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Existence and blow up of solutions for a class of parabolic problems with dynamical boundary condition

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Abstract In this paper, we study weak solutions for a class of parabolic problems with dynamical boundary condition. We establish the existence of a weak solution to the corresponding Dynamical problem. Moreover, we will show that the existence time T of solution is finite when the initial energy satisfies certain condition.

Keywords Parabolic problem; Dynamic boundary condition; Global existence; Blow-up

AMS 2010 subject classifications 35K55, 35K61, 35J05

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1. Introduction

In this paper, we consider a nonlinear parabolic problem with dynamical boundary conditions:

$$\begin{cases} \partial_t u - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t>0, \\ \sigma \partial_t u + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \partial\Omega,\ t>0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with Lipschitz boundary $\partial\Omega$, and where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the well known p-Laplacian operator defined in $W^{1,p}(\Omega)$. Here, u_t or $\partial_t u$ respectively denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x. Furthermore, for the sake of simplicity, the dynamical coefficient σ is assumed to be a nonnegative constant, $\lambda > 0$, p and q satisfy

$$(H) \qquad \frac{2n}{n+1} \leq p < +\infty, \quad p < 2+q \quad \text{and} \quad \begin{cases} 1 \leq q+2 \leq p^\partial & \text{if } p \neq n, \\ 1 \leq q+2 < \infty & \text{if } p = n. \end{cases}$$

Recall that

$$p^\partial := \begin{cases} \frac{p(n-1)}{n-p} & \text{if } 1 < p < n, \\ \infty & \text{if } p \geq n. \end{cases}$$

Parabolic equation and systems with dynamical boundary conditions have been extensively studied in the literature (see for instance, [1, 4, 5, 6, 7, 8, 3]). In particular, local existence and uniqueness of solution to general

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quasilinear parabolic equation (systems) with dynamical boundary condition has been established in a series of papers by Escher [5, 6, 7] (see also [9] for a semigroup approach in the $H_p^2(\Omega)$ -setting, and [4] for the solvability result in a weighted H?lder space). These boundary conditions of a domain relate the time derivative to the potential and the spatial exterior normal derivative of it. They are applied in many modellings with Parabolic Partial Differential Equations, notably in control theory and chemistry. In the case of heat diffusion, the dynamical boundary conditions model the heat input from to the conductivity of the wall of the medium in which the heat flow is observed. In the case of chemical reactions, these dynamic conditions mean t that the species in the reaction need energy to leave the medium in which the chemical reaction is taking place, making it difficult for the species to escape.

This paper consists of four sections: In Section 2, we present the basic preliminary results. The proofs of our main theorems are given in section 3 and section 4.

2. Preliminaries

The Lebesgue norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$, and the Lebesgue norm of $L^p(\partial\Omega,\rho)$ by $\|\cdot\|_{p,\partial\Omega}$, for $p\in[1,\infty]$, where $d\rho$ denotes the restriction to $\partial\Omega$. Especially for $p=2$, the scalar product of $L^2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$ and the scalar product of $L^2(\partial\Omega,\rho)$ will be denoted by $\langle\cdot,\cdot\rangle_0$:

$$\langle u,v\rangle=\int_\Omega uv\,dx,\quad \langle u,v\rangle_0=\oint_{\partial\Omega} uv\,d\rho.$$

Moreover, usual Sobolev space on Ω is defined by

$$W^{1,p}(\Omega)=\{u\in L^p(\Omega):|\nabla u|\in L^p(\Omega)\},$$

and it is equipped with the norm

$$\|u\|_{1,p}=\|u\|_p+\|\nabla u\|_p,$$

or to the equivalent norm

$$\|u\|_{1,p}=\left(\|u\|_p^p+\|\nabla u\|_p^p\right)^{\frac{1}{p}},\ \text{ if } 1\le p<+\infty.$$

