STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, June 2023, pp 709–718.
Published online in International Academic Press (www.IAPress.org)

Comparison of E-Bayesian Estimators in Burr XII Model Using E-PMSE Based on Record Values

Alaa Alhamaidah [1], M. Naghizadeh Qomi [1,*], A. Kiapour [2]

[1] *Department of Statistics, University of Mazandaran, Babolsar, Iran*
[2] *Department of Statistics, Babol branch, Islamic Azad University, Babol, Iran*

Abstract In this paper, we consider the problem of E-Bayesian estimation and its expected posterior mean squared error (E-PMSE) in a Burr type XII model on the basis of record values. The Bayesian and E-Bayesian estimators are computed under different prior distributions for hyperparameters. The E-PMSE of E-Bayesian estimators are calculated in order to measure the estimated risk. Performances of the E-Bayesian estimators are compared using a Monte Carlo simulation. A real data set is analyzed for illustrating the estimation results.

Keywords Bayesian Estimation, Burr XII Distribution, E-Bayesian Estimation, E-PMSE, Record Values

AMS 2010 subject classifications 62F15, 62N05

DOI: 10.19139/soic-2310-5070-1687.

1. Introduction

Consider X has a Burr type XII ($BurrXII(\alpha,\beta)$) distribution with probability density function (pdf) and cumulative density function (cdf), respectively,

$$\begin{aligned} f(x) &= \alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)}, \ \ x>0, \\ F(x) &= 1-\frac{1}{(1+x^{\alpha})^{\beta}}, \ \ x>0, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape parameters. The Burr type XII has been proposed as a lifetime model in quality control, reliability analysis and failure time modeling. [17] obtained the maximum likelihood and interval estimation based on censored and uncensored data. [15] derived maximum likelihood, Bayesian and empirical Bayesian estimators based on progressive censored samples using various loss functions and [16] considered empirical Bayesian inference based on record values. For more references in this area, see [1].

An observation X_j is said to be an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. By convention X_1 is a record value. An analogous definition deals with lower record values. Data of this type arise in a wide variety of practical situations. Examples of application areas include industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys; see [2] for these types of applications. We denote the nth upper record value by X_n. The joint density of the first n-records $\mathbf{X} = (X_1, ..., X_n)$ is given by

$$f_{X_1,\cdots,X_n}(x_1,...,x_n) = f(x_n)\prod_{i=1}^{n-1}\frac{f(x_i)}{1-F(x_i)}, \quad x_1 < x_2 < \cdots < x_n. \tag{2}$$

*Correspondence to: M. Naghizadeh Qomi (Email: m.naghizadeh@umz.ac.ir). Department of Statistics, University of Mazandaran, Babolsar, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Also, the marginal pdf. of the nth record, X_n, is given by

$$f_{X_n}(x) = \frac{[-\log(1-F(x))]^{n-1}}{(n-1)!} f(x). \tag{3}$$

Therefore, from (1) and (2), the likelihood function of β based on $\mathbf{X} = (X_1, ..., X_n)$ at $\mathbf{x} = (x_1, ..., x_n)$ is given by

$$\begin{aligned} L(\beta) &= \frac{\alpha\beta x_n^{\alpha-1}}{(1+x_n^{\alpha})^{\beta+1}} \times \prod_{j=1}^{n-1} \frac{\alpha\beta x_j^{\alpha-1}}{(1+x_j^{\alpha})} \\ &= (\alpha\beta)^n (\prod_{j=1}^{n} \frac{x_j^{\alpha-1}}{1+x_j^{\alpha}}) e^{-\beta y}, \end{aligned} \tag{4}$$

where $y = \ln(1+x_n^{\alpha})$.

A classical method for estimating β is based on sample imformation, for example, calculating the maximum likelihood estimator (MLE) which can be derived from the equation $\frac{\partial L(\beta)}{\partial \beta} = 0$, which is given by $\hat{\beta}_{ml} = n/y$. A Bayesian approach to a statistical problem requires defining a prior distribution over the parameter space and loss function. Many Bayesians believe that just one prior can be elicited. In practice, the prior knowledge is vague and any elicited prior distribution is only an approximation to the true one. Various solutions to this problem have been proposed. The E-Bayesian estimator which was first introduced by [4], is the expectation of the Bayesian estimator of unknown parameter over the hyperparameter(s). E-Bayesian estimation is investigated by [5, 6, 7], [9], [13], [10], [12], [11] and [14]. Recently, [8] proposed E-posterior risk of E-Bayesian estimation for measuring the estimated risk of E-Bayesian estimations.

