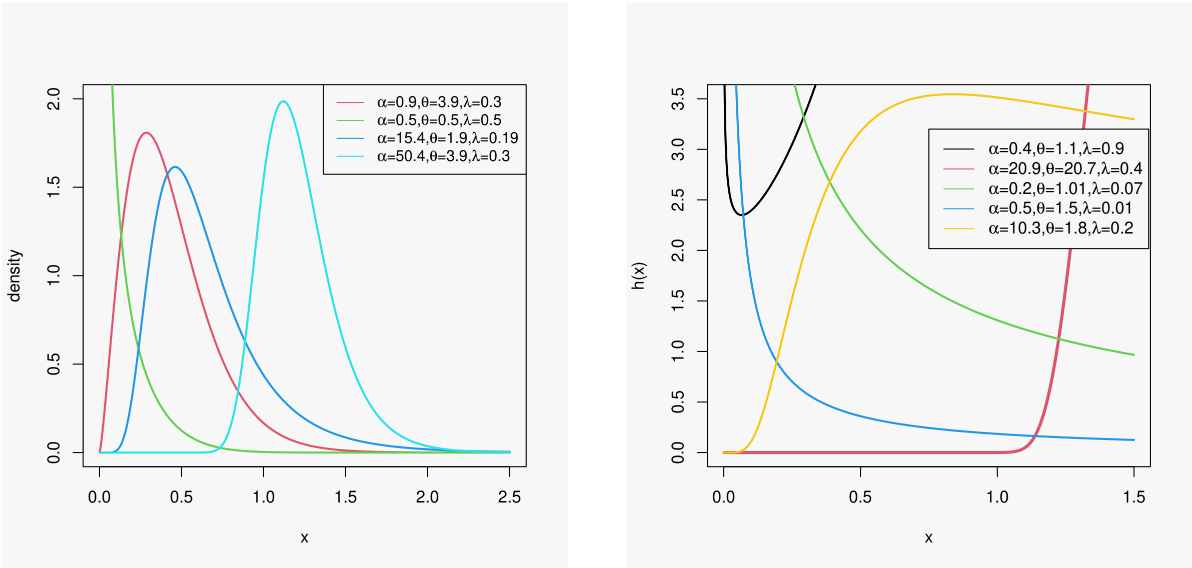


Figure 3. Pdf and hrf plots for TLOBX-W distribution

From Figure 3, the pdf of TLOBX-W distribution follows different shapes which include the reverse-J, right-skewed and almost symmetric. The hrf exhibit decreasing, increasing, bathtub and up-side down bathtub shapes.

4.2. Topp-Leone odd Burr X-Log-Logistic (TLOBX-L) Distribution

Taking the baseline distribution as the ℓog-ℓogistic distribution with pdf and cdf $\mathrm{g}(x;\beta)=\beta x^{\beta-1}(1+x^{\beta})^{-2}$ and $\mathrm{G}(x;\beta)=1-(1+x^{\beta})^{-1}$, we get the pdf, cdf and hrf of the TLOBX-L distribution as

$$\mathrm{f}(x;\alpha,\theta,\beta)=4\alpha\theta\beta x^{\beta-1}(1+x^{\beta})(1-(1+x^{\beta})^{-1})\exp\left[-x^{2\beta}\right]\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta-1}$$
$$\times\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]\left[1-\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha-1}, \quad (14)$$

$$\mathrm{F}(x;\alpha,\theta,\beta)=\left[1-\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha} \tag{15}$$

and

$$\begin{aligned}\mathrm{h}(x;\alpha,\theta,\beta)=&\ 4\alpha\theta\beta x^{\beta-1}(1+x^{\beta})(1-(1+x^{\beta})^{-1})\exp\left[-x^{2\beta}\right]\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta-1}\\ &\times\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]\left[1-\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha-1}\\ &\times\left\{1-\left[1-\left[1-\left\{1-\exp\left[-x^{2\beta}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha}\right\}^{-1},\end{aligned}$$

respectively, for $x,\alpha,\theta,\beta>0$. Figure 4 gives the plots of the pdf and hrf of TLOBX-L distribution for different parameter values.

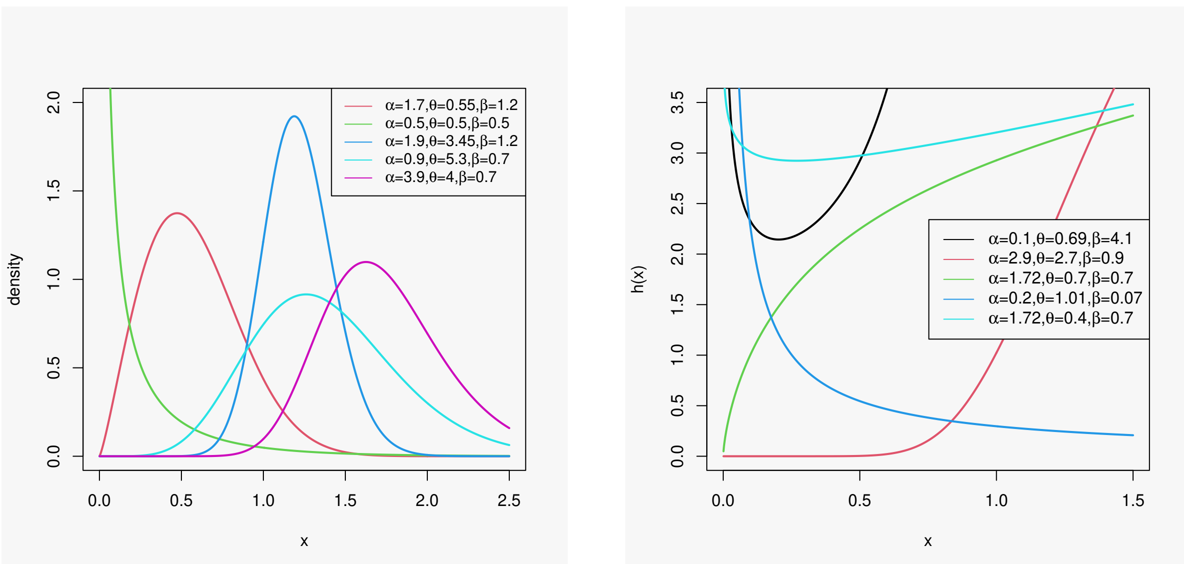


Figure 4. Pdf and hrf plots for TLOBX-L distribution

Figure 4 shows that the pdf of TLOBX-L distribution can be unimodal, decreasing, right-skewed and almost symmetric. The hrf can also can be decreasing, increasing and bathtub shaped depending on the selected parameter values.