Set

$$\mathcal{X}^q=L^q(\Omega)\times L^q(\partial\Omega,\rho),\ \text{ for } 1\le q\le\infty,$$

and

$$U=(u,\varphi)\in\mathcal{X}^q,\quad \|U\|_{\mathcal{X}^q}:=\left(\|u\|_q^q+\sigma\|\varphi\|_{q,\partial\Omega}^q\right)^{1/q},$$

and for $q=2$ and $U=(u,\varphi),V=(v,\psi)\in\mathcal{X}^2$

$$\langle U,V\rangle_{\mathcal{X}^2}:=\langle u,v\rangle+\sigma\langle\varphi,\psi\rangle_0.$$

$$\langle\partial_t u,\varphi\rangle_{\mathcal{X}^2}:=\langle\partial_t u,\varphi\rangle+\sigma\langle\partial_t u_{|\partial\Omega},\varphi\rangle_0,$$

for any $\varphi\in W^{1,p}(\Omega)$, and

$$\|\partial_t u\|_{\mathcal{X}^2}^2:=\|\partial_t u\|_2^2+\sigma\left\|\partial_t u_{|\partial\Omega}\right\|_{2,\partial\Omega}^2.$$

Remark 2.1
The trace $u_{|\partial\Omega}$ of any function $u\in W^{1,p}(\Omega)$ is well defined since $\partial\Omega$ is regular enough.

Next, for a reflexive Banach space $(X,\|\cdot\|_X)$ and $q\in[1,\infty)$, the classical Bochner space $L^q((0,T);X)$ will be endowed with the norm

$$\|u\|_{L^q((0,T);X)}:=\left(\int_0^T\|u\|_X^q\,dt\right)^{1/q}.$$

Proposition 2.1
(See [2])
The critical Sobolev exponent for the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ will be denoted by p^*, where

$$p^* := \begin{cases} \frac{pn}{n-p} & \text{if } 1 < p < n, \\ \infty & \text{if } p \geq n. \end{cases}$$

It is worth noting that

$$W^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \Longleftrightarrow p \geq p_0 := \frac{2n}{n+2}.$$

Proposition 2.2
(See [2])
The trace operator $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ is continuous if and only if $1 \leq q \leq p^\partial$ if $p \neq n$ and for $1 \leq q < \infty$ if $p = n$. Note that for $q = 2$, the trace operator is well-defined and continuous under the following condition:

$$W^{1,p}(\Omega) \to L^2(\partial\Omega, \rho) \Longleftrightarrow p \geq p_1 := \frac{2n}{n+1}.$$

Finally, let us introduce some functionals and sets as follows

$$E(u) = \frac{1}{p}\|u\|_{1,p}^p - \frac{\lambda}{2+q}\|u\|_{2+q,\partial\Omega}^{2+q}. \tag{2}$$

$$X = \left\{u \in W^{1,p}(\Omega) \mid F(u) > 0, E(u) < d\right\} \cup \{0\},$$

where

$$F(u) = \|u\|_{1,p}^p - \lambda\|u\|_{2+q,\partial\Omega}^{2+q},$$

and the depth of potential well

$$d = \inf_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \sup_{\beta \geq 0} E(\beta u). \tag{3}$$

We define the auxiliary functional

$$E_\delta(u) = \frac{\delta}{p}\|u\|_{1,p}^p - \frac{\lambda}{2+q}\|u\|_{2+q,\partial\Omega}^{2+q}, \quad \forall \delta \in (0,1),$$

and the depth function of potential wells

$$d(\delta) = \frac{1-\delta}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}, \tag{4}$$

where C_* is the embedding constant form $W^{1,p}(\Omega)$ into $L^{2+q}(\partial\Omega)$, i.e.,

$$C_* = \sup \frac{\|u\|_{2+q,\partial\Omega}}{\|u\|_{1,p}}. \tag{5}$$

In addition, we define

$$\begin{aligned}
X_\delta &= \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) > 0, E(u) < d(\delta)\right\} \cup \{0\}, \quad \forall\, 0 < \delta < 1, \\
\overline{X}_\delta &= X_\delta \cup \partial X_\delta = \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) \geq 0, E(u) \leq d(\delta)\right\}, \\
Y_\delta &= \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) < 0, E(u) < d(\delta)\right\}, \quad \forall\, 0 < \delta < 1, \\
\overline{Y}_\delta &= Y_\delta \cup \partial Y_\delta = \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) \leq 0, E(u) \leq d(\delta)\right\}, \\
Y &= \left\{u \in W^{1,p}(\Omega) \mid F(u) < 0, E(u) < d\right\},
\end{aligned}$$

