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Nonparametric tests of independence using copula-based Rényi and Tsallis divergence measures

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Abstract We introduce new nonparametric independence tests based on Rényi and Tsallis divergence measures and copula density function. These tests reduce the complexity of calculations because they only depend on the copula density. The copula density estimated using the local likelihood probit-transformation method is appropriate for the identification of independence. Also, we present the consistency of the copula-based Rényi and Tsallis divergence measures estimators that are considered as test statistics. A simulation study is provided to compare the empirical power of these new tests with the independence test based on the empirical copula. The simulation results show that the suggested tests outperform in weak dependency. Finally, an application in hydrology is presented.

Keywords Independence test, Rényi divergence, Tsallis divergence, Copula density, Probit-transformation.

AMS 2010 subject classifications 62G10, 62H15

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1. Introduction

One of the most important tasks faced by experimenters is to assert independence in the data. Most of the classical independence testing was based on a dependence measure, such as the Pearson linear correlation, Kendall's τ, and Spearman's ρ. These tests are usually inconsistent, so Blum et al. [3] used the Cramér-von Mises (CvM) distance to compare the joint empirical distribution function with the product of its corresponding marginal empirical distributions. Genest and Remillard [11] presented the test of independence based on the CvM distance and empirical copula.

Belalia et al. [2] showed that the empirical copula-based independent test fails when the dependency occurs only at the tails. They suggested that the copula density is appropriate for the detection of independence. So, they proposed the independence test based on Bernstein copula density and Kullback–Leibler (KL) divergence. Mohammadi et al. [32] introduced nonparametric tests of independence via copula-based α-divergence. They showed that the Hellinger distance, as a special case of the α-divergence, outperforms in comparison of the existing test based on empirical copula.

The idea of divergence measure has been widely employed in probability, statistics, information theory, and related fields. The KL divergence which is introduced in [25], also regarded as Relative Entropy or Mutual Information, is a nonsymmetrical measure of the distinction between two probability density functions. Rényi [36]

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introduced a generalized entropy and divergence measure that naturally extend the KL divergence. Tsallis entropy was firstly described by Havrda and Charvat [15] and unearthed by Tsallis [42]. Generalizations of Shannon's entropy have attracted the attention of many researchers. For recent properties of these generalization measures, we refer to [17, 19, 22, 23, 24, 33, 14, 40]. Also, Tsallis entropy extensions have been performed by some researchers, among which we can mention [41, 20, 21]. Some of these measures can be considered as a statistic for the independence test.

Ma and Sun [27] introduced the concept of copula entropy by combining the KL divergence and the copula density. They demonstrated that the KL divergence is equal to the negative of copula entropy. The copula entropy was considered as a measure of multivariate association by Blumentritt and Schmid [4]. In this paper, we provide Rényi and Tsallis divergence measures based on copula density together with their basic properties. We use these measures to perform two nonparametric tests the independence. These tests are simple to implement and reduce the complexity because they depend only on the copula density. Also, the copula-based Rényi and Tsallis divergence measure independence tests provide a bigger power compared to the empirical copula-based test in weak dependency.

The rest of the paper is arranged as follows. In Section 2, the copula-based Rényi and Tsallis divergence measures together with their basic properties are provided. Estimators of the copula-based Rényi and Tsallis divergence measures are considered as test statistics for testing independence, and their consistency is established in Section 3. In Section 4, the simulation results are provided to compare the empirical power of independence tests. Finally, an application of new methods in hydrology is presented in Section 5.

2. Copula-based Rényi and Tsallis divergence measures

The KL divergence between two density functions f_1 and f_2 is defined as

$$KL(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x) \log \frac{f_1(x)}{f_2(x)} dx, \qquad \alpha > 0, \alpha \neq 1.$$

This divergence is nonnegative, and $KL(f_1, f_2) = 0$ if and only if $f_1(x) = f_2(x)$. The Rényi divergence (or relative Rényi entropy) of order α between two density functions f_1 and f_2 is defined as

$$R_\alpha(f_1 \parallel f_2) = \frac{1}{\alpha - 1} \log \int_{\mathbb{R}} f_1^{\alpha}(x)\, f_2^{1-\alpha}(x) dx, \qquad \alpha > 0, \alpha \neq 1, \tag{1}$$

and the Tsallis divergence of order α is defined as

$$T_\alpha(f_1 \parallel f_2) = \frac{1}{\alpha - 1} \Big(\int_{\mathbb{R}} f_1^{\alpha}(x)\, f_2^{1-\alpha}(x) dx - 1 \Big), \qquad \alpha > 0, \alpha \neq 1. \tag{2}$$

The larger α values give the Rényi and Tsallis divergence measures dominated by the greatest ratio between the two functions. One of the interesting special cases of the Rényi and Tsallis divergence measures occurs for $\alpha \to 1$, which gives the KL divergence. We also get the well-known Bhattacharyya distance in the special case where $\alpha = 0.5$ for the Rényi divergence. On the other hand, it can be noted that the special case $\alpha = 0.5$ for the Tsallis divergence is equal to the double Hellinger distance between probability distributions.

