STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, January 2023, pp 29–43.
Published online in International Academic Press (www.IAPress.org)

An improved hybrid defuzzification method for fuzzy controllers

Mourad Nachaoui[1], Abdeljalil Nachaoui[2], R. Y. Shikhlinskaya[3], Abdelali Elmoufidi[1,*]

[1]*Equipe de Mathématiques et Interactions Université Sultan Moulay slimane, Faculté des Sciences et Techniques,Maroc*
[2]*Laboratoire de Mathématiques Jean Leray CNRS / Université de Nantes UMR6629 2 rue de la Houssinière, France*
[3]*AZ1148, Z.Khalilov, 23, Baku State University, Institute Applied Mathematics, Baku, Azerbaijan,*

Abstract This paper deals with fuzzy logic controllers. Our interest is in studying different techniques of defuzzification methods. In particular, we propose a generalization of the WABL method to the multiple output system controller. We give a numerical realization, and we show the advantages of this method compared to other defuzzification methods (MOM, COG, WAF,. . .) through an application for intelligent control of air conditioners.

Keywords Fuzzy Sets, Fuzzy Logic, Fuzzy Controllers, Inference, Defuzzification,WABL Method

AMS 2010 subject classifications 93C42

DOI: 10.19139/soic-2310-5070-1706

.

1. Introduction

Artificial Intelligence is one of the most investigated scientific fields [5, 12, 13, 14]. It consists of enabling the computer to think and decide in some tasks that require intelligence. One of the most used concept in artificial intelligence is the fuzzy logic. The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [32, 33, 34, 35, 36, 20]. We refer the reader to [11] for more information on fuzzy numbers and fuzzy arithmetic. Currently, fuzzy logic becomes a prototype for an alternative design technique that can be implemented in the elaboration of both linear and non-linear systems for embedded control. By using fuzzy logic, designers can accomplish lower development costs, better features, and improved end-product performance. In addition, products can be brought to market faster and more cost-effectively.

Aiming at improving the performance of fuzzy controllers, and providing systematic design procedures for the translation of the expert's knowledge in the form of fuzzy inference systems, various concepts have, so far, been developed. We state, for instance, the advent of the notion of self-organizing controllers [29, 21], and the use of artificial neural networks and genetic algorithms in the design of adaptive fuzzy controllers [3, 17, 26] and others [2, 9, 24]. However, no performance enhancement nor systematic design technique has been sought, so far, by constructing a defuzzification method that integrates defuzzification into the overall setting of the controller components. There are different defuzzification methods: COG(Center of gravity) [4], MOM(Mean of Maxima) [6], WAF(Weighted average formulae), QM(Quality Method) [8], WABL(Weighted Average Based on Levels) [18, 19], etc. The problem is that each one gives a different value for the same problem. The most important question is: which one is good? The main objectives of this study are to describe and justify a defuzzification method based on the global structure of a fuzzy controller and show how it can be used to help the designer achieve his goals in a simple and systematic manner. The remainder of the paper is organized as follows. Section 2 deals

*Correspondence to: elmoufidiabdelali9@gmail.com

ISSN 2310-5070 (online) ISSN 2311-004X (print)

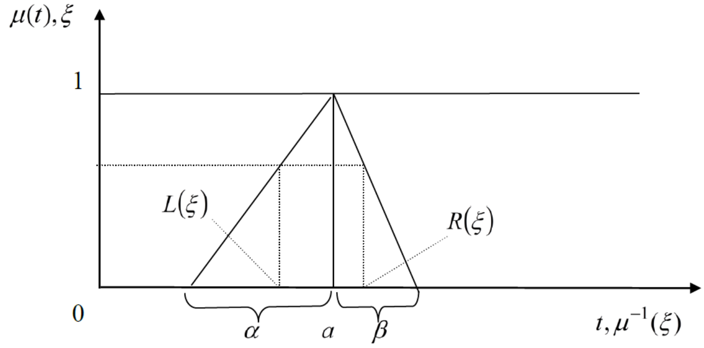

Figure 1. The used membership

with a setting problem and presents a test for a single input problem. Section 3 presents common defuzzification methods for a control system with multiple inputs. In particular, the proposed methodology is presented and the obtained results are presented and discussed.

2. Preliminary results

Fuzzy numbers can be seen as a generalization of regular, real numbers in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its weight between 0 and 1. Zadeh has introduced fuzzy numbers to deal with imprecise numerical quantities in a practical way [32]. There exist many classes of fuzzy numbers often used. For the sake of simplicity, we recall the definition of the used fuzzy number in our case. For more details about fuzzy numbers, and their operations for the addition, please see [25]. Given a real number $a \in \mathbb{R}$, we can consider its corresponding set denoted by $\{a\}$. Let's $\mathfrak{S} \subset \mathbb{R}$ be a subset that belongs to $\mathbb{R}$. It is well known that $\mathfrak{S}$ cab uniquely determined trough its indicator function: $\chi_{\mathfrak{S}} : \mathbb{R} \mapsto \{0, 1\}$, with

$$x \in \mathfrak{S} \Leftrightarrow \chi_{\mathfrak{S}}(x) = 1, \text{ and } x \notin \mathfrak{S} \Leftrightarrow \chi_{\mathfrak{S}}(x) = 0.$$

We can define the notion of fuzzy numbers in the same way. In particular, the indicator function will be replaced by a membership function $\mu : \mathbb{R} \mapsto [0, 1]$. As mentioned above, there exist many way to define fuzzy numbers. In this paper we use a general form of membership function depending on some parameters $\alpha, \beta > 0$. For a given $a \in \mathbb{R}$, the considered membership function is defined as follows

$$\mu(t) = \begin{cases} 1 - \left(\frac{a-t}{\alpha}\right), & t \in [a-\alpha, a] \\ 1 - \left(\frac{t-a}{\beta}\right), & t \in [a, a+\beta] \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The inverse functions corresponding to the increasing and decreasing parts of are defined as:

$$\mu_{\uparrow}^{-1}(\xi) := L(\xi) = a - \alpha(1-\xi), \tag{2}$$
$$\mu_{\downarrow}^{-1}(\xi) := R(\xi) = a + \beta(1-\xi). \tag{3}$$

3. Setting of the problem

A Fuzzy Inference System (FIS) has as its goal to transform input data into output data from the evaluation of a set of rules. The entries come from the fuzzification process and a set of rules are normally defined by the expertise of the expert. A FIS (see Figure 3.3) consists of three steps: a) Fuzzification, b) Inference, and c) Defuzzification. The first step is fuzzification, which consists in characterizing the linguistic variables used in the system. Hence, it's a transformation of real inputs, to a fuzzy set, defined over a representation space, related to the entry. This representation space is typically a fuzzy subset. During the fuzzification step, each input variable and output is connected with fuzzy sets. The second step is the inference engine, which is a mechanism to condense the information system through a set of rules defined for the representation of any problem. Each rule issues a partial conclusion, which is then aggregated with other rules to provide a conclusion (aggregation). The rules are the fuzzy inference system, it's expressed in the form of "IF-THEN". The third step is the defuzzification, this operation is the inverse of the fuzzification, it allows the transformation of the fuzzy output of inference into a non-fuzzy as the final response of FIS. We start by a simple one-input temperature controller example studied in [27]. We have