This paper is the first attempt to compute the E-Bayesian estimation in Burr type XII model and corresponding E-PMSE based on record observations. To do this, we obtain Bayesian and E-Bayesian estimators of β under the squared error loss (SEL) function in Section 2. The E-PMSE of E-Bayesian estimators are derived in Section 3. In Section 4, we perform a simulation study for comparing the performances of proposed estimators. Ilustration of the proposed estimators is given in Section 5. We end the paper by a concluding remark.

2. Bayesian estimation

Let $\mathbf{x} = (x_1, ..., x_n)$ be record observations from the $BurrXII(\alpha, \beta)$ distribution with likelihood function given in (4). By considering gamma conjugate prior density, $Ga(a, b)$ for parameter β with pdf

$$\pi(\beta|a,b) = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-b\beta}, \beta > 0, a > 0, b > 0, \tag{5}$$

the posterior density of β given $\mathbf{x}$ is again $Ga(a+n, b+y)$ with pdf

$$\pi(\beta|\mathbf{x}) = \frac{(b+y)^{n+a}}{\Gamma(a+n)} \beta^{a+n-1} e^{-(b+y)\beta}, \tag{6}$$

where Γ stands for complete gamma function as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

In the following theorem, we obtain the Bayesian estimator of β and its posterior MSE (PMSE) under the SEL function.

Theorem 2.1. *Let $\mathbf{x} = (x_1, ..., x_n)$ be record observations from $BurrXII(\alpha, \beta)$ distribution. Then, we have the following results:*

(i) The Bayesian estimate of β *is* $\hat{\beta}_{BS} = \dfrac{a+n}{b+y}$.

(ii) The PMSE of the Bayesian estimate is given by $PR(\hat{\beta}_{BS}) = \frac{a+n}{(b+y)^2}$.

Proof

(i) Considering posterior density (6), it is clear that

$$\hat{\beta}_{BS} = E[\beta|\mathbf{x}] = \frac{a+n}{b+y}.$$

(ii) The PMSE of the Bayesian estimator under the SEL function is given by

$$PR(\hat{\beta}_{BS}) = Var[\beta|\mathbf{x}] = \frac{a+n}{(b+y)^2}.$$

□

3. E-Bayesian estimation

Consider prior $\pi(\beta|a,b)$ for β with hyperparameters a and b given in (5). According to [4] the prior parameters a and b should be selected to guarantee that $\pi(\beta|a,b)$ is a decreasing function of β. Therefore, hyperparameters a and b should be in the ranges $0 < a < 1$ and $b > 0$, respectively, due to $\frac{d\pi(\beta|a,b)}{d\beta} < 0$. A prior distribution with thinner tail reduces the robustness of Bayesian estimate, see [3]. Accordingly, b should not be too big while $0 < a < 1$. For $b > 0$, there is a constant, say c, that $0 < b < c$.

The E-Bayesian estimator of β is the expectation of the Bayesian estimator for the all hyperparameters which is defined as

$$\hat{\beta}_{EB}(\mathbf{x}) = \int\int_D \hat{\beta}_{BS}(\mathbf{x})\pi(a,b)dadb = E^{a,b}(\hat{\beta}_{EB}(\mathbf{X})),$$

where D is the domain of a and b and $\pi(a,b)$ is the joint prior density function of a and b.

Assuming that a and b are independent with bivariate density function $\pi(a,b) = \pi(a)\pi(b)$. We consider three prior distributions of the hyperparameters a and b as follows:

$$\pi_1(a,b) = \frac{a^{u-1}(1-a)^{v-1}}{cB(u,v)}, \ \ 0 < a < 1, \ 0 < b < c, \tag{7}$$

$$\pi_2(a,b) = \frac{2(c-b)a^{u-1}(1-a)^{v-1}}{c^2B(u,v)}, \ \ 0 < a < 1, \ 0 < b < c, \tag{8}$$

$$\pi_3(a,b) = \frac{2ba^{u-1}(1-a)^{v-1}}{c^2B(u,v)}, \ \ 0 < a < 1, \ 0 < b < c, \tag{9}$$

where $B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$ is the beta function. These distributions are used to investigate the influence of the different prior distributions on the E-Bayesian estimation, see [9] and [12].

