STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, July 2024, pp 962–981.
Published online in International Academic Press (www.IAPress.org)

E-Bayesian Estimation and the Corresponding E-MSE under Progressive Type-II Censored Data for Some Characteristics of Weibull Distribution

Omid Shojaee [1,*], Hassan Zarei [2], Fatemeh Naruei [2]

[1] *Department of Statistics, University of Zabol, Zabol, Sistan and Baluchestan, Iran*
[2] *Department of Statistics, University of Sistan and Baluchestan, Zahedan, Sistan and Baluchestan, Iran*

Abstract Estimating the parameters (or characteristics) of a distribution, from the availability of censored samples, is one of the most important topics in statistical inference over the past decades. This study is concerned about the E-Bayesian estimation method to compute the estimates of the parameter, the hazard rate function and the reliability function of the Weibull distribution when the progressive type-2 censored samples are available. The estimations are obtained based on the Squared error loss function (as a symmetric loss) and General Entropy and LINEX loss functions (as asymmetric losses). In addition, the asymptotic behaviour of the derived E-Bayesian estimators is discussed. Moreover, the E-Bayesian estimators under the different loss functions have been compared through Monte Carlo simulation studies by calculating the E-MSE of the resulting estimators, which is a new measure to compare the E-Bayesian estimators. As an application, we analyzed two real data sets that follow from the Weibull distribution.

Keywords E-Bayesian estimation, Weibull distribution, Progressive Type-II censored data, Survival functions, General Entropy loss function, Monte Carlo simulation

AMS 2010 subject classifications 62F15, 62N05

DOI: 10.19139/soic-2310-5070-1709

1. Introduction

In statistical inference, usually the problem of estimating the parameters (or characteristics) of a distribution is one of the favorite situations of researchers in this field. In the theory of reliability, to estimate aging indicators of lifetime distributions such as survival functions and failure rates, available failure times from designed engineering experiments are often used. On the other hand, there is usually some additional information, such as experts' previous beliefs about parameter values and observation of internal parameters, etc, which can be useful for more accurate estimation. Bayesian methods are often useful tools for modeling redundant information as prior distributions.

Weibull distribution is one of the most popular distribution in analyzing skewed data. The probability density and distribution functions of the Weibull distribution are defined by

$$f(x) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}} \tag{1}$$

and

$$F(x) = 1 - e^{-\lambda x^{\alpha}} \tag{2}$$

*Correspondence to: Omid Shojaee (Email: o_shojaee@uoz.ac.ir). Department of Statistics, University of Zabol, Zabol, Sistan and Baluchestan, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

respectively. The reliability function $R(t)$ and the hazard rate function $h(t)$ at mission time t, respectively, are given by

$$R(t) = e^{-\lambda t^{\alpha}} \tag{3}$$

and

$$h(t) = \alpha\lambda t^{\alpha-1}, \tag{4}$$

where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter.

The Weibull distribution is a beneficial distribution for modeling and analyzing in different aspects of statistics, such as lifetime studies and reliability theory; because it is a very flexible distribution and can cover all types of aging for different values of its shape parameter. For example, as shown in Figure 1, it has an increasing, constant and decreasing failure rate for $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$, respectively. It should also be noted that when $\alpha = 1$, we arrive at Exponential distribution.