The Rényi and Tsallis divergence measures between a joint density function and the product of its corresponding marginal density functions can be rewritten in terms of the copula density function. The copula function suggested by Sklar [38] has been implemented in a broad spectrum of scientific fields such as hydrology and finance. Let random variables X and Y follow arbitrary marginal cumulative distribution functions F_X and F_Y, respectively. Then there is a copula function C that combines these marginal distribution functions to give the joint distribution function F as $F(x, y) = C(F_X(x), F_Y(y); \theta)$, where $(x, y) \in \mathbb{R}^2$ and θ is a copula parameter. Recently, semiparametric methods for the estimation of copula parameter based on minimum Alpha-Divergence are presented in [31], which perform well in small sample size and weak dependency. If C is an absolutely continuous copula distribution on $[0,1]^2$, then its density function is $c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v}$ where (U,V) is a random vector defined in the unit square with uniform marginal distributions as $U=F(X)$ and $V=F(Y)$. It is obvious that the relationship between the copula density c and the joint density function f of (X,Y) can be represented as

$$f(x,y)=c(F_X(x),F_Y(y))f_X(x)f_Y(y), \qquad (x,y)\in\mathbb{R}^2, \tag{3}$$

where f_X and f_Y are the marginal density function of X and Y, respectively.

The KL divergence based on copula density using equation (3) can be written as

$$\begin{aligned} KL(c) &\equiv KL(f \parallel f_X f_Y) \\ &= \int_{\mathbb{R}^2} f(x,y)\log\Big(\frac{f(x,y)}{f_X(x)f_Y(y)}\Big)dxdy \\ &= \int_{[0,1]^2} c(u,v)\log c(u,v)\, dudv \\ &= E\Big(\log c(U,V)\Big). \end{aligned} \tag{4}$$

Now, we focus on representing the Rényi and Tsallis divergence measures based on the copula density to separate the dependence structure of the marginal distributions. Let X and Y be two random variables with density functions f_X and f_Y, respectively, joint density function f, and copula density function c. Using equations (1) and (3), for $\alpha>0$ and $\alpha\neq 1$, the copula-based Rényi divergence measure is defined as

$$\begin{aligned} R_\alpha(c) &\equiv R_\alpha(f \parallel f_X f_Y) \\ &= \frac{1}{\alpha-1}\log\int_{\mathbb{R}^2} f^\alpha(x,y)\,(f_X(x)f_Y(y))^{1-\alpha}dxdy \\ &= \frac{1}{\alpha-1}\log\int_{[0,1]^2} c^\alpha(u,v)dudv \\ &= \frac{1}{\alpha-1}\log E\Big(c^{\alpha-1}(U,V)\Big), \qquad \alpha>0,\alpha\neq 1. \end{aligned} \tag{5}$$

On the other hand, the copula-based Tsallis divergence measure is given as

$$\begin{aligned} T_\alpha(c) &\equiv T_\alpha(f \parallel f_X f_Y) \\ &= \frac{1}{\alpha-1}\Big(\int_{\mathbb{R}^2} f^\alpha(x,y)\,(f_X(x)f_Y(y))^{1-\alpha}dxdy-1\Big) \\ &= \frac{1}{\alpha-1}\Big(\int_{[0,1]^2} c^\alpha(u,v)dudv-1\Big) \\ &= \frac{1}{\alpha-1}E\Big(c^{\alpha-1}(U,V)-1\Big) \qquad \alpha>0,\alpha\neq 1. \end{aligned} \tag{6}$$

Remark 1

According to equations (5) and (6), it can be easy to conclude that the relationship between the Rényi and Tsallis divergence is as follows:

$$T_\alpha(c)=\frac{1}{\alpha-1}\Big(e^{(\alpha-1)R_\alpha(c)}-1\Big), \qquad \alpha>0,\alpha\neq 1. \tag{7}$$

The measures in equations (5) and (6) only depend on the copula density and are independent from marginal distributions. Now, we review some theoretical properties of these measures in Proposition 1.

Proposition 1

Let C denote the copula function of random variables (X,Y) and let c be the corresponding copula density function. For $\alpha>0$ and $\alpha\neq 1$, the following statements hold:

(a) $R_\alpha(c) \geq 0$, $T_\alpha(c) \geq 0$, and $R_\alpha(c) = T_\alpha(c) = 0$ if and only if X and Y are independent.
(b) $R_\alpha(c) = R_\alpha(c)$ and $T_\alpha(c) = T_\alpha(c)$.
(c) $R_\alpha(c)$ and $T_\alpha(c)$ are invariant under the strictly increasing and decreasing transformations of one or two components of (X, Y).

Proof

(a) Using the Jensen inequality,

$$\begin{aligned} R_\alpha(c) &= \frac{1}{\alpha - 1} \log \int_{[0,1]^2} c^\alpha(u, v) dudv \\ &= \frac{1}{\alpha - 1} \log\ E\Big(c^{\alpha-1}(U, V)\Big) \\ &\geq \frac{1}{\alpha - 1} \log\ E^{1-\alpha}\Big(\frac{1}{c(U, V)}\Big) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} T_\alpha(c) &= \frac{1}{\alpha - 1} \Big(\int_{[0,1]^2} c^\alpha(u, v) dudv - 1\Big) \\ &= \frac{1}{\alpha - 1}\ E\Big(c^{\alpha-1}(U, V) - 1\Big) \\ &\geq \frac{1}{\alpha - 1}\ E^{1-\alpha}\Big(\frac{1}{c(U, V)} - 1\Big) \\ &= 0, \end{aligned}$$

where $(u, v) \in [0, 1]^2$. If X and Y are independent random variables, then $c(u, v) = 1$. Therefore $R_\alpha(c) = T_\alpha(c) = 0$.
(b) By applying Fubini's theorem, this property holds.
(c) The copula function is invariant under strictly increasing transformations of the random variables underlying (see [35]), so the copula-based Rényi and Tsallis divergence measures are invariant under strictly increasing transformations. Now, let δ and η be strictly decreasing transformations. Then relations $C_{\delta(X),Y}(u, v) = v - C_{X,Y}(1 - u, v)$ and $C_{\delta(X),\eta(Y)}(u, v) = u + v - 1 - C_{X,Y}(1 - u, 1 - v)$ are established; see [35]. Thus, we are able to write $c_{\delta(X),Y}(u, v) = c_{X,Y}(1 - u, v)$ and $c_{\delta(X),\eta(Y)}(u, v) = c_{X,Y}(1 - u, 1 - v)$. Hence, for a strictly decreasing transformation on X, we have

$$\begin{aligned} R_{\alpha,(\delta(X),Y)}(c(U, V)) &= \frac{1}{\alpha - 1} \log \int_{[0,1]^2} c^\alpha(1 - u, v) dudv \\ &= \frac{1}{\alpha - 1} \log \int_{[0,1]^2} c^\alpha(z, v) dzdv \qquad \text{(substituting 1-u=z)} \\ &= R_{\alpha,(X,Y)}(c). \end{aligned}$$

