STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. 11, March 2023, pp 186–195.
Published online in International Academic Press (www.IAPress.org)

Estimates for Distributions of Suprema of Spherical Random Fields

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Abstract Bounds for distributions of suprema of φ-sub-Gaussian random fields defined over the N-dimensional unit sphere are stated. Applications of the results to the spherical fractional Brownian motion, isotropic Gaussian fields and some other models are presented.

Keywords Spherical random fields, φ-sub-Gaussian random fields, Distribution of supremum, Spherical fractional Brownian motion, Isotropic Gaussian random fields

AMS 2010 subject classifications 60G60, 60G22, 60G70

DOI: 10.19139/soic-2310-5070-1705

1. Introduction

The constantly growing interest in studying spherical random fields is motivated by the strong demand from various applied areas such as geophysics, geodesy, planetary sciences, astronomy, cosmology and others, especially, in view of accumulation of huge amounts of experimental data which should be analyzed by appropriate statistical methods and tools. Numerous research papers have been devoted to the characterization of covariance structure, spectral and statistical analysis, modelling and simulation of random fields on the sphere. We mention two classical monographs by Yadrenko [28] and Marinucci and Peccati [24] as the excellent fundamental sources in the area.

The present paper is related to the investigations of extreme values and excursion probabilities of random fields on the sphere. The problem of evaluation of probabilities $\mathsf{P}\big\{\sup_{t\in\mathbf{T}} X(t) \geq u\big\}$ have been intensively studied for various classes of stochastic processes considered over different parameter spaces $\mathbf{T}$. In particular, for Gaussian case, a number of approaches and techniques have been developed for deriving approximations for excursion probabilities (tail probabilities) for large u (see, e.g., [1], [2] and references therein). Distributions of suprema under more general non-Gaussian settings were treated in [3]. For spherical random fields the properties of sample paths, excursion sets and excursion probabilities were recently studied in [4, 5, 21, 22, 23, 25, 26]. Asymtotics for excursion probabilities $\mathsf{P}\big\{\sup_{t\in\mathbb{S}^N} X(t) \geq u\big\}$, as $u\to\infty$, for a (locally) isotropic Gaussian random field X over N-dimensional unit sphere $\mathbb{S}^N$, were stated in [4, 5] by applying different methods for the cases of smooth and non-smooth sample paths and appropriately adapting for the spherical case the techniques developed for the fields in $\mathbb{R}^N$.

In the paper we present non asymptotic bounds for $\mathsf{P}\big\{\sup_{t\in\mathbb{S}^N} |X(t)| \geq u\big\}$ assuming X to be φ-sub-Gaussian random field (to be defined below in Section 2). Note that it is important for applications to

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

go beyond the Gaussianity assumption and possible extensions are provided by sub-Gaussian and φ-sub-Gaussian random fields. Powerful tools and techniques for investigation of sample paths properties of these classes of fields by means of entropy methods have been elaborated in the literature (see, e.g., [3]). Here we apply the results from [3] for spherical random fields. To the best of our knowledge, the specification of the results from [3] for spherical random fields was not presented in the literature before.

In Section 2 we give necessary definitions and facts on φ-sub-Gaussian random fields. Bounds for distributions of suprema are stated in Section 3. Applications of the results to spherical fractional Brownian motion, isotropic Gaussian fields and some other models are presented in Section 4.

2. Preliminaries

The main theory for the spaces of φ-sub-Gaussian random variables and stochastic processes was presented in [3, 9, 20] and has numerous further developments in the recent literature. Such spaces can be considered as exponential type Orlicz spaces of random variables and provide generalizations of Gaussian and sub-Gaussian random variables and processes (see, [3, Ch.2]).

We present the main definitions and facts needed in our exposition.

Definition 1. [9, 20] A continuous even convex function φ is called an Orlicz N-function if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and $\lim\limits_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim\limits_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

Condition Q. Let φ be an N-function which satisfies $\lim\inf\limits_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible.

Definition 2. [9, 20] Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathsf{P}\}$ be a standard probability space. The random variable ζ is φ-sub-Gaussian, or belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, if $\mathbb{E}\zeta = 0$, $\mathbb{E}\exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$\mathbb{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(\lambda a)\}.$$

The random process $\zeta = \{\zeta(t), t \in T\}$ is called φ-sub-Gaussian if the random variables $\{\zeta(t), t \in T\}$ are φ-sub-Gaussian.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm (see [9, 20]):

$$\tau_\varphi(\zeta) = \inf\{a > 0 : \mathbb{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(a\lambda)\}.$$

Definition 3. [9, 20] The function φ^* defined by $\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$ is called the Young-Fenchel transform (or convex conjugate) of the function φ.