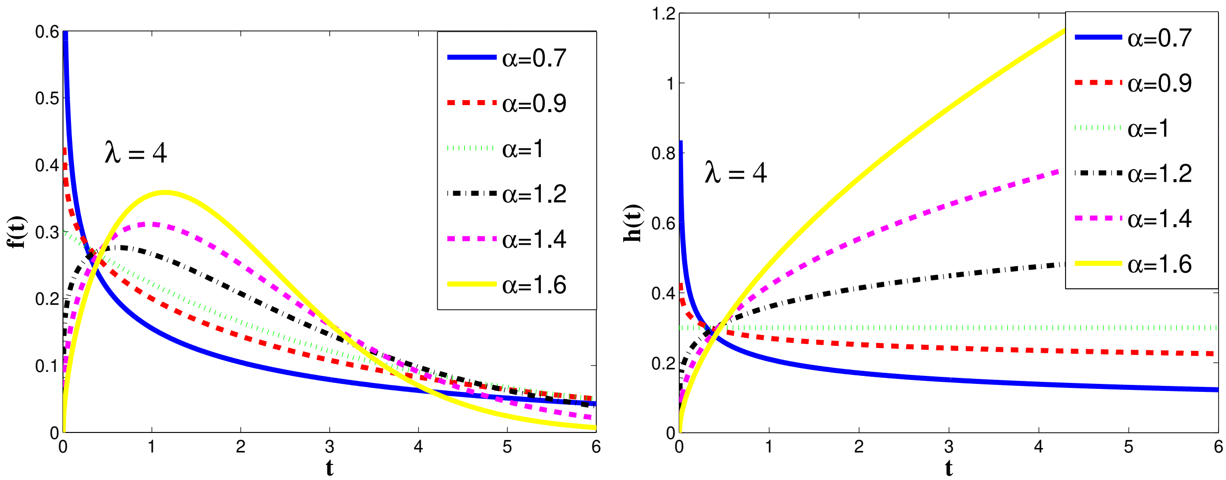

Figure 1. The plot of probability density (left) and hazard rate (right) functions of Weibull distribution for different values of parameter α.

Many authors have investigated various aspects of Weibull distribution. For instance, Kundu and Gupta [14], by considering independent strength and stress random variables distributed as Weibull model, have estimated the stress-strength reliability. Pak et al. [30] have discussed different estimation procedures for the Weibull distribution when the obtained data are fuzzy numbers. Muhammed and Almetwally [24], by considering the bivariate inverse Weibull distribution, have obtained Maximum likelihood and Bayesian estimation for the unknown parameters of this distribution in the presence of progressive type-II censored data. For some applications and different methods of estimation of the family of Weibull distribution, interested readers may refer to Almetwally et al. [3], Mastor et al. [23], Vila and Niyazi Çankaya [33], Jana and Bera [18] and Jiang [19], to name a few.

The survival and the hazard rate functions, as two important indicators to measure the aging of systems or components, have many applications in statistics, specially in lifetime testing and engineering. For example, Asgharzadeh et al. [4] have considered the two-parameter generalized exponential distribution, and have studied the stress-strength reliability of the distribution under the record data via the Bayes and maximum likelihood methods. A reliability analysis of the proportional hazard rate models (PHM) in aeronautics have been studied by Diamoutene et al. [11] . Demiray and Kızılaslan [10] have considered classical and Bayesian approaches to estimate the stress-strength reliability in consecutive k-out of-n systems when the two variables follow the PHM.

In reliability and life testing analysis, the data are often censored (Balakrishnan [6]; Balakrishnan and Aggarwala [7]).

Suppose that n independent items are put on a test and that the lifetime distribution of each item is given by the probability density function of (1) (Jianhua et al. [20]). The ordered m-failures are observed under the type-II progressively censoring plan $(R_1, ..., R_m)$ where each $R_i \geq 0$ and $\sum_{j=1}^{m} R_j + m = n$. It should be noted that, if, in the progressive type-II censoring scheme, we set $R_i = 0$, $i = 1, \dots, m$, we arrive at the type-II censored data. If the ordered m-failures are denoted by $x_{(1)} < x_{(2)} < ... < x_{(m)}$, then the likelihood function based on the observed sample $x_{(1)} < x_{(2)} < ... < x_{(m)}$ (for convenience, notations are denoted by $x_1 < x_2 < ... < x_m$) is

$$L(\alpha, \lambda) = c \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i} \tag{5}$$

where $c = n(n-1-R_1)...(n-R_1-...-R_{m-1}-m+1)$. For more details, interested readers may refer to Balakrishnan et al. [7] and Balakrishnan and Cramer [8]. Substituting (1), (2) in (5), the latter expression can be obtained as follows,

$$L(\alpha, \lambda) = \alpha^m \lambda^m e^{(\alpha-1)\sum_{i=1}^{m} \ln x_i} e^{-\lambda \sum_{i=1}^{m} (R_i+1) x_i^{\alpha}}. \tag{6}$$