and

$$B_\delta=\left\{u\in W^{1,p}(\Omega)\mid \|u\|_{1,p}<\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\},$$

$$\overline{B}_\delta=B_\delta\cup\partial B_\delta=\left\{u\in W^{1,p}(\Omega)\mid \|u\|_{1,p}\leq\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\},$$

$$B_\delta^c=\left\{u\in W^{1,p}(\Omega)\mid \|u\|_{1,p}>\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\}.$$

3. Global existence of solutions

In this section, we prove our main existence result. We prepare the proof by a series of auxiliary results:

Lemma 3.1
As a function of $\delta,\ d(\delta)$ satisfies the following properties on $[0,1]$.

(i) $d(0)=d(1)=0$;

(ii) $d(\delta)$ is increasing for $0\leq\delta\leq\delta_0$, decreasing for $\delta_0\leq\delta\leq 1$, and takes the maximum $d(\delta_0)$ at $\delta_0=\frac{p}{2+q}$;

(iii) The equation $d(\delta)=e$ has two roots $\delta_1\in(0,\delta_0)$ and $\delta_2\in(\delta_0,1)$, for any given $e\in(0,d(\delta_0))$.

Proof
This lemma follows directly from

$$\begin{aligned}
d'(\delta)&=\frac{-1}{p}\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}+\frac{(2+q)(1-\delta)}{(2+q-p)\lambda pC_*^{2+q}}\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}-1}\\
&=\frac{1}{p}\left(\frac{2+q}{\lambda pC_*^{2+q}}\right)^{\frac{p}{2+q-p}}\delta^{\frac{p}{2+q-p}}\left(\frac{p}{2+q-p}\cdot\frac{1-\delta}{\delta}-1\right)\\
&=\frac{1}{2+q-p}\left(\frac{2+q}{\lambda pC_*^{2+q}}\right)^{\frac{p}{2+q-p}}\delta^{\frac{p}{2+q-p}}\left(\frac{1}{\delta}-\frac{2+q}{p}\right).
\end{aligned}$$

□

Theorem 3.1
If $u\in W^{1,p}(\Omega),\ \|u\|_{1,p}\neq 0$ and $E_\delta(u)=0$, then $d(\delta)=\inf E(u)$. Moreover,
$d=d(\delta_0)$.

Proof
By (5) and $E_\delta(u)=0$, we obtain

$$\frac{2+q}{\lambda p}\delta\|u\|_{1,p}^p=\|u\|_{2+q,\partial\Omega}^{2+q}\leq C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^p,$$

consequently, if $\|u\|_{1,p}\neq 0$, we get

$$\|u\|_{1,p}\geq\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}},$$

which along with

$$E(u)=\frac{1-\delta}{p}\|u\|_{1,p}^p+E_\delta(u)=\frac{1-\delta}{p}\|u\|_{1,p}^p,$$

gives

$$E(u) \geq \frac{1-\delta}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}$$
$$= d(\delta).$$

These give the conclusion of first assertion.
Note that

$$\frac{1}{2+q}F(u) = \left(\frac{1}{2+q} - \frac{\delta}{p}\right)\|u\|_{1,p}^p + E_\delta(u).$$

From (ii) in Lemma (3.1) we have $E_{\delta_0}(u) = 0$ if and only if $F(u) = 0$.
On the other hand, in view of Liu and Zhao [[10], Theorem 2.1], the depth of potential well given by (3) can be characterized as $d = \inf E(u)$ subject to the conditions $u \in W^{1,p}(\Omega)$, $\|u\|_{1,p} \neq 0$ and $F(u) = 0$. Hence, from the first conclusion of Theorem (3.1), we obtain $d = d(\delta_0)$. □

Lemma 3.2
Assume that $0 < E(u) < d$ for some $u \in W^{1,p}(\Omega)$, and $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u)$. Then the sign of $E_\delta(u)$ does not change for $\delta_1 < \delta < \delta_2$.