The function φ^* (known also as the Legendre or Legendre-Fenchel transform) plays an important role in the theory of φ-sub-Gaussian random variables and processes, in particular, one can estimate the 'tail' probabilities in terms of the function φ^*. Namely, if ζ is a φ-sub-Gaussian random variable, then for all $u > 0$ we have

$$P\{|\zeta| > u\} \leq 2\exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}. \tag{1}$$

Moreover, it is stated in [3] (see, Corollary 4.1, p. 68) that a random variable ζ is a φ-sub-Gaussian if and only if $\mathbb{E}\zeta = 0$ and there exist constants $C > 0$, $D > 0$ such that

$$P\{|\zeta| > u\} \leq C\exp\left\{-\varphi^*\left(\frac{u}{D}\right)\right\}. \tag{2}$$

As one can see, the property of φ-sub-Gaussianity can be characterized in a double way: by introducing a bound on the exponential moment of a random variable as prescribed by Definition 2, or by the tail behavior of the form (1) or (2), which is even more essential from the practical point of view.

The class of φ-sub-Gaussian random variables is rather wide and comprises, for example, centered compactly supported distributions, reflected Weibull distributions, centered bounded distributions, Gaussian, Poisson distributions. In the case when $\varphi = \frac{x^2}{2}$, the notion of φ-sub-Gaussianity reduces to the classical sub-Gaussianity. Various classes of φ-sub-Gaussian processes and fields were studied, in particular, in [6, 15, 17, 18, 19, 27] (see also references therein).

The example below demonstrates one particular way to construct φ-sub-Gaussian processes and fields.

Example 1. [14] Let $\{\xi_k,\ k = \overline{1,\infty}\}$ be a family of independent φ-sub-Gaussian random variables and φ be a such function that $\varphi(\sqrt{x})$, $x > 0$, is convex. If there exists $C > 0$ such that $\tau_\varphi(\xi_n) \le C(\mathbb{E}\xi_k^2)^{1/2}$ for any $k \ge 1$, and for a sequence of nonrandom functions$f_k(t)$, $t \in T$, $k \ge 1$, the series $\sum_{k=1}^{\infty} \mathbb{E}\xi_k^2 f_k^2(t)$ converges for all $t \in T$, then $X(t) = \sum_{k=1}^{\infty} \xi_k f_k(t)$, $t \in T$, is a φ-sub-Gaussian random process and $\tau_\varphi^2(X(t) - X(s)) \le C^2\mathbb{E}(X(t) - X(s))^2$, $t, s \in T$.

Sample paths properties of φ-sub-Gaussian processes and fields can be characterized by means of entropy methods. We will use the following well known result.

Let us consider the metric space $(\mathbf{T}, \rho)$, $\mathbf{T} = \{a_i \le t_i \le b_i, i = 1, ..., N\}$, $\rho(t, s) = \max_{i=\overline{1,N}} |t_i - s_i|$ and $X = \{X(t), t \in \mathbf{T}\}$ be a φ-sub-Gaussian process.

Introduce the following conditions.

A.1 $\varepsilon_0 = \sup_{t\in\mathbf{T}} \tau_\varphi(X(t)) < \infty$.

A.2 The process X is separable on the space $(\mathbf{T}, \rho)$.

A.3 There exists a strictly increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and

$$\sup_{\rho(t,s)<h} \tau_\varphi(X(t) - X(s)) \le \sigma(h).$$

A.4 Let $r = \{r(x), x \ge 1\}$ be a non-negative, nondecreasing function such that $r(e^y)$, $y \ge 0$, is convex.

Denote

$$I_r(\delta) = \int_0^\delta r\Big(\prod_{i=1}^{N}\Big(\frac{b_i - a_i}{2\sigma^{(-1)}(u)} + 1\Big)\Big)du,\ \ \delta > 0. \tag{3}$$

Denote $\gamma_0 = \sigma(\max_{i=\overline{1,N}} |b_i - a_i|)$. For a function $f(t)$, $t \ge 0$, we denote by $f^{(-1)}(u), u \ge 0$, the inverse function.

Theorem 1 below is a corollary of the result stated in [3, Theorem 4.4, p. 107] (see also [19, Theorem 3.1], [15]).