Estimating the ageing characteristics such as the survival and the hazard rate functions in the presence of incomplete data (such as censored and recorded data) is an attractive topic for researchers. For example, Kayal et al. [21] have studied some inferences for Chen distribution under type-I progressive hybrid censored data. Al-Zahrani and Ali [2] have considered the Lindley distribution, and in the presence of multiple type-II censored samples of order statistics, have obtained the recurrence formulas for the moments. Estimating the stress-strength reliability for two Weibull independent random variables with different shape parameters under progressive type-II censored data have been done by Valiollahi et al. [27] . Similar results for the inverted exponential Rayleigh distribution have been extracted by Ma et al. [22]. Ren and Gui [26], in the presence of progressive type-II censored data, have studied the inference of competing risks model for generalized Rayleigh distribution. Feroze et al. [13], by considering progressive type-II censored data, have obtained Bayesian estimation for some reliability characteristics of the Topp-Leone distribution. Panahi [29], by considering independent strength and stress random variables distributed as inverted exponentiated Rayleigh model, have estimated the stress-strength reliability parameter when type II hybrid censored samples are available.

The aim of the paper is to estimate the unknown parameter and the ageing indices of the Weibull distribution using the E-Bayesian (EB) estimation method and compare them. As we have already known, Bayes is a method that uses experts' previous beliefs about parameter variations, called the prior distribution, to more exactly estimate of the parameter. But in practice, since the prior distribution parameters depend on the other parameters (hyperparameters), the E-Bayesian method is employed to model and estimate such condition. Thus, the E-Bayesian estimation method, as an alternative method to Bayesian approach, has been proposed by Han [15]. Han [15] have derived the E-Bayesian estimation and its expected mean squared error (E-MSE) for exponential distribution based on the scaled squared error loss function. Estimating the scale parameter and the reversed hazard rate function of an inverse Rayleigh distribution using E-Bayesian and Hierarchical Bayesian methods under the squared error, precautionary and entropy loss functions have been done by Athirakrishnan and Abdul-Sathar [5]. Okasha et al. [25] have derived the E-Bayesian estimations for some characteristics of the Lomax distribution under adaptive type-I progressive hybrid censoring scheme. Recently, Shojaee et al. [31] have compared the E-Bayesian estimations of the parameter, the hazard rate function and the survival function of the Compound Rayleigh distribution using the values of their E-MSE's when progressive type-II censored data are available. See, also, Shojaee et al. [32], where the authors have derived Bayesian estimator for Compound Rayleigh distribution under the record data.

In this paper, we have employed the E-Bayesian estimation method to estimate the unknown parameter, the survival function, and the hazard function of the Weibull distribution. Since, it is shown that the E-Bayesian estimation is a more precise estimation than the Bayesian estimation, and also the progressive type-II censoring is a more cost-effective plan, we obtained these estimates based on the asymmetric linear exponential (LINEX) and general entropy (GE) and the symmetric squared error loss functions (SE) under the progressive type-II censored data. In addition, we showed that the asymptotic behaviors of the E-Bayesian estimators based on different priors are the same. Moreover, to compare the E-Bayesian estimators, we used the "**E-MSE**"of the resulting estimators, which is a "**new measure to compare the E-Bayesian estimators**"(Han [15]).

The main motivations of the paper are as follows:

- The two-parameter Weibull distribution is one of the most important and popular statistical lifetime distributions in reliability engineering due to its simplicity and flexibility, and many components or systems can be cited as examples whose lifetimes follow this distribution.
- The two-parameter Weibull distribution for different values of its shape parameter can cover the lifetime distribution of systems or components with increasing, decreasing or constant failure rates.
- In some survival studies, the Weibull distribution has a better level of fit than the other distributions.
- It is proven that the E-Bayesian estimator has a higher precision than the older estimators including maximum likelihood (ML) and Bayesian estimators.
- E-Bayesian estimators cover the uncertainty in parameters more than other estimators such as maximum likelihood and Bayesian estimators, and in fact, they are robust to parameters uncertainties.