In the following theorem, we obtain the E-Bayesian estimates of β under the SEL function.

Theorem 3.1. *Let* $\boldsymbol{x} = (x_1, ..., x_n)$ *be record observations from the* $BurrXII(\alpha,\beta)$ *distribution. Then, the E-Bayesian estimates of parameter* β *corresponding to the priors given in* (7)-(9) *under the SEL function are given* *respectively by*

$$\hat{\beta}_{EB1} = \frac{u+n(u+v)}{c(u+v)}\ln(\frac{c+y}{y}), \quad (10)$$

$$\hat{\beta}_{EB2} = 2\frac{u+n(u+v)}{c^2(u+v)}[(c+y)\ln(\frac{c+y}{y})-c], \quad (11)$$

$$\hat{\beta}_{EB3} = 2\frac{u+n(u+v)}{c^2(u+v)}[y\ln(\frac{y}{c+y})+c]. \quad (12)$$

Proof

For the joint prior density function $\pi_1(a,b)$ given in (7), the E-Bayesian estimation of β is obtained as

$$\begin{aligned}
\hat{\beta}_{EB1} &= \int_0^c\int_0^1 \hat{\beta}_{BS}\ \pi_1(a,b)dadb \\
&= \int_0^c\int_0^1 \frac{a+n}{b+y}\frac{a^{u-1}(1-a)^{v-1}}{cB(u,v)}dadb \\
&= \frac{1}{cB(u,v)}\int_0^1 (a+n)a^{u-1}(1-a)^{v-1}da[\int_0^c \frac{1}{b+y}db] \\
&= \frac{1}{cB(u,v)}\ln(\frac{c+y}{y})\times[B(u+1,v)+nB(u,v)] \\
&= \frac{u+n(u+v)}{c(u+v)}\ln(\frac{c+y}{y}).
\end{aligned}$$

If the joint prior density function is $\pi_2(a,b)$, then the E-Bayesian estimate of β is given by

$$\begin{aligned}
\hat{\beta}_{EB2} &= \int_0^c\int_0^1 \hat{\beta}_{BS}\ \pi_2(a,b)dadb \\
&= \int_0^c\int_0^1 \frac{a+n}{b+y}\frac{2(c-b)a^{u-1}(1-a)^{v-1}}{c^2B(u,v)}dadb \\
&= \frac{2}{c^2B(u,v)}\int_0^1 (a+n)a^{u-1}(1-a)^{v-1}da\int_0^c \frac{c-b}{b+y}db \\
&= 2\frac{u+n(u+v)}{c^2(u+v)}[(c+y)\log(\frac{c+y}{y})-c].
\end{aligned}$$

Similarly, for the prior density function $\pi_3(a,b)$, we get

$$\begin{aligned}
\hat{\beta}_{EB3} &= \int_0^c\int_0^1 \hat{\beta}_{BS}\ \pi_3(a,b)dadb \\
&= \int_0^c\int_0^1 \frac{a+n}{b+y}\frac{2ba^{u-1}(1-a)^{v-1}}{c^2B(u,v)}dadb \\
&= \frac{2}{c^2B(u,v)}\int_0^1 (a+n)a^{u-1}(1-a)^{v-1}da\int_0^c \frac{b}{b+y}db \\
&= 2\frac{u+n(u+v)}{c^2(u+v)}[y\log(\frac{y}{c+y})+c].
\end{aligned}$$

□

In the following theorem, we provide a relationship between the proposed E-Bayesian estimators.