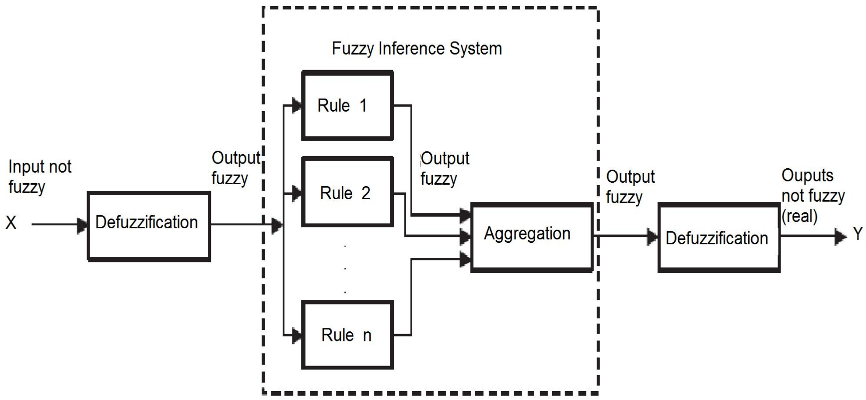

Figure 2. FIS structure.

3 rules:

$$\begin{array}{c}\text{IF temperature IS cool THEN fan speed IS low}\\ \text{IF temperature IS warm THEN fan speed IS medium}\\ \text{IF temperature IS hot THEN fan speed IS high}\end{array} \tag{4}$$

The membership of the input variable (temperature) and the output variable (fan speed) are presented in figure 3,4.
The WABL defuzzification is given by:

$$I_w = \sum_{i=1}^{n}\sum_{j=1}^{m} k_i I_w(l_j). \tag{5}$$

Where $k_i = \mu_i(x)$ is the validity degree of the input variable x by the membership function μ_i, and $I_w(l_j)$ denoted the WABL aggregation of the output fuzzy state ($l_j =$ Low, Medium or High), defined by:

$$I_w(l_j) = c_l \int_0^1 L_{l_j}(\xi)P(\xi)d\xi + (1-c_l)\int_0^1 R_{l_j}(\xi)P(\xi)d\xi \tag{6}$$

With $c_l \in [0,1]$ indicate the important degree of the left and the right side of the fuzzy numbers, $L_{l_j}(\xi) = \mu_{\uparrow}^{-1}$,$R_{l_j}(\xi) = \mu_{\downarrow}^{-1}$. $\mu_{\uparrow}^{-1}$ and $\mu_{\downarrow}^{-1}$ are the inverse functions of the left and right side of the output membership

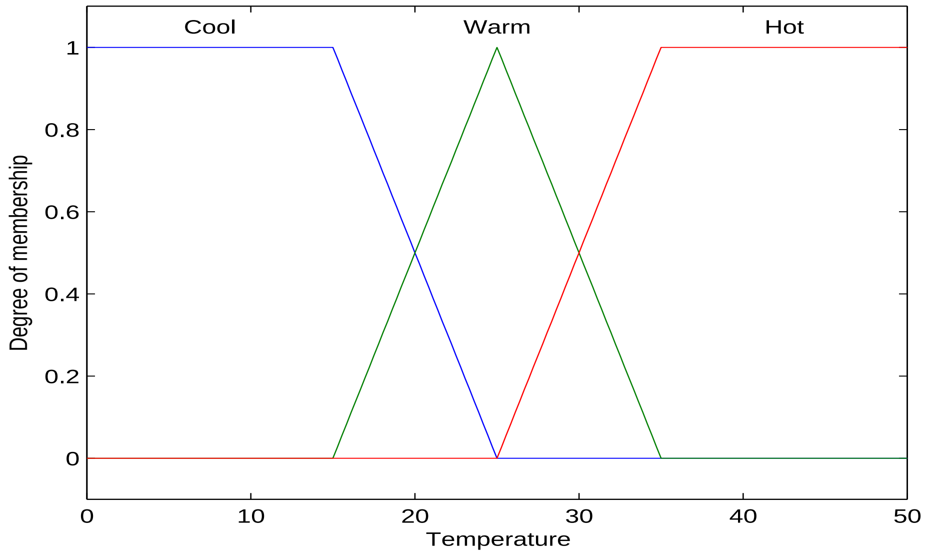

Figure 3. Input fuzzy sets.

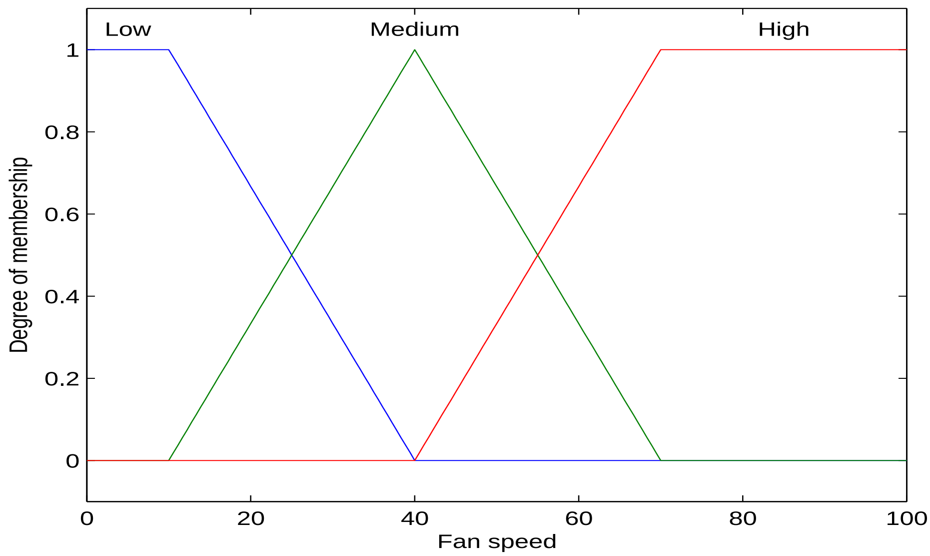

Figure 4. Output fuzzy sets.

function, and $P(\xi)$ is the distribution function of importance degree:

$$\begin{array}{c} P : [0,1] \longmapsto [0,+\infty] \ \text{ verify} \\ \int_0^1 P(\xi)d\xi = 1 \end{array} \tag{7}$$

The principal advantages of the WABL method are still in the free parameters c_l and the distribution $P(\xi)$ which allow adaptation of the defuzzification method to obtain more accurate result for a specific problem. Figure 3 illustrates different why to decrease the temperature from 37 to 20 with $P(\xi) = (k+1)\xi^k, k > 0$. In figure 6 we