Likewise, the result of copula-based Tsallis divergence can be proved under a strictly decreasing transformation on X or Y. In the same way, for strictly decreasing transformations on X and Y, for copula-based Rényi divergence, we can write

$$R_{\alpha,(\delta(X),\eta(Y))}(c) = \frac{1}{\alpha - 1} \log \int_{[0,1]^2} c^\alpha(1 - u, 1 - v) dudv.$$

By replacing $1-u=z$ and $1-v=w$, we get

$$\begin{aligned}R_{\alpha,(\delta(X),\eta(Y))}(c) &= \frac{1}{\alpha-1}\log\int_{[0,1]^2} c^{\alpha}(z,w)dzdw\\ &= R_{\alpha,(X,Y)}(c).\end{aligned}$$

Similarly, the result can be proved for the copula-based Tsallis divergence under strictly decreasing transformation on X and Y.

□

In Example 1, the copula-based Rényi and Tsallis divergence measures for the Gaussian (Normal) copula will be considered, and its properties will be illustrated.

Example 1
Let (U,V) be a random vector from the bivariate Gaussian copula with parameter ρ (see [35]) as

$$C(u,v;\rho)=\Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\rho),\quad (u,v)\in[0,1]^2,\quad \rho\in[-1,1],$$

where Φ_2 is the bivariate normal distribution function with zero means, variances one, and the correlation matrix $R=\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$. Moreover Φ^{-1} denotes the univariate standard normal distribution quantile function. The density of the Gaussian copula is determined by

$$c(u,v;\rho)=\frac{1}{\sqrt{1-\rho^2}}\exp\left\{\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v)-\rho^2(\Phi^{-1}(u)^2+\Phi^{-1}(v)^2)}{2(1-\rho^2)}\right\},\quad (u,v)\in[0,1]^2.$$

Gil et al. [12] demonstrated the general form of the Rényi divergence for the multivariate Gaussian distribution. Now by using part (c) of Proposition 1 and equation (5), we can show that

$$\begin{aligned}R_\alpha(c) &= R_\alpha^c(\Phi^{-1}(u),\Phi^{-1}(v))\\ &= -\frac{1}{2}\log(1-\rho^2)-\frac{1}{2(\alpha-1)}\log(1-(1-\alpha)^2\rho^2),\quad \alpha>0,\alpha\neq 1,\ \rho\in[-1,1].\end{aligned}$$

By using equation (7), we can write

$$T_\alpha(c)=\frac{1}{\alpha-1}\Big((1-\rho^2)^{\frac{1-\alpha}{2}}(1-(1-\alpha)^2\rho^2)^{-\frac{1}{2}}-1\Big),\quad \alpha>0,\alpha\neq 1,\ \rho\in[-1,1].$$

By using the relationship between Kendall's τ and Spearman's ρ for the bivariate Gaussian copula, $\tau=\frac{2}{\pi}\arcsin(\rho)$, the behavior of the Rényi and Tsallis divergence measures for different values of Kendall's τ are given in Figure 1. Note that the Rényi and Tsallis divergence measures for all values considered for α take their minimum value if $\tau=0$. The values of $R_\alpha(c)$ and $T_\alpha(c)$ increase by increasing the absolute value of the Kendall's τ. According to Figure 1 and the study of [29], we can give the result that among the Rényi and Tsallis divergence measures, $R_{0.1}(c)$ and $T_{0.1}(c)$ are outperforming, respectively, because they have a smaller value than the other measures for different values of α. Also, it is observed that the maximum value of $T_{0.1}(c)$ is bounded, and this is a reason for its superiority, too.

3. Test of independence using copula density

The null hypothesis of copula-based independence test can be expressed as

$$H_0: C(u,v)=uv,\qquad (u,v)\in[0,1]^2.$$

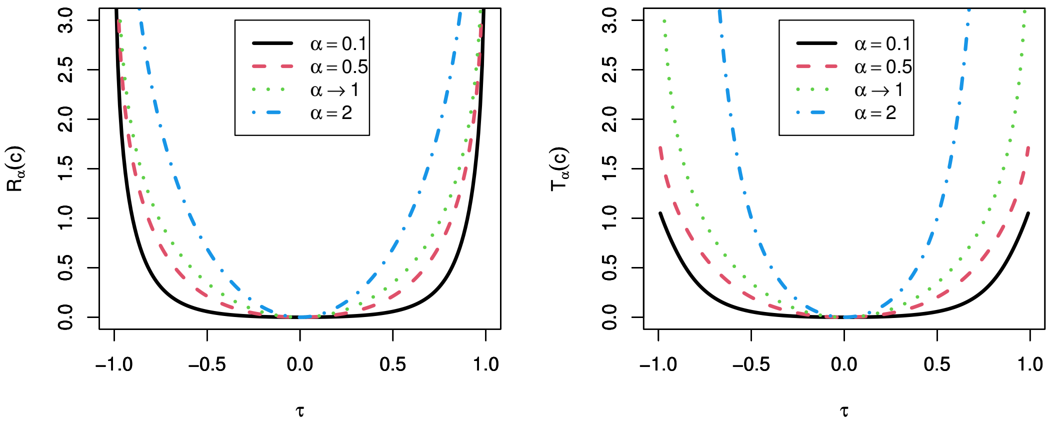

Figure 1. Copula-based Rényi and Tsallis divergence measures for the bivariate Gaussian copula