4.3. Topp-Leone odd Burr X-Uniform (TLOBX-U) Distribution

Suppose the baseline distribution is uniform distribution with pdf and cdf $\mathrm{g}(x;\mu) = 1/\mu$ and $\mathrm{G}(x;\mu) = x/\mu$, for $x \leq \mu$, then the pdf, cdf and hrf of the TLOBX-U distribution are given by

$$\begin{aligned}\mathrm{f}(x;\alpha,\theta,\mu) = \frac{4\alpha\theta\mu x}{(x-\mu)^3}\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta-1} \\ \times\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right] \\ \times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha-1},\end{aligned} \tag{16}$$

$$\mathrm{F}(x;\alpha,\theta,\mu) = \left[1-\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha} \tag{17}$$

and

$$\begin{aligned}\mathrm{h}(x;\alpha,\theta,\mu) = \frac{4\alpha\theta\mu x}{(x-\mu)^3}\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta-1} \\ \times\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right] \\ \times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha-1} \\ \times\left\{1-\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha}\right\}^{-1},\end{aligned}$$

respectively, where $x, \alpha, \theta, \mu > 0$. Figure 5 gives the plots of the pdf and hrf of TLOBX-U distribution for different parameter values.

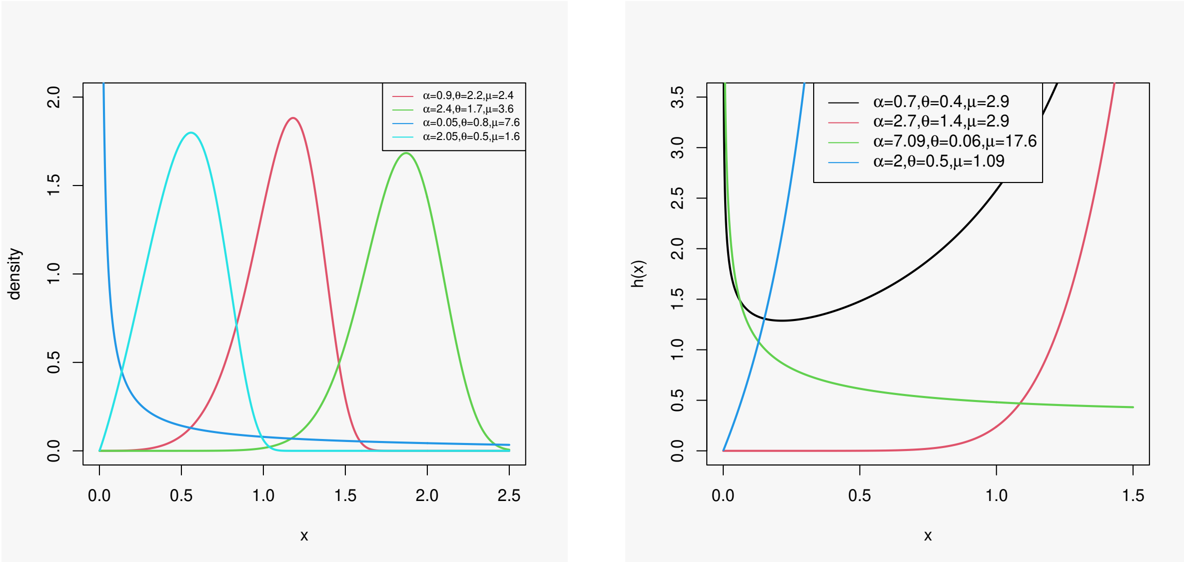


Figure 5. Pdf and hrf plots for TLOBX-U distribution

Figure 5 shows that the pdf of TLOBX-U distribution can be reverse-J, left-skewed, right-skewed and almost symmetric. The hrf can also be bathtub shaped, increasing and decreasing depending on the selected parameter values.

5. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) method is used to estimate the parameters of TLOBX-G family. If we let X_1, X_2, X_3,...,X_n represent a random sample of size n from the TLOBX-G family, we get the log-likelihood as

$$\begin{aligned}\ell n(L(\mathbf{\Psi})) =& n\ln(4\alpha\theta) + \sum_{i=1}^{n}\ln[\mathrm{g}(x_i;\xi)] + \sum_{i=1}^{n}\ln[\mathrm{G}(x_i;\xi)] - 3\sum_{i=1}^{n}\ln[\overline{\mathrm{G}}(x_i;\xi)] \\ & - \sum_{i=1}^{n} Z_i^2 + (\theta-1)\sum_{i=1}^{n}\ln\left(1-e^{-Z_i^2}\right) + \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{-Z_i^2}\right\}^{\theta}\right] \\ & + (\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left[1-\left\{1-e^{-Z_i^2}\right\}^{\theta}\right]^2\right],\end{aligned}$$

where $Z_i = \mathrm{G}(x_i;\xi)/\overline{\mathrm{G}}(x_i;\xi)$. The elements of the score vector given as $\frac{\ell n(L(\mathbf{\Psi}))}{\partial\alpha}$, $\frac{\ell n(L(\mathbf{\Psi}))}{\partial\theta}$ and $\frac{\ell n(L(\mathbf{\Psi}))}{\partial\xi_j}$ are provided in the appendix and R software via the *nlm* function will be used to estimate this parameters using a specific baseline distribution. If $\underline{\mathbf{0}} = (0,0,0)^T$ and $J(\hat{\mathbf{\Psi}})^{-1}$ represent the mean vector and the observed information matrix evaluated at $\hat{\mathbf{\Psi}}$, we can derive the confidence intervals and regions for our model parameters using the multivariate normal distribution $N_p(\underline{\mathbf{0}}, J(\hat{\mathbf{\Psi}})^{-1})$. The approximate $100(1-\varphi)\%$ confidence intervals (CIs) for α, θ and ξ_j are $\hat{\alpha} \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Psi}})}$, $\hat{\alpha} \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Psi}})}$ and $\hat{\xi}_j \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\xi_j\xi_j}(\hat{\mathbf{\Psi}})}$, respectively. The $\left(\frac{\varphi}{2}\right)^{th}$ percentile of the standard normal is represented with $\mathrm{Z}_{\frac{\varphi}{2}}$, and $I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Psi}})$, $I^{-1}_{\theta\theta}(\hat{\mathbf{\Psi}})$ and $I^{-1}_{\xi_j\xi_j}(\hat{\mathbf{\Psi}})$ are the diagonal elements of $I_n^{-1}(\hat{\mathbf{\Psi}}) = (nI(\hat{\mathbf{\Psi}}))^{-1}$.