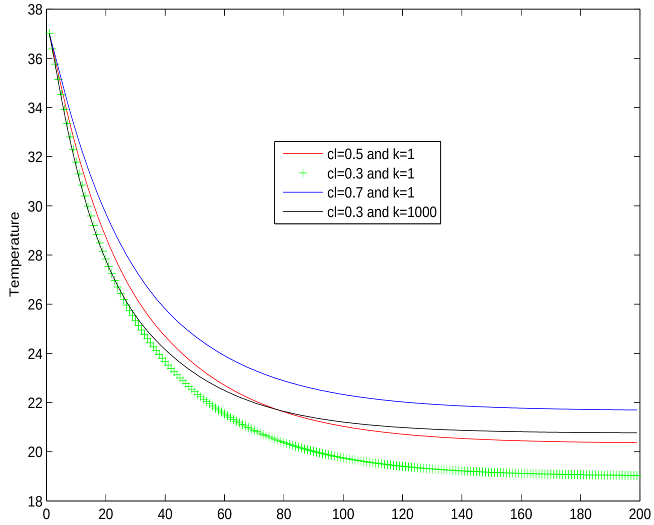

Figure 5. decreasing of temperature with inference rules. different values of WABL defuzzification with different parameters.

present the comparison between the WABL method, the center gravity (COG), and The mean of maxima (MOM). Clearly, the WABL method offers the possibility to obtain a curve with desired properties. Indeed if we want a speed decreasing the temperature of rom we can choose $cl = 0.9, k = 1$, and conversely, if we want a slow decreasing the temperature we can choose $cl = 0.1, k = 1$. This is not possible with the classical method of defuzzification (MOM, COG,...)

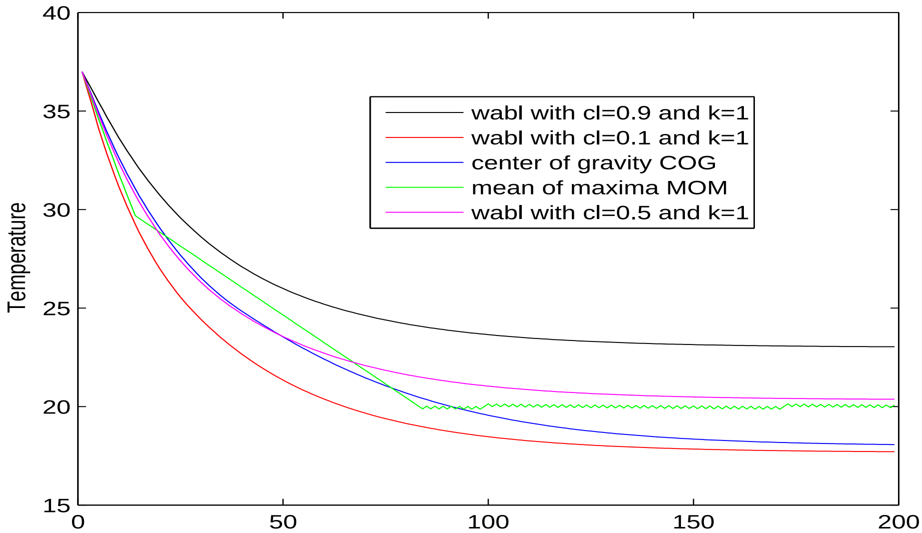

Figure 6. decreasing of temperature with inference rules. different values of WABL defuzzification with different parameters.

Remark 1
The author in [27] tell that with $cl = 0.5, k = 1$, the WABL method coincides with the COG method for some class of membership functions, but our numerical test shows that this is not correct, as the figure 6 shows.

4. Common defuzzification methods for controller system with multiple input

The single input temperature controller presented so far has helped us illustrate some fundamental concepts. Nevertheless, real-life control is so far more complex in nature. Almost control applications have numerous inputs and need modeling and adjustment of a large number of parameters which makes realization very tedious and time-consuming. In the sequel, we explain the inference for a system with two input variables and one output variable.

4.1. Inference rules

A collection of N inference rules for a system with two input variables and one output variable whose form is typical in fuzzy controllers are such that the $j-th$ rule for $1 \leq j \leq N$ is expressed as follows [31]:

$$R_j: \text{ IF } x \text{ is } A_j \text{ AND } y \text{ is } B_j, \text{ THEN } z \text{ is } C_j. \tag{8}$$

In the aforementioned rules, x and y designate the input variables of the fuzzy controller, and z is the output variable. Of course, more than two inputs can be considered and the aforementioned rules can be reformulated accordingly. $A_1, A_2, ..., A_N$ is the linguistic or fuzzy values entrusted over the space, say I_1, of the first input variable. While $B_1, B_2, ..., B_N$ are those entrusted over the space I_2 of the second input variable, and $C_1, C_2, ..., C_N$ are the fuzzy sets entrusted over the space S of the output variable. The "IF" element of a rule is commonly called the rule "antecedent" and the "THEN" component is the rule "consequent". The realization of the inference rules in (8) is usually done according to the compositional rule of inference [33]. Actually, the inference rules in (8) can be represented by the fuzzy relation

$$R = \cup_{i=1}^{N}[(A_i \cap B_i) \times C_i] \tag{9}$$

In (9), the symbol $\cup$ represents the OR operator introduced between the rules. The symbol $\cap$ represents the AND operator used in the antecedent parts of the rules and $\times$ represents the THEN or fuzzy implication operator. The fuzzy controller output that corresponds to a crisp input pair (x_0, y_0) is given by

$$C(z) = R(x_0, y_0, z). \tag{10}$$

If the minimum ("min" or $(\wedge)$) operation is adopted for AND and for the fuzzy implication (FI) and the maximum ("max") operation is adopted for OR, then (10) with R as in (9) can be expressed as:

$$C(z) = \max_{1 \leq i \leq N}[A_i(x_0) \wedge A_i(B_0) \wedge C_i(z)]. \tag{11}$$

It is worth noting here that other than maximum and minimum have respectively been suggested for the OR, AND, and FI operators [10, 37]. Consider a modified version of the temperature controller example, with two inputs, temperature (Figure 3) and humidity (Figure 7) and the same output, fan speed (Figure 4). This example can be described with a small set of rules as follows

$$\begin{array}{lll}
R_1: & \text{IF temperature IS cool and Humidity is Low} & \text{THEN fan speed IS low} \\
R_2: & \text{IF temperature IS cool and Humidity is Med} & \text{THEN fan speed IS low} \\
R_3: & \text{IF temperature IS cool and Humidity is High} & \text{THEN fan speed IS Medium} \\
R_4: & \text{IF temperature IS Warm and Humidity is Low} & \text{THEN fan speed IS Medium} \\
R_5: & \text{IF temperature IS Warm and Humidity is Med} & \text{THEN fan speed IS Medium} \\
R_6: & \text{IF temperature IS Warm and Humidity is High} & \text{THEN fan speed IS High} \\
R_7: & \text{IF temperature IS hot and Humidity is Low} & \text{THEN fan speed IS High} \\
R_8: & \text{IF temperature IS hot and Humidity is Med} & \text{THEN fan speed IS High} \\
R_9: & \text{IF temperature IS hot and Humidity is High} & \text{THEN fan speed IS High}
\end{array} \tag{12}$$