Theorem 1. Let for a φ-sub-Gaussian process $X = \{X(t), t \in \mathbf{T}\}$ conditions A.1-A.4 hold and suppose that $I_r(\gamma_0) < \infty$. Then for all $0 < \mu < 1$ and $u > 0, \lambda > 0$

$$\mathbb{E}\exp\Big\{\lambda \sup_{t\in\mathbf{T}} |X(t)|\Big\} \le 2\exp\Big\{\varphi\Big(\frac{\lambda\varepsilon_0}{1-\mu}\Big)\Big\}A(\mu), \tag{4}$$

$$\mathsf{P}\Big\{\sup_{t\in\mathbf{T}} |X(t)| \ge u\Big\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\mu)}{\varepsilon_0}\Big)\Big\}A(\mu), \tag{5}$$

where

$$A(\mu) = r^{(-1)}\left(\frac{I_r(\mu\varepsilon_0)}{\mu\varepsilon_0}\right). \tag{6}$$

For a particular form of σ, by choosing an appropriate function r, the expression (6) can be calculated in the closed form.

Remark 1. We comment shortly on the conditions of Theorem 1. Let $\xi(x)$, $x \in \mathbf{T}$, be a φ-sub-Gaussian process and $\rho_\xi(x,y) = \tau_\varphi(\xi(x) - \xi(y))$, $x, y \in \mathbf{T}$ (which is a pseudometric on $\mathbf{T}$, see [3]). The integrals of the form

$$I(\varepsilon) := \int_0^\varepsilon g(N(v))\, dv,\ \varepsilon > 0, \tag{7}$$

are called entropy integrals, where $g(v)$, $v \ge 1$, is a nonnegative nondecreasing function and $N(v)$, $v > 0$, is the metric massiveness of the pseudometric space $(\mathbf{T}, \rho_X)$, that is, the smallest number of elements in a v-covering of $\mathbf{T}$ formed by closed balls of radius of at most v. Entropy characteristics of the parametric set $\mathbf{T}$ with respect to the pseudometrics ρ_ξ generated by the process ξ, and the rate of growth of metric massiveness $N(v)$, or metric entropy $H(v) := \ln(N(v))$, are closely related to the properties of the process ξ (see [3] for details).

The integrals (7) play an important role in studying properties of sample paths and estimating modulii of continuity and distribution of supremum of a process. General results of this kind for φ-sub-Gaussian processes are related to the convergence of the integrals (7), where for $g(v)$ one takes $\Psi(\ln(v))$ with $\Psi(v) = \frac{v}{\varphi^{(-1)}(v)}, v > 0$, for a sub-Gaussian case ($\varphi(x) = \frac{x^2}{2}$) they reduce to the Dudly integrals $\int \sqrt{\ln(N(v))}dv$.

As pointed out in [3], integrals (7) with g satisfying condition A.4 are more suitable for the case of "moderate" growth of the metric entropy and can lead to improved inequalities for upper bound for the distribution of supremum, in comparison with more general inequalities involving the integrals based on the above function Ψ. Specifically for $\mathbf{T} = \{a_i \le t_i \le b_i, i = 1, ..., N\}$, and in view of condition A.3, the integral (7) becomes of the form (3) (for more detail, see [3, 15, 19]).

Remark 2. As for the practical use, we mention that Theorem 1 was applied, for example, in [15, 16] for developing uniform approximation schemes for φ-sub-Gaussian processes. In [8] Theorem 1 was proved to be effective in studying increments of multifractional Brownian motion and some its functionals, and in developing statistical estimation methods.

Example 2. Returning to Example 1, let $(\mathbf{T}, \rho)$ be as in Theorem 1, suppose additionally that functions f_k are such that for some $c_k > 0$, $k \ge 1$, and strictly increasing continuous function $\sigma(h)$, $h \ge 0$, $\sigma(0) = 0$, we have $\sup_{\rho(t,s)<h} |f_k(t) - f_k(s)| \le c_k \sigma(h)$ and $\sum\limits_{k=1}^{\infty} \mathbb{E}\xi_k^2 c_k^2 < \infty$. Then condition A.3 holds.

3. Results

Let $\xi = \{\xi(x), x \in \mathbb{S}^N\}$ be a φ-sub-Gaussian random field on the unit sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$. We are interested in evaluation of $P\Big\{\sup\limits_{x \in \mathbb{S}^N} |\xi(x)| \ge u\Big\}$. For $x = (x_1, \dots, x_{N+1}) \in \mathbb{S}^N$ introduce the spherical coordinates:

$$x_1 = \cos\theta_1, x_2 = \sin\theta_1 \cos\theta_2, \dots, x_N = \Big(\prod_{i=1}^{N-1} \sin\theta_i\Big) \cos\theta_N, x_{N+1} = \prod_{i=1}^{N} \sin\theta_i, \tag{8}$$

$\theta \in \Theta = [0,\pi]^{N-1} \times [0, 2\pi)$.

Denote by $d(x,y) = \arccos\langle x, y\rangle$, $x, y \in \mathbb{S}^N$, the spherical distance on $\mathbb{S}^N$.