6. Simulation Study

Taking Weibull as the baseline distribution, we present the results for the simulation study in this section which was performed $N = 3000$ times for sample size $n = 100, 200, 400, 800$ and 1600. The accuracy and consistency of the maximum likelihood estimators are evaluated using the mean estimates, root mean square errors (RMSEs) and average bias (Abias). Consistent estimators should have mean estimates getting closer to the actual parameter values and both RSMEs and Abias converges towards zero as n increases. RMSE and Abias for a parameter, say $\boldsymbol{\Psi}$, are given as:

$$\text{RMSE}(\hat{\boldsymbol{\Psi}}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\boldsymbol{\Psi}}_i - \boldsymbol{\Psi})^{\mathbf{2}}}{N}}, \quad \text{and} \quad \text{Abias}(\hat{\boldsymbol{\Psi}}) = \frac{\sum_{i=1}^{N}\hat{\boldsymbol{\Psi}}_i}{N} - \boldsymbol{\Psi}\,, \boldsymbol{\Psi} > \mathbf{0}.$$

Table 2. Monte Carlo Simulation Results for TLOBX-W Distribution: Mean Estimate, RMSE and Abias

		$(0.6, 0.8, 0.9)$			$(0.9, 0.6, 0.9)$		
Parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
α	100	1.0840	1.4568	0.4840	1.8359	3.3116	0.9359
	200	0.8564	0.7159	0.2564	1.2932	1.1967	0.3932
	400	0.7772	0.4834	0.1772	1.1568	0.7745	0.2568
	800	0.7242	0.3044	0.1242	1.0702	0.4865	0.1702
	1600	0.6805	0.2072	0.0805	1.0058	0.3225	0.1058
θ	100	0.9350	0.8698	0.1350	0.7397	0.7231	0.1397
	200	0.9048	0.6970	0.1048	0.7196	0.5962	0.1196
	400	0.8321	0.5201	0.0321	0.6515	0.4181	0.0515
	800	0.7844	0.3979	-0.0156	0.6109	0.3158	0.0109
	1600	0.7692	0.2875	-0.0308	0.5920	0.2256	-0.0080
λ	100	1.0851	0.5642	0.1851	1.0980	0.6070	0.1980
	200	1.0422	0.4590	0.1422	1.0349	0.4707	0.1349
	400	1.0220	0.3808	0.1220	1.0102	0.3758	0.1102
	800	0.9863	0.2835	0.0863	0.9788	0.2782	0.0788
	1600	0.9561	0.2047	0.0561	0.9531	0.2033	0.0531

Table 3. Monte Carlo Simulation Results for TLOBX-W Distribution: Mean Estimate, RMSE and Abias

		$(0.6, 0.6, 0.9)$			$(0.6, 0.9, 0.9)$		
Parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
α	100	1.0048	1.0610	0.4048	1.2062	2.1876	0.6062
	200	0.8081	0.5750	0.2081	0.8873	0.7768	0.2873
	400	0.7689	0.4118	0.1689	0.7784	0.5045	0.1784
	800	0.7472	0.3010	0.1472	0.7140	0.3061	0.1140
	1600	0.7303	0.2450	0.1303	0.6671	0.1943	0.0671
θ	100	0.7684	0.7862	0.1684	0.9927	0.9030	0.0927
	200	0.7517	0.6509	0.1517	0.9677	0.7262	0.0677
	400	0.6597	0.4816	0.0597	0.9260	0.5453	0.0260
	800	0.5852	0.3382	-0.0148	0.8845	0.4133	-0.0155
	1600	0.5489	0.2533	-0.0511	0.8726	0.2993	-0.0274
λ	100	1.0229	0.4847	0.1229	1.1285	0.6290	0.2285
	200	0.9843	0.3906	0.0843	1.0781	0.5151	0.1781
	400	0.9931	0.3400	0.0931	1.0236	0.3892	0.1236
	800	0.9873	0.2757	0.0873	0.9799	0.2773	0.0799
	1600	0.9858	0.2296	0.0858	0.9481	0.1899	0.0481

From Table 2 and 3, we can verify that as the sample size n increases, the mean estimate approaches the true parameter value whereas the RSME and Abias decrease with increasing sample size which is an indication that the parameter estimators are consistent.

7. Applications

In this section, we use two examples to show how TLOBX-W distribution is flexible in handling real life data. We evaluate the performance of TLOBX-W distribution against TLOBX-L distribution and other distributions using the goodness-of-fit statistics: -2 log-likelihood ($-2\ln(L)$), Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$, Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where the value of the likelihood function is represented with $L = L(\hat{\mathbf{\Psi}})$ and p is the number of estimated parameters. We also use the Cramér-Von Mises (W^*), Andersen-Darling (A^*), Sum of Squares (SS) and Kolmogorov-Smirnov (K-S) statistics together with its p-value for comparing the models. Sum of squares is given as:

$$SS = \sum_{j=1}^{n}\left[F(x_{(j)}; \hat{\alpha}, \hat{\theta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25}\right)\right]^2.$$

The best model in terms of performance should have lower values of -2LogL, AIC, AICC, BIC, W^*, A^*, K-S and SS together with a large p-value of K-S statistic. Evaluation of the models is also carried out using graphical presentations such as the fitted densities, empirical cdf, Kaplan-Meier and probability plots. TLOBX-W distribution was compared with TLOBX-L distribution (in Equation (14)), Gamma Weibull (GW) distribution [21], Weibull Lomax (WL) distribution [22], Type I Half Logistic Weibull (TIHLW) distribution [23], Topp-Leone Generalized Exponential (TLGE) distribution [24], Topp-Leone Exponential (TLE) distribution [25] and Generalized odd Burr X-Weibull (GOBXW) distribution [26]. The pdf of the non-nested models are given in Appendix E.