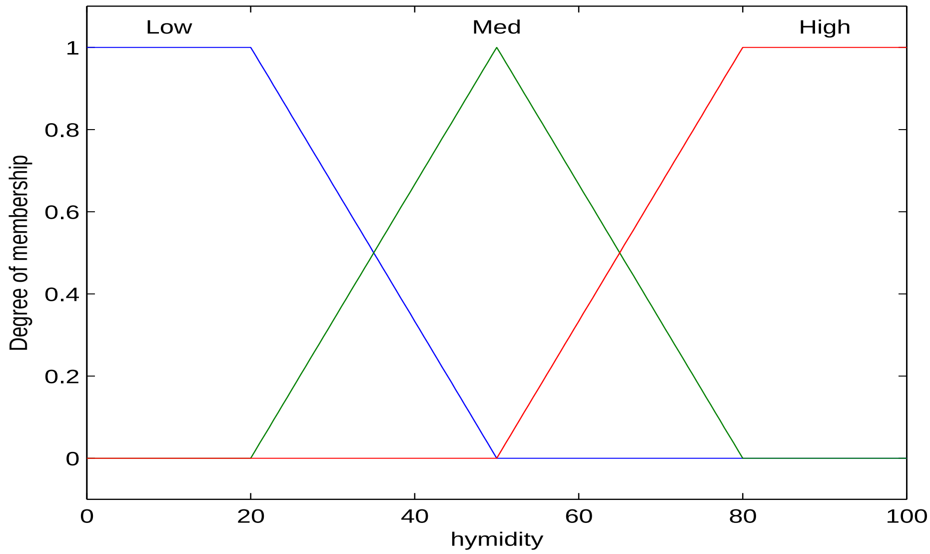

Figure 7. Input fuzzy sets.

We suppose that the inference system is done with (11). In order to transform this output into a crisp one we must apply a defuzzification technic, in the following we present different defuzzification methods and the result obtained with each method.

4.2. The mean of Maxima (MOM)

The MOM applies to the fuzzy output $C(z)$ by taking the mean of the z values at which $C(z)$ is maximized. Suppose that $z1, z2, ..., z_p$ are the maximizing points of $C(z)$, then

$$MOM[C(z)] = \frac{z1 + z2 + ... + z_p}{p} \tag{13}$$

The output fuzzy set shown in Fig. 4 is defuzzified by the MOM method, to give a fan speed presented in Figure 8. The MOM accounts only for rules, which are triggered at the maximum membership level. Although this leads to a considerable computational simplification, it is generally felt that ignoring rules which are triggered below the maximum level of membership is not properly fuzzy [15] as we can see in the plot.

4.3. Center of gravity (COG)

It consists of finding the centroid of the area bounded by the controller output MF and its abscissa is taken as the crisp controlling value [31, 7, 28]. Hence,

$$COG[C(z)] = \frac{\int_{-\infty}^{\infty} zC(z)dz}{\int_{-\infty}^{\infty} C(z)dz} \tag{14}$$

In figure 9, we present the output of the fuzzy controller defuzzed by GOG method. Compared to the MOM method, The COG takes into account the rules, which are triggered below and at the maximum membership level. On the other hand, it has the disadvantage of not allowing control actions towards the extremes of the action (output) range [15].

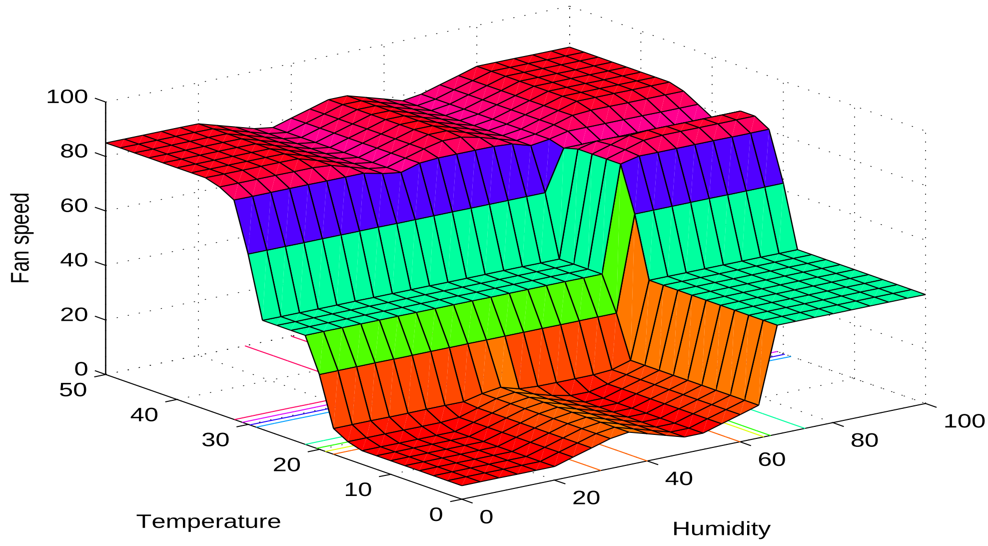

Figure 8. Control surface of the fuzzy controller with inference rules as in (12). The MOM defuzzification is applied

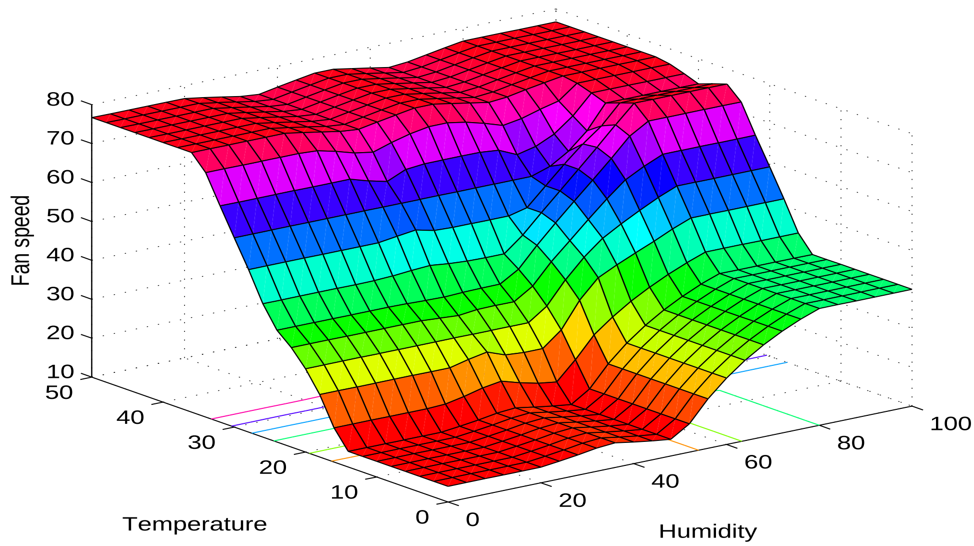

Figure 9. Control surface of the fuzzy controller with inference rules as in (12). The COG defuzzification is applied

4.4. The WAF, MAX-WAF and QM

$[A_j(x0) \wedge B_j(y0)] = \mu_j$, then with c_j denoting the the crisp output of rule j in (8), the WAF formula applies as follows to produce the crisp output c

$$WAF[c] = \frac{\sum_{j=1}^{N} \mu_j c_j}{\sum_{j=1}^{N} \mu_j} \tag{15}$$

It is to be noted here that according to Berenji [1], (15) was first suggested by Tsukamoto and it has a modified version [30], that permits structure and parameter identification of fuzzy systems. Applying (15) to the fan example with rules (12) we obtain the control surface presented in the figure 10.