Define the field $\widetilde{\xi} = \{\widetilde{\xi}(\theta), \theta \in \Theta\}$ as $\widetilde{\xi}(\theta) := \xi(x)$, that is, ξ under the spherical coordinates given above.

Introduce the following conditions.

B.1 $\varepsilon_0 = \sup_{x \in \mathbb{S}^N} \tau_\varphi(\xi(x)) < \infty$.

B.2 There exists a strictly increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and

$$\tau_\varphi(\xi(x) - \xi(y)) \le \sigma(d(x,y)),\ x, y \in \mathbb{S}^N.$$

We note that $P\Big\{\sup\limits_{x\in\mathbb{S}^N}|\xi(x)|\geq u\Big\}=P\Big\{\sup\limits_{\theta\in\Theta}|\widetilde{\xi}(\theta)|\geq u\Big\}$, and therefore, we can apply Theorem 1 for the field $\widetilde{\xi}$ to obtain the result for the field ξ. Consider $\widetilde{\xi}(\theta)$ on the metric space (Θ,ρ) with the metric $\rho(\theta,\eta)=\max\limits_{i=\overline{1,N}}|\theta_i-\eta_i|$, $\theta,\eta\in\Theta$, and suppose that the next condition holds.

B.3 The field $\widetilde{\xi}$ is separable on the space (Θ,ρ).

Denote

$$\widetilde{I}_r(\delta)=\int_0^\delta r\Big(\Big(\frac{\pi}{2\widetilde{\sigma}^{(-1)}(u)}+1\Big)^{N-1}\Big(\frac{2\pi}{2\widetilde{\sigma}^{(-1)}(u)}+1\Big)\Big)du, \tag{9}$$

where $\widetilde{\sigma}(u)=\sigma\Big(\frac{\pi}{2}\sqrt{N}u\Big)$.

Theorem 2. Let for a φ-sub-Gaussian field $\xi=\{\xi(x),x\in\mathbb{S}^N\}$ Conditions B.1-B.3 and A.4 hold. Suppose that $\widetilde{I}_r(\gamma_0)<\infty$ for $\gamma_0=\widetilde{\sigma}(2\pi)$.

Then for all $0<\mu<1$ and $u>0,\lambda>0$

$$\mathbb{E}\exp\Big\{\lambda\sup_{x\in\mathbb{S}^N}|\xi(x)|\Big\}\leq 2\exp\Big\{\varphi\Big(\frac{\lambda\varepsilon_0}{1-\mu}\Big)\Big\}A_1(\mu\varepsilon_0), \tag{10}$$

$$\mathsf{P}\Big\{\sup_{x\in\mathbb{S}^N}|\xi(x)|\geq u\Big\}\leq 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\mu)}{\varepsilon_0}\Big)\Big\}A_1(\mu\varepsilon_0), \tag{11}$$

where

$$A_1(\mu\varepsilon_0)=r^{(-1)}\left(\frac{\widetilde{I}_r(\mu\varepsilon_0)}{\mu\varepsilon_0}\right). \tag{12}$$

Proof
Since $P\Big\{\sup\limits_{x\in\mathbb{S}^N}|\xi(x)|\geq u\Big\}=P\Big\{\sup\limits_{\theta\in\Theta}|\widetilde{\xi}(\theta)|\geq u\Big\}$, we apply Theorem 1 for the field $\widetilde{\xi}$.

We have

$$\tau_\varphi(\widetilde{\xi}(\theta)-\widetilde{\xi}(\eta))=\tau_\varphi(\xi(x)-\xi(y))\leq\sigma(d(x,y)). \tag{13}$$

Using the relation between Euclidean and spherical distances $\|x-y\|=2\sin\left(\frac{d(x,y)}{2}\right)$ and the estimate $\frac{2}{\pi}t\leq\sin t\leq t$ for $0\leq t\leq\frac{\pi}{2}$, we can write the estimate $d(x,y)\leq\frac{\pi}{2}\|x-y\|$. Consider

$$\begin{aligned}
\|x-y\|^2 &= (\cos\eta_1-\cos\theta_2)^2+(\sin\eta_1\cos\eta_2-\sin\theta_1\cos\theta_2)^2+\dots\\
&\quad+\Big(\prod_{i=1}^N\sin\eta_i-\prod_{i=1}^N\sin\theta_i\Big)^2\\
&= 2-2\cos(\eta_1-\theta_1)+2(\sin\eta_1\sin\theta_1)[1-\cos(\eta_2-\theta_2)]+\dots\\
&\quad+2\Big(\prod_{i=1}^{N-1}\sin\eta_i\sin\theta_i\Big)[1-\cos(\eta_N-\theta_N)]\leq 2\sum_{i=1}^N 2\sin^2\left(\frac{\eta_i-\theta_i}{2}\right)\\
&\leq \sum_{i=1}^N(\eta_i-\theta_i)^2=\|\theta-\eta\|^2\leq N\max_{i=\overline{1,N}}|\theta_i-\eta_i|^2=N\rho^2(\theta,\eta).
\end{aligned}$$