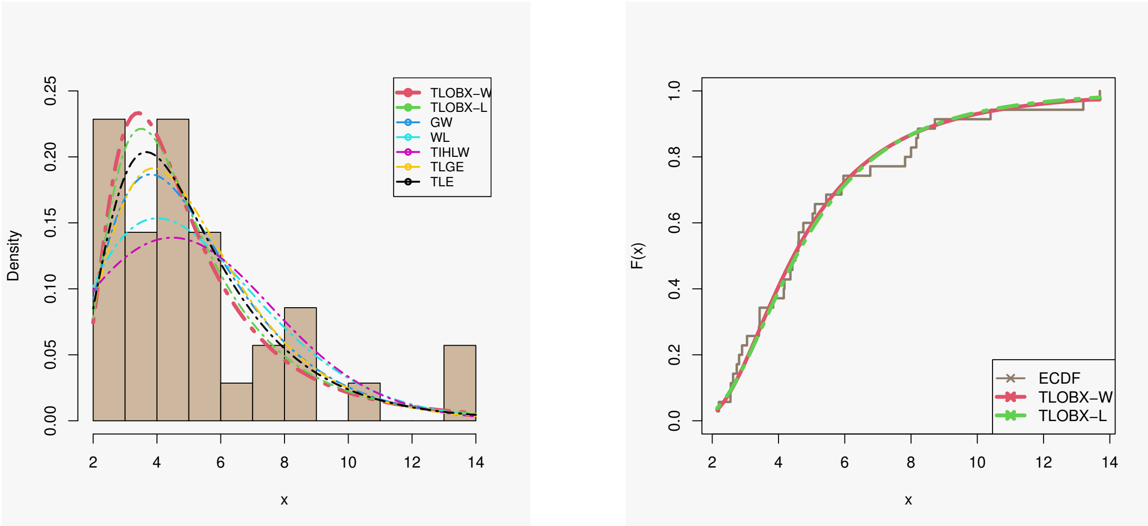
7.1. Growth Hormone Times Dataset

The first dataset has $n = 35$ observations of the estimated time since growth hormone medication until the children reached the target age [27]. The dataset is given as: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70. Table 4 presents the TLOBX-W model, TLOBX-L model and several existing non-nested models parameter estimates and the goodness-of-fit statistics for growth hormone times dataset.

Table 4. Parameter estimates and goodness-of-fit statistics of the TLOBX-W model and various models for the growth hormone times dataset

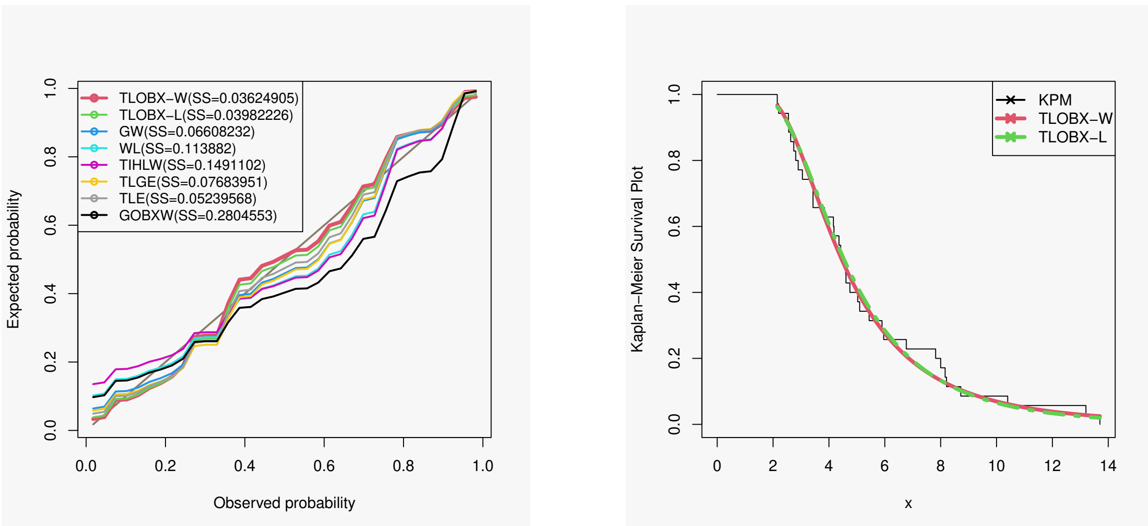
Distribution		Estimates		-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	α	θ	λ									
TLOBX-W	9.972×10^{2}	2.220	8.335×10^{-2}	155.2	161.2	161.9	165.8	0.0343	0.2475	0.0867	0.9547	0.0362
	(3.803×10^{-4})	(3.322×10^{-1})	(7.659×10^{-3})									
	α	θ	β									
TLOBX-L	4.943×10^{3}	1.022×10^{-1}	2.618×10^{-1}	155.5	161.5	162.2	166.1	0.0368	0.2669	0.0868	0.9544	0.0398
	(1.456×10^{-7})	(1.366×10^{-2})	(2.106×10^{-2})									
	k	β	α									
GW	3.876×10^{-1}	1.023×10^{1}	9.578×10^{-4}	158.0	164.0	164.8	168.6	0.0699	0.4711	0.0995	0.8788	0.0660
	(5.621×10^{-2})	(4.748×10^{-4})	(1.537×10^{-3})									
	a	b	α									
WL	0.2547	2.4038	4.2176	162.2	168.2	168.9	172.8	0.1265	0.8061	0.1329	0.5663	0.1138
	(0.1824)	(5.0114)	(2.4856)									
	a	b	α									
TIHLW	0.0920	0.8314	1.6583	166.6	172.6	173.4	177.3	0.1860	1.1441	0.1415	0.4846	0.1491
	(0.0422)	(0.0046)	(0.2148)									
	α	θ	β									
TLGE	2.320×10^{-3}	2.774×10^{3}	4.785×10^{-1}	158.1	164.1	164.8	168.7	0.0723	0.4875	0.1024	0.8561	0.0768
	(7.516×10^{-4})	(3.161×10^{-7})	(7.207×10^{-2})									
	α	θ	λ									
TLE	1.182×10^{4}	1.529×10^{-2}	5.0073	156.5	162.5	163.2	167.1	0.0505	0.3537	0.0869	0.9541	0.0523
	(1.517×10^{-6})	(3.290×10^{-3})	(1.0088)									
	α	β	λ									
GOBXW	56.0289	0.0897	0.5941	163.3	169.3	170.1	173.9	0.1232	0.8543	0.1835	0.1890	0.2804
	(30.2347)	(0.0422)	(0.0403)									

Both TLOBX-W and TLOBX-L models have the lowest values of -2LogL, AIC, AICC, BIC, W^*, A^*, K-S and SS together with the largest value of K-S p-value compared to other models as shown in Table 4. As a result, we can verify that TLOBX-W and TLOBX-L provides a better fit for the growth hormone times dataset. Figure 6 and 7 presents the fitted density plot, empirical cdf plot, probability plot and Kaplan-Meier survival plot for TLOBX-W model constructed from the growth hormone times data.