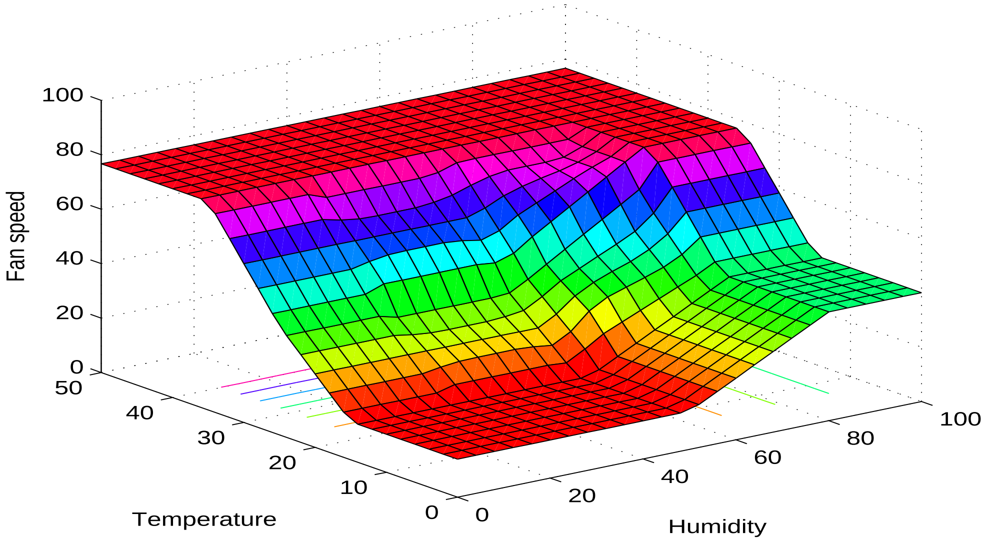

Figure 10. Control surface of the fuzzy controller with inference rules as in (12). The WAF defuzzification is applied

4.5. MAX-WAF method

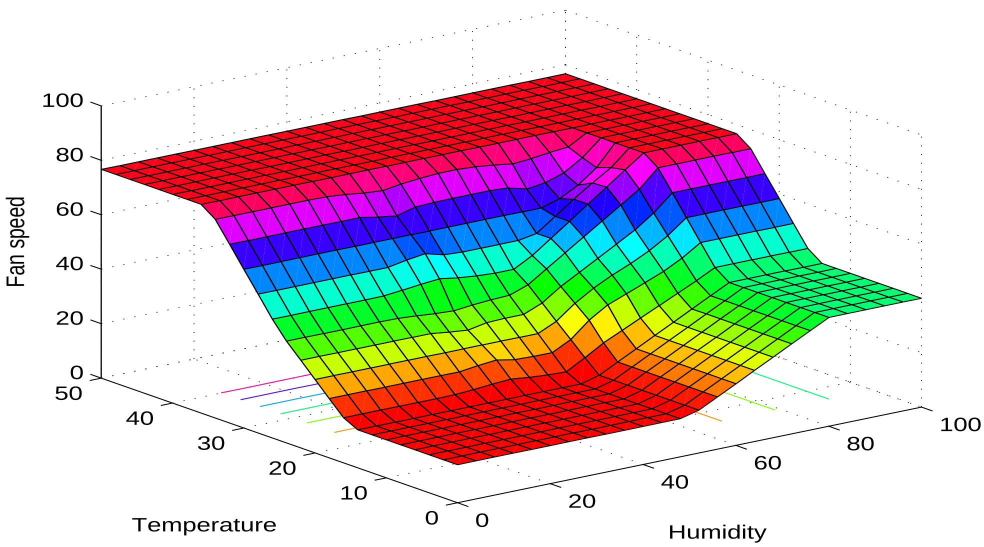

Figure 11. Control surface of the fuzzy controller with inference rules as in (12). The MAX-WAF defuzzification is applied

In the case where more than one rule gives the same crisp consequent, then the application of (15) can be done considering the"OR" operator between such conflicting rules reflected through the use of the maximum operation applied to the membership grades resulting from these rules. Let

$$\begin{aligned} \mu_{1max} &= \max(\mu_1, \mu_2, ..., \mu_i) \\ \mu_{(i+1)max} &= \max(\mu_{(i+1)}, \mu_{(i+2)}, ..., \mu_p), \\ \mu_{(p+1)max} &= \max(\mu_{(p+1)}, \mu_{(p+2)}, ..., \mu_N) \end{aligned}$$

and c_1, c_2, c_3 be the crisp consequents corresponding respectively to rules 1 to i, $i+1$ to p and $p+1$ to N. Equation (15) becomes then a MAX-WAF formula that applies as follows

$$MAX - WAF[c] = \frac{(\mu_{1max} \times c_1) + (\mu_{(i+1)max} \times c_2) + (\mu_{(P+1)max} \times c_3)}{\mu_{1max} \times + \mu_{(i+1)max} \times + \mu_{(P+1)max}} \tag{16}$$

The Figure 11 illustrate the result of the fan speed versus temperature and humidity using (16) with minimum used for "AND".

4.6. QM defuzzification

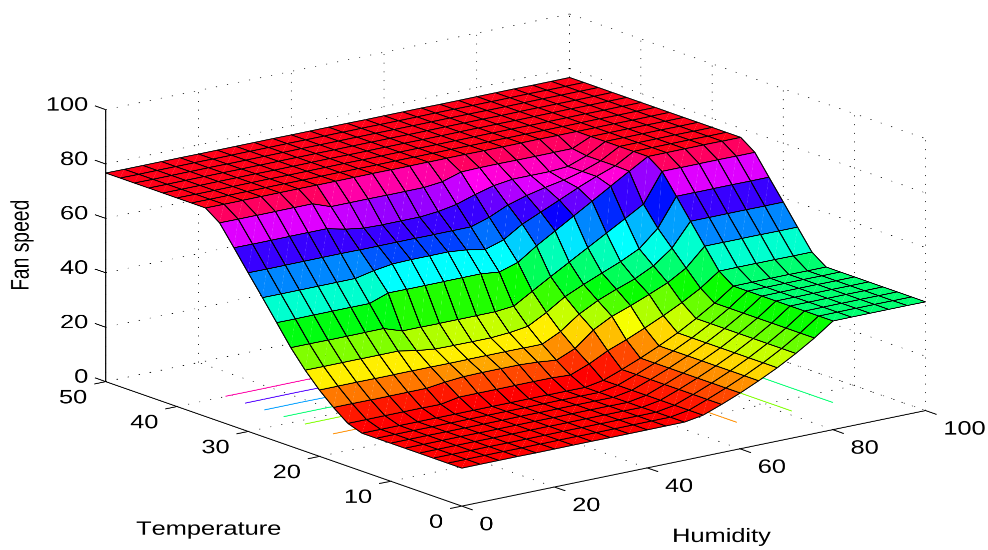

Figure 12. Control surface of the fuzzy controller with inference rules as in (12). The QM defuzzification is applied

