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Estimation of the Multicomponent Stress-Strength Reliability Model Under the Topp-Leone Distribution: Applications, Bayesian and Non-Bayesian Assessment

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Abstract The advantages of applying multicomponent stress-strength models lie in their ability to provide a comprehensive and accurate analysis of system reliability under real-world conditions. By accounting for the interactions between different stress components and identifying critical weaknesses, engineers can make informed decisions, leading to safer and more reliable designs. The primary emphasis of this research is placed on the Bayesian and classical estimations of a multicomponent stress-strength reliability model that is derived from the bounded Topp Leone distribution. It is presumable that both stress and strength follow a Topp Leone distribution, but the shape parameters of each variable differ, and the scale parameters (which determine where the variable is bounded) remain the same. Statisticians utilize approaches such as maximum likelihood paired with parametric and non-parametric bootstrap, as well as Bayesian methods, in order to evaluate the dependability of a system. Bayesian methods are also utilized. Simulation studies are carried out with the intention of establishing the degree of precision that may be achieved by employing the various methods of estimating. For the sake of this example, two genuine data sets are dissected and examined in detail.

Keywords Topp Leone distribution, Multicomponent stress-strength, Maximum likelihood estimation, Bootstrap, Gibbs sampling.

AMS 2010 subject classifications 62F10, 62F40, 62F15

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1. Introduction

In engineering, industry, and various real-life applications, ensuring the reliability and safety of complex systems is of paramount importance. The multicomponent stress-strength reliability model is a powerful statistical framework utilized to assess the likelihood of a system's performance meeting specified safety criteria under various stress and strength conditions. This model provides a robust means of understanding the interplay between stress and strength variables, thus enabling accurate predictions of system reliability. The classical stress-strength reliability model, often based on standard distribution assumptions, has proven to be effective in many scenarios. However, its limitation in accommodating more complex and diverse data distributions has spurred the exploration of novel statistical approaches. One such promising extension is the utilization of the Topp-Leone (TL) distribution,

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a versatile family of distributions that encompasses several well-known distributions as special cases. The TL distribution presents a flexible framework to capture various data characteristics, making it an attractive choice for modeling complex systems. The primary objective of this paper is to propose and investigate the estimation of the multicomponent stressstrength reliability model under the TL distribution. This study seeks to address the limitations of traditional methods and enhance the accuracy and applicability of reliability assessments in a wide range of disciplines. The first goal of this research is to develop a comprehensive multicomponent stress-strength reliability model based on the TL distribution. The formulation will aim to capture the joint distribution of stress and strength variables, allowing for a more nuanced and realistic representation of system behavior under different loading conditions. By leveraging the TL distribution, this study will explore the practical applications of the proposed model across diverse industries. From civil engineering and aerospace to material science and manufacturing, the research will investigate how the model can be effectively employed to evaluate system reliability and safety in critical applications. Another critical aspect of this research is to conduct a comparative analysis of Bayesian and non-Bayesian approaches for estimating the parameters of the multicomponent stress-strength reliability model. The investigation will consider the strengths, limitations, and computational efficiency of each approach, aiding decision-makers in choosing the most suitable methodology for their specific needs. The significance of this research lies in its potential to advance the field of reliability modeling by offering a more sophisticated and flexible statistical framework. The multicomponent stress-strength reliability model under the TL distribution can better accommodate diverse data distributions and improve the accuracy of reliability estimates, thereby enhancing the safety and performance evaluation of complex systems. The outcomes of this study will not only contribute to the theoretical aspects of reliability modeling but also have practical implications in engineering, industrial settings, and beyond. The proposed model's versatility and applicability may lead to more reliable design decisions, risk assessments, and optimization strategies, ultimately contributing to improved performance, safety, and resource efficiency in various domains. In the subsequent sections of this paper, we will delve into the methodology, data analysis, results, and discussions, providing a comprehensive understanding of the estimation of the multicomponent stress-strength reliability model under the TL distribution and its wide-ranging applications.

The Topp-Leone (TL) distribution is a univariate continuous model with bounded support which was first proposed by [1]. Many authors studied this distribution. This distribution has received a lot of attention from researchers recently, some of them made generalizations for the distribution itself based on the baseline TL distribution, some of them derived new G families based on the basic distribution, and some of them applied it in the areas of statistical modeling and tests of statistical hypotheses in the case of complete data and in the case of censored data.

Stress-strength analysis examines the interaction between the stresses produced on the materials and their strength; the term ”materials” here refers not just to finished products or component components but also to a complete system. Reliability engineering uses a technology called stress-strength analysis. The distribution of environmental stressors and the distribution of component strengths have a mean and a standard deviation, respectively. The likelihood of failure is determined by how these distributions overlap. Stress-strength interference is another name for this overlapping phenomenon.

Recently, [2] studied it's hazard rate, reversed hazard rate, mean residual life and the expected inactivity time, [3] presented the asymptotic distribution of it's order statistics, [4] and [5] obtained expression for the moments of it's order statistics, [6] introduced an admissible minimax estimator for the shape parameter of the TL distribution and [7] obtained estimation of the shape parameter of the TL distribution based on progressive type-**II** censored samples. Furthermore, [10] studied the maximum likelihood and uniformly minimum variance unbiased estimations of the reliability $\mathbf{R} = \Pr(X > Y)$ when X and Y are independent random variables (rv) following the TL model with scale parameter equal to 1 percent and different shape parameters. [11] used maximum likelihood and Bayesian approaches to obtain the estimation of $\Pr(X > Y)$ based on a set of lower record values from TL model with scale parameter equal to 1 and different shape parameters. In fact, the flexibility of the TL distribution was the main motivation that prompted many researchers to study and analyze this distribution, and to provide new generalizations of the TL model that have high flexibility, as is the case in the baseline distribution. The statistical literature is not rich in works on the topic of multicomponent stress strength. However, we will present a survey of all the works that dealt with this problem through study and analysis, from near or from afar. For the Bayesian

and classical inference of the reliability model in multicomponent stress-strength utilixing the generalized logistic distribution see [13]. For the Bayesian and classical inference for generalized stress-strength parameter under generalized logistic distribution with simulations and applications see [14]. For the reliability estimation for the remained stress-strength model under the generalized exponential lifetime distribution with applications see [15]. For the generalized stress-strength and generalized multicomponent stress-strength Models with applications to real-life data sets see [16]
A stress strength model, which is largely used in reliability engineering but is also applied in economics, psychology, and medicine, compares the strength and stresses on a system. One of the simple stress-strength models for a system is defined as strength is greater than the stress imposed on the system. Thus, the probability of this event a good deal of, $\mathbf{R} = \Pr(X > Y)$, where X and Y are the random strength and the random stress applied to an same intervals, is of interest. An excellent review of theory and applications in this area can be found in [19].
Upon studying the multicomponent system (system with more than one component), three methods of stress-strengh models can be used. These systems are a series system or a parallel system or a complex combination of series and parallel systems. The series and parallel systems are special cases of a general class of systems called k-out-of-n systems which are of the following two types:

1. The failure of a system occurs by the failure of the k^{th} components, denoted by k-out-of-n.
2. The survive of a system continues where at least k components are working and is denoted by k-out-of-n.

Some interesting examples of multicomponent system are presented in [20] Suppose that we have a system with k identical components functions when at least $s|_{1\leq s\leq k}$ components operate simultaneously. In practice a common stress Y which is a rv with cumulative distribution function (cdf) $G_Y(.)$ exposes to the system. Let the strengths of the components be independent and identically rvs (iid rvs) distributed with cdf $F_Y(.)$ and be exposed to the common random stress Y having cdf $G_Y(.)$. Thus the reliability of the system i.e. the probability of survival of the system, $\mathbf{R}_{s,k}$ is given by

$$\begin{aligned}\mathbf{R}_{s,k} &= \Pr(\text{at least s of the } X_1, X_2, ..., X_k \text{ exceed Y}) \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1-F(y)]^{\,i} \, [F(y)]^{\,k-i} dG(y).\end{aligned} \tag{1}$$

The model in (1) has been firstly pioneered by [21]. Applying multicomponent stress-strength models in reliability analysis offers several advantages, making it a valuable approach for assessing the safety and performance of complex engineering systems. Some of the key advantages include:

1. Multicomponent stress-strength models allow engineers to consider the interaction of multiple stress components that a system may experience during its operational life. This provides a more realistic representation of real-world scenarios, where structures and equipment are subjected to a combination of loads (e.g., mechanical, thermal, environmental) simultaneously.
2. By considering the weakest component's strength, multicomponent stress-strength models provide a more accurate estimation of the probability of system failure. This is crucial for engineering designs where safety and reliability are paramount concerns.
3. The models help identify the weakest components in the system. By knowing which components are more susceptible to failure, engineers can focus their efforts on improving or monitoring those specific areas to enhance overall system reliability.
4. Multicomponent stress-strength models naturally handle variations in stress and strength data, making them more robust in dealing with uncertainties and data variability. This is especially important in engineering, where material properties and environmental conditions often have inherent variability.
5. The models do not impose strict assumptions about the underlying probability distributions of stress and strength components, making them versatile for various applications. They can work with different types of probability distributions, including normal, log-normal, Weibull, and others.
6. Multicomponent stress-strength models find application in diverse industries such as aerospace, automotive, civil engineering, electronics, and more. They can be used to analyze the reliability of bridges, aircraft, electronic circuits, and other critical systems.