Proof
Arguing by contradiction, we suppose that the sign of $E_\delta(u)$ is changeable for $\delta_1 < \delta < \delta_2$, thus there exists a $\delta^* \in (\delta_1, \delta_2)$ such that $E_{\delta^*}(u) = 0$. On the other hand, $E(u) > 0$ implies $\|u\|_{1,p} \neq 0$. Combining Theorem (3.1) and Lemma (3.1) we obtain

$$E(u) \geq d(\delta^*) > d(\delta_1) = d(\delta_2),$$

which contradicts $E(u) = d(\delta_1) = d(\delta_2)$. □

Corollary 3.1
Assume that $0 < E(u) < d$ for some $u \in H^1(\Omega)$, and $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u)$. Then $E_\delta(u) > 0$ (or < 0) for all $\delta \in (\delta_1, \delta_2)$ if and only if there exists a $\bar{\delta} \in [\delta_1, \delta_2]$ such that $E_{\bar{\delta}}(u) > 0$ (or < 0).

We now give the definition of the solutions to our problem.

Definition 3.1
A function $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is called a weak solution of problem (1) if

(i) $u \in L^\infty\left(0, T; W^{1,p}(\Omega)\right) \cap C\left(0, T; \mathcal{X}^2\right)$,

(ii) $\partial_t u \in L^2\left(0, T; L^2(\Omega)\right)$; $\partial_t u_{|\partial\Omega} \in L^2\left(0, T; L^2(\partial\Omega, \rho)\right)$,

(iii) for any $v \in W^{1,p}(\Omega)$ and for almost all $t \in [0, T]$ it holds

$$\langle \partial_t u, v\rangle + \sigma\langle \partial_t u_{|\partial\Omega}, v\rangle_0 + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \lambda\langle |u_{|\partial\Omega}|^q u_{|\partial\Omega}, v\rangle_0,$$

(iv) $u(x, 0) = u_0(x)$ in $W^{1,p}(\Omega)$.

Remark 3.1
By writing $u \in \mathcal{X}^q$ we mean that $u : \overline{\Omega} \to \mathbb{R}$ is such that $u_{|\Omega} \in L^q(\Omega)$ and also $u_{|\partial\Omega} \in L^q(\partial\Omega, \rho)$.

Here, we have our main first result

Theorem 3.2
Let $u_0(x) \in W^{1,p}(\Omega)$, p and q satisfy (H). Assume that $0 < E(u_0) < d$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u_0)$ and $E_{\delta_2}(u_0) > 0$. Then problem (1) admits a global solution $u \in L^\infty\left(0, T; W^{1,p}(\Omega)\right) \cap C\left(0, T; \mathcal{X}^2\right)$, with $\partial_t u \in L^2\left(0, T; L^2(\Omega)\right)$, $\partial_t u_{|\partial\Omega} \in L^2\left(0, T; L^2(\partial\Omega, \rho)\right)$, and $u(t) \in X_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in [0, \infty)$.

Proof of Theorcm 3.2. We start by constructing a sequence such that its limit equal to the solution of (1). Let $\{\varphi_j(x)\}_{j=1}^{\infty}$ be a system of base functions in $W^{1,p}(\Omega)$, define the approximate solution to (1) as follows:

$$u_m(x,t)=\sum_{j=1}^{m} f_{jm}(t)\varphi_j(x), \quad m=1,2,\ldots$$

satisfying

$$\langle \partial_t u_m, \varphi_s\rangle + \sigma\langle \partial_t u_m, \varphi_s\rangle_0 + \langle |u_m|^{p-2}u_m, \varphi_s\rangle + \langle |\nabla u_m|^{p-2}\nabla u_m, \nabla\varphi_s\rangle = \lambda\langle |u_m|^q u_m, \varphi_s\rangle_0\ , \quad 1\leq s\leq m, \tag{6}$$

$$u_m(0)=\sum_{j=1}^{m} f_{jm}(0)\varphi_j(x) \to u_0(x) \quad \text{in } W^{1,p}(\Omega). \tag{7}$$