(a) (b)

Figure 6. Fitted density (a) and empirical cdf plot (b) for growth hormone times dataset



(a) (b)

Figure 7. Probability plot (a) and Kaplan-Meier plot (b) for growth hormone times dataset

Figures 6 and 7 show that TLOBX-W distribution fit the growth hormone times data better than other competing models. The estimated variance-covariance matrix of TLOBX-W model for the growth hormone times dataset is given by

$$\begin{bmatrix} 1.44 \times 10^{-7} & 1.26 \times 10^{-4} & 2.37 \times 10^{-6} \\ 1.26 \times 10^{-4} & 1.10 \times 10^{-1} & 2.07 \times 10^{-3} \\ 2.37 \times 10^{-6} & 2.07 \times 10^{-3} & 5.86 \times 10^{-5} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [9.97 \times 10^2 \pm 7.45 \times 10^{-4}], \theta \in [2.22 \pm 6.51 \times 10^{-1}]$ and $\lambda \in [8.33 \times 10^{-2} \pm 1.50 \times 10^{-2}]$.

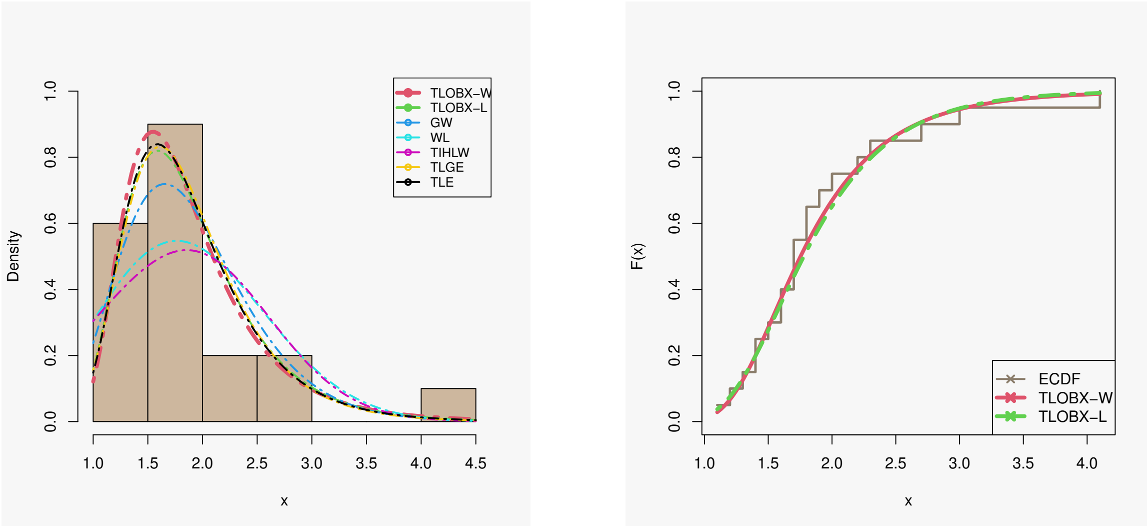
7.2. Relief Times Dataset

The second dataset that was analysed by [28] and deals with the relief times (in minutes) for 20 patients that were receiving analgesic given as: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. Table 5 presents the TLOBX-W model, TLOBX-L model and existing non-nested models parameter estimates and the goodness-of-fit statistics for relief times dataset.

Table 5. Parameter estimates and goodness-of-fit statistics of the TLOBX-W model and various models for the relief times dataset