Therefore, we can evaluate

$$d(x,y)\leq\frac{\pi}{2}\|x-y\|\leq\frac{\pi}{2}\|\theta-\eta\|\leq\frac{\pi}{2}\sqrt{N}\rho(\theta,\eta)$$

and obtain

$$\sup_{\rho(\theta,\eta)<h}\tau_\varphi(\widetilde{\xi}(\theta)-\widetilde{\xi}(\eta))\leq\widetilde{\sigma}(h),$$

with $\widetilde{\sigma}(u)=\sigma\Big(\frac{\pi}{2}\sqrt{N}u\Big)$. We conclude that conditions of Theorem 1 are satisfied for the field $\widetilde{\xi}$ and applying this theorem we obtain (10) and (11).

□

The next result presents the case where the bound can be calculated in the closed form.

Theorem 3. Let under the conditions of Theorem 2 $\sigma(h)=Ch^{\beta}$ with $C>0, 0<\beta\le 1$. Then for any $\mu\in(0,1)$ such that $\mu\varepsilon_0<\left(\frac{\pi}{2}\right)^{2\beta}CN^{\beta/2}$ and any $u>0,\lambda>0$

$$\mathbb{E}\exp\Big\{\lambda\sup_{x\in\mathbb{S}^N}|\xi(x)|\Big\}\le 2\exp\Big\{\varphi\Big(\frac{\lambda\varepsilon_0}{1-\mu}\Big)\Big\}\widetilde{A}_1(\mu\varepsilon_0), \tag{14}$$

$$\mathsf{P}\Big\{\sup_{x\in\mathbb{S}^N}|\xi(x)|\ge u\Big\}\le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\mu)}{\varepsilon_0}\Big)\Big\}\widetilde{A}_1(\mu\varepsilon_0), \tag{15}$$

where

$$\widetilde{A}_1(\mu\varepsilon_0)=2^{1-N}\pi^{2N}N^{N/2}(eC)^{N/\beta}(\mu\varepsilon_0)^{-N/\beta}. \tag{16}$$

Proof

We evaluate the expression $r^{(-1)}\Big(\frac{\widetilde{I}_r(\mu\varepsilon_0)}{\mu\varepsilon_0}\Big)$ by choosing $r(x)=x^{\alpha}-1,\ 0<\alpha<\beta/2$. We have $\widetilde{\sigma}(h)=\sigma\Big(\frac{\pi}{2}\sqrt{N}h\Big)=C\left(\frac{\pi}{2}\sqrt{N}\right)^{\beta}h^{\beta}=\widetilde{c}\,h^{\beta}$, $\widetilde{\sigma}^{(-1)}(h)=\left(\frac{h}{\widetilde{c}}\right)^{1/\beta}$ and $r^{(-1)}(x)=(x+1)^{1/\alpha}$.

Consider

$$\widetilde{I}_r(\mu\varepsilon_0)=\int_0^{\mu\varepsilon_0}\Big(\Big(\frac{\pi\widetilde{c}^{1/\beta}}{2u^{1/\beta}}+1\Big)^{(N-1)\alpha}\Big(\frac{2\pi\widetilde{c}^{1/\beta}}{2u^{1/\beta}}+1\Big)^{\alpha}-1\Big)du,$$

let us choose μ such that $\frac{\pi\widetilde{c}^{1/\beta}}{2(\mu\varepsilon_0)^{1/\beta}}>1$ then we can write

$$\begin{aligned}\hat{I}_r(\mu\varepsilon_0) &\le \int_0^{\mu\varepsilon_0}\Big(\Big(\frac{\pi\widetilde{c}^{1/\beta}}{u^{1/\beta}}\Big)^{(N-1)\alpha}\Big(\frac{2\pi\widetilde{c}^{1/\beta}}{u^{1/\beta}}\Big)^{\alpha}-1\Big)du=\int_0^{\mu\varepsilon_0}\Big(2^{\alpha}\Big(\frac{\pi\widetilde{c}^{1/\beta}}{u^{1/\beta}}\Big)^{N\alpha}-1\Big)du\\ &= 2^{\alpha}\pi^{N\alpha}\,\widetilde{c}^{N\alpha/\beta}\left(1-\frac{N\alpha}{\beta}\right)^{-1}(\mu\varepsilon_0)^{1-N\alpha/\beta}-\mu\varepsilon_0\end{aligned}$$

and

$$r^{(-1)}\Big(\frac{\widetilde{I}_r(\mu\varepsilon_0)}{\mu\varepsilon_0}\Big)\le 2\pi^N\,\widetilde{c}^{N/\beta}\left(1-\frac{N\alpha}{\beta}\right)^{-1/\alpha}(\mu\varepsilon_0)^{-N/\beta}.$$