7. Understanding the weakest components and their contribution to system reliability helps engineers optimize designs more efficiently. By focusing resources on improving critical areas, unnecessary costs can be avoided while maintaining the required safety margins.
8. Multicomponent stress-strength models aid in risk assessment by quantifying the likelihood of system failure under different operating conditions. This information is valuable for designing appropriate risk mitigation strategies.
9. In various industries, compliance with safety standards and regulations is essential. Multicomponent stressstrength models provide a rigorous approach for demonstrating compliance and meeting safety requirements.

In this work, we illustrate both the traditional and Bayesian approaches for estimating R at the point and interval levels. The conventional method is shown first, followed by the Bayesian way. Priors are considered in the Bayesian process, even when they do not contribute any new information. The approach of estimating that has the greatest possibility of being correct takes into account the multicomponent stress-strength that is being imposed on the structure. The multicomponent stress-strength under parametric bootstrap sampling and the multicomponent stressstrength under non-parametric bootstrap sampling are both calculated when that step is completed. After that, we look at the differences between the outcomes. In this final section, we will talk about how to estimate the multicomponent stress strength using the Bayesian method. An exhaustive research project based on simulation was carried out in order to compare and contrast the efficacy of each of these tactics. As part of this research, the simulation was given specific settings and controls, which were then analyzed. In addition, an application was provided on two distinct kinds of real data since it was difficult to ignore the application on real data because it is a necessary first step in reviewing the models, approaches, and strategies that were utilized, as well as a crucial first step in weighting them. This was done because it was impossible to ignore the application on real data because it is a crucial first step in assessing the models, approaches, and strategies that were utilized. In addition, an application was supplied on two different types of real data because it was hard to ignore the significance of the application on genuine data. The non-parametric bootstrap sampling makes use of three basic approaches, which are the bias-corrected and accelerated bootstrap confidence intervals, the percentile bootstrap confidence intervals, and the Student's t bootstrap confidence intervals. Each of these ways is referred to as a confidence interval.
In this paper, we study the estimation of reliability of a system with a second type of s-out-of-k system when the stress and strength both have the TL distribution with three unknown parameters. In Section 2, we present a brief introduction of TL distribution and obtained an explicit expression for $\mathbf{R}_{s,k}$. Section 3 deals with classical and Bayesian methods of estimation and confidence-credible intervals. Numerical results based on simulation study are presented in Section 4. An example based on two real data sets is presented for demonstrating benefit of the proposed model in Section 5 and finally, some concluding remarks are given in the last section.

2. The methodology

The estimation of the multicomponent stress-strength reliability under the TL distribution holds significant importance in various fields and industries. This statistical approach offers several advantages over traditional methods and can address specific challenges associated with complex systems. Below are some key reasons highlighting the importance of this estimation technique:

- The TL distribution provides a flexible and robust framework to model the joint distribution of stress and strength variables. By considering a wide range of data characteristics, this approach can lead to more accurate and reliable reliability assessments of complex systems. Such precise estimations are crucial for ensuring the safety and performance of critical structures, components, and devices.
- Many real-life scenarios involve stress and strength variables that do not adhere to traditional distribution assumptions. The TL distribution's ability to encompass various distributions as special cases enables it to handle diverse data, making it applicable to a wide range of practical applications where non-standard data patterns are prevalent.

- The estimation of the multicomponent stress-strength reliability under the TL distribution has wide-ranging applications in different industries, such as civil engineering, aerospace, material science, manufacturing, healthcare, and more. Its versatility allows engineers, researchers, and decision-makers from various domains to assess reliability with tailored precision.
- Accurate reliability estimations are fundamental for making informed decisions in system design, maintenance, and risk management. The TL-based multicomponent stress-strength model empowers stakeholders with reliable data to optimize design choices, allocate resources effectively, and develop costefficient safety strategies.
- The Bayesian approach within the TL framework offers a powerful means to quantify uncertainty in reliability estimations. Uncertainty quantification is crucial in evaluating the robustness of systems and identifying critical components that need further investigation or strengthening.
- In the context of emerging technologies, such as autonomous systems, the Internet of Things (IoT), and advanced materials, the estimation of multicomponent stress-strength reliability under the TL distribution can provide valuable insights into system performance and safety. It plays a vital role in the development and deployment of cutting-edge technologies with complex interaction patterns.
- With increasing concerns about resilience and risk mitigation in the face of natural disasters, climate change, and global challenges, accurate reliability assessments are essential for building robust and resilient infrastructures. The TL-based approach contributes to proactive risk mitigation strategies.
- As the estimation technique gains prominence, it may lead to the establishment of standard practices and guidelines for its application in specific industries. Such standardization fosters consistency and reliability in reliability assessments across different applications and sectors.

A rv X is said to have the TL distribution if its probability density function (pdf) and cdf are given by

$$f_X(x) = 2\frac{\alpha}{\sigma}\left(1-\frac{x}{\sigma}\right)\left[\left(\frac{x}{\sigma}\right)\left(2-\frac{x}{\sigma}\right)\right]^{\alpha-1}|_{(0<x<\sigma \text{ and } \alpha>0)}, \tag{2}$$

and

$$F_X(x) = \left[\left(\frac{x}{\sigma}\right)\left(2-\frac{x}{\sigma}\right)\right]^{\alpha}, \; 0<x<\sigma. \tag{3}$$

respectively. For more TL extensions see 12, 8 and 9. We shall use the notation $\mathrm{TL}(\alpha,\sigma)$ is used for TL distribution given in (2) and (3). Let $X \sim\mathrm{TL}(\alpha_1,\sigma)$ and $Y \sim\mathrm{TL}(\alpha_2,\sigma)$ with unknown parameters α_1, α_2 and σ, where X and Y are independently distributed. The reliability of a multicomponent stress-strength model under TL distribution can be derived from (1), (2) and (3) as follows

$$\begin{aligned}\mathbf{R}_{s,k} &= \sum_{i=s}^{k}\binom{k}{i}\int_0^\sigma \left\{1-\left[\left(\tfrac{y}{\sigma}\right)\left(2-\tfrac{y}{\sigma}\right)\right]^{\alpha_1}\right\}^i \\ &\times\left\{\left[\left(\tfrac{y}{\sigma}\right)\left(2-\tfrac{y}{\sigma}\right)\right]^{\alpha_1}\right\}^{k-i} \\ &\times\tfrac{2\alpha_2}{\sigma}\left(1-\tfrac{y}{\sigma}\right)\left[\left(\tfrac{y}{\sigma}\right)\left(2-\tfrac{y}{\sigma}\right)\right]^{\alpha_2-1}dy \\ &= \tfrac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\binom{k}{i}\int_0^1 t^{k+\left(\frac{\alpha_2}{\alpha_1}\right)-i-1}(1-t)^i dt \\ &= \tfrac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\binom{k}{i}\frac{\Gamma\left(k+\left(\frac{\alpha_2}{\alpha_1}\right)-i\right)}{\Gamma\left(k+\left(\frac{\alpha_2}{\alpha_1}\right)+1\right)}\Gamma(i+1)\end{aligned} \tag{4}$$

where

$$t = \left[\left(\frac{y}{\sigma}\right)\left(2-\frac{y}{\sigma}\right)\right]^{\alpha_1}$$

and $\Gamma(.)$ is the gamma function. Thus, the $\mathbf{R}_{s,k}$ in (4) is not related to the parameter σ.

3. Estimation of $\mathbf{R}_{s,k}$

In this section, we present classical and Bayesian methods for obtaining point and interval estimations of $\mathbf{R}_{s,k}$. The Bayesian technique takes into account historical data that does not provide any further information. When employing the maximum likelihood estimate methodology, it is critical to take into account the multicomponent stress-strength. After that, the multicomponent stress-strength under the non-parametric bootstrap sampling and the multicomponent stress-strength under the parametric bootstrap sampling are both computed simultaneously. This is done to ensure accuracy. In conclusion, we are going to talk about how Bayesian estimate can be used to determine the multicomponent stress-strength. All of these methods have been assessed by making use of a comprehensive simulation study. This study was carried out with certain variables and controls with the intention of analyzing and comparing the various degrees of performance that each strategy exhibited. Due to the fact that it was impossible to ignore the application on actual data, it was also displayed on two separate forms of real data. This is due to the fact that the use of the models, approaches, and procedures that were employed on actual data is a big entry to evaluating them, as well as an important entrance to weighing them. It was impossible to disregard the impact the application had on the actual data.