Theorem 3.2. *We have the following relation between E-Bayesian estimators given in* (10), (11) *and* (12)*:*

$$\hat{\beta}_{EB3} < \hat{\beta}_{EB1} < \hat{\beta}_{EB2}. \tag{13}$$

Proof
We have

$$\begin{aligned}\hat{\beta}_{EB_2} - \hat{\beta}_{EB_1} &= \hat{\beta}_{EB_1} - \hat{\beta}_{EB_3} \\ &= \frac{r(u+v)+u}{c(u+v)}\left\{\frac{2y+c}{c}\ln\left(1+\frac{c}{y}\right)-2\right\}.\end{aligned} \tag{14}$$

Using the fact $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^4}{4} + ...$, for $-1 < x < 1$, we can write

$$\begin{aligned}\frac{2y+c}{c}\ln\left(1+\frac{c}{y}\right)-2 &= \frac{2y+c}{c}\left\{\frac{c}{y}-\frac{1}{2}\left(\frac{c}{y}\right)^2+\frac{1}{3}\left(\frac{c}{y}\right)^3-\frac{1}{4}\left(\frac{c}{y}\right)^4+\dots\right\}-2 \\ &= \left(2-\frac{c}{y}+\frac{2}{3}\left(\frac{c}{y}\right)^2-\frac{1}{2}\left(\frac{c}{y}\right)^2+\frac{2}{5}\left(\frac{c}{y}\right)^4+\dots\right) \\ &+ \left(\frac{c}{y}-\frac{1}{2}\left(\frac{c}{y}\right)^2+\frac{1}{3}\left(\frac{c}{y}\right)^3-\frac{1}{4}\left(\frac{c}{y}\right)^4+\dots\right)-2 \\ &= \left(\frac{c^2}{6y^2}-\frac{c^3}{6y^3}\right)+\left(\frac{3c^4}{6y^4}-\frac{2c^5}{15y^5}\right)+\dots \\ &= \frac{c^2}{6y^2}\left(1-\frac{c}{y}\right)+\frac{c^4}{60y^4}\left(9-\frac{8c}{y}\right)+\dots.\end{aligned} \tag{15}$$

Therefore, we get

$$\hat{\beta}_{EB_2} - \hat{\beta}_{EB_1} = \hat{\beta}_{EB_1} - \hat{\beta}_{EB_3} > 0,$$

which completes the proof. □

It is noticed here that we can conclude from (13) that

$$\hat{\beta}_{EB3} - \beta < \hat{\beta}_{EB1} - \beta < \hat{\beta}_{EB2} - \beta,$$

which gives us the following relation for the bias of E-Bayesian estimators:

$$Bias_\beta(\hat{\beta}_{EB3}) < Bias_\beta(\hat{\beta}_{EB1}) < Bias_\beta(\hat{\beta}_{EB2}).$$

4. The E-PMSE of E-Bayesian estimation

The concept of E-PMSE of E-Bayesian estimator is introduced by [8] for measuring the estimated risk of E-Bayesian estimators. The E-PMSE of E-Bayesian estimator is defined as

$$EP(\hat{\beta}_{EB}) = \int\int_D PR(\hat{\beta}_{BS})\,\pi(a,b)dadb,$$

where D is the domain of a and b, $PR(\hat{\beta}_{BS})$ is the posterior risk of Bayesian estimation and $\pi(a,b)$ is the joint prior density of a and b.

In the following theorem, we present the formulas for E-PMSE of E-Bayesian estimators of β under the SEL function.

Theorem 4.1. *Let* $\mathbf{x} = (x_1, ..., x_n)$ *be record observations from* $BurrXII(\alpha, \beta)$ *distribution. For the prior density functions of* $\pi_i(a, b),\ i = 1, 2, 3,$ *given in* (7), (8) *and* (9)*, the E-PMSE of E-Bayesian estimates* $\hat{\beta}_{EBi},\ i = 1, 2, 3,$ *are given, respectively, by*

$$\begin{aligned} EP(\hat{\beta}_{EB1}) &= \frac{u + n(u+v)}{(u+v)y(c+y)}, \\ EP(\hat{\beta}_{EB2}) &= 2\frac{u + n(u+v)}{c^2(u+v)}[\frac{c}{y} + \ln(\frac{y}{c+y})], \\ EP(\hat{\beta}_{EB3}) &= 2\frac{u + n(u+v)}{c^2(u+v)}[\ln(\frac{c+y}{y}) - \frac{c}{c+y}]. \end{aligned}$$