We now let $\alpha\to 0$ so that $\left(1-\frac{N\alpha}{\beta}\right)^{-1/\alpha}\to e^{N/\beta}$ and insert the expression for $\widetilde{c}$ to obtain (16). □

Remark 3. An important natural generalization of Gaussian fields is obtained with $\varphi(x)=\frac{|x|^{\varkappa}}{\varkappa}, 1<\varkappa\le 2$. For this case $\varphi^*(x)=\frac{|x|^{\gamma}}{\gamma}$, where $\gamma\ge 2$, and $\frac{1}{\varkappa}+\frac{1}{\gamma}=1$. For such φ-sub-Gaussian field the exponential term in (15) takes the form $\exp\Big\{-\frac{u^{\gamma}(1-\mu)^{\gamma}}{\gamma\varepsilon_0^{\gamma}}\Big\}$ and in the case of a Gaussian centered field ξ we have the term $\exp\Big\{-\frac{u^2(1-\mu)^2}{2\varepsilon_0^2}\Big\}$ with $\varepsilon_0^2=\sup_{x\in\mathbb{S}^N}E(\xi(x))^2$.

We next present the bounds for the distribution of supremum for the random field ξ over a geodesic disc on $\mathbb{S}^N$ with radius $a\in(0,\pi)$ at the center $o=(1,0,\dots,0)\in\mathbb{R}^{N+1}$ defined as $\mathbb{D}_a=\{x\in\mathbb{S}^N: d(x,o)\le a\}$. With such choice of a center, the set $\mathbb{D}_a$ under the spherical coordinates is $\Theta_a=[0,a]\times[0,\pi]^{N-2}\times[0,2\pi)$.

Denote

$$\widetilde{I}_r^a(\delta)=\int_0^{\delta}r\Big(\Big(\frac{a}{2\widetilde{\sigma}^{(-1)}(u)}+1\Big)\Big(\frac{\pi}{2\widetilde{\sigma}^{(-1)}(u)}+1\Big)^{N-2}\Big(\frac{2\pi}{2\widetilde{\sigma}^{(-1)}(u)}+1\Big)\Big)du, \tag{17}$$

$\widetilde{\varepsilon}_0=\sup_{x\in\mathbb{D}_a}\tau_{\varphi}(\xi(x))$.

Theorem 4. Under the conditions of Theorem 2 for all $0<\mu<1$ and $u>0, \lambda>0$

$$\mathbb{E}\exp\Big\{\lambda \sup_{x\in\mathbb{D}_a}|\xi(x)|\Big\} \le 2\exp\Big\{\varphi\Big(\frac{\lambda\widetilde{\varepsilon}_0}{1-\mu}\Big)\Big\}A_2(\mu\widetilde{\varepsilon}_0), \tag{18}$$

$$\mathsf{P}\Big\{\sup_{x\in\mathbb{D}_a}|\xi(x)|\ge u\Big\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\mu)}{\widetilde{\varepsilon}_0}\Big)\Big\}A_2(\mu\widetilde{\varepsilon}_0), \tag{19}$$

where

$$A_2(\mu\widetilde{\varepsilon}_0) = r^{(-1)}\left(\frac{\widetilde{I}_r^a(\mu\widetilde{\varepsilon}_0)}{\mu\widetilde{\varepsilon}_0}\right). \tag{20}$$

Let, moreover, $\sigma(h)=Ch^\beta$ with $C>0, 0<\beta\le 1$. Then for any $\mu\in(0,1)$ such that $\mu\widetilde{\varepsilon}_0<\left(\frac{a\pi}{4}\right)^\beta CN^{\beta/2}$ and any $u>0, \lambda>0$ the bounds (18) and (19) hold with the term A_2 replaced by the following one:

$$\widetilde{A}_2(\mu\widetilde{\varepsilon}_0) = 2^{1-N}a\pi^{2N-1}N^{N/2}(eC)^{N/\beta}(\mu\widetilde{\varepsilon}_0)^{-N/\beta}. \tag{21}$$

Remark 4. The results of this section are obtained basing on Theorem 1. This theorem has been applied in different contexts, e.g., in [8, 11, 15, 16, 19], etc. However, to the best of the author's knowledge, specification of this theorem for the case of spherical random fields was not presented in the literature before. We believe that such results can be useful from the practical point of view. In the next section we present several particular examples. The interesting and important question would be to compare the obtained results with the asymptotic results presented in [4, 5] and to check under what conditions the obtained bounds could be close to optimal. We address these topics to further investigation, in particular, by numerical methods and considering the examples presented in the next section.