3.1. Multicomponent stress-strength under the the maximum likelihood estimation method

The multicomponent stress-strength reliability model under the maximum likelihood estimation (MLE) method is a statistical approach used to estimate the parameters of the model based on observed data. In this context, the model aims to assess the reliability of a system with multiple stress and strength variables and determine the probability of the system meeting specified safety criteria. The MLE method is a widely used statistical technique for estimating the parameters of a statistical model. The main idea behind MLE is to find the values of the model parameters that maximize the likelihood function, which measures the probability of observing the given data under the assumed model. The MLE method is crucial in multicomponent stress-strength reliability analysis because it provides a principled and statistically rigorous way to estimate the model parameters based on observed data. By maximizing the likelihood function, MLE finds the most plausible values of the parameters, leading to reliable and data-driven reliability assessments. he multicomponent stress-strength reliability model involves two main components: stress and strength. The stress component represents the loads, forces, or demands that the system experiences, while the strength component characterizes the resistance or capacity of the system to with stand those loads. In the context of multicomponent stress-strength reliability, the goal is to assess the probability that the strength of the system exceeds the applied stress. This probability can be expressed as the reliability of the system, which is crucial in determining the safety and performance of various engineering and industrial systems. This approach allows engineers and researchers to make informed decisions, optimize system design, and ensure safety and performance in a wide range of practical applications. Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ be random samples from TL(α_1, σ) and TL(α_2, σ), respectively. Then, the log-likelihood function of the observed sample values based on $\mathbf{\Theta} = (\alpha_1, \alpha_2, \sigma)$ can be written as:

$$\begin{aligned}\ell(\mathbf{\Theta}) &= n\log\left(\tfrac{2\alpha_1}{\sigma}\right) + m\log\left(\tfrac{2\alpha_2}{\sigma}\right) + \sum_{i=1}^{n}\log\left(1-\tfrac{x_i}{\sigma}\right) + \sum_{j=1}^{m}\log\left(1-\tfrac{y_j}{\sigma}\right)\\ &+(\alpha_1-1)\sum_{i=1}^{n}\log\left[\left(\tfrac{x_i}{\sigma}\right)\left(2-\tfrac{x_i}{\sigma}\right)\right] + (\alpha_2-1)\sum_{j=1}^{m}\log\left[\left(\tfrac{y_j}{\sigma}\right)\left(2-\tfrac{y_j}{\sigma}\right)\right],\end{aligned}$$

The Maximum Likelihood Estimates (MLE) of σ is

$$\hat{\sigma} = \max(x_{(n)}, y_{(n)})$$

and MLEs of α_1, α_2 are $\hat{\alpha}_1$ and $\hat{\alpha}_2$ respectively, can be obtained by solving following equations simultaneously

$$\begin{aligned}\tfrac{\partial}{\partial\alpha_1}\ell(\alpha_1,\alpha_2,\hat{\sigma}) = 0 \quad &\Rightarrow \hat{\alpha}_1 = -n\frac{1}{\sum_{i=1}^{n}\log\left[\left(\frac{x_i}{\hat{\sigma}}\right)\left(2-\frac{x_i}{\hat{\sigma}}\right)\right]},\\ \tfrac{\partial}{\partial\alpha_2}\ell(\alpha_1,\alpha_2,\hat{\sigma}) = 0 \quad &\Rightarrow \hat{\alpha}_2 = -m\frac{1}{\sum_{j=1}^{m}\log\left[\left(\frac{y_j}{\hat{\sigma}}\right)\left(2-\frac{y_j}{\hat{\sigma}}\right)\right]}.\end{aligned}$$

Thus, based on invariance property of MLE, the MLE of $\mathbf{R}_{s,k}$ is

$$\hat{\mathbf{R}}_{s,k} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1} \sum_{i=s}^{k} \binom{k}{i} \frac{\Gamma(k + \left(\frac{\alpha_2}{\alpha_1}\right) - i)\Gamma(i+1)}{\Gamma(k + \left(\frac{\alpha_2}{\alpha_1}\right) + 1)} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \frac{1}{\prod_{j=0}^{i} \left[k + \left(\frac{\alpha_2}{\alpha_1}\right) - j\right]} \tag{5}$$

Because the support of rv X with TL(α, σ) distribution is depend on σ parameter, regularity conditions of MLE is not established. Thus for obtaining confidence interval, following non-parametric and parametric bootstrap methods are used.

3.2. Multicomponent stress-strength under the non-parametric bootstrap sampling

The non-parametric bootstrap sampling is a resampling technique used to estimate the uncertainty associated with a statistical model's parameter estimates without making strong distributional assumptions. In the context of the multicomponent stress-strength reliability model, the non-parametric bootstrap sampling can be applied to assess the reliability of the system and provide confidence intervals for parameter estimates. This approach is particularly useful when the underlying data distribution is not well-defined or when parametric assumptions are difficult to satisfy. The non-parametric bootstrap sampling method involves creating multiple resamples (bootstrap samples) from the observed data by randomly sampling with replacement. Each bootstrap sample is of the same size as the original data, and it is considered a surrogate for the population from which the data were drawn. By repeatedly resampling, new datasets are generated, and statistical quantities of interest, such as parameter estimates, can be computed from each bootstrap sample. To apply the non-parametric bootstrap sampling to the multicomponent stress-strength reliability model, the following steps are typically followed:

- The observed data consisting of stress and strength variables are collected from the system of interest.
- Using the observed data, a large number of bootstrap samples are generated by randomly sampling with replacement. Each bootstrap sample is of the same size as the original data.
- For each bootstrap sample, the parameters of the multicomponent stress-strength reliability model are estimated using any appropriate estimation method, such as MLE method or Bayesian estimation.
- With parameter estimates from each bootstrap sample, the reliability of the system can be computed for different stress conditions. This results in a distribution of reliability estimates, providing insights into the variability and uncertainty associated with the system's performance.
- From the distribution of parameter estimates obtained from the bootstrap samples, confidence intervals can be constructed for each parameter. These intervals give an indication of the range within which the true population parameter value is likely to lie with a specified level of confidence.

The non-parametric bootstrap sampling technique offers several advantages in the context of the multicomponent stress-strength reliability model:

- The non-parametric bootstrap does not rely on distributional assumptions, making it suitable for cases where the data distribution is unknown or difficult to model accurately.
- By resampling from the observed data, the non-parametric bootstrap provides a more robust estimate of the uncertainty associated with the model parameters, as it does not rely on assumptions about the data distribution.
- The non-parametric bootstrap considers the variability in the observed data, leading to more realistic reliability assessments that account for the inherent randomness in the system.
- The bootstrap approach allows for the quantification of uncertainty in parameter estimates through the construction of confidence intervals, providing more comprehensive insights into the reliability model's performance.

In this Subsection, the multicomponent stress-strength under the non-parametric bootstrap sampling, the non-parametric bootstrap sampling contains three main methods:

1. Percentile bootstrap confidence intervals.
2. Student's t bootstrap confidence intervals.
3. Bias-corrected and accelerated bootstrap confidence intervals.

In step 1, we generate independent bootstrap samples $x_1^*, x_2^*, ..., x_n^*$ and $y_1^*, y_2^*, ..., y_m^*$ taken with replacement from the given samples $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$ with distributions (TL(α_1, σ) and TL(α_2, σ)), respectively. Based on the bootstrap samples, compute the MLE $(\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\sigma}^*)$ of $(\alpha_1, \alpha_2, \sigma)$ as well as $\hat{\mathbf{R}}_{s,k}^* = \mathbf{R}_{s,k}(\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\sigma}^*)$ of $\mathbf{R}_{s,k}$.
Step 2: Repeat Step 1, B times to obtain a set of bootstrap samples of $\mathbf{R}_{s,k}$, say

$$\left(\hat{\mathbf{R}}_{s,k}^*\right)_j \mid_{(j=1,2,...,B)}.$$

With the above bootstrap samples of $\mathbf{R}_{s,k}$ we obtain three different bootstrap confidence intervals of $\mathbf{R}_{s,k}$. The ordered $\left(\hat{\mathbf{R}}_{s,k}^*\right)_j \; \forall \, j = 1, ..., B$ will be denoted as:

$$\hat{R}_{s,k}^{*(1)} < \hat{R}_{s,k}^{*(2)} < \cdots < \hat{R}_{s,k}^{*(B)}$$

3.3. Percentile bootstrap (p-boot) confidence interval

Percentile bootstrap confidence intervals are a type of non-parametric statistical method used to estimate the confidence interval for a parameter (such as the mean, median, standard deviation, etc.) of a population when the underlying data distribution is unknown or difficult to model. The percentile bootstrap is a resampling technique that involves repeatedly sampling from the observed data to estimate the sampling distribution of the parameter of interest. The confidence interval is then constructed based on percentiles of this sampling distribution. Here's how the percentile bootstrap method works:

1. Sample Resampling: Given a dataset with n observations, the percentile bootstrap begins by randomly sampling n observations (with replacement) from the original data to create a bootstrap sample. This new sample will have the same size as the original data but may contain duplicate observations.
2. Parameter Estimation:With each bootstrap sample, the parameter of interest (e.g., the mean, median, standard deviation) is calculated.
3. Repeat Sampling: Steps 1 and 2 are repeated a large number of times (typically several thousand) to create a distribution of parameter estimates, known as the bootstrap sampling distribution.
4. Confidence Interval: The confidence interval is then obtained by determining the lower and upper bounds of the desired confidence level (e.g., 95 percent) of the bootstrap sampling distribution. The bounds are determined by the corresponding percentiles of the distribution.