Multiplying (6) by $f'_{sm}(t)$, summing for s and integrating with respect to t, we get

$$\int_0^t \|\partial_\tau u_m(\tau)\|^2_{\mathcal{X}^2}\ \mathrm{d}\tau + E\left(u_m(t)\right) = E\left(u_m(0)\right), \quad \forall t\in[0,\infty). \tag{8}$$

Next, if $0<E\left(u_0\right)<d$ and $E_{\delta_2}\left(u_0\right)>0$, then by Corollary (3.1) we have $E_\delta\left(u_0\right)>0$ and $E\left(u_0\right)<d(\delta)$ for all $\delta\in(\delta_1,\delta_2)$, consequently $u_0(x)\in X_\delta$ for all $\delta\in(\delta_1,\delta_2)$. For any fixed $\delta\in(\delta_1,\delta_2)$, we get $u_m(0)\in X_\delta$ for sufficiently large m.
Next, we prove that

$$u_m(t)\in X_\delta,\ \forall t\in[0,\infty). \tag{9}$$

Arguing by contradiction, we assume that there exist a $t_0>0$ such that $u_m\left(t_0\right)\in\partial X_\delta$, i.e., $E_\delta\left(u_m\left(t_0\right)\right)=0$ and $\left\|u_m\left(t_0\right)\right\|_{1,p}\neq 0$ or $E\left(u_m\left(t_0\right)\right)=d(\delta)$. By (8) we obtain

$$E\left(u_m(t)\right)\leq E\left(u_m(0)\right)<d(\delta), \quad \forall t\in[0,\infty). \tag{10}$$

From (10) we can see that $E\left(u_m\left(t_0\right)\right)\neq d(\delta)$. If $E_\delta\left(u_m\left(t_0\right)\right)=0$ and $\left\|u_m\left(t_0\right)\right\|_{1,p}\neq 0$, then it follows from Theorem (3.1) that $E_\delta\left(u_m\left(t_0\right)\right)\geq d(\delta)$, which contradicts (10). Thus assertion (9) follows as desired.
From (8), (9) and

$$E\left(u_m(t)\right)=\frac{1-\delta}{p}\left\|u_m(t)\right\|^p_{1,p}+E_\delta\left(u_m(t)\right)$$

we see that

$$\int_0^t \|\partial_\tau u_m\|^2_{\mathcal{X}^2}\ \mathrm{d}\tau < d(\delta),$$

and

$$\|u_m(t)\|_{1,p}<\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}.$$

Then

$$\|\ |u_m(t)|^{p-2}u_m(t)\ \|^s_s=\|u_m(t)\|^p_p<\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}, \qquad s=\frac{p}{p-1},\ 0\leq t<\infty.$$

Moreover, from (5) we deduce

$$\|u_m(t)\|_{q+2,\partial\Omega}\leq C_*\|u_m(t)\|_{1,p}<\left(\frac{2+q}{\lambda p C_*^{p}}\delta\right)^{\frac{1}{2+q-p}},$$

thus

$$\|\ |u_m(t)|^{q}u_m(t)\ \|^r_{r,\partial\Omega}=\|u_m(t)\|^{q+2}_{q+2,\partial\Omega}<\left(\frac{2+q}{\lambda p C_*^{p}}\delta\right)^{\frac{q+2}{2+q-p}}, \qquad r=\frac{q+2}{q+1},\ 0\leq t<\infty,$$

for sufficiently large m and $t \in [0, \infty)$.
Then, there exist a u and a subsequence $\{u_v\}$ of $\{u_m\}$ such that as $v \to \infty$,

$$\begin{aligned}
&u_v \to u \quad \text{weakly star in } L^\infty\left(0,\infty; W^{1,p}(\Omega)\right), \\
&\partial_t u_v \to \partial_t u \quad \text{weakly in } L^2\left(0,\infty; L^2(\Omega)\right), \\
&\partial_t u_{v_{|\partial\Omega}} \to \partial_t u_{|\partial\Omega} \quad \text{weakly in } L^2\left(0,\infty; L^2(\partial\Omega)\right), \\
&|u_v|^{p-2}u_v \to |u|^{p-2}u \quad \text{weakly star in } L^\infty\left(0,\infty; L^s(\Omega)\right), \\
&|u_{v_{|\partial\Omega}}|^q u_{v_{|\partial\Omega}} \to |u_{|\partial\Omega}|^q u_{|\partial\Omega} \quad \text{weakly star in } L^\infty\left(0,\infty; L^r(\Omega)\right) \cap C\left(0,\infty; \mathcal{X}^2\right).
\end{aligned}$$