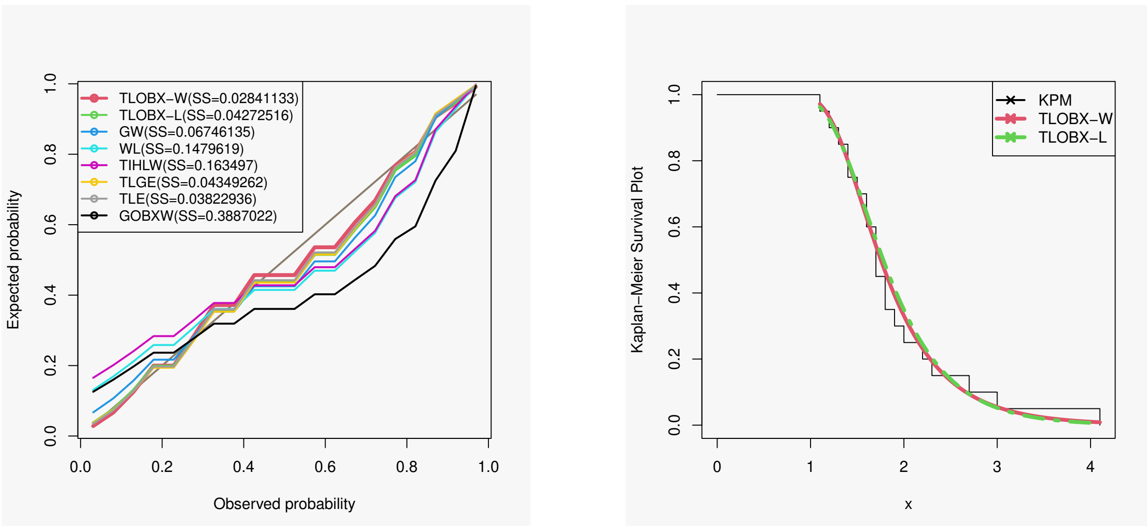
Distribution	Estimates			-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	α	θ	λ									
TLOBX-W	9.575×10^3	4.197×10^{-1}	1.544×10^{-1}	31.3	37.3	38.7	40.2	0.0364	0.2093	0.1141	0.9569	0.0362
	(1.343×10^{-6})	(6.090×10^{-2})	(2.088×10^{-2})									
	α	θ	β									
TLOBX-L	1.228×10^4	4.071×10^{-2}	4.992×10^{-1}	32.1	38.0	39.5	41.0	0.0492	0.2876	0.1343	0.8631	0.0427
	(9.700×10^{-9})	(5.607×10^{-3})	(6.008×10^{-2})									
	k	β	α									
GW	4.692×10^{-1}	2.063×10^1	5.513×10^{-4}	34.4	40.4	41.9	43.4	0.0866	0.5123	0.1541	0.7285	0.0674
	(4.562×10^{-2})	(1.151×10^{-4})	(5.463×10^{-4})									
	a	b	α									
WL	0.2652	5.9968	5.7855	39.5	45.5	47.0	48.5	0.1610	0.9508	0.1805	0.5319	0.1479
	(0.2482)	(15.8262)	(4.3755)									
	a	b	α									
TIHLW	0.1365	1.9594	2.3168	42.1	48.0	49.5	51.0	0.1982	1.1570	0.1707	0.1707	0.1634
	(0.0541)	(0.0037)	(0.3740)									
	α	θ	β									
TLGE	6.798×10^{-4}	5.361×10^4	2.230	32.5	38.5	40.0	41.4	0.0556	0.3272	0.1347	0.8609	0.0434
	(4.415×10^{-4})	(2.146×10^{-7})	(4.080×10^{-1})									
	α	θ	λ									
TLE	1.222×10^4	4.285×10^{-2}	9.733×10^{-1}	32.1	38.0	39.5	41.0	0.0483	0.2823	0.1297	0.8896	0.0382
	(6.728×10^{-8})	(1.716×10^{-2})	(2.189×10^{-1})									
	α	β	λ									
GOBXW	31.2727	0.0820	1.0156	43.1	49.0	50.5	52.0	0.1844	1.1389	0.2668	0.1160	0.3887
	(17.8250)	(0.0455)	(0.0914)									

As shown in Table 5, TLOBX-W model have the lowest values of -2LogL, AIC, AICC, BIC, W^*, A^*, K-S and SS together with the largest value of K-S p-value compared to other models. As for the TLOBX-L model, TLE model had lower values than TLOBX-L model on some of the goodness-of-fit statistics such as the W^*, A^*, K-S but it also provided an excellent fit. From Table 4, we can conclude that TLOBX-W provided a better fit for the relief times dataset. In Figures 8 and 9, we present the fitted density plot, empirical cdf plot, probability plot and Kaplan-Meier survival plot for TLOBX-W model constructed from this dataset.



(a) (b)

Figure 8. Fitted density (a) and empirical cdf plot (b) for relief times dataset



(a) (b)

Figure 9. Probability plot (a) and Kaplan-Meier plot (b) for relief times dataset

Figures 6 and 7 show that TLOBX-W distribution provides a better fit to the relief times dataset better than other competing models. The estimated variance-covariance matrix of TLOBX-W model for the relief times dataset is given by

$$\begin{bmatrix} 1.80 \times 10^{-12} & 8.18 \times 10^{-8} & 1.76 \times 10^{-8} \\ 8.18 \times 10^{-8} & 3.70 \times 10^{-3} & 8.01 \times 10^{-4} \\ 1.76 \times 10^{-8} & 8.01 \times 10^{-4} & 4.36 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [9.58 \times 10^3 \pm 2.63 \times 10^{-6}], \theta \in [4.20 \times 10^{-1} \pm 1.19 \times 10^{-1}]$ and $\lambda \in [1.54 \times 10^{-1} \pm 4.09 \times 10^{-2}]$.

8. Concluding Remarks

In this paper, we have developed a new family of distributions called the Topp-Leone odd Burr X-G (TLOBX-G) family of distributions from the Topp-Leone-G (TL-G) and odd Burr X-G (OBX-G) families of distributions. We provided some of its structural properties which include the quantile function, density expansion, ordinary moments, central moments, moment generating function, incomplete moments, Rényi entropy, order statistics, and probability weighted moments. We also considered some special cases by taking the baseline as the Weibull distribution, ℓog-ℓogistic distribution, and uniform distribution. The parameters of the TLOBX-G family of distributions were estimated using the maximum likelihood technique. Based on Topp-Leone odd Burr X-Weibull (TLOBX-W) distribution, we showed that its parameters were consistent and accurate. Monte Carlo simulation results showed that mean estimates of the results approached the true parameters and both root mean square errors (RMSEs) and average bias (Abias) decreased with increasing sample size. We illustrated the flexibility, applicability, and usefulness of this family of distributions using TLOBX-W distribution and two real-life datasets. The TLOBX-W distribution had the lowest values of -2LogL, AIC, AICC, BIC, W^*, A^*, K-S, and SS together with the largest value of K-S p-value compared to the Topp-Leone odd Burr X-Log-ℓogistic (TLOBX-L) model and other compared non-nested models indicating that it provides a better fit in real life.

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A. Components of the Score Vector

The elements of the score vector for the TLOBX-G family are given as:

$$\frac{\ell n(L(\boldsymbol{\Psi}))}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^{\theta} \right]^2 \right] ,$$

$$\begin{aligned}\frac{\ell n(L(\boldsymbol{\Psi}))}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln \left(1 - e^{-Z_i^2} \right) - \sum_{i=1}^{n} \frac{\left\{ 1 - e^{-Z_i^2} \right\}^{\theta} \ln \left\{ 1 - e^{-Z_i^2} \right\}}{1 - \left\{ 1 - e^{-Z_i^2} \right\}^{\theta}} \\ + (\alpha - 1) \sum_{i=1}^{n} \frac{2 \left\{ 1 - e^{-Z_i^2} \right\}^{\theta} \ln \left\{ 1 - e^{-Z_i^2} \right\} \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^{\theta} \right]}{1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^{\theta} \right]^2} ,\end{aligned}$$