For the test of independence, Genest and Remillard [11] suggested a test statistics based on the CvM distance as follows:

$$S_n = n \int_{[0,1]^2} (C_n(u,v) - uv)^2 du dv, \tag{8}$$

where $C_n(u,v)$ is the empirical copula, which was initially introduced by Deheuvels [7]. The empirical copula was defined as $C_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} I\{\widehat{U}_i \leq u, \widehat{V}_i \leq v\}$, where $\widehat{U}_i = \frac{R_i}{n+1}$ and $\widehat{V}_i = \frac{S_i}{n+1}$, $i = 1, \ldots, n$, are the pseudo observations and R_i and S_i are the ranks of the observation, respectively.

It is obvious that the null hypothesis of independence test based on copula density is equivalent to

$$H_0 : c(u,v) = 1, \qquad (u,v) \in [0,1]^2. \tag{9}$$

In order to test this null hypothesis, we define nonparametric estimators of the Rényi and Tsallis divergence measures, which will be used as the test statistics.

3.1. Estimators as test statistics

We consider plug-in estimators of the copula-based Rényi and Tsallis divergence measures as test statistics for the test of independence. For this purpose, the copula density must be estimated. A specific class of nonparametric copula density estimators is kernel estimators. Charpentier et al. [5] presented different methods for kernel-based estimation of the copula density, such as the mirror-reflection, beta kernel, and probit-transformation. In this paper, the modified probit-transformation method known as the local likelihood probit-transformation ($\mathcal{LLPT}$) method, suggested by Geenens et al. [9], will be used to estimate the copula density. This method is easy to implement estimators, fix boundary problems, and able to cope with unbounded copula density functions. In the $\mathcal{LLPT}$ method, a polynomial locally fits the log-density of the transformed sample. A comprehensive simulation study by Nagler [34] showed that the $\mathcal{LLPT}$ method yields very good performance among different copula density estimation methods.

Let $(U_j, V_j)_{j=1,\ldots,n}$ be independent and identically distributed (iid) observations from the bivariate copula C. Then the purpose is to estimate the corresponding copula density function. The probit-transformation vector $(X_i, Y_i) = (\Phi^{-1}(U_i), \Phi^{-1}(V_i))$ is a random vector with Gaussian margins and copula C, where Φ is the standard Gaussian distribution. According to equation (3), the joint density function f of the transformed sample (X_i, Y_i) should be estimated for copula density estimation. For this aim, a local log-quadratic estimation of $\log f$ around

$(x, y) \in \mathbb{R}^2$ and (x', y') close to (x, y) is considered as follows:

$$\begin{aligned}\log f(x', y') =& a_{2,0}(x, y) + a_{2,1}(x, y)(x' - x) + a_{2,2}(x, y)(y' - y)\\ &+ a_{2,3}(x, y)(x' - x)^2 + a_{2,4}(x, y)(y' - y)^2 + a_{2,5}(x, y)(x' - x)(y' - y)\\ \equiv& P_{a_2}(x' - x, y' - y),\end{aligned}$$

where the vector $a_2(x, y) \equiv (a_{2,0}(x, y), \ldots, a_{2,5}(x, y))$ is estimated by solving a weighted maximum likelihood problem. So, the estimation of $f(x, y)$ is $\tilde{f}^p(x, y) = \exp\{\widehat{a}_2(x, y)\}$ and using equation (3), the $\mathcal{LLPT}$ estimation of the copula density is

$$\widehat{c}_n^{(\mathcal{LLPT})}(u, v) = \frac{\tilde{f}^p(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \qquad (u, v) \in [0, 1]^2, \tag{10}$$

where the nearest-neighbor method is used for the bandwidth selection; see [9],.

Thus, the plug-in estimators of the copula-based Rényi and Tsallis divergence measures as follows:

$$\widehat{R}_\alpha(c) = \frac{1}{\alpha - 1} \log \left(\frac{1}{n}\sum_{i=1}^{n} \widehat{c}_n^{(\mathcal{LLPT})}(U_i, V_i)^{\alpha-1}\right), \quad \alpha > 0, \alpha \neq 1, \tag{11}$$

and

$$\widehat{T}_\alpha(c) = \frac{1}{\alpha - 1} \left(\frac{1}{n}\sum_{i=1}^{n} \widehat{c}_n^{(\mathcal{LLPT})}(U_i, V_i)^{\alpha-1} - 1\right), \quad \alpha > 0, \alpha \neq 1, \tag{12}$$

where $\widehat{c}_n^{(\mathcal{LLPT})}$ is the $\mathcal{LLPT}$ estimation of the copula density as in equation (10).

3.2. *Asymptotically consistent*

Two different approaches were proposed in [1, 18] to prove the consistency of the entropy estimation. By taking ideas from these approaches, we prove the asymptotic second-order consistency of Rényi and Tsallis divergence measures estimators in equations (11) and (12) in Proposition 2.