For example, to construct a 95 $\%$ percentile bootstrap confidence interval for the mean of a dataset, one would find the 2.5th and 97.5th percentiles of the bootstrap sampling distribution. These values correspond to the lower and upper bounds of the confidence interval, respectively. The percentile bootstrap method is particularly useful when the data distribution is non-normal or when the assumptions required for parametric methods (such as t tests or confidence intervals) are violated. It makes fewer assumptions about the underlying data distribution, making it more robust in many cases. However, it is essential to keep in mind that the percentile bootstrap requires a sufficiently large number of bootstrap samples to achieve reliable results. Let $\hat{\mathbf{R}}_{s,k}^{*(\tau)}$ be the τ percentile of $\left[\left(\hat{\mathbf{R}}_{s,k}^*\right)_j \; \mid_{(j=1,2,...,B)}\right]$, i.e. $\hat{\mathbf{R}}_{s,k}^{*(\tau)}$ is such that

$$\tau = \frac{1}{B}\sum_{j=1}^{B} \mathbf{I}\left(\left(\hat{\mathbf{R}}_{s,k}^*\right)_j < \hat{\mathbf{R}}_{s,k}^{*(\tau)}\right) \mid_{(0<\tau<1)},$$

where $\mathbf{I}(.)$ is the indicator function. A $100(1-\tau)\%$ p-boot confidence interval of $R_{s,k}$ is given by

$$\left(\hat{\mathbf{R}}_{s,k}^{*(\frac{\tau}{2})}, \hat{\mathbf{R}}_{s,k}^{*(1-\frac{\tau}{2})}\right). \tag{6}$$

3.4. Student's t bootstrap (t-boot) confidence interval

The Student's t bootstrap, also known as t-boot, is a variant of the bootstrap method used to construct confidence intervals for a population parameter, typically the mean, when the underlying data may not follow a normal distribution and the sample size is small. The t-boot is particularly useful when dealing with data where the assumptions of normality required by traditional parametric methods are not met, or when the sample size is limited. Here's how the t-boot confidence interval is constructed:

1. Sample Resampling: Like in other bootstrap methods, the t-boot starts by randomly sampling with replacement from the original dataset to create a bootstrap sample. The size of the bootstrap sample is the same as the original dataset.
2. Parameter Estimation:With each bootstrap sample, the parameter of interest (usually the mean) is calculated. Since the t-boot is used when the population distribution may be non-normal and the sample size is small, the sample mean is often the parameter of choice.
3. T-Distribution: Instead of using the normal distribution, the t-distribution is used to build the confidence interval. The t-distribution takes into account the variability in the sample mean due to the uncertainty caused by the small sample size. It has fatter tails than the normal distribution, making it more appropriate for small samples.
4. Repeat Sampling: The resampling and parameter estimation process is repeated a large number of times (e.g., thousands of times) to create a distribution of sample means, which follows a t-distribution.
5. Confidence Interval: The confidence interval is then obtained from the t-distribution. The lower and upper bounds of the confidence interval are determined by the appropriate percentiles of the t-distribution based on the desired confidence level (e.g., 95%).

The t-boot is considered more robust for small sample sizes and non-normal data compared to traditional methods that rely on assumptions of normality. It accounts for the additional uncertainty introduced by small samples and provides more accurate confidence intervals in such cases. Let $\bar{\hat{\mathbf{R}}}^*_{s,k}$ and $SE^*(\hat{\mathbf{R}}^*_{s,k})$ be the sample mean and sample standard deviation of

$$\left[\left(\hat{\mathbf{R}}^*_{s,k}\right)_j |_{(j=1,2,\ldots,B)}\right],$$

where $\left(\hat{\mathbf{R}}^*_{s,k}\right)_j$ is the MLE of $R_{s,k}$ for the j^{th} bootstrap sample. Also, let $\hat{t}^{*(\tau)}$ be the τ percentile of

$$\frac{\left(\hat{\mathbf{R}}^*_{s,k}\right)_j - \bar{\hat{\mathbf{R}}}^*_{s,k}}{SE^*(\hat{\mathbf{R}}_{s,k})} |_{(j=1,2,\ldots,B)},$$

i.e., $\hat{t}^{*(\tau)}$ is such that

$$\tau = \frac{1}{B}\sum_{j=1}^{B}\mathbf{I}\left(\frac{\left(\hat{\mathbf{R}}^*_{s,k}\right)_j - \bar{\hat{\mathbf{R}}}^*_{s,k}}{SE^*(\hat{R}_{s,k})} \leq \hat{t}^{*(\tau)}\right) |_{(0<\tau<1)}. \tag{7}$$

A $100(1-\tau)\%$ t-boot confidence interval of $\mathbf{R}_{s,k}$ is given by

$$\left(\hat{\mathbf{R}}_{s,k} - \hat{t}^{*(\frac{\tau}{2})}SE^*(\hat{\mathbf{R}}^*_{s,k}), \hat{\mathbf{R}}_{s,k} + \hat{t}^{*(\frac{\tau}{2})}SE^*(\hat{\mathbf{R}}^*_{s,k})\right). \tag{8}$$

3.5. Bias-corrected and accelerated bootstrap (BC-boot) confidence interval

The Bias-Corrected and Accelerated Bootstrap (BC-boot) is a modification of the basic bootstrap method that addresses two potential issues with the traditional percentile bootstrap: bias and skewness in the bootstrap distribution. The BC-boot provides more accurate and robust confidence intervals, especially for small sample sizes or skewed data. Here's how the BC-boot confidence interval is constructed:

1. Sample Resampling: Similar to other bootstrap methods, the BC-boot starts by resampling with replacement from the original dataset to create a bootstrap sample. The size of the bootstrap sample is the same as the original dataset.
2. Parameter Estimation: With each bootstrap sample, the parameter of interest (e.g., mean, median, etc.) is calculated, just as in the basic bootstrap.
3. Bias Correction: The BC-boot estimates the bias of the parameter by comparing the average of the bootstrap sample estimates to the estimate from the original data. The bias is then subtracted from the percentile-based estimate to reduce bias in the bootstrap distribution.
4. Acceleration Correction: The BC-boot also estimates the acceleration factor to account for the skewness in the bootstrap distribution. The acceleration factor quantifies the departure from the normality assumption. It is used to adjust the percentile-based confidence interval and make it more symmetric.
5. Repeat Sampling: As with other bootstrap methods, the resampling and parameter estimation process is repeated a large number of times (e.g., thousands of times) to create a distribution of parameter estimates.
6. Confidence Interval: The confidence interval is then constructed using the bias-corrected and accelerated bootstrap estimates. It involves adjusting the traditional percentile-based interval using the bias and acceleration factor. The BC-boot is especially useful when the underlying data distribution is not symmetric or when the sample size is small. It provides more accurate confidence intervals and reduces the potential bias in the estimates, making it more reliable for a wide range of scenarios.