Hence, for fixed s, taking $m = v \to \infty$ in (6), we obtain

$$\langle \partial_t u, \varphi_s\rangle + \sigma\langle \partial_t u_{|\partial\Omega}, \varphi_s\rangle_0 + \langle |u|^{p-2}u, \varphi_s\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi_s\rangle = \lambda\langle |u_{|\partial\Omega}|^q u_{|\partial\Omega}, \varphi_s\rangle_0.$$

Furthermore, by (7) we get $u(x,0) = u_0(x)$ in $W^{1,p}(\Omega)$. Then, problem (1) admits a global solution $u \in L^\infty\left(0,T; W^{1,p}(\Omega)\right) \cap C\left(0,T; \mathcal{X}^2\right)$, with $\partial_t u \in L^2\left(0,T; L^2(\Omega)\right)$, $\partial_t u_{|\partial\Omega} \in L^2\left(0,T; L^2(\partial\Omega,\rho)\right)$, and $u(t) \in X_\delta$ for all $t \in [0,\infty)$. Since δ is arbitrary, then $u(t) \in X_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0,\infty)$.

4. Blow up in finite time

In this section, we prove the blow-up of solutions to problem (1) when the initial energy satisfies certain condition. In order to prove our main result, we will use the following auxiliary results.

Lemma 4.1
If $E(u) \le d(\delta)$, then

(i) $E_\delta(u) > 0$ if and only if

$$0 < \|u\|_{1,p} < \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}; \tag{11}$$

(ii) $E_\delta(u) < 0$ if and only if

$$\|u\|_{1,p} > \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}. \tag{12}$$

Proof

(i) If (11) holds, then we have

$$\|u\|_{2+q,\partial\Omega}^{2+q} \le C_*^{2+q}\|u\|_{1,p}^{2+q} = C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^p < \frac{2+q}{\lambda p}\delta\|u\|_{1,p}^p.$$

Consequently, $E_\delta(u) > 0$.
If $E_\delta(u) > 0$, then $\|u\|_{1,p} > 0$. Thus, from

$$E(u) = \frac{1-\delta}{p}\|u\|_{1,p}^p + E_\delta(u) \le d(\delta) \tag{13}$$

we get (11).

(ii) It is easy to see $\|u\|_{1,p} \neq 0$ from $E_\delta(u) < 0$. Hence, by

$$\frac{2+q}{\lambda p}\delta\|u\|_{1,p}^{p} < \|u\|_{2+q,\partial\Omega}^{2+q} \leq C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^{p}$$

we obtain (12).
On the other hand, combining (12) and (13) we obtain $E_\delta(u) < 0$.

□

Theorem 4.1
If $E(u) \leq d(\delta)$, then $X_\delta \subset B_\delta$ and $Y_\delta \subset B_\delta^c$.

Proof
This Theorem follows from Lemma (4.1). □

From Theorem (4.1) and Lemma (4.1) we have the following

Corollary 4.1
Assume that $E(u) \leq d(\delta)$. Then,

(i) $u \in X_\delta$ if and only if $u \in B_\delta$;

(ii) $u \in Y_\delta$ if and only if $u \in B_\delta^c$.

Corollary 4.2
Let $u_0(x) \in W^{1,p}(\Omega)$, p and q satisfy (H). Assume that $0 < e < d$ and $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = e$. Then,

(i) Solutions of problem (1) with $0 < E(u_0) \leq e$ belong to $\overline{X}_{\delta_1}$, provided $F(u_0) > 0$;

(ii) Solutions of problem (1) with $0 < E(u_0) \leq e$ belong to $\overline{Y}_{\delta_2}$, provided $F(u_0) < 0$.