Proposition 2
Suppose that $(U_j, V_j)_{j=1,\ldots,n}$ is an iid sample from copula C and that the corresponding copula density c is twice continuously differentiable. Let $E\left(c(U, V)\right) < \infty$, $E\Big(c(U, V)^2\Big) < \infty$, $(u, v) \in (0, 1)^2$, ${b_n}^2 \to 0$, and $n{b_n}^2 \to \infty$. Then

(i) $E\Big[\widehat{T}_\alpha(c) - T_\alpha(c)\Big]^2 \longrightarrow 0, \qquad n \to \infty;$

(ii) $E\Big[\widehat{R}_\alpha(c) - R_\alpha(c)\Big]^2 \longrightarrow 0, \qquad n \to \infty.$

Proof
The asymptotic normality of the $\mathcal{LLPT}$ estimator was demonstrated in [9] as

$$\widehat{c}_n^{(\mathcal{LLPT})}(u, v) \; is \; AN\Big(\mu(u, v), \frac{\sigma^2(u, v)}{nb_n^2}\Big),$$

where

$$\sigma^2(u, v) = \frac{5c(u, v)}{8\pi\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \tag{13}$$

$$\begin{aligned}\mu(u, v) = c(u, v) - \frac{b_n^4}{8}\frac{c(u, v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}&\Big\{\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial y^4} + 2\frac{\partial^4 g}{\partial x^2 \partial y^2}\\ &+ 4\Big(\frac{\partial^3 g}{\partial x^3}\frac{\partial g}{\partial x} + \frac{\partial^3 g}{\partial y^3}\frac{\partial g}{\partial y} + \frac{\partial^3 g}{\partial x^2 \partial y}\frac{\partial g}{\partial y} + \frac{\partial^3 g}{\partial x \partial y^2}\frac{\partial g}{\partial x}\Big)\Big\}(x, y),\end{aligned} \tag{14}$$

and $g(x,y) = \log c(\Phi(x), \Phi(y)) + \log \phi(x) + \log \phi(y)$ in $x = \Phi^{-1}(u)$, $y = \Phi^{-1}(v)$. Bias and variance of the copula density estimation using $\mathcal{LLPT}$ method will be used to prove the consistency of estimators.

(i) We write T_α for $(\alpha-1)T_\alpha(c)+1$ and $\widehat{T}_\alpha$ for $(\alpha-1)\widehat{T}_\alpha(c)-1$ on $I=[0,1]$. Define

$$T_\alpha = \int_{I^2} c^{\alpha-1}(u,v)dC(u,v) = E(c^{\alpha-1}(U,V)), \tag{15}$$

$$\widehat{T}_\alpha = \int_{I^2} \widehat{c}^{\alpha-1}(u,v)dC_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} \widehat{c}^{\alpha-1}(U_i,V_i), \tag{16}$$

$$\theta_\alpha = \int_{I^2} c^{\alpha-1}(u,v)dC_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} c^{\alpha-1}(U_i,V_i). \tag{17}$$

By using the Minkowski inequality, we have

$$\begin{aligned} E\Big[\widehat{T}_\alpha(c) - T_\alpha(c)\Big]^2 &= \frac{1}{(\alpha-1)^2}E[\widehat{T}_\alpha - T_\alpha]^2 \\ &= \frac{1}{(\alpha-1)^2}E[(\widehat{T}_\alpha - \theta_\alpha) + (\theta_\alpha - T_\alpha)]^2 \\ &\leq \frac{1}{(\alpha-1)^2}\Big(E^{1/2}(\widehat{T}_\alpha - \theta_\alpha)^2 + E^{1/2}(\theta_\alpha - T_\alpha)^2\Big)^2 \\ &= \frac{1}{(\alpha-1)^2}(I_1^{1/2} + I_2^{1/2})^2. \end{aligned}$$

To prove part (i) of the proposition, it suffices to show that $I_j \to 0$ as $n \to \infty$ for $j = 1,2$.

By substituting (16) and (17) into I_1, the Taylor expansion of function $\widehat{c}^{\alpha-1}(u,v)$ about $c(u,v)$ can be written as

$$\begin{aligned} I_1 &= E\Big[\int_{I^2}(\widehat{c}^{\alpha-1}(u,v) - c^{\alpha-1}(u,v))dC_n(u,v)\Big]^2 \\ &= E\Big[\int_{I^2}\Big((\alpha-1)c^{\alpha-2}(u,v)(\widehat{c}(u,v) - c(u,v))\Big)dC_n(u,v) + O(\frac{b_n^8}{n})\Big]^2 \\ &= E\Big[\frac{1}{n}\sum_{i=1}^{n}(\alpha-1)c^{\alpha-2}(U_i,V_i)(\widehat{c}(U_i,V_i) - c(U_i,V_i)) + O(\frac{b_n^8}{n})\Big]^2 \\ &= \frac{1}{n^2}E\Big[\sum_{i=1}^{n}(\alpha-1)^2c^{2(\alpha-2)}(U_i,V_i)\Big(\widehat{c}(U_i,V_i) - c(U_i,V_i)\Big)^2\Big] \\ &\quad + \frac{1}{n^2}E\Big[\sum_{i\neq j}^{n}(\alpha-1)^2c^{\alpha-2}(U_i,V_i)\Big(\widehat{c}(U_i,V_i) - c(U_i,V_i)\Big)c^{\alpha-2}(U_j,V_j)\Big(\widehat{c}(U_j,V_j) - c(U_j,V_j)\Big)\Big] \\ &\quad + O(\frac{b_n^{16}}{n^2}) \\ &= A_1 + A_2 + O(\frac{b_n^{16}}{n^2}). \end{aligned}$$

Now, we evaluate expression A_1 and A_2. From the fact that $\widehat{c}(u_i,v_i)$ for $i = 1,\ldots,n$ are identically distributed, we can write

$$\begin{aligned} A_1 &= \frac{1}{n}E_{(U,V)}\Big\{E_{\widehat{c}}\Big[(\alpha-1)^2c^{2(\alpha-2)}(U,V)\Big(\widehat{c}(U,V) - c(U,V)\Big)^2\Big]|U,V\Big\} \\ &= \frac{1}{n}\int_{I^2}E_{\widehat{c}}\Big[(\alpha-1)^2c^{2(\alpha-2)}(U,V)\Big(\widehat{c}(U,V) - c(U,V)\Big)^2|U=u,V=v\Big]c(u,v)dudv. \end{aligned}$$