4. Examples

4.1. Spherical fractional Brownian motion

Istas [12], [13] studied the spherical fractional Brownian motion (SFBM) $B_\beta=\{B_\beta(x), x\in\mathbb{S}^N\}$ defined as a centered real-valued Gaussian random field such that $B_\beta(o)=0$ for some fixed point $o\in\mathbb{S}^N$ and

$$\mathbb{E}(B_\beta(x)-B_\beta(y))^2 = d^{2\beta}(x,y),\ x,y\in\mathbb{S}^N,\ \beta\in(0,1/2]. \tag{22}$$

Note that the covariance function of B_β is of the form

$$R_\beta(x,y) = Cov(B_\beta(x),B_\beta(y)) = \frac{1}{2}(d^{2\beta}(x,o)+d^{2\beta}(y,o)-d^{2\beta}(x,y)),\ x,y\in\mathbb{S}^N. \tag{23}$$

Since we have the Gaussian case, $\varphi(x)=\frac{x^2}{2}$, $\tau_\varphi^2(B_\beta(x))=\mathbb{E}B_\beta^2(x)$, and (22) implies the fulfilment of Condition B.2 with $\sigma(h)=h^\beta$. Applying Theorems 3, 4 we obtain the following bounds for the excursion probabilities of B_β over $\mathbb{S}^N$ and $\mathbb{D}_a$ for all $u>0$ and $0<\mu<1$ (we set the point $o=(1,0,\dots,0)\in\mathbb{R}^{N+1}$):

$$\mathsf{P}\Big\{\sup_{x\in\mathbb{S}^N}|B_\beta(x)|\ge u\Big\} \le 2\exp\Big\{-\frac{u^2(1-\mu)^2}{2\pi^{2\beta}}\Big\}2^{1-N}\pi^N N^{N/2}e^{N/\beta}\mu^{-N/\beta},$$

$$\mathsf{P}\Big\{\sup_{x\in\mathbb{D}_a}|B_\beta(x)|\ge u\Big\} \le 2\exp\Big\{-\frac{u^2(1-\mu)^2}{2a^{2\beta}}\Big\}(2a)^{1-N}\pi^{2N-1}N^{N/2}e^{N/\beta}\mu^{-N/\beta}.$$

We used that $\mathbb{E}B_\beta^2(x)=d^{2\beta}(x,o)=\theta_1^{2\beta}$, in spherical coordinates, therefore, $\varepsilon_0=\sup_{x\in\mathbb{S}^N}(\mathbb{E}B_\beta^2(x))^{1/2}=\pi^\beta$ and $\widetilde{\varepsilon}_0=\sup_{x\in\mathbb{D}_a}(\mathbb{E}B_\beta^2(x))^{1/2}=a^\beta$.

4.2. Isotropic Gaussian random fields

Let us consider a zero-mean isotropic Gaussian random field T on the sphere $\mathbb{S}^2$ with spectral representation (in the mean square sense):

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(x), \; x \in \mathbb{S}^2, \tag{24}$$

where

$$a_{lm} = \int_{\mathbb{S}^2} T(x) Y_{lm}^*(x) \lambda(dx) \tag{25}$$

are Fourier random coefficients, $\{Y_{lm} : l \geq 0, \; m = -l, \dots, l\}$ are spherical harmonics. The coefficients (25) are zero-mean Gaussian complex random variables such that $\mathbf{E}[a_{lm} a_{l'm'}^*] = \delta_l^{l'} \delta_m^{m'} A_l$, where A_l, $l \geq 0$, is the angular power spectrum of the random field T (Here δ_a^b is the Kronecker's delta and the symbol "$*$" stands for complex conjugation.) We refer to [24] for more details and results for such fields.

The angular power spectrum A_l, $l \geq 0$, fully characterizes, under Gaussianity, the dependence structure of T. The behavior of sample paths and other properties can be characterized basing on the decay of the angular spectrum (see, e.g., [24], [23]). So, it would be useful to provide the conditions in spectral terms for Theorem 2 to hold for isotropic Gaussian fields.