and

$$\frac{\ell n(L(\boldsymbol{\Psi}))}{\partial\xi_j} = \sum_{i=1}^{n}\frac{\mathrm{g}'(x;\xi)}{\mathrm{g}(x;\xi)} + \sum_{i=1}^{n}\frac{\mathrm{G}'(x;\xi)}{\mathrm{G}(x;\xi)} - 3\sum_{i=1}^{n}\frac{\overline{\mathrm{G}}\,'(x;\xi)}{\overline{\mathrm{G}}(x;\xi)} - 2\sum_{i=1}^{n}Z_i\frac{\partial Z_i}{\partial\xi_j}$$
$$+ (\theta-1)\sum_{i=1}^{n}\frac{2Z_i e^{-Z_i^2}}{1-e^{-Z_i^2}}\frac{\partial Z_i}{\partial\xi_j} - \frac{2\theta Z_i\, e^{-Z_i^2}\left\{1-e^{-Z_i^2}\right\}^{\theta-1}}{1-\left\{1-e^{-Z_i^2}\right\}^{\theta}}\frac{\partial Z_i}{\partial\xi_j}$$
$$+ \frac{4\theta Z_i\, e^{-Z_i^2}\left\{1-e^{-Z_i^2}\right\}^{\theta-1}\left[1-\left\{1-e^{-Z_i^2}\right\}^{\theta}\right]}{1-\left[1-\left\{1-e^{-Z_i^2}\right\}^{\theta}\right]^2}\frac{\partial Z_i}{\partial\xi_j},$$

where $Z_i = \mathrm{G}(x_i;\xi)/\overline{\mathrm{G}}(x_i;\xi)$, $\mathrm{g}'(x;\xi) = \partial\mathrm{g}(x;\xi)/\partial\xi_j$, $\mathrm{G}'(x;\xi) = \partial\mathrm{G}(x;\xi)/\partial\xi_j$ and $\overline{\mathrm{G}}\,'(x;\xi) = \partial\overline{\mathrm{G}}(x;\xi)/\partial\xi_j$ for ξ_j representing the j^{th} element of the vector of baseline parameters ξ.

B. Series Expansion Proofs

Note that

$$\mathrm{f}(x;\alpha,\theta,\xi) = \frac{4\alpha\theta\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1}$$
$$\times\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^{2i+1}$$
$$= \frac{4\alpha\theta\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\alpha-1}{i}\binom{2i+1}{j}$$
$$\times\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta(j+1)-1}$$
$$= \frac{4\alpha\theta\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}^3(x;\xi)}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{2i+1}{j}\binom{\theta(j+1)-1}{k}$$
$$\times\exp\left[-(k+1)\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right].$$

Applying the exponential series representation $\exp(-z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}$, the density further expands to

$$\mathrm{f}(x;\alpha,\theta,\xi) = 4\alpha\theta\sum_{i,j,k,m=0}^{\infty}\frac{(-1)^{i+j+k+m}(k+1)^m}{m!}\binom{\alpha-1}{i}$$
$$\times\binom{2i+1}{j}\binom{\theta(j+1)-1}{k}\frac{\mathrm{g}(x;\xi)\mathrm{G}^{2m+1}(x;\xi)}{\bar{\mathrm{G}}^{2m+3}(x;\xi)}.$$

Note that $\mathrm{G}^{2m+1}(x;\xi) = \left[1-\bar{\mathrm{G}}(x;\xi)\right]^{2m+1}$, hence applying the generalized binomial series representation, we get

$$\begin{aligned}
\mathrm{f}(x;\alpha,\theta,\xi) &= 4\alpha\theta \sum_{i,j,k,m,n=0}^{\infty} \frac{(-1)^{i+j+k+m+n}(k+1)^m}{m!} \binom{\alpha-1}{i}\binom{2i+1}{j} \\
&\quad \times \binom{\theta(j+1)-1}{k}\binom{2m+1}{n} \mathrm{g}(x;\xi)\bar{\mathrm{G}}^{n-(2m+3)}(x;\xi) \\
&= 4\alpha\theta \sum_{i,j,k,m,n,p=0}^{\infty} \frac{(-1)^{i+j+k+m+n+p}(k+1)^m}{m!} \binom{\alpha-1}{i}\binom{2i+1}{j} \\
&\quad \times \binom{\theta(j+1)-1}{k}\binom{2m+1}{n}\binom{n-(2m+3)}{p} \mathrm{g}(x;\xi)\mathrm{G}^{p}(x;\xi). \qquad (18)
\end{aligned}$$

C. Rényi Entropy Proofs

Note that

$$\begin{aligned}
\mathrm{f}^{v}(x;\alpha,\theta,\xi) &= \frac{(4\alpha\theta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{\bar{\mathrm{G}}^{3v}(x;\xi)} \exp\left[-v\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right] \\
&\quad \times \left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{v(\theta-1)} \sum_{i=0}^{\infty}(-1)^i\binom{v(\alpha-1)}{i} \\
&\quad \times \left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^{2i+v} \\
&= \frac{(4\alpha\theta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{\bar{\mathrm{G}}^{3v}(x;\xi)} \exp\left[-v\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right] \sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{v(\alpha-1)}{i} \\
&\quad \times \binom{2i+v}{j}\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta(j+v)-v} \\
&= \frac{(4\alpha\theta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{\bar{\mathrm{G}}^{3v}(x;\xi)} \exp\left[-v\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right] \sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{v(\alpha-1)}{i} \\
&\quad \times \binom{2i+v}{j}\binom{\theta(j+v)-v}{k}\exp\left[-(k+v)\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^2\right] \\
&= (4\alpha\theta)^v \sum_{i,j,k,m=0}^{\infty} \frac{(-1)^{i+j+k+m}(k+v)^m}{m!}\binom{v(\alpha-1)}{i} \\
&\quad \times \binom{2i+v}{j}\binom{\theta(j+v)-v}{k}\frac{\mathrm{g}^v(x;\xi)\mathrm{G}^{2m+v}(x;\xi)}{\bar{\mathrm{G}}^{2m+3v}(x;\xi)}.
\end{aligned}$$