From equations (13) and (14) for all $(u,v) \in I^2$, we have

$$\begin{aligned} E_{\widehat{c}}\Big[(\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(\widehat{c}(u,v)-c(u,v)\Big)^2\Big] &= (\alpha-1)^2 c^{2(\alpha-2)}(u,v) E_{\widehat{c}}\Big(\widehat{c}(u,v)-c(u,v)\Big)^2 \\ &= (\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(Var_{\widehat{c}}(\widehat{c}(u,v)) + Bias^2_{\widehat{c}}(\widehat{c}(u,v))\Big) \\ &= (\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(O(\frac{1}{nb_n^2}) + O(b_n^8)\Big) \\ &\longrightarrow 0, \qquad n \to \infty. \end{aligned}$$

Furthermore,

$$\begin{aligned} & E_{\widehat{c}}\Big[(\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(\widehat{c}(u,v)-c(u,v)\Big)^2\Big] c(u,v) \\ &\leq (\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(Var_{\widehat{c}}(\widehat{c}(u,v)) + E^2_{\widehat{c}}(\widehat{c}(u,v)) + c^2(u,v)\Big) c(u,v) \\ &\leq (\alpha-1)^2 c^{2(\alpha-2)}(u,v)\Big(O(\frac{1}{nb_n^2}) + c^2(u,v) + O(b_n^8) + c^2(u,v)\Big) c(u,v) \\ &\longrightarrow 2(\alpha-1)^2 c^{2\alpha-1}(u,v), \qquad n \to \infty. \end{aligned}$$

Thus, for all (u,v), the integrand of expression A_1 is bounded above by $2(\alpha-1)^2 c^{2\alpha-1}(u,v)$ as $n \to \infty$, which is integrable. Then by the Lebesgue dominated convergence theorem, $A_1 \longrightarrow 0$ as $n \to \infty$. On the other hands, from the Cauchy–Schwarz inequality, the expression A_2 can be evaluated as follows:

$$A_2 \leq \frac{1}{n^2} n(n-1)(\sqrt{A_1})^2 \tag{18}$$

$$= \frac{n-1}{n} A_1 \longrightarrow 0, \qquad n \to \infty. \tag{19}$$

Therefore, $A_2 \longrightarrow 0$ and so $I_1 \longrightarrow 0$.

As for I_2, by substituting (15) and (17), we have

$$\begin{aligned} I_2 &= E\Big[\frac{1}{n}\sum_{i=1}^{n} c^{\alpha-1}(U_i,V_i) - \int_{I^2} c^{\alpha-1}(u,v)dC(u,v)\Big]^2 \\ &= \frac{1}{n} Var(c^{\alpha-1}(U,V)) = O(\frac{1}{n}). \end{aligned}$$

Then, $I_2 \longrightarrow 0 \; as \; n \to \infty$, and this completes the proof of the part (i).

(ii) The proof of this part can be easily completed in the same way as part (i) by using the Taylor expansion of the logarithm function. □

4. Simulation study

A simulation study was performed to evaluate the finite sample properties of the suggested tests of independence. The Clayton, Gumbel, and Frank copulas from the Archimedean class and Gaussian and T copulas from the Elliptical class are considered under the alternative hypothesis. These copulas cover different degrees of dependency as measured by Kendall's τ. The **copula** package provided by Hofert et al. [16] is used to simulate the copula functions. We use a procedure proposed in [10] to calculate the critical value (CV), p-value, and empirical power (EP) of the suggested test statistics at a 5% significance level. This procedure was also used in [32].

The empirical size and power of the suggested test statistics are compared with the classical test based on the empirical copula (S_n) considered in [11]. A Monte Carlo experiment with 1000 replications is performed on test statistics $S_n, \widehat{KL}(c), \widehat{R}_\alpha(c)$, and $\widehat{T}_\alpha(c)$ for special value of α ($\alpha = 0.1, 0.5, 2$), various sample sizes

($n = 50, 100, 200$), and different degrees of dependency based on Kendall's τ coefficient ($\tau = 0, 0.1, 0.175, 0.25$). We assume that the marginal distributions have standard uniform distributions. For Kendall's τ coefficient greater than 0.5, all tests provide very good performance. Simulation results for considered test statistics are reported in Table **??** for Archimedean copulas and Table 1 for Elliptical copulas.

The empirical size of all tests is obtained when Kendall's τ is equal to zero. Indeed, in T copula with a small degree of freedom ($df = 2$), dependency occurs in the tail areas when $\tau = 0$; see [8]. Consequently, independence does not exist in this case. However, by increasing the degree of freedom in the T copula, dependency is less in tail areas. Results in Tables **??** and 1 show that all tests generally control the size.

As seen in Tables **??** and 1, the empirical power of all tests increases by increasing the sample size or Kendall's τ. The empirical powers of $\widehat{T}_\alpha(c)$ increase by decreasing $alpha$ values. Indeed, the empirical powers of $\widehat{R}_\alpha(c)$ do not show a monotone behavior with respect to $alpha$ values. In both tables, it can be seen that $\widehat{T}_{0.1}(c)$ has a better performance than $\widehat{T}_{0.5}(c)$ and $\widehat{T}_2(c)$. Also, we know that $\widehat{T}_{0.5}(c)$ is equal to double Hellinger distance ($\widehat{He}(c)$) proposed in [32] as a good test statistic in weak dependency or small sample size. Simulation results show that $\widehat{T}_{0.1}(c)$ has the best performance among all the considered statistics in weak dependency ($\tau = 0.1, 0.175$). It can be seen in Table 1 that the suggested independence tests based on Rényi and Tsallis divergence measures do much better than the empirical copula-based test S_n for T copula with a small degree of freedom ($df = 2$) in terms of power. For example, the empirical power of $\widehat{R}_{0.1}(c)$ and $\widehat{T}_{0.1}(c)$ for T copula with 2 degrees of freedom in $n = 200$ and $\tau = 0$ are equal to 0.841 and 0.942, respectively, whereas the empirical power of S_n is equal to 0.165.