Proof
Using Theorem 2.1(ii), the definition of E-PMSE, and for $\pi_1(a, b)$ given in (7), we get

$$\begin{aligned} EP(\hat{\beta}_{EB1}) &= \int_0^c \int_0^1 PR(\hat{\beta}_{BS})\pi_1(a, b) da db \\ &= \int_0^c \int_0^1 \frac{a+n}{(b+y)^2} \frac{a^{u-1}(1-a)^{v-1}}{cB(u, v)} da db \\ &= \frac{1}{cB(u, v)} \int_0^1 (a+n)a^{u-1}(1-a)^{v-1} da \int_0^c \frac{1}{(b+y)^2} db \\ &= \frac{u + n(u+v)}{(u+v)y(c+y)} \end{aligned}$$

Also, for $\pi_2(a, b)$ given in (8), we obtain

$$\begin{aligned} EP(\hat{\beta}_{EB2}) &= \int_0^c \int_0^1 PR(\hat{\beta}_{BS})\pi_2(a, b) da db \\ &= \int_0^c \int_0^1 \frac{a+n}{(b+y)^2} \frac{2(c-b)a^{u-1}(1-a)^{v-1}}{c^2B(u, v)} da db \\ &= \frac{2}{c^2B(u, v)} \int_0^1 (a+n)a^{u-1}(1-a)^{v-1} da \int_0^c \frac{c-b}{(b+y)^2} db \\ &= 2\frac{u + n(u+v)}{c^2(u+v)}[\frac{c}{y} + \ln(\frac{y}{c+y})]. \end{aligned}$$

Similarly, for $\pi_3(a, b)$ given in (9), we obtain

$$\begin{aligned} EP(\hat{\beta}_{EB3}) &= \int_0^c \int_0^1 PR(\hat{\beta}_{BS})\pi_3(a, b) da db \\ &= \int_0^c \int_0^1 \frac{a+n}{b+y} \frac{2ba^{u-1}(1-a)^{v-1}}{c^2B(u, v)} da db \\ &= \frac{2}{c^2B(u, v)} \int_0^1 (a+n)a^{u-1}(1-a)^{v-1} da \int_0^c \frac{b}{(b+y)^2} db \\ &= 2\frac{u + n(u+v)}{c^2(u+v)}[\ln(\frac{c+y}{y}) - \frac{c}{c+y}]. \end{aligned}$$

□

5. Simulation study

In this section, we perform a simulation study for comparison of proposed estimators of β. For this purpose, we generate record observations $x_1, ..., x_n$ of size $n \in \{2, 3, 5, 7, 10, 15, 30\}$ from $BurrXII(\alpha, \beta)$ distribution with $\alpha = 1$ and $\beta = 2, 3, 5$ and compute the E-Bayesian estimates and corresponding E-PMSEs for the selected values $c = 0.5, 1, 1.5$, $u = 2$ and $v = 3$. The performances of the E-Bayesian estimates and corresponding E-PMSEs are compared for repeated $M = 10^5$ times simulation runs and are summarized in Tables 1-3.

The following conclusions can be drawn from the results:

- In all considered situations of parameter β, E-Bayesian estimate and its E-PMSE are robust.
- For fixed values of c and the same values of n, we observe the following relationship between E-Bayesian estimates and corresponding E-PMSEs:

$$\hat{\beta}_{EB3} < \hat{\beta}_{EB1} < \hat{\beta}_{EB2},$$

and

$$EP(\hat{\beta}_{EB3}) < EP(\hat{\beta}_{EB1}) < EP(\hat{\beta}_{EB2}). \tag{16}$$

If we use E-PMSE for evaluating the E-Bayesian estimates, then from (16) we conclude that $\hat{\beta}_{EB3}$ is superior to $\hat{\beta}_{EB1}$ and $\hat{\beta}_{EB1}$ is superior to $\hat{\beta}_{EB2}$.

- For fixed c, the performances of E-Bayesian estimates and corresponding E-PMSEs improve by increasing n.