If we let $\mathrm{G}^{2m+v}(x;\xi) = \left[1-\bar{\mathrm{G}}(x;\xi)\right]^{2m+v}$ and apply the generalized binomial series representation we get

$$
\begin{aligned}
\mathrm{f}^{v}(x;\alpha,\theta,\xi) &= (4\alpha\theta)^{v}\sum_{i,j,k,m,n=0}^{\infty}\frac{(-1)^{i+j+k+m+n}(k+v)^{m}}{m!}\binom{v(\alpha-1)}{i}\binom{2i+v}{j}\\
&\quad\times\binom{\theta(j+v)-v}{k}\binom{2m+v}{n}\mathrm{g}^{v}(x;\xi)\bar{\mathrm{G}}^{n-(2m+3v)}(x;\xi)\\
&= (4\alpha\theta)^{v}\sum_{i,j,k,m,n,p=0}^{\infty}\frac{(-1)^{i+j+k+m+n+p}(k+v)^{m}}{m!}\binom{v(\alpha-1)}{i}\binom{2i+v}{j}\\
&\quad\times\binom{\theta(j+v)-v}{k}\binom{2m+v}{n}\binom{n-(2m+3v)}{p}\mathrm{g}^{v}(x;\xi)\mathrm{G}^{p}(x;\xi)\,.
\end{aligned}
$$

Plugging the above expansion of $\mathrm{f}^{v}(x;\alpha,\theta,\xi)$ in $\mathrm{I}_R(v)$, we get

$$
\begin{aligned}
\mathrm{I}_R(v) &= \frac{1}{1-v}\log\Bigg[(4\alpha\theta)^{v}\sum_{i,j,k,m,n,p=0}^{\infty}\frac{(-1)^{i+j+k+m+n+p}(k+v)^{m}}{\left[\frac{p}{v}+1\right]^{v}m!}\binom{v(\alpha-1)}{i}\binom{2i+v}{j}\\
&\quad\times\binom{\theta(j+v)-v}{k}\binom{2m+v}{n}\binom{n-(2m+3v)}{p}\int_0^{\infty}\left(\left[\frac{p}{v}+1\right]\mathrm{g}(x;\xi)\mathrm{G}^{\frac{p}{v}}(x;\xi)\right)^{v}dx\Bigg]\,.
\end{aligned}
$$

D. Distribution of Order Statistics Proofs

The pdf of the i^{th} order statistic is given by

$$
\begin{aligned}
\mathrm{f}_{i:n}(x;\alpha,\theta,\xi) &= \frac{n!\mathrm{f}(x;\alpha,\theta,\xi)}{(i-1)!(n-i)!}\sum_{t=0}^{n-i}(-1)^{t}\binom{n-i}{t}\mathrm{F}^{t+i-1}(x;\alpha,\theta,\xi)\\
&= \frac{4\alpha\theta n!\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{(i-1)!(n-i)!\bar{\mathrm{G}}^{3}(x;\xi)}\sum_{t=0}^{n-i}(-1)^{t}\binom{n-i}{t}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\\
&\quad\times\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta-1}\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta}\right]\\
&\quad\times\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\bar{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha(t+i)-1}\,.
\end{aligned}
$$

The expansion of the order statistics is exactly the same as that of series expansion in Section 3.2. Thus, we can express the order statistics in the form of Equation (18) in Appendix B as

$$
\begin{aligned}
\mathrm{f}_{i:n}(x;\alpha,\theta,\xi) &= \frac{4\alpha\theta n!}{(i-1)!(n-i)!}\sum_{h,j,k,m,p,q=0}^{\infty}\sum_{t=0}^{n-i}\frac{(-1)^{h+j+k+m+p+q+t}(k+1)^{m}}{m!}\\
&\quad\times\binom{n-i}{t}\binom{\alpha(t+i)-1}{h}\binom{2h+1}{j}\binom{\theta(j+1)-1}{k}\binom{2m+1}{p}\\
&\quad\times\binom{p-(2m+3)}{q}\mathrm{g}(x;\xi)\mathrm{G}^{q}(x;\xi)\,.
\end{aligned}
$$

E. Non-nested Distributions

The non-nested models used for comparisons are:

$$f_{GW}(x)=\frac{k\lambda^{-k-\beta}x^{\beta+k-1}e^{-\lambda^{-k}x^{k}}}{\Gamma(1+\frac{\beta}{k})}\ ,$$

for $k,\beta,\lambda,x>0$,

$$f_{WL}(x)=\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1}\exp\left(-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right),$$

for $a,b,\alpha,x>0$,

$$f_{TIHLW}(x) \quad = \quad \frac{2\lambda\delta\gamma x^{\gamma-1}e^{-\lambda\delta x^{\gamma}}}{[1+e^{-\lambda\delta x^{\gamma}}]^{2}},$$

for $\lambda,\delta,\gamma,x>0$,

$$f_{TLGE}(x;\alpha,\beta,\lambda)=2\alpha\beta\lambda e^{-\lambda x}(1-(1-e^{-\lambda x})^{\beta}(1-e^{-\lambda x})^{\beta\alpha-1}(2-(1-e^{-\lambda x})^{\beta}))^{\alpha-1},$$

for $\alpha,\beta,\lambda,x>0$,

$$f_{TLE}(x;\alpha,\theta,\lambda)=\frac{\frac{2\alpha\theta}{\lambda}\exp\left[-\frac{x}{\lambda}\right]\left[1-\exp\left[-\frac{x}{\lambda}\right]\right]^{\theta\alpha-1}\left[1-\left[1-\exp\left[-\frac{x}{\lambda}\right]\right]^{\theta}\right]}{\left[2-\left[1-\exp\left[-\frac{x}{\lambda}\right]\right]^{\theta}\right]^{-(\alpha-1)}}$$

for $\alpha,\theta,\lambda,x>0$, and

$$\begin{aligned}f_{GOBXW}(x;\alpha,\beta,\lambda)&=\frac{2\alpha\beta\lambda x^{\lambda-1}\exp -x^{\lambda}\left[1-\exp -x^{\lambda}\right]^{2\alpha-1}}{\left[1-\left[1-\exp -x^{\lambda}\right]^{\alpha}\right]^{3}}\\&\times\exp\left\{-\left[\frac{\left[1-\exp -x^{\lambda}\right]^{\alpha}}{1-\left[1-\exp -x^{\lambda}\right]^{\alpha}}\right]^{2}\right\}\\&\times\left\{1-\exp\left\{-\left[\frac{\left[1-\exp -x^{\lambda}\right]^{\alpha}}{1-\left[1-\exp -x^{\lambda}\right]^{\alpha}}\right]^{2}\right\}\right\}^{\beta-1},\end{aligned}$$

for $\alpha,\beta,\lambda,x>0$.