5. Application in hydrology

In this section, an application of suggested independence tests on a real dataset is presented for analysis of hydrological drought. McKee et al. [28] proposed the concept of standardized precipitation index (SPI) based on the long-term precipitation record for a specific period. Guttman [13] recommended using the SPI as a primary drought index because it is simple, spatially invariant in its interpretation, and probabilistic. The SPI series is used for this paper. Fitting this long-term precipitation record to a probability distribution is the first step in calculating the SPI series. Once the probability distribution is determined, the cumulative probability of observed precipitation is computed and then inverse transformed by a standard normal distribution with mean 0 and variance 1. The resulting quantile is SPI. A drought event is thus defined as a continuous period in which the SPI is below 0.

Drought characteristics (events) based on SPI include drought duration (D_d), drought severity (S_d), and drought interval time (L_d). Drought duration D_d is defined as the number of consecutive intervals (months) where SPI remains below the threshold value of 0; see [37]. Drought severity S_d is defined as a cumulative SPI value during a drought period, $S_d = \sum_{i=1}^{D_d} SPI_i$, where SPI_i means the SPI value in the ith month; see [30]. The drought interval time L_d is defined as the period elapsing from the initiation of drought to the beginning of the next drought; see [39].

The objective of this section is to test of independence between drought characteristics. Wong et al. [43] established a joint distribution function of drought intensity, duration, and severity using Gaussian and Gumbel copulas. Ma et al. [26] investigated the drought events in the Weihe river basin and selected the Gaussian and T copulas to model the joint distribution among drought duration, severity, and peaks. Recently, a comprehensive book on the application of copula in hydrology was published by Chen and Guo [6], and the concepts in this section are taken from this book.

The monthly precipitation data of Mashhad station, located in Iran, from 1951 to 2017 (http://www.irimo.ir/eng/index.php) are used as real data to illustrate the proposed methodology. The monthly precipitation can be fitted as a gamma distribution. The monthly SPI series is then calculated and demonstrated in Figure 2 for this 67-year period.

The spearman's ρ and Kendall's τ coefficients for pairs of drought variables are given in Table 2. Results confirm that the pairs (S_d, D_d) and (L_d, D_d) show positive and significant (at 5% level) correlations. The pair (S_d, L_d) has a positive and weak dependency.

Table 1. Empirical size and power of the test statistics for Elliptical copulas

Copula	Sample Size	Kendall's τ	S_n	$\widehat{KL}(c)$	$\widehat{R}_{0.1}(c)$	$\widehat{R}_{0.5}(c)$	$\widehat{R}_2(c)$	$\widehat{T}_{0.1}(c)$	$\widehat{T}_{0.5}(c)$	$\widehat{T}_2(c)$
Gaussian	50	τ=0	0.045	0.047	0.052	0.044	0.062	0.048	0.049	0.040
		τ=0.1	0.180	0.185	0.136	0.134	0.124	0.198	0.191	0.150
		τ=0.175	0.441	0.453	0.376	0.371	0.354	0.484	0.474	0.354
		τ=0.25	0.695	0.711	0.658	0.634	0.630	0.737	0.723	0.624
	100	τ=0	0.060	0.052	0.044	0.052	0.044	0.048	0.051	0.040
		τ=0.1	0.261	0.269	0.218	0.200	0.184	0.298	0.274	0.200
		τ=0.175	0.480	0.461	0.428	0.459	0.460	0.550	0.484	0.409
		τ=0.25	0.960	0.972	0.880	0.874	0.828	0.983	0.978	0.862
	200	τ=0	0.050	0.053	0.048	0.053	0.054	0.049	0.050	0.041
		τ=0.1	0.463	0.456	0.341	0.330	0.258	0.486	0.471	0.292
		τ=0.175	0.679	0.673	0.601	0.535	0.520	0.699	0.687	0.504
		τ=0.25	1.000	0.998	0.991	0.983	0.971	1.000	1.000	0.990
$T(df=2)$	50	τ=0	0.120	0.291	0.324	0.272	0.258	0.361	0.296	0.263
		τ=0.1	0.215	0.338	0.337	0.318	0.318	0.367	0.345	0.321
		τ=0.175	0.424	0.568	0.571	0.562	0.553	0.595	0.572	0.512
		τ=0.25	0.730	0.782	0.794	0.751	0.732	0.816	0.797	0.718
	100	τ=0	0.132	0.551	0.586	0.598	0.604	0.636	0.598	0.602
		τ=0.1	0.410	0.791	0.786	0.770	0.712	0.818	0.813	0.698
		τ=0.175	0.686	0.912	0.935	0.923	0.911	0.976	0.951	0.903
		τ=0.25	0.945	0.975	0.964	0.952	0.938	0.989	0.979	0.964
	200	τ=0	0.165	0.891	0.926	0.921	0.894	0.942	0.909	0.912
		τ=0.1	0.480	0.976	0.981	0.968	0.963	0.987	0.982	0.966
		τ=0.175	0.700	0.988	0.986	0.977	0.969	0.991	0.988	0.981
		τ=0.25	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$T(df=10)$	50	τ=0	0.061	0.052	0.067	0.059	0.054	0.059	0.052	0.059
		τ=0.1	0.171	0.212	0.181	0.158	0.151	0.229	0.221	0.156
		τ=0.175	0.402	0.425	0.410	0.364	0.408	0.480	0.472	0.413
		τ=0.25	0.686	0.706	0.662	0.648	0.626	0.718	0.713	0.661
	100	τ=0	0.055	0.056	0.078	0.070	0.070	0.148	0.051	0.088
		τ=0.1	0.365	0.383	0.266	0.270	0.216	0.399	0.395	0.227
		τ=0.175	0.598	0.604	0.565	0.545	0.443	0.665	0.623	0.505
		τ=0.25	0.960	0.975	0.920	0.898	0.820	0.984	0.979	0.868
	200	τ=0	0.055	0.053	0.064	0.075	0.062	0.057	0.050	0.068
		τ=0.1	0.521	0.509	0.521	0.505	0.443	0.619	0.526	0.469
		τ=0.175	0.721	0.799	0.807	0.744	0.688	0.910	0.809	0.710
		τ=0.25	1.000	1.000	0.992	0.984	0.980	1.000	1.000	0.991