Let $z^{(\tau)}$ and $\hat{z}_0$, respectively, be such that $z^{(\tau)} = \Phi^{-1}(\tau)$ and

$$\hat{z}_0 = \Phi^{-1}\left[\frac{1}{B}\sum_{i=1}^{B}\mathbf{I}\Big(\big(\hat{\mathbf{R}}^*_{s,k}\big)_j < \hat{\mathbf{R}}_{s,k}\Big)\right]$$

where $\Phi^{-1}(.)$ is the inverse cdf of the standard normal distribution. The value $\hat{z}_0$ is called bias-correction. Also, let

$$\hat{a} = \frac{\sum_{i=1}^{n}\left[\big(\hat{\mathbf{R}}_{s,k}\big)_{(.)} - \big(\hat{\mathbf{R}}_{s,k}\big)_{(i)}\right]^3}{6\left\{\sum_{i=1}^{n}\left[\big(\hat{\mathbf{R}}_{s,k}\big)_{(.)} - \big(\hat{\mathbf{R}}_{s,k}\big)_{(i)}\right]^2\right\}^{\frac{3}{2}}},$$

where $\big(\hat{\mathbf{R}}_{s,k}\big)_{(i)}$ is the MLE of $\mathbf{R}_{s,k}$ based of observations after excluding the i^{th} observation and

$$\big(\hat{\mathbf{R}}_{s,k}\big)_{(.)} = \frac{1}{n}\sum_{i=1}^{n}\big(\hat{\mathbf{R}}_{s,k}\big)_{(i)}.$$

The value $\hat{a}$ is called acceleration factor. A $100(1-\tau)\%$ BC-boot confidence interval of $\mathbf{R}_{s,k}$ is given by

$$\left(\hat{\mathbf{R}}^{*(\nu_1)}_{s,k}, \hat{\mathbf{R}}^{*(\nu_2)}_{s,k}\right), \tag{9}$$

where

$$\nu_1 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 + z^{(\frac{\tau}{2})}}{1 - \hat{a}\left(\hat{z}_0 + z^{(\frac{\tau}{2})}\right)}\right]$$

and

$$\nu_2 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\frac{\tau}{2})}}{1 - \hat{a}\left(\hat{z}_0 + z^{(1-\frac{\tau}{2})}\right)}\right].$$

3.6. Multicomponent stress-strength under the parametric bootstrap sampling

The parametric bootstrap sampling is a resampling technique used to estimate the uncertainty of parameter estimates in a statistical model by generating bootstrap samples from a fitted parametric model. In the context of the multicomponent stress-strength reliability model, the parametric bootstrap can be employed to assess the reliability of the system and provide confidence intervals for parameter estimates while still assuming a specific parametric form for the data distribution. The parametric bootstrap sampling involves the following steps:

- The observed data, consisting of stress and strength variables, are collected from the system of interest.
- A parametric model is fitted to the observed data, typically using MLE or another appropriate method. This model represents the assumed distribution for the data.
- Bootstrap samples are generated by randomly drawing data points from the fitted parametric model. Unlike the non-parametric bootstrap, which samples directly from the observed data, the parametric bootstrap resamples from the fitted model's estimated parameters.
- For each bootstrap sample, the parameters of the multicomponent stress-strength reliability model are estimated using the same parametric model that was fitted to the observed data in step 2.
- With parameter estimates from each parametric bootstrap sample, the reliability of the system can be computed for different stress conditions. This provides a distribution of reliability estimates, offering insights into the variability and uncertainty associated with the system's performance.
- From the distribution of parameter estimates obtained from the parametric bootstrap samples, confidence intervals can be constructed for each parameter. These intervals give an indication of the range within which the true population parameter value is likely to lie with a specified level of confidence.

The parametric bootstrap sampling technique offers several advantages in the context of the multicomponent stress-strength reliability model:

- Unlike the non-parametric bootstrap, which makes no distributional assumptions, the parametric bootstrap allows for the incorporation of specific parametric models that may better fit the observed data. This can lead to more accurate parameter estimates and reliability assessments.
- By resampling from the fitted parametric model, the parametric bootstrap captures the uncertainty associated with the model's parameters. This accounts for the variability in the estimated parameters and leads to more robust reliability assessments.
- When there are valid reasons to assume a certain parametric form for the data distribution, the parametric bootstrap aligns with these assumptions and produces reliable estimates within that context.
- The parametric bootstrap may be computationally more efficient than the non-parametric bootstrap, especially for complex models, as it involves resampling from a smaller set of model parameters rather than the entire observed dataset.

In this Subsection, we present the following method to generate parametric bootstrap samples from the given random samples with TL(α, σ) distribution, and they can be used to create different parametric bootstrap confidence intervals.

Step 1: Compute the MLEs of α_1, α_2 and σ from the given random samples, say $\hat{\alpha}_1, \hat{\alpha}_2$, and $\hat{\sigma}$, respectively. Generate independent bootstrap samples $x_1^*, x_2^*, ..., x_n^*$ and $y_1^*, y_2^*, ..., y_m^*$, from TL$(\hat{\alpha}_1, \hat{\sigma})$ and TL$(\hat{\alpha}_2, \hat{\sigma})$, respectively. Based on the bootstrap samples, compute the MLE $(\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\sigma}^*)$ of $(\alpha_1, \alpha_2, \sigma)$ as well as $\hat{\mathbf{R}}_{s,k}^* = \mathbf{R}_{s,k}(\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\sigma}^*)$ of $\mathbf{R}_{s,k}$.

Step 2: Repeat Step 1, B times to obtain a set of bootstrap estimates of $\mathbf{R}_{s,k}$, say $\left[\left(\hat{R}_{s,k}^*\right)_j, j = 1, 2, ..., B\right]$. Using the above bootstrap samples of $\mathbf{R}_{s,k}$ we can obtain three parametric bootstrap confidence intervals of $\mathbf{R}_{s,k}$ similar to the non-parametric ones.

3.7. Multicomponent stress-strength under Bayesian estimation

Bayesian estimation involves the use of Bayes' theorem to update our prior beliefs about model parameters with new data, yielding a posterior distribution that represents our updated beliefs after observing the data. The key components of Bayesian estimation are:

- This represents our initial beliefs or knowledge about the model parameters before observing any data. It encapsulates any existing information or uncertainty about the parameters.

- This function describes the probability of observing the data given specific parameter values. It represents the likelihood of the data under the assumed model.
- The posterior distribution combines the prior distribution and the likelihood function to represent the updated beliefs about the model parameters after incorporating the observed data. It provides a probabilistic representation of the uncertainty in the parameter estimates.

To apply Bayesian estimation to the multicomponent stress-strength reliability model, the following steps are typically followed:

1. The multicomponent stress-strength reliability model is formulated, and appropriate prior distributions for the model parameters are selected. The prior distributions may be chosen based on prior knowledge or expert opinions if available, or non-informative priors can be used when little prior information is known.
2. Using the likelihood function and the chosen prior distributions, Bayes' theorem is applied to calculate the posterior distribution of the model parameters given the observed data.
3. In many cases, it is computationally challenging to obtain the posterior distribution analytically. Therefore, MCMC sampling techniques, such as Gibbs sampling or Metropolis-Hastings algorithm, are commonly employed to draw samples from the posterior distribution.
4. From the MCMC samples, Bayesian point estimates (e.g., posterior mean, median) and credible intervals (Bayesian analog of confidence intervals) for the model parameters are computed. These provide estimates of the parameters' most likely values and quantify the uncertainty around these estimates.
5. Using the Bayesian parameter estimates, the reliability of the system under various stress conditions can be computed. The Bayesian framework allows for the incorporation of uncertainty in the parameter estimates, providing more informative and robust reliability assessments.

The Bayesian estimation approach has several advantages in the context of multicomponent stress-strength reliability analysis:

- Bayesian estimation allows for the incorporation of prior knowledge or beliefs about the model parameters, making it valuable when prior information is available from previous studies or expert opinions.
- Bayesian estimation provides a probabilistic representation of parameter estimates, allowing for more informative and interpretable results. It quantifies uncertainty in estimates, which is essential in decisionmaking and risk assessment.
- The Bayesian framework can handle complex models and data structures, making it suitable for the analysis of real-life systems with intricate stress-strength relationships.
- Bayesian estimation is particularly useful when dealing with small sample sizes or data that deviates from normality, as it can still provide meaningful results and reliable parameter estimates.