Table 1. Performances of E-Bayesian estimates and their E-PMSEs for $\beta = 2$.

n	c	$\hat{\beta}_{EB1}$	$\hat{\beta}_{EB2}$	$\hat{\beta}_{EB3}$	$EP(\hat{\beta}_{EB1})$	$EP(\hat{\beta}_{EB2})$	$EP(\hat{\beta}_{EB3})$
	0.5	2.7653	3.1535	2.3772	5.7970	8.2097	3.3843
2	1	2.1713	2.5899	1.7527	3.4991	5.2928	1.7054
	1.5	1.8322	2.2498	1.4146	2.5336	3.9806	1.0866
	0.5	2.5708	2.7909	2.3506	2.8431	3.5330	2.1533
3	1	2.1578	2.4407	1.8750	1.9395	2.5924	1.2867
	1.5	1.8903	2.1998	1.5809	1.4898	2.0849	0.8948
	0.5	2.3481	2.4542	2.2420	1.2947	1.4476	1.1418
5	1	2.1065	2.2691	1.9440	1.0152	1.2068	0.8235
	1.5	1.9256	2.1230	1.7281	0.8421	1.0459	0.6384
	0.5	2.2503	2.3168	2.1837	0.8033	0.8598	0.7468
7	1	2.0838	2.1952	1.9724	0.6790	0.7627	0.5953
	1.5	1.9486	2.0921	1.8051	0.5905	0.6888	0.4923
	0.5	2.1843	2.2270	2.1416	0.5130	0.5356	0.4905
10	1	2.0708	2.1466	1.9951	0.4575	0.4947	0.4204
	1.5	1.9727	2.0748	1.8706	0.4136	0.46066	0.3666
	0.5	2.1272	2.1535	2.1008	0.3162	0.3246	0.3077
15	1	2.0539	2.1028	2.0051	0.2937	0.3087	0.2787
	1.5	1.9874	2.0556	1.9191	0.2744	0.2945	0.2543
	0.5	2.0614	2.0735	2.0493	0.1449	0.1467	0.1431
30	1	2.0264	2.0497	2.0031	0.1399	0.1432	0.1366
	1.5	1.9930	2.0268	1.9593	0.1352	0.1400	0.1305

Table 2. Performances of E-Bayesian estimates and their E-PMSEs for $\beta = 3$.

n	c	$\hat{\beta}_{EB1}$	$\hat{\beta}_{EB2}$	$\hat{\beta}_{EB3}$	$EP(\hat{\beta}_{EB1})$	$EP(\hat{\beta}_{EB2})$	$EP(\hat{\beta}_{EB3})$
	0.5	3.6273	4.2432	3.0114	9.7817	14.3973	5.1662
2	1	2.7483	3.3748	2.1219	5.7007	8.9565	2.4449
	1.5	2.2710	2.8746	1.6673	4.0547	6.6075	1.5019
	0.5	3.5070	3.8949	3.1191	5.1694	6.6962	3.6426
3	1	2.8355	3.2997	2.3713	3.3522	4.6911	2.0133
	1.5	2.4266	2.9130	1.9403	2.5044	3.6725	1.3364
	0.5	3.3263	3.5332	3.1193	2.5552	2.9549	2.1555
5	1	2.8884	3.1846	2.5921	1.8949	2.3532	1.4366
	1.5	2.5808	2.9245	2.2371	1.5182	1.9789	1.0575
	0.5	3.2435	3.3800	3.1071	1.6545	1.8173	1.4918
7	1	2.9229	3.1382	2.7077	1.3287	1.5498	1.1076
	1.5	2.6777	2.9432	2.4122	1.1155	1.3602	0.8708
	0.5	3.1881	3.2784	3.097	1.0880	1.1569	1.0191
10	1	2.9591	3.1123	2.8060	0.9306	1.0364	0.8249
	1.5	2.7709	2.9698	2.5719	0.8151	0.9419	0.6882
	0.5	3.1345	3.1915	3.0775	0.6852	0.7121	0.6584
15	1	2.9811	3.0835	2.8787	0.6175	0.6628	0.5722
	1.5	2.8468	2.9860	2.7077	0.5625	0.6208	0.5042
	0.5	3.0656	3.0923	3.0389	0.3204	0.3262	0.3146
30	1	2.9896	3.0403	2.9389	0.3044	0.3150	0.2937
	1.5	2.9185	2.9908	2.8462	0.2899	0.30481	0.2751

Table 3. Performances of E-Bayesian estimates and their E-PMSEs for $\beta = 5$.