If the centers of gravity of the output fuzzy sets are taken as the crisp output values and the degree of activation of each rule (μ_j) is divided by the measure of the support of the rule consequent then (15) becomes the QM method introduced in [29]

$$QM(c) = \frac{\sum_{j=1}^{N} \mu_j c_j / d_j}{\sum_{j=1}^{N} \mu_j / d_j} \tag{17}$$

where c_j is the center of gravity of the fuzzy consequent of rule j and d_j is the measure of the support of the consequent of rule j. The application of (17) to the fan example with rules as in (12) provides the plot (12). Obviously, the result is more accurate than that achieved with the other defuzzification technique (WAF, MAX-WAF...). Nevertheless, the plot still contains a few undesirable parts. Having regions in which, the fan speed decreases when temperature increases and humidity is kept fixed and vice versa does not sound reasonable since it

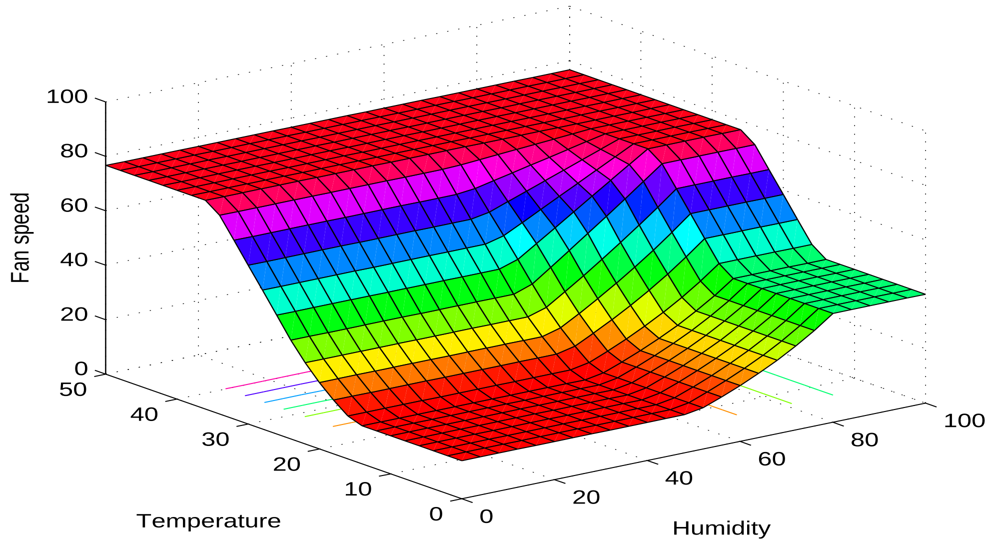

Figure 13. Control surface of the fuzzy controller with inference rules as in (12). The “product” operation is used for “AND” for QM defuzzification

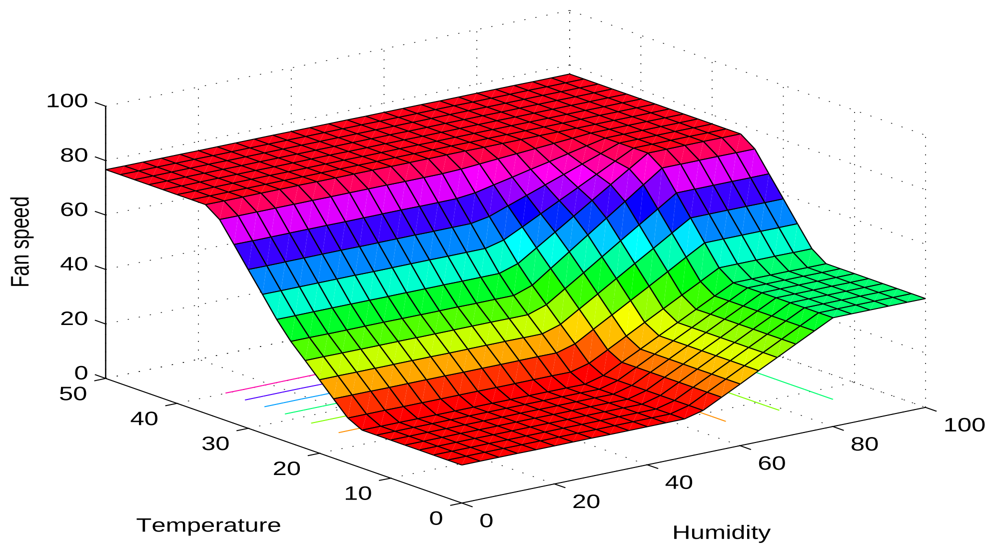

Figure 14. Control surface of the fuzzy controller with inference rules as in (12). The “product” operation is used for “AND” for WAF defuzzification

does not satisfy the previously specified designer’s objective. In addition, the application of the WAF and QM to the fan example, using the rules given in (ref eq5). And again using the “min” for “AND” results in similar undesirable parts in the fan controller surfaces. The plot, however, becomes a smooth and satisfactory design target when the “min” for “AND” is replaced by the product (Fig. 13). Thus, the combinations of product-sum-product and product-max-product for AND-OR-F.I. operators are advantageous over min-sum-product and min-max-product.

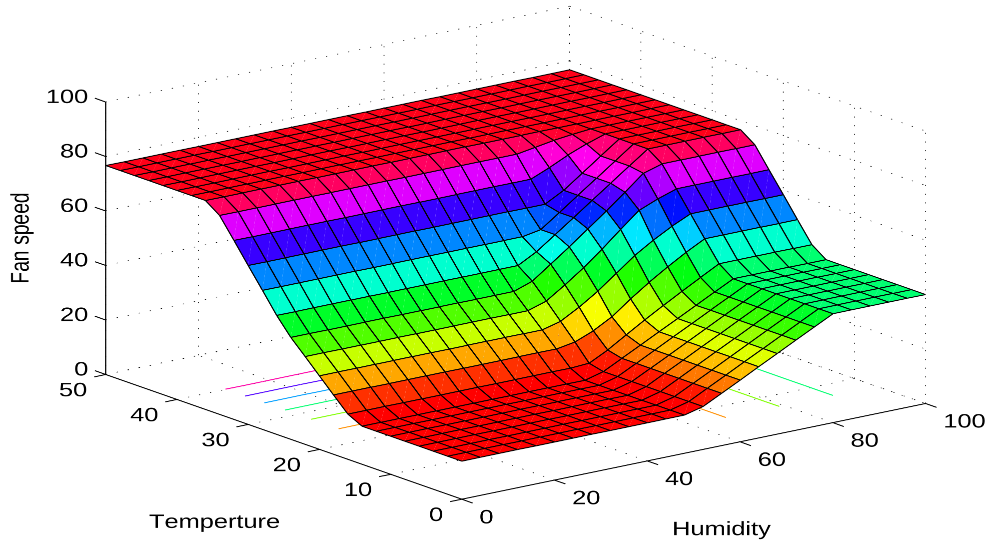

Figure 15. Control surface of the fuzzy controller with inference rules as in (12). The "product" operation is used for "AND" for MAX-WAF defuzzification

Yet, product-sum- product seems better than product-max-product. This can be seen by comparing Fig.13 with Figs. 14 and 15.