Table 2. Kendall's τ and Spearman's ρ coefficients for drought variables

pairs	Kendall's τ		Spearman's ρ	
	tau	p-value	rho	p-value
(S_d, D_d)	0.4747	<2.2E-16	0.6611	<2.2E-16
(S_d, L_d)	0.1569	0.0034	0.2265	0.0041
(L_d, D_d)	0.2170	4.97E-05	0.3075	8.07E-05

The test statistics $S_n, \widehat{KL}(c), \widehat{R}_\alpha(c)$, and $\widehat{T}_\alpha(c)$ $(\alpha = 0.5, 2, 3)$ are computed to test independence between drought events, and the results are shown in Table 3. The empirical marginal distributions are considered for observed drought events. The results show that the independence hypothesis is rejected based on all considered tests for pairs (S_d, D_d) and (L_d, D_d). Furthermore, the independence hypothesis for pair (S_d, L_d) is rejected based on all tests except test statistics $\widehat{T}_{0.1}(c)$. For this pair, the value of test statistics $\widehat{T}_{0.1}(c) = 0.0514$ is less than its critical value (0.0523), and $p - value = 0.0602 \geq 0.05$. This result shows that the proposed test statistics $\widehat{T}_{0.1}(c)$ has a good ability to detect weak dependency.

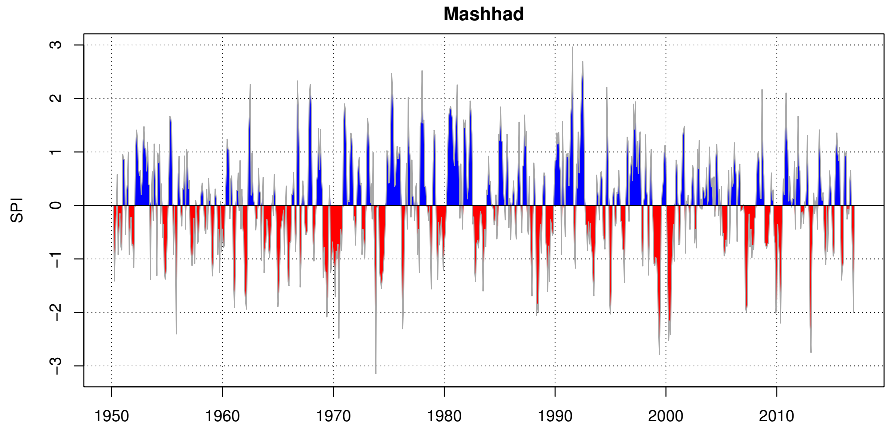

Figure 2. The monthly SPI time series for the Mashhad station

Table 3. Results of the independence tests for the SPI of Mashhad

Test statistics	C.V. 95 %	(S_d,D_d)		(S_d,L_d)		(L_d,D_d)	
		value of statistics	p-value	value of statistics	p-value	value of statistics	p-value
S_n	0.1212	1.8940	<0.001	0.2549	0.0010	0.4874	<0.001
$\widehat{KL}(c)$	0.0775	0.3756	<0.001	0.0830	0.0413	0.2467	<0.001
$\widehat{R}_{0.1}(c)$	0.0552	0.2417	<0.001	0.0558	0.0486	0.1696	<0.001
$\widehat{R}_{0.5}(c)$	0.0634	0.2985	<0.001	0.0672	0.0441	0.2035	<0.001
$\widehat{R}_{2}(c)$	0.1149	0.5914	<0.001	0.1251	0.0282	0.3453	<0.001
$\widehat{T}_{0.1}(c)$	0.0523	0.2173	<0.001	0.0514	0.0602	0.1573	<0.001
$\widehat{T}_{0.5}(c)$	0.0629	0.2773	<0.001	0.0662	0.0491	0.1935	<0.001
$\widehat{T}_{2}(c)$	0.1204	0.8066	<0.001	0.1332	0.0334	0.4124	<0.001

Conclusion

Nonparametric independence tests between continuous random variables were proposed based on the copula density estimator and Rényi and Tsallis divergence measures. These tests were able to look at the weak dependency. The comparison of empirical powers in a simulation study showed that the suggested test $\widehat{T}_{0.1}(c)$ performs better than the test based on the empirical copula in weak dependency. In particular, the independent tests based on Rényi and Tsallis divergence measures have a greater ability than the empirical copula-based test for T copula with a small degree of freedom. Finally, a real example was performed in hydrology to show the usefulness of the proposed tests. Multivariate independence tests via copula-based divergence measures will be investigated in the future.

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