Proof
Let $u(t)$ be any solution of problem (1) with $0 < E(u_0) \leq e$, and T be the maximum existence time of $u(t)$. Multiplying the first equation of (1) by u_t and integrating on Ω implies

$$\|u_t\|_\sigma^2 = -\frac{1}{p}\frac{d}{dt}\|\nabla u\|_{1,p}^{p} + \frac{\lambda}{q+2}\frac{d}{dt}\|u\|_{q+2,\partial\Omega}^{q+2}.$$

This equality along with (2) gives

$$\frac{d}{dt}E(u) = -\|u_t\|_{\mathcal{X}^2}^2,$$

then

$$E(u) + \int_0^t \|u_\tau\|_{\mathcal{X}^2}^2 d\tau = E(u_0), \quad \forall t \in [0,\infty). \tag{14}$$

By (14) we get $E(u) \leq d(\delta_1) = d(\delta_2)$. For fixed $t \in [0,T)$, taking $\delta \to \delta_1$ ($\delta \to \delta_2$) in $E_\delta(u) > 0$ ($E_\delta(u) < 0$), we obtain $E_{\delta_1}(u) \geq 0$ ($E_{\delta_2}(u) \leq 0$) for all $t \in [0,T)$. This shows the conclusions of the corollary (4.2). □

Here, we have our main result.

Theorem 4.2
Let $u_0(x) \in W^{1,p}(\Omega)$, p and q satisfy (H), and $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = E(u_0)$.

(i) Assume that $E(u_0) < d$ and $E_{\delta_1}(u_0) < 0$. Then solutions of problem (1) blow up in finite time.

(ii) Assume that $E(u_0) = d$ and $E_{\delta_0}(u_0) < 0$. Then the conclusion of (i) remains valid.

Proof of Theorem 4.2.

(i) Let $u(t)$ be any solution of problem (1) and T be the maximum existence time of $u(t)$. Next we prove $T < \infty$. Arguing by contradiction, we suppose that $T = \infty$.
Set

$$H(t) = \frac{1}{2}\int_0^t \|u\|_{\mathcal{X}^2}^2 \, d\tau.$$

Then

$$H'(t) = \frac{1}{2}\|u\|_{\mathcal{X}^2}^2,$$

and

$$H''(t) = \langle u, u_t \rangle_{\mathcal{X}^2} = -F(u). \tag{15}$$

By (14) and

$$E(u) = \frac{2+q-p}{p(2+q)}\|u\|_{1,p}^p + \frac{1}{2+q}F(u), \tag{16}$$

we obtain

$$F(u) = -\frac{2+q-p}{p}\|u\|_{1,p}^p + (2+q)E(u_0) - (2+q)\int_0^t \|u_\tau\|_{\mathcal{X}^2}^2 \, d\tau.$$

Now, from (15), we can write

$$H''(t) = \frac{2+q-p}{p}\|u\|_{1,p}^p - (2+q)E(u_0) + (2+q)\int_0^t \|u_\tau\|_{\mathcal{X}^2}^2 \, d\tau. \tag{17}$$

Next, we show that

$$H''(t) \geq (2+q)\int_0^t \|u_\tau\|_{\mathcal{X}^2}^2 \, d\tau. \tag{18}$$

To see this, we distinguish the following two cases.
Case 1. The case $E(u_0) \leq 0$.
Assertion (18) follows directly from (17).
Case 2. The case $0 < E(u_0) < d$.
By $E_{\delta_1}(u_0) < 0$ and Corollary (3.1) we get $E_{\delta_0}(u_0) < 0$. Note that $E(u) \leq E(u_0) < d$. Hence, by recaling the definition of Y_δ, we obtain $u \in Y_{\delta_0}$. Consequently, from (ii) in Corollary (4.1), we obtain $u \in B_{\delta_0}^c$, i.e.,

$$\|u\|_{1,p} > \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta_0\right)^{\frac{1}{2+q-p}}.$$

Which together with (ii) in Lemma (3.1) and Theorem (3.1), we can deduce

$$\|u\|_{1,p}^p > C_*^{-\frac{(2+q)p}{2+q-p}} = \frac{(2+q)p}{2+q-p}d > \frac{(2+q)p}{2+q-p}E(u_0).$$