In this Subsection, we use a Bayesian approach based on the MCMC technique. In this part, we suppose σ parameter is known because this parameter is bounded parameter and we can replace this parameter with maximum likelihood estimation of it. To obtain the posterior distribution for quantities of interest in multicomponent stress-strength based on a TL distribution, we use non-informative prior for θ as

$$\pi(\theta) \propto 1, \tag{10}$$

which means, for each parameters non-informative prior is selected. The likelihood function of the multicomponent stress-strength model is

$$\begin{aligned} L(\theta) &= \left(\tfrac{2\alpha_1}{\hat{\sigma}}\right)^n \left(\tfrac{2\alpha_2}{\hat{\sigma}}\right)^m \prod_{i=1}^{n} \left(1 - \tfrac{x_i}{\hat{\sigma}}\right) \prod_{j=1}^{m} \left(1 - \tfrac{y_j}{\hat{\sigma}}\right) \\ &\times \left\{ \prod_{i=1}^{n} \left[\left(\tfrac{x_i}{\hat{\sigma}}\right)\left(2 - \tfrac{x_i}{\hat{\sigma}}\right)\right] \right\}^{\alpha_1 - 1} \\ &\times \left\{ \prod_{j=1}^{m} \left[\left(\tfrac{y_j}{\hat{\sigma}}\right)\left(2 - \tfrac{y_j}{\hat{\sigma}}\right)\right] \right\}^{\alpha_2 - 1} \end{aligned} \tag{11}$$

The joint posterior density for $\theta = (\alpha_1, \alpha_2)$ is

$$\pi(\theta|\mathbf{x},\mathbf{y}) \propto \alpha_1^n \alpha_2^m \exp\begin{pmatrix} -\alpha_1 \left\{ -\sum_{i=1}^{n} \ln\left[\left(\frac{x_i}{\hat{\sigma}}\right)\left(2-\frac{x_i}{\hat{\sigma}}\right)\right]\right\} \\ -\alpha_2 \left\{ -\sum_{j=1}^{m} \ln\left[\left(\frac{y_j}{\hat{\sigma}}\right)\left(2-\frac{y_j}{\hat{\sigma}}\right)\right]\right\} \end{pmatrix}. \quad (12)$$

From (12) the full conditional distributions needed for the Gibbs sampling algorithm [?] are

$$\begin{cases} \alpha_1\,|\alpha_2,\mathbf{x},\mathbf{y} \sim \Gamma(n+1,\ -\sum_{i=1}^{n} \ln\left(\left(\frac{x_i}{\hat{\sigma}}\right)\left(2-\frac{x_i}{\hat{\sigma}}\right)\right)) \\ \alpha_2\,|\alpha_1,\mathbf{x},\mathbf{y} \sim \Gamma(m+1,\ -\sum_{j=1}^{m} \ln\left(\left(\frac{y_j}{\hat{\sigma}}\right)\left(2-\frac{y_j}{\hat{\sigma}}\right)\right)). \end{cases} \quad (13)$$

where $\Gamma(.,.)$ is gamma distribution with shape and scale parameters. Since (13) does not have simple form, the conditional posterior densities for the multicomponent stress-strength TL models can not be derived directly. Thus, the posterior estimation of $R_{s,k}$ is obtained based on monte carlo method and the credible interval is obtained by generating sample from posterior distribution.

4. Simulation study

Simulation studies play a crucial role in the analysis and assessment of multicomponent stress-strength reliability models. These studies involve generating synthetic data and performing computational experiments to investigate the behavior and performance of the reliability model under various scenarios. The main importance of simulation studies in multicomponent stress-strength reliability models includes:

- Simulation studies allow researchers to evaluate the performance of the multicomponent stress-strength reliability model under controlled conditions. By comparing the true reliability values with the model's estimates, one can assess the accuracy and reliability of the model in capturing the system's behavior.
- Simulation studies help in understanding how changes in input parameters and model assumptions affect the reliability estimates. This sensitivity analysis provides insights into the factors that significantly influence the system's reliability, allowing engineers to focus on critical components or stress conditions.
- Simulated data can be used to validate the multicomponent stress-strength reliability model against known theoretical results or experimental data. Furthermore, calibration of model parameters can be performed to ensure that the model aligns with the observed data and reflects the real-life system's behavior.
- In real-life applications, it may be challenging to obtain data from extreme stress or strength conditions. Simulation studies enable the exploration of scenarios beyond the observed data range, facilitating the assessment of reliability under extreme situations or rare events.
- Simulation studies can aid in determining the required sample size for reliability analysis. By investigating the effects of different sample sizes on the precision and accuracy of parameter estimates, researchers can optimize data collection strategies for future studies.
- Simulation studies allow for the comparison of different multicomponent stress-strength reliability models under the same conditions. This comparative analysis can help identify the most appropriate model for a specific application based on its performance and reliability estimates.
- Simulation studies provide an opportunity to examine the validity of model assumptions. If the data generation process deviates from the assumed distributions, researchers can explore the impact on reliability estimates and make necessary adjustments to enhance model validity.
- Some engineering systems are designed to handle rare, high-stress events. Simulation studies allow for the investigation of such rare events and their impact on system reliability, providing valuable insights into extreme event management and risk assessment.

- By systematically varying stress and strength levels in the simulated data, researchers can optimize experimental designs for real-life testing, maximizing the efficiency of data collection and minimizing costs.
- Simulation studies can be valuable for educational purposes and for communicating the intricacies of multicomponent stress-strength reliability models to practitioners, stakeholders, and decision-makers.

In this section, some Monte Carlo simulation results are presented to compare the performances of different methods (MLE, parametric bootstrap, non-parametric bootstrap, Bayesian) for different sample sizes and for different parameter values. The performance of the point and interval estimation of the multicomponent reliability $\mathbf{R}_{1,3}$ based on maximum likelihood procedure when all the parameters are unknown and Bayesian estimation when all the parameters except σ are unknown is investigated. Specifically, the bias, mean square error (MSE), the coverage probability and the length of the simulated 95% confidence intervals are studied.
Its worth to mention again, because the fisher information matrix of TL distribution can not defined, for MLE procedure we just present bias and MSE criteria. For this aim, we generated 3,000 samples with 4,000 bootstrap samples from each of independent TL(α_1, σ) and TL(α_2, σ) distributions where $(\alpha_1, \alpha_2, \sigma)$, (1.5, 2, 1), (3,1.7, 2) and (7, 1, 1.5). These parameter values correspond to $\mathbf{R}_{1,3} = 0.178, 0.418$ and 0.779, respectively. We used different sample sizes (n, m), (20,20), (20,30), (20,50), (30,20), (30,30),(30,50), (50,20), (50,30), (50,50).
In Table 1, Table 2 and Table 3, the average biases and the mean squared errors of the estimates of $\mathbf{R}_{1,3}$ based on MLE, parametric bootstrap, non-parametric bootstrap and Bayesian methods are obtained. Table 4, Table 5 and Table 6 present the coverage percentages and the average lengths of the confidence intervals(credible interval) of $R_{1,3}$ based on MLE with different bootstrap methods and Bayesian estimation of $\mathbf{R}_{1,3}$.
Some of the results of these experiments are presented as follows:

- Based on small sample sizes, biases and MSEs of all estimation methods are near to zero.
- The MSE for all methods decreases as n and m increase. It verifies the consistency property of the MLE of $\mathbf{R}_{1,3}$.
- Based on MSE and bias, MLE and Bayesian estimation has better performance than other methods.
- The coverage probability of nonparametric bootstrap t and percentile parametric bootstrap better work than other methods.
- With length of confidence interval, there is not a winner method among others.

Table 1. Average bias (mean square error) of different estimators of $R_{1,3}$ where $\alpha_1 = 1.5$, $\alpha_2 = 2$, $\sigma = 1$. ($R_{1,3} = 0.178$)

(n, m)	MLE	P-Boot	NP-Boot	Bayes
	Bias(MSE)	Bias(MSE)	Bias(MSE)	Bias(MSE)
(20,20)	0.0044(0.0064)	0.0030(0.0139)	0.0031(0.0109)	0.0083(0.0108)
(20,30)	0.0037(0.0053)	0.0090(0.0100)	0.0118(0.0099)	0.0108(0.0093)
(20,50)	0.0059(0.0040)	0.0142(0.0094)	0.0141(0.0087)	0.0151(0.0078)
(30,20)	0.0011(0.0051)	0.0040(0.0106)	0.0039(0.0092)	0.0006(0.0085)
(30,30)	0.0021(0.0041)	0.0025(0.0091)	0.0056(0.0080)	0.0039(0.0071)
(30,50)	0.0024(0.0032)	0.0041(0.0065)	0.0085(0.0064)	0.0069(0.0059)
(50,20)	-0.0025(0.0038)	-0.0014(0.0082)	0.0022(0.0081)	0.0012(0.0072)
(50,30)	0.0005(0.0031)	0.0005(0.0066)	-0.0007(0.0059)	-0.0003(0.0056)
(50,50)	-0.0006(0.0022)	-0.0029(0.0049)	-0.0016(0.0046)	0.0026(0.0044)

Table 2. Average bias (mean square error) of different estimators of $R_{1,3}$ where $\alpha_1 = 3$, $\alpha_2 = 1.7$, $\sigma = 2$. ($R_{1,3} = 0.418$)

(n, m)	MLE	P-Boot	NP-Boot	Bayes
	Bias(MSE)	Bias(MSE)	Bias(MSE)	Bias(MSE)
(20,20)	0.0184(0.0113)	0.0433(0.0242)	0.0290(0.0216)	0.0221(0.0194)
(20,30)	0.0194(0.0095)	0.0471(0.0193)	0.0316(0.0177)	0.0248(0.0172)
(20,50)	0.0228(0.0080)	0.0458(0.0160)	0.0317(0.0152)	0.0235(0.0136)
(30,20)	0.0145(0.0093)	0.0235(0.0178)	0.0208(0.0168)	0.0160(0.0162)
(30,30)	0.0137(0.0074)	0.0378(0.0141)	0.0248(0.0148)	0.0145(0.0130)
(30,50)	0.0190(0.0059)	0.0287(0.0118)	0.0296(0.0127)	0.0193(0.0105)
(50,20)	0.0082(0.0077)	0.0185(0.0153)	0.0086(0.0141)	0.0056(0.0132)
(50,30)	0.0109(0.0058)	0.0229(0.0122)	0.0128(0.0106)	0.0104(0.0101)
(50,50)	0.0006(0.0042)	0.0309(0.0087)	0.0173(0.0082)	0.0089(0.0082)