n	c	$\hat{\beta}_{EB1}$	$\hat{\beta}_{EB2}$	$\hat{\beta}_{EB3}$	$EP(\hat{\beta}_{EB1})$	$EP(\hat{\beta}_{EB2})$	$EP(\hat{\beta}_{EB3})$
	0.5	4.9570	6.0066	3.9073	18.3526	28.3497	8.3554
2	1	3.5898	4.5807	2.5989	10.2817	16.9092	3.6543
	1.5	2.8918	3.8063	1.9772	7.1756	12.2035	2.1478
	0.5	5.0300	5.7770	4.2831	10.5218	14.4205	6.6231
3	1	3.8691	4.6837	3.0546	6.4221	9.5319	3.3123
	1.5	3.2154	4.0235	2.4073	4.6517	7.2263	2.0771
	0.5	5.0258	5.4806	4.5710	5.7496	6.9968	4.5024
5	1	4.1609	4.7516	3.5701	3.9588	5.2282	2.6893
	1.5	3.6044	4.2490	2.9599	3.0390	4.2274	1.8507
	0.5	5.0326	5.3543	4.7109	3.9467	4.5221	3.3713
7	1	4.3462	4.8099	3.8825	2.9431	3.6339	2.2524
	1.5	3.8648	4.4028	3.3268	2.3581	3.0644	1.6518
	0.5	5.0504	5.2745	4.8263	2.7151	2.9809	2.4492
10	1	4.5252	4.8777	4.1727	2.1750	2.5408	1.8091
	1.5	4.1259	4.5590	3.6927	1.8198	2.2263	1.4134
	0.5	5.0498	5.1971	4.9026	1.7736	1.8843	1.6629
15	1	4.6761	4.9260	4.4262	1.5178	1.6895	1.3461
	1.5	4.3692	4.6936	4.0447	1.3285	1.5356	1.1214
	0.5	5.0239	5.0955	4.9523	0.85982	0.8852	0.8345
30	1	4.8264	4.9582	4.6946	0.79294	0.8376	0.7482
	1.5	4.6486	4.8315	4.4658	0.7359	0.7955	0.6763

6. Real data analysis

In this section, a real data set of testing the efficacy of an analgesic has been analyzed and corresponding parameters have been estimated. The data are Relief Times (in Hours) for 50 Patients patients receiving a fixed dosage of a medication. The data set is provided by [18] and presented in Table 4.

The MLE's of α and β from data are 5.0008 and 8.2689, respectively. A Kolmogrov-Smirnov test was used for checking the validity of the Burr type XII distribution based on these parameters. The test statistic K-S= 0.1031 with a corresponding p-value= 0.6247 implies that the Burr XII distribution have a good fit to the above data.

The observed record values $\mathbf{x} = (x_1, x_2, x_3, x_4)$ are obtained to be

$$\mathbf{x} = (0.7, 0.84, 0.85, 0.87).$$

We get $y = \ln(1 + (0.87)^{5.0008})$=0.4044, and then, the MLE of β is $\frac{n}{y} = 9.8918$.

We summarize the E-Bayesian estimates and their E-PMSEs in Table 5 for selected values of $c = 0.1, 0.3, 0.4, 0.5$, $u = 2$ and $\nu = 3$. It is observed that $\hat{\beta}_{EB3}$ has smaller E-PMSE and therefore is suggested. Also, the numerical results are consistent with the numerical simulation results.

Table 4. Relief Times (in Hours) for 50 Patients.

0.70	0.84	0.58	0.50	0.55	0.82	0.59	0.71	0.72	0.61
0.62	0.49	0.54	0.72	0.36	0.71	0.35	0.64	0.85	0.55
0.59	0.29	0.75	0.53	0.46	0.60	0.60	0.36	0.52	0.68
0.80	0.55	0.84	0.70	0.34	0.70	0.49	0.56	0.71	0.61
0.57	0.73	0.75	0.58	0.44	0.81	0.80	0.87	0.29	0.50.

Table 5. Results for E-Bayesian estimates and corresponding E-PMSEs.

c	$\hat{\beta}_{EB1}$	$\hat{\beta}_{EB2}$	$\hat{\beta}_{EB3}$	$EP(\hat{\beta}_{EB1})$	$EP(\hat{\beta}_{EB2})$	$EP(\hat{\beta}_{EB3})$
0.1	9.7230	10.0808	9.3652	21.5732	23.1597	19.9867
0.3	8.1395	8.8886	7.3905	15.4477	18.2764	12.6191
0.4	7.5650	8.4253	6.7046	13.5273	16.5802	10.4743
0.5	7.0831	8.0232	6.1430	12.0315	15.1914	8.8716
Range	2.6399	2.0576	3.2222	9.5417	7.9683	11.1152