4.7. Generalization of the WABL method

A general form of the WABL method for a controller with two input variables and one output variable can now be written as follows to produce the crisp output c:

$$c = \sum_{i=1}^{n} \sum_{j=1}^{p} [A_i(x_0) \cap B_j(y_0)] I_w(C_{ij}). \tag{18}$$

$A_1, A_2, ..., A_n$ are the fuzzy sets defined over the first input variable of the fuzzy controller. $B_1, B_2, ..., B_p$ are the fuzzy sets defined over the second input variable. $A_i(x_0)$ and $B_j(y_0)$ are, respectively, the membership grades of the crisp inputs x_0 and y_0 in the fuzzy set A_i and B_j with (x_0, y_0) being the crisp input pair for which the crisp output is to be determined. C_{ij} is the crisp consequence for the rule whose antecedent part is formed by A_i and B_j. I_w have the same definition as in (6).

For the two input temperature controller, the crisp fan speed for a given temperature t_0 and humidity h_0, defined by the WABL method using the product operation for the operator "AND" is represented by :

$$WABL[c(t_0, h_0)] = \sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{ij} I_w(c_{ij}) \tag{19}$$

where : $\mu_{ij} = \mu_{lt_i}(t_0) \times \mu_{lh_j}(h_0),\ lt_1 = cool, lt_2 = warm, lt_3 = hot, lh_1 = low, lh_2 = med, lh_3 = high$ and c_{ij} is the result correspond to rule "IF Temperature is lt_i AND Humidity is lh_j THEN c_{ij}".
If we use the "min" for the operator "AND", we define μ_{ij} like this $\mu_{ij} = min(\mu_{lt_i}(t_0), \mu_{lh_j}(h_0))$. The plot 16 presents the control surface obtained with the WABL method (cl=0.9, k=1) using the "min" operation for the operator "AND", we that it contains a non-desired region. The plot 17 presents the control surface obtained with the WABL method (cl=0.9, k=1) using the "product" operation for the operator "AND". This control surface is

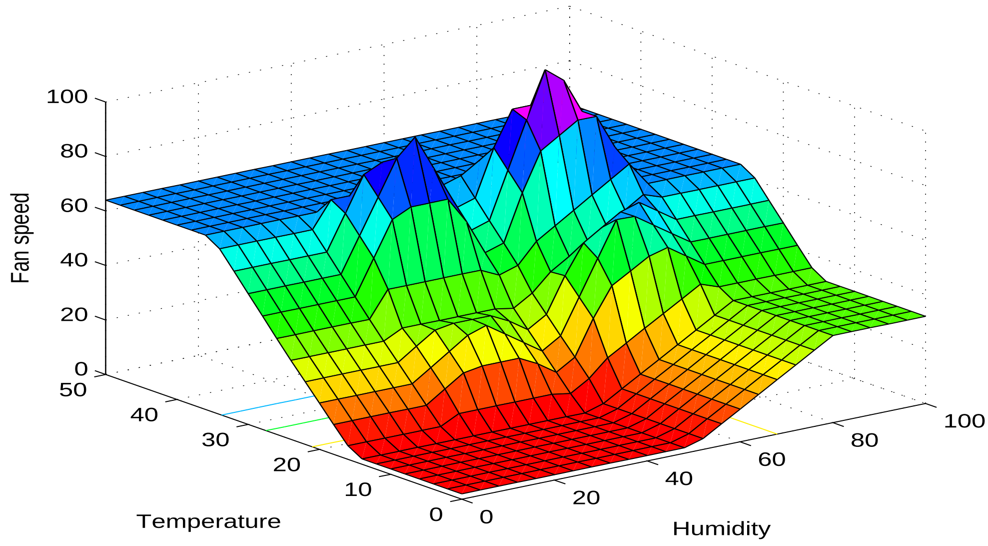

Figure 16. Control surface of the fuzzy controller with inference rules as in (12). The "min" operation is used for "AND" for WABL defuzzification

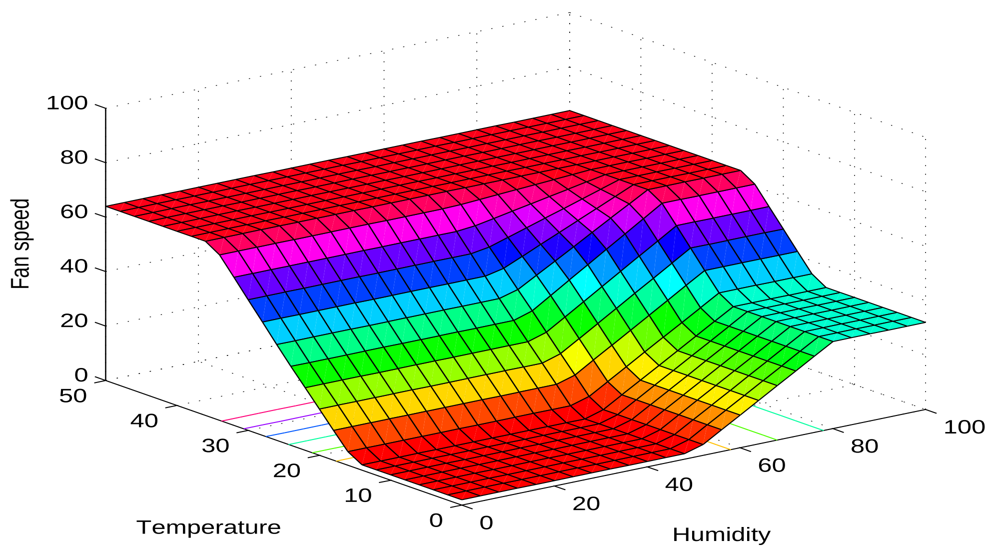

Figure 17. Control surface of the fuzzy controller with inference rules as in (12). The "product" operation is used for "AND" for WABL defuzzification

close to the ones obtained using the WAF and QM when the product is used for "AND" and rules as in the Figures (13) and (14). The plot obtained using the WABL method with products, however, is better than those obtained using the other ones. Indeed, we have a reduction in the size of the areas in the input space over which the fan speed remains constant. Furthermore, the WABL method reduces the gradient of the control curves.

5. Conclusion

This paper provides a comparison of various defuzzification techniques with numerical tests on a model of an input temperature controller. The results obtained show that traditional methods of defuzzification present a major handicap, that they offer no possibility of adaptation, and that they then give a reliable result for a specific problem. We have demonstrated through the experimental results that the WABL method avoids this disadvantage and provides the possibility to achieve the desired result for a specific problem. Generalization of the WABL method for multiple entries has been carried out. Numerical tests show that the WABL methods proved to be more efficient compared to the classical methods.