Table 3. Average bias (mean square error) of different estimators of $R_{1,3}$ where $\alpha_1 = 7$, $\alpha_2 = 1$, $\sigma = 1.5$. ($R_{1,3} = 0.779$)

(n, m)	MLE	P-Boot	NP-Boot	Bayes
	Bias(MSE)	Bias(MSE)	Bias(MSE)	Bias(MSE)
(20,20)	0.0301(0.0035)	0.0541(0.0064)	0.0343(0.0076)	0.0183(0.0069)
(20,30)	0.0274(0.0029)	0.0571(0.0053)	0.0358(0.0053)	0.0258(0.0060)
(20,50)	0.0271(0.0025)	0.0584(0.0042)	0.0306(0.0049)	0.0285(0.0050)
(30,20)	0.0190(0.0031)	0.0385(0.0054)	0.0223(0.0056)	0.0159(0.0054)
(30,30)	0.0226(0.0024)	0.0413(0.0041)	0.0301(0.0044)	0.0171(0.0048)
(30,50)	0.0222(0.0019)	0.0455(0.0034)	0.0330(0.0032)	0.0238(0.0040)
(50,20)	0.0139(0.0025)	0.0287(0.0044)	0.0114(0.0048)	0.0066(0.0048)
(50,30)	0.0162(0.0018)	0.0334(0.0036)	0.0199(0.0038)	0.0140(0.0038)
(50,50)	0.0190(0.0014)	0.0324(0.0026)	0.0218(0.0028)	0.0159(0.0028)

5. Multicomponent stress-strength under real data

In engineering, the concept of ”multicomponent stress-strength” refers to a situation where a system or structure is subjected to multiple types of stresses, and the strength of the system is determined by the weakest component’s ability to withstand those stresses. This is a crucial consideration in the design and analysis of various engineering systems and structures. Let’s explore the importance and provide some examples to illustrate the concept:

- In many real-life applications, engineering systems and structures are exposed to multiple stresses simultaneously. For example, a bridge may experience a combination of dead loads (its own weight), live loads (traffic), wind loads, and seismic loads. Understanding the interaction of these stresses is essential for the safe and efficient design of such structures.
- By considering the weakest component, engineers can identify potential failure points and critical areas in a system. This allows for targeted design improvements and risk mitigation measures. Reliability and Safety: Accounting for multicomponent stress-strength ensures that the system is designed with an appropriate safety margin. It helps avoid catastrophic failures and enhances the reliability of the structure or equipment. Analyzing multicomponent stress-strength aids in selecting appropriate materials for each component of the system, optimizing performance, and minimizing costs. On the other hand, the following examples provide a few illustrations of multicomponent stress-strength analysis:
- Aircraft structures need to withstand various stress components during their operation. This includes aerodynamic forces, mechanical loads during takeoff and landing, and thermal stresses due to temperature

Table 4. Coverage probability (confidence length) for $R_{1,3} = 0.178$ based on bayesian, parametric bootstrap (Pboot) and non-parametric bootstrap (NPboot) methods

Sample size n, m	Bayes	percentile-Pboot	t-Pboot	BC-Pboot	percentile-NPboot	t-NPboot	BC-NPboot
(20,20)	0.8903	0.9158	0.9219	0.9741	0.9811	0.9406	0.9479
	(0.4550)	(0.4441)	(0.4472)	(0.5068)	(0.5073)	(0.4553)	(0.4569)
(20,30)	0.8906	0.9169	0.9312	0.9778	0.9836	0.9459	0.9559
	(0.4365)	(0.4182)	(0.4236)	(0.4761)	(0.4794)	(0.4279)	(0.4348)
(20,50)	0.9040	0.9108	0.9232	0.9648	0.9752	0.9279	0.9384
	(0.4132)	(0.3971)	(0.4016)	(0.4431)	(0.4439)	(0.4025)	(0.4088)
(30,20)	0.9025	0.9284	0.9339	0.9761	0.9792	0.9532	0.9584
	(0.4145)	(0.4247)	(0.4276)	(0.4692)	(0.4723)	(0.4338)	(0.4374)
(30,30)	0.9046	0.9189	0.9276	0.9725	0.9763	0.9455	0.9481
	(0.4109)	(0.4019)	(0.4027)	(0.4423)	(0.4412)	(0.4075)	(0.4114)
(30,50)	0.9093	0.9221	0.9352	0.9684	0.9771	0.9426	0.9482
	(0.3961)	(0.3764)	(0.3787)	(0.4082)	(0.3123)	(0.3815)	(0.3859)
(50,20)	0.9056	0.9311	0.9383	0.9613	0.9679	0.9448	0.9531
	(0.3974)	(0.3987)	(0.4107)	(0.4321)	(0.4345)	(0.4119)	(0.4141)
(50,30)	0.9127	0.9371	0.9414	0.9702	0.9748	0.9512	0.9561
	(0.3903)	(0.3821)	(0.3818)	(0.4046)	(0.4069)	(0.3831)	(0.3857)
(50,50)	0.9208	0.9389	0.9421	0.9670	0.9712	0.9508	0.9527
	(0.3719)	(0.3569)	(0.3542)	(0.3781)	(0.3819)	(0.3623)	(0.3639)

Table 5. Coverage probability (confidence length) for $R_{1,3} = 0.418$ based on bayesian, parametric bootstrap (Pboot) and non-parametric bootstrap (NPboot) methods

Sample size n, m	Bayes	percentile-Pboot	t-Pboot	BC-Pboot	percentile-NPboot	t-NPboot	BC-NPboot
(20,20)	0.8853	0.9218	0.9264	0.9454	0.9539	0.9412	0.9532
	(0.5195)	(0.5263)	(0.5284)	(0.5311)	(0.5329)	(0.5291)	(0.5329)
(20,30)	0.8901	0.9354	0.9372	0.9586	0.9603	0.9547	0.9564
	(0.4917)	(0.4981)	(0.4975)	(0.5063)	(0.5042)	(0.4985)	(0.4989)
(20,50)	0.8968	0.9294	0.9336	0.9519	0.9572	0.9381	0.9459
	(0.4736)	(0.4635)	(0.4723)	(0.4853)	(0.4860)	(0.4682)	(0.4769)
(30,20)	0.8919	0.9346	0.9361	0.9448	0.9477	0.9519	0.9546
	(0.4871)	(0.4954)	(0.4972)	(0.4909)	(0.4936)	(0.4961)	(0.5010)
(30,30)	0.8936	0.9339	0.9357	0.9486	0.9519	0.9489	0.9570
	(0.4749)	(0.4660)	(0.4659)	(0.4671)	(0.4679)	(0.4680)	(0.4671)
(30,50)	0.9048	0.9361	0.9427	0.9531	0.9582	0.9511	0.9558
	(0.4593)	(0.4361)	(0.4386)	(0.4434)	(0.4459)	(0.4378)	(0.4362)
(50,20)	0.9037	0.9274	0.9385	0.9392	0.9470	0.9469	0.9567
	(0.4421)	(0.4635)	(0.4697)	(0.4508)	(0.4614)	(0.4649)	(0.4701)
(50,30)	0.9102	0.9413	0.9475	0.9509	0.9539	0.9477	0.9590
	(0.4321)	(0.4363)	(0.4381)	(0.4336)	(0.4359)	(0.4382)	(0.4351)
(50,50)	0.9211	0.9422	0.9469	0.9487	0.9513	0.9504	0.9540
	(0.4174)	(0.4092)	(0.4107)	(0.4071)	(0.4058)	(0.4084)	(0.4097)

Table 6. Coverage probability (confidence length) for $R_{1,3} = 0.779$ based on bayesian, parametric bootstrap (Pboot) and non-parametric bootstrap (NPboot) methods