7. Conclusion

The problem of E-Bayesian estimation in a Burr type XII distribution and its E-PMSE based on record values are considered. Using different joint prior distributions of hyperparameters, the Bayesian and E-Bayesian estimates of β are computed. The formulas for the E-PMSE of E-Bayesian estimators are presented for comparing the error of E-Bayesian estimators. A Monte Carlo simulation study is performed for comparison of the E-Bayesian estimators. Our findings show that the performances of estimates improve in terms of E-PMSE values when n increases. Moreover, The results of the real data analysis agree with the simulation results, where the estimator $\hat{\beta}_{EB3}$ is better than other E-Bayesian estimators in terms of their E-PMSEs.

REFERENCES

1. R. Arabi Belaghi, M. Arashi, and S. M. M. Tabatabaey, *On the construction of preliminary test estimator based on record values for the Burr XII model*, Communications in Statistics-Theory and Methods, vol. 44, pp. 1–23, 2015.
2. B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *Records*, John Wiley, New York, 1998.

3. J. O. Berger, *Statistical decision theory and Bayesian analysis*, 2nd edition, NY: Springer-Verlag, New York, 1985.
4. M. Han, *The structure of hierarchical prior distribution and its applications*, Chinese Operations Research and Management Science, vol. 6, pp. 31–40, 1997.
5. M. Han, *E-Bayesian estimation and hierarchical Bayesian estimation of failure rate*, Applied Mathematical Modeling, pp. 1915–1922, 2009.
6. M. Han, *E-Bayesian estimation of the reliability derived from binomial distribution*, Applied Mathematical Modeling, vol. 35, no. 5, pp. 2419–2424, 2011.
7. M. Han, *The E-Bayesian and hierarchical Bayesian estimations of Pareto distribution parameter under different loss functions*, Journal of Statistical Computation and Simulation, vol. 87, pp. 577–593, 2017.
8. M. Han, *E-Bayesian estimation and its E-MSE under the scaled squared error loss function, for exponential distribution as example*, Communications in Statistics-Simulation and Computation, vol. 48, no. 6, pp. 1880–1890, 2019.
9. Z. F. Jaheen, and H. M. Okasha, *E-Bayesian estimation for the burr type XII model based on type-II censoring*, Applied Mathematical Modeling, vol. 35, pp. 4730–4737, 2011.
10. A. Kiapour, *Bayes, E-Bayes and robust Bayes premium estimation and prediction under the squared log error loss function*, Journal of the Iranian Statistical Society, vol. 17, pp. 33–47, 2018.
11. A. Kiapour, *Bayesian estimation of the expected queue length of a system M/M/1 with certain and uncertain priors*, Communications in Statistics-Theory and Methods, vol. 51, pp. 5310–5317, 2022.
12. M. Naghizadeh Qomi, and A. Kiapour, *E-Bayesian and hierarchical Bayesian estimation of risk premium*, 15th Iranian Statistics Conference, Yazd, pp. 336–343, 2020.
13. H. Okasha, and J. Wang, *E-Bayesian estimation for the geometric model based on record statistics*, Applied Mathematical Modeling, vol. 40, pp. 658–670, 2016.
14. O. Shojaee, H. Piriaei, and M. Babanezhad, *E-Bayesian estimations and Its E-MSE For compound Rayleigh progressive type-II censored data*, Statistics, Optimization & Information Computing, vol. 10, pp. 1056–1071, 2022.
15. A. A. Soliman, *Estimation of life from proressively censored data using Burr XII model*, IEEE. Trans. Rel., vol. 54, no. 1, pp. 34–42, 200).
16. L. Wang, and Y. Shi, *Empirical Bayes inference for the Burr model based on records*, Appl. Math. Sci., vol. 4, no. 34, pp. 1663–1670, 2010.
17. F. K. Wang, J. B. Keats, and W. J. Zimmer, *Maximum likelihood estimation of the Burr XII distribution with censored and uncensored data*, Microelectron Reliability, vol. 36, pp. 359–362, 1996.
18. D. R. Wingo, *Maximum likelihood methods for fitting the Burr type XII distribution to life test data*, Biometric Journal, vol. 1, pp. 77–83, 1983.