Combining this with (17), thus assertion (18) follows as desired.
Next, from (18), there exists a $t^* > 0$ such that $H'(t) \geq H'(t^*) > 0$ and $H(t) \geq H'(t^*)(t - t^*) + H(t^*)$ for all $t \in [t^*, \infty)$. Consequently

$$\lim_{t\to\infty} H(t) = \infty. \tag{19}$$

Combining (18) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} H(t)H''(t) &\geq \frac{2+q}{p}\int_0^t \|u\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau \int_0^t \|u_\tau\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau \\ &\geq \frac{2+q}{p}\left(\int_0^t \langle u, u_\tau\rangle_{\mathcal{X}^2}\mathrm{d}\tau\right)^2 \\ &= \frac{2+q}{p}\left(H'(t)-H'(0)\right)^2. \end{aligned}$$

Then there exists a $\alpha > 0$ such that

$$H(t)H''(t) \geq (1+\alpha)H'(t)^2.$$

For all $t \in [t^*, \infty)$, consequently

$$\left(H^{-\alpha}(t)\right)' = -\frac{\alpha H'(t)}{H^{\alpha+1}(t)} < 0,$$

and

$$\left(H^{-\alpha}(t)\right)'' = -\frac{\alpha}{H^{\alpha+2}(t)}\left[H(t)H''(t) - (\alpha+1)\left(H'(t)\right)^2\right] \leq 0.$$

Therefore, $H^{-\alpha}(t) > 0$ is decreasing and concave on $[t^*, \infty)$, which contradicts (19), then $T < \infty$. Hence the conclusion of (i) holds.

(ii) First, we show that

$$E_{\delta_0}(u) < 0, \quad \forall t \in [0, \infty). \tag{20}$$

Arguing by contradiction, we assume that there exist a first time $t_0 > 0$ such that $E_{\delta_0}(u(t_0)) = 0$ and $E_{\delta_0}(u) < 0$ for all $t \in [0, t_0)$. By (ii) in Lemmas (4.1) and (3.1), we can deduce

$$\|u\|^p_{1,p} > C_*^{-\frac{(2+q)p}{2+q-p}}, \quad \forall t \in [0, t_0)$$

which together with Theorem (3.1) gives

$$\|u\|^p_{1,p} > \frac{(2+q)p}{2+q-p}d, \quad \forall t \in [0, t_0),$$

consequently

$$\|u(t_0)\|^p_{1,p} \geq \frac{(2+q)p}{2+q-p}d.$$

Which together with (16), we obtain

$$E(u(t_0)) \geq d. \tag{21}$$

At the same time, by (15), we have $\langle u, u_t\rangle_{\mathcal{X}^2} > 0$, which implies that $\int_0^t \|u_\tau\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau$ is increasing in time. Consequently

$$\int_0^{t_0} \|u_\tau\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau > 0.$$

Combining this with (14) and $E(u_0) = d$, we get

$$E(u(t_0)) < d.$$

which contradicts (21), then assertion (20) holds.
For any $\tilde{t} > 0$, let

$$d_1 := d - \int_0^{\tilde{t}} \|u_t\|^2_{\mathcal{X}^2}\,\mathrm{d}t,$$

Thus

$$0 < E(u) \le d_1 < d \text{ for all } t \in [\tilde{t}, \infty),$$

which together with assertion (20) and (ii) in Corollary (4.2) gives

$$u \in \overline{Y}_{\tilde{\delta}_2} \text{ for all } t \in [\tilde{t}, \infty),$$

where $\tilde{\delta}_1 < \tilde{\delta}_2$ are two roots of equation $d(\delta) = d_1$.
Consequently

$$E_{\tilde{\delta}_2}(u) \le 0 \text{ for all } t \in [\tilde{t}, \infty).$$

We also obtain

$$E_{\tilde{\delta}_1}(u) < 0 \text{ for all } t \in [\tilde{t}, \infty).$$

The remainder of proof of (ii) can be performed by a repetition of the arguments in the proof of Case 2 in (i).

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