Sample size n, m	Bayes	percentile-Pboot	t-Pboot	BC-Pboot	percentile-NPboot	t-NPboot	BC-NPboot
(20,20)	0.8803	0.9313	0.8972	0.9578	0.9015	0.8767	0.9291
	(0.3276)	(0.3473)	(0.3315)	(0.3721)	(0.3425)	(0.3253)	(0.3641)
(20,30)	0.8862	0.9274	0.8958	0.9662	0.9027	0.8702	0.9431
	(0.3229)	(0.3359)	(0.3262)	(0.3534)	(0.3363)	(0.3211)	(0.3517)
(20,50)	0.9016	0.9335	0.9130	0.9523	0.9371	0.9003	0.9509
	(0.3139)	(0.3224)	(0.3148)	(0.3397)	(0.3251)	(0.3143)	(0.3307)
(30,20)	0.8649	0.9137	0.8841	0.9465	0.9019	0.8630	0.9390
	(0.3157)	(0.3349)	(0.3192)	(0.3548)	(0.3309)	(0.3142)	(0.3471)
(30,30)	0.8833	0.9269	0.8979	0.9538	0.9149	0.8821	0.9348
	(0.3067)	(0.3247)	(0.3112)	(0.3368)	(0.3215)	(0.3084)	(0.3326)
(30,50)	0.8981	0.9321	0.9064	0.9511	0.9284	0.8978	0.9493
	(0.2982)	(0.3111)	(0.3016)	(0.3225)	(0.3132)	(0.2998)	(0.3172)
(50,20)	0.8869	0.92270	0.8902	0.9445	0.9135	0.8871	0.9327
	(0.3062)	(0.3216)	(0.3089)	(0.3381)	(0.3189)	(0.3037)	(0.3328)
(50,30)	0.8921	0.9334	0.9039	0.9501	0.9202	0.8910	0.9475
	(0.2976)	(0.3078)	(0.2992)	(0.3212)	(0.3098)	(0.2989)	(0.3184)
(50,50)	0.9093	0.9372	0.9084	0.9493	0.9336	0.9089	0.9482
	(0.2859)	(0.2987)	(0.2875)	(0.3091)	(0.2971)	(0.2849)	(0.3032)

changes. The weakest component's strength, such as the material used in critical parts like wings or fuselage, will determine the overall structural integrity of the aircraft.

- When designing a car, engineers must consider the multicomponent stresses that the vehicle will encounter during its lifespan, such as static loads, dynamic loads from road conditions, and thermal stresses due to engine operation. The design of safety-critical components like the chassis or braking system relies on understanding the weakest link in the system.
- In the design of buildings, bridges, and other civil structures, engineers must consider the combination of stresses they will experience during their lifetime, including dead loads, live loads, wind loads, and seismic forces. The weakest component in the structure, such as a particular beam or column, will determine the overall safety and stability of the entire structure.
- In the electronics industry, multicomponent stress-strength analysis is crucial for the design of electronic circuits and microelectronics. The combination of mechanical stress, thermal stress, and electrical stress can impact the performance and reliability of the components, and the weakest component will determine the overall functionality of the circuit.

Recently, [23] used two real data sets for comparing different estimation methods for multicomponent stress strength model under the Kumarswamy distribution. They introduced these data sets as follow: For the first data set of 50 observations on burr (in the unit of millimeter), the hole diameter is 12 mm and the sheet thickness is 3:15 mm. For the second data set of 50 observations, hole diameter and sheet thickness are 9 mm and 2 mm respectively. Hole diameter readings are taken on jobs with respect to one hole, selected and fixed as per a predetermined orientation. The two data sets relate to two different machines under comparison see [24]. These data sets are:
Data set I: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16

Table 7. Statistical inference of $R_{(1,3)}$ via different estimation methods

Estimation Methods	$\hat{R}_{(1,3)}$	95% CI of $R_{(1,3)}$	Confidence Interval Length
MLE	0.289	$----$	$-----$
⇒ Parametric Bootstrap			
p-boot	0.295	(0.1800,0.4217)	0.2417
t-boot	0.295	(0.1749,0.4165)	0.2416
BC-boot	0.295	(0.1694,0.4070)	0.2376
⇒ Non Parametric Bootstrap			
p-boot	0.292	(0.1736,0.4250)	0.2514
t-boot	0.292	(0.1713,0.4227)	0.2514
BC-boot	0.292	(0.1657,0.4175)	0.2518
Bayes	0.291	(0.1865,0.4092)	0.2227

and
Data set II:
0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16 .

The ML estimations of parameters for first and second parameters are $\hat{\alpha} = 2.1581, \hat{\sigma} = 0.3200$ and $\hat{\alpha} = 1.8708, \hat{\sigma} = 0.3200$, respectively. TL distribution has a good fit to these datasets similar Kumaraswamy distribution. Also we compute Cramer-Von Mises statistics (W^*) for the TL and the Kumaraswamy distributions based on data sets **I** and **II**. TheW^*s for these distributions are near to equal with 0.1 value.
Table 7 presents estimations, confidence intervals, highest posterior distribution (HPD) credible interval, confidence intervals length based on different methods that are presented in Section 3. ML estimation does not include Fisher information, thus bootstrap methods are used for this aim. Also Bayesian estimation based on square error loss function (posterior mean) with HPD interval are presented. The length of HPD interval is smaller than other confidence interval, but Bayesian estimate is near to other estimations. Thus Bayesian method is selected for this model.

6. Concluding remarks

Stress-strength analysis examines the interaction between the stresses produced on the materials and their strength; the term "materials" here refers not just to finished products or component components but also to a complete system. Reliability engineering uses a technology called stress-strength analysis. The distribution of environmental stressors and the distribution of component strengths have a mean and a standard deviation, respectively. The likelihood of failure is determined by how these distributions overlap. Stress-strength interference is another name for this overlapping phenomenon. In this article, first we introduce the multicomponent stress-strength reliability for the two parameters Topp-Leone distribution when both the stress and strength follows the same distribution. We investigate maximum likelihood, bootstrapping estimation methods (point and interval) of $\mathbf{R}_{(s,k)}$ and their performances are examined by extensive simulations. Also Bayesian estimate based on a non informative prior with Gibbs sampling are obtained. By bias and mean squared errors, maximum likelihood estimation and Bayesian estimation are better than bootstrap methods. But based on coverage probability or confidence length different estimations are selected as a best method. An example is provided to illustrate these results. It is hoped that these investigation will be useful for researchers dealing with multicomponent stress strength with the bounded data. The non-informative priors are taken into consideration by the Bayesian approach. The maximum likelihood estimation method takes into account the multicomponent stress-strength. The multicomponent stress-strength

under parametric bootstrap sampling and the multicomponent stress-strength under non-parametric bootstrap sampling are both calculated after that. The Bayesian calculated multicomponent stress strength is shown last. To assess and compare the effectiveness of each of these strategies, a complete simulation study was conducted using particular conditions and controls. As real data applications are a critical initial step in evaluating the models, methods, and approaches used, as well as a crucial first step in weighting them, it was difficult to ignore them. As a result, an application on two distinct types of real data was supplied. The percentile bootstrap confidence intervals, Student's t bootstrap confidence intervals, and bias-corrected and accelerated bootstrap confidence intervals are the three primary techniques used in non-parametric bootstrap sampling.
Here are some potential future points related to our topic:

- Future research could focus on developing more efficient and scalable computational techniques for estimating parameters in the multicomponent stress-strength reliability model under the TL distribution. This could involve leveraging advances in numerical methods, parallel computing, or even exploring quantum computing for complex systems.
- As new technologies emerge, such as quantum computing, nanotechnology, or advanced materials, the application of the multicomponent stress-strength reliability model under the TL distribution may expand into these cutting-edge fields. This could help ensure the reliability and safety of innovative technologies.
- Future research might include more extensive case studies and the incorporation of real-life field data to validate the proposed model's performance. This empirical evidence could strengthen the model's credibility and highlight its practical applications across diverse industries.
- With the increasing prevalence of autonomous systems like self-driving cars, drones, and robots, there may be a need to adapt the multicomponent stress-strength reliability model under the TL distribution to address the unique challenges of ensuring reliability in such systems.
- Research could explore how the multicomponent stress-strength reliability model can be integrated into resilience engineering and risk mitigation strategies. Understanding system vulnerabilities and designing robust systems can be crucial in critical applications like disaster response and infrastructure development.
- The Internet of Things and big data technologies are becoming more prevalent, generating vast amounts of data. Future developments may focus on how to integrate these data sources with the multicomponent stress-strength reliability model to improve accuracy and real-time monitoring.
- Combining Bayesian techniques with deep learning could be explored to handle complex, high-dimensional data and improve the estimation of reliability parameters in challenging scenarios. This hybrid approach may lead to more accurate predictions and uncertainty quantification.
- The multicomponent stress-strength reliability model could find applications in healthcare and medicine, particularly in medical device development, drug testing, and patient safety assessments. Ensuring reliability is crucial for enhancing healthcare outcomes.
- As the model gains prominence, there may be efforts to develop standardization and guidelines for its application in specific industries or sectors. These guidelines could facilitate consistent and reliable reliability assessments.
- Future research might involve collaborations between reliability engineers, statisticians, computer scientists, and domain experts from various industries to tackle complex challenges and explore new applications of the model.

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