REFERENCES

1. Berenji, Hamid R. “Fuzzy logic controllers.” An introduction to fuzzy logic applications in intelligent systems. Springer, Boston, MA, 1992. 69-96.
2. Bounabi, M., Karim EL Moutaouakil, and Khalid Satori. The Optimal Inference Rules Selection for Unstructured Data Multi-Classification. Statistics, Optimization & Information Computing, Vol. 10, no. 1, Feb. 2022, pp. 225-3, doi:10.19139/soic-2310-5070-1131.
3. B. Carse, T. C. Fogarty, and A. Munro,“Evolving fuzzy rule based controllers using genetic algorithms,” Fuzzy Sets and Systems, vol. 80, no. 3, pp. 273-293, Jun. 1996.
4. Sain, Debdoot, and B. M. Mohan. “Modeling, simulation and experimental realization of a new nonlinear fuzzy PID controller using Center of Gravity defuzzification.” ISA transactions 110 (2021): 319-327.
5. Ganascia, Jean-Gabriel. Le Mythe de la Singularit? Faut-il craindre l’intelligence artificielle?. Le Seuil, 2017.
6. Gilda, K. S., and S. L. Satarkar. “Maxima-Based Defuzzification Methods Proposed for Fuzzy Systems.” Proceedings of Integrated Intelligence Enable Networks and Computing. Springer, Singapore, 2021. 629-639.
7. H. Hellendoorn and C. Thomas,“Defuzzification in fuzzy controllers,” Journal of Intelligent and Fuzzy Systems, vol. 1, no. 2, pp. 109-123, 1993.
8. Iqbal, Shafqat, et al. “A new fuzzy time series forecasting method based on clustering and weighted average approach.” Journal of Intelligent & Fuzzy Systems 38.5 (2020): 6089-6098.
9. Khanali, H., and B. Vaziri. Similarity Technique Effectiveness of Optimized Fuzzy C-Means Clustering Based on Fuzzy Support Vector Machine for Noisy Data. Statistics, Optimization & Information Computing, Vol. 9, no. 3, July 2021, pp. 618-29, doi:10.19139/soic-2310-5070-1035.
10. G. Klir and T. Folgers, Fuzzy Sets, Uncertainty and Information, Prentice Hall, Englewood Cliffs, 1988.
11. Y. Lin, G. A. Cunningham III, and S. V. Coggeshall, “Using fuzzy partitions to create fuzzy systems from input-output data and set the initial weights in a fuzzy neural network,” IEEE Trans. Fuzzy Systems, vol. 5, no. 4, pp. 614-621, Nov. 1997.
12. Lyaqini, S., M. Nachaoui, and M. Quafafou.“Non-smooth classification model based on new smoothing technique.” Journal of Physics: Conference Series. Vol. 1743. No. 1. IOP Publishing, 2021.
13. Lyaqini, S., M. Nachaoui, and A. Hadri. “An efficient primal-dual method for solving non-smooth machine learning problem.” Chaos, Solitons & Fractals 155 (2022): 111754.
14. Lyaqini, S., and M. Nachaoui.“Identification of Genuine from Fake Banknotes Using an Enhanced Machine Learning Approach.” International Conference on Numerical Analysis and Optimization Days. Springer, Cham, 2021.
15. E. H. Mamdani, H. J. Efstathiou, and K. Sugiyama, “Developments in fuzzy logic control, ” in Proc. 23rd Conf. on Decision and Control, pp. 888-893, Las Vegas, V, Dec. 1984.
16. Nachaoui, M. . Étude théorique et approximation numérique d’un problème inverse de transfert de la chaleur (Doctoral dissertation, Université de Nantes (2011)).
17. Nachaoui, M., A. Chakib, and A. Nachaoui. “An efficient evolutionary algorithm for a shape optimization problem.” Applied and Computational Mathematics 19.2 (2020): 220-244.
18. Nasiboglu, Resmiye, and Rahila Abdullayeva. “Analytical Formulations for the Level Based Weighted Average Value of Discrete Trapezoidal Fuzzy Numbers.” arXiv preprint arXiv:1810.05110 (2018).
19. Nasibov, Efendi N., and Ali Mert. “On methods of defuzzification of parametrically represented fuzzy numbers.” Automatic Control and Computer Sciences 41.5 (2007): 265-273.
20. Yager, Ronald R., and Lotfi A. Zadeh, eds. An introduction to fuzzy logic applications in intelligent systems. Vol. 165. Springer Science & Business Media, 2012.
21. K. Sugiyama, “Rule-based self-organizing controller,” Fuzzy Computing Theory, Hardware and Applications, by M.M. Gupta and T. Yamakawa, Editors, Elsevier Science Pubs., pp. 341-353,1988.
22. C. Von Altrock, B. Krause, and H. -J. Zimmermann, “Advanced fuzzy logic control of a model car in extreme situations,” Fuzzy Sets and Systems, vol. 48, no. 1, pp. 41-52, 26 May 1992.
23. M. Delgado, A. F. G. Skarmita, and F. Martin, “A fuzzy clusteringbased rapid prototyping for fuzzy rule-based modelling,” IEEE Trans. Fuzzy Systems, vol. 5, no. 2, pp. 223-233, May 1997.
24. Pattnaik, M. Fuzzy Economic Order Quantity (FEOQ) Model With Units Lost Due to Deterioration. Statistics, Optimization & Information Computing, Vol. 2, no. 3, Aug. 2014, pp. 257-73, doi:10.19139/soic.v2i3.40.
25. Dijkman, J. G.; van Haeringen, H.; de Lange, S. J. “Fuzzy numbers.” J. Math. Anal. Appl. 92 (1983), no. 2, 301-341.

26. R. R. Yager, "Implementing fuzzy logic controllers using a neural network framework," Fuzzy Sets and Systems, vol. 48, pp. 53-64, 1999.
27. SHIKHLINSKAYA R. Y. ON A SOLUTION OF ONE FUZZY LOGIC PROBLEM, arXiv:0811.3809v1 [math.LO] 24 Nov 2008
28. S. Roychowdhury and B. -H. Wang, "Cooperative neighbors in defuzzification," Fuzzy Sets and Systems, vol. 78, no. 1, pp. 37-49, 26 Feb. 1996.
29. T. J. Procyk and E. H. Mamdani, "A linguistic self-organizing process controller," Automatica, vol. 15, pp. 15-30, 1979.
30. T. Takagi and M. Sugeno, "Fuzzy identification of systems and its application to modeling and control," IEEE Trans. Systems, Man and Cybernetics, vol. 15, no. 1, pp. 116-132, Jan./Feb. 1985.
31. W. Pedrycz, Fuzzy Control and Fuzzy Systems, Second Extended Edition, Research Studies Press, England, 1993.
32. L. A. Zadeh, "Fuzzy sets," Information and Control, vol. 8, pp. 338-353, 1965.
33. L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," IEEE Trans. Systems, Man and Cybernetics, vol. 3, no. 1, pp. 28-44, Jan. 1973.
34. L. A. Zadeh,"Fuzzy logic and approximate reasoning," Synthese, vol. 30pp. 407-428, 1975.
35. L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning," Information Sciences, vol. 8, pp. 199- 249, 1975.
36. L. A. Zadeh, "Making computers think like people," IEEE Spectrum, vol. 21, no. 8, pp. 26-32, Aug. 1984.
37. H. -J. Zimmermann, Fuzzy Set Theory and its Applications, 2nd Edition, Kluwer Academic Publishers, 1991.