From [23, Lemma 4.3] it follows that if the angular power spectrum satisfies the summability condition

$$\sum_{l=0}^{\infty} A_l l^{\beta+1} < +\infty \tag{26}$$

for some $\beta \in [0, 2]$, then there exists a constant C_β such that for all $x, y \in \mathbb{S}^2$

$$\mathbb{E}(T(x) - T(y))^2 \leq C_\beta d^\beta(x, y).$$

Therefore, for the field T with the angular spectrum satisfying condition (26) we can write the following bounds for the probabilities of excursions over the whole sphere $\mathbb{S}^2$ and over a geodesic disc $\mathbb{D}_a$ of a radius a with an arbitrary center:

$$\mathsf{P}\Big\{ \sup_{x \in \mathbb{S}^N} |T(x)| \geq u \Big\} \leq 2 \exp\Big\{ -\frac{u^2 (1-\mu)^2}{2\varepsilon_0^2} \Big\} \pi^4 C_\beta^{2/\beta} e^{4/\beta} (\mu \varepsilon_0)^{-4/\beta},$$

$$\mathsf{P}\Big\{ \sup_{x \in \mathbb{D}_a} |T(x)| \geq u \Big\} \leq 2 \exp\Big\{ -\frac{u^2 (1-\mu)^2}{2\varepsilon_0^2} \Big\} a\pi^3 C_\beta^{2/\beta} e^{4/\beta} (\mu \varepsilon_0)^{-4/\beta},$$

for arbitrary $u > 0$, $0 < \mu < 1$ such that $\mu \varepsilon_0 < \big(\frac{\pi}{\sqrt{2}}\big)^\beta C_\beta^{\beta/2}$, where $\varepsilon_0^2 = \mathbb{E} T^2(x) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} A_l$. The constant C_β can be represented in terms of the algular spectrum as $C_\beta = \pi^{-1} \sum_{l=0}^{\infty} A_l (2l+1)(l(l+1))^{\beta/2}$ (see the proof of Lemma 4.2 in [23]). Therefore, the above bounds are completely representable in terms of the angular power spectrum of the field T.

4.3. Class of φ-sub-Gaussian spherical random fields with the Karhunen–Loève type representation

Consider the covariance function (23), and let us take $N = 2$. It was shown in [13] that the following Mercer's representation can be written for this covariance:

$$R_\beta(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} (-\pi d_l)(Y_{lm}(x) - Y_{lm}(o))(Y_{lm}^*(x) - Y_{lm}^*(o)), \; x, y \in \mathbb{S}^2,$$

where $\{Y_{lm} : l \geq 0,\ m = -l, \dots, +l\}$ are spherical harmonics, $d_l = \int_{-1}^{1} \arccos^{2\beta}(x) P_l(x)\, dx$ with P_l, $l \geq 0$, being the Legendre polynomials. And the SFBM has the following Karhunen–Loève representation:

$$B_\beta(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sqrt{-\pi d_l}\varepsilon_{lm}(Y_{lm}(x) - Y_{lm}(o)),\ x \in \mathbb{S}^2,$$

with ε_{lm} being independent centered standard normal variables (see [13] for more detail).

Let us consider the spherical random field ξ defined by means of the following expansion:

$$\xi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sqrt{-\pi d_l}\xi_{lm}(Y_{lm}(x) - Y_{lm}(o)),\ x \in \mathbb{S}^2,$$

where ξ_{lm}, $l \geq 0$, $m = -l, \dots, l$, are independent identically distributed φ-sub-Gaussian random variables and suppose that $\varphi(\sqrt{\cdot})$ is a convex function. In this case, in view of Example 1, we conclude that $\xi(x)$ is a φ-sub-Gaussian field and $\tau_\varphi^2(\xi(x) - \xi(y)) \leq C\mathbb{E}(\xi(x) - \xi(y))^2 = C^2 d^{2\beta}(x, y),\ x, y \in \mathbb{S}^2,\ C = \tau_\varphi(\xi_{lm})$. Therefore, Condition B.2 is satisfied with $\sigma(h) = h^\beta$ and Theorems 3, 4 can be applied.

5. Conclusion

In this paper, bounds for distributions of suprema of φ-sub-Gaussian random fields defined over the N-dimensional unit sphere are presented. Powerful techniques for investigation of sample paths properties of φ-sub-Gaussian processes and fields have been elaborated in the literature (see, e.g., [3]). Applications of these techniques in various theoretical and practical contexts can be found in the recent papers [10, 11, 15, 17, 18, 19, 27], among others. Here we apply the classical results from [3] for spherical random fields. We believe that such results can be useful from the practical point of view. For illustation, several particular examples are given in Section 4, the results obtained can be also applied to other models, such as those presented in [7]. The interesting and important question for future research would be to compare the obtained results with the asymptotic results presented in [4, 5], in particular, by numerical methods.

Acknowledgement

The author is grateful to the Editor for helpful suggestions to improve the presentation of the paper.

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