The rest of the paper is organized as follows. In sections 2 and 3, the Bayesian and E-Bayesian estimations for some interested characteristics of the Weibull distribution, respectively, are derived. We investigated the asymptotic behaviour of the E-Bayesian estimators in section 4. A simulation study of the resulting E-Bayesian estimators is carried out via the Monte Carlo simulation in section 5. In section 6, as an application, we analyzed two real data sets following the Weibull distribution. Finally, conclusions and concluding remarks are given in section 7.

2. Baysian Estimators

To avoid complex numerical methods that are possible only through simulation and to obtain the theoretical results presented in Section 4, when parameter α is known, we consider the natural conjugate family of prior densities for parameter λ as follows (Han [17]; Zeinhum and Hassan [12]).

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}. \tag{7}$$

Combining the likelihood function (6) with the prior pdf (7), one can obtain the posterior density function of λ as follows:

$$\pi(\lambda|\mathbf{x}) = \frac{(b+T)^{m+a}}{\Gamma(m+a)} \lambda^{m+a-1} e^{-\lambda(b+T)} \tag{8}$$

where $T = \sum_{i=1}^{m}(R_i+1)x_i^{\alpha}$.

As we know, Bayesian estimators are obtained under different loss functions. Therefore, in this article, we will use the squared error loss function as a symmetric loss function and general entropy and LINEX loss functions as asymmetric loss functions to estimate the desired characteristics. Also, one can compare the obtained Bayesian estimators by considering different loss functions. Under the squared error loss function $\left(L(\hat{\lambda}-\lambda) = (\hat{\lambda}-\lambda)^2\right)$, the Bayes estimator of λ can be driven

$$\hat{\lambda}_S = \frac{m+a}{b+T}. \tag{9}$$

The LINEX loss function for λ can be expressed as the following proportional

$$L(\Delta) \propto \exp(k\Delta) - k\Delta - 1; \qquad k \neq 0,$$

where $\Delta = (\hat{\lambda}-\lambda)$ and $\hat{\lambda}$ is an estimate of λ. The Bayes estimator of λ, denoted by $\hat{\lambda_L}$ under the LINEX loss function is given by

$$\hat{\lambda_L} = -\frac{1}{k} \ln E_{\lambda}[\exp(-k\lambda)].$$

Under LINEX loss function, we obtain Bayesian estimator of the parameter λ as the following form

$$\hat{\lambda}_L = \frac{m+a}{k} \ln\left(1+\frac{k}{b+T}\right). \tag{10}$$

The Bayes estimator of the reliability, R_{BS}, based on the squared error loss function is obtained from (3) and (8) as

$$\hat{R}_S = (\frac{b+T}{b+T+T^*})^{m+a} \tag{11}$$

where

$$T^* = t^\alpha.$$

The Bayes estimator of the hazard rate function, h_{BS}, under the squared error loss function is obtained from (4) and (8) as follows

$$\hat{h}_S = \alpha t^\alpha \frac{m+a}{T+b}. \tag{12}$$

Also, based on the LINEX loss function, the Bayes estimator of the reliability, R_L , and the hazard rate function, h_L are given by

$$\begin{aligned}\hat{R}_L = \frac{-1}{k}(\ln E(e^{-ke^{-\lambda t^\alpha}})) &= \frac{-1}{k}\ln\left(\frac{(b+T)^{m+a}}{\Gamma(m+a)}\int_0^\infty e^{-ke^{-\lambda T^*}}\lambda^{m+a-1}e^{-(b+T)\lambda}d\lambda\right) \\ &= \frac{-1}{k}\ln\left(\sum_{i=0}^{\infty}\frac{(-k)^i}{\Gamma(i)}(\frac{b+T}{b+T+iT^*})^{m+a}\right),\end{aligned} \tag{13}$$

and

$$\hat{h}_L = \alpha t^{\alpha-1}\frac{m+a}{k}\ln\left(1+\frac{k}{b+T}\right). \tag{14}$$

respectively.

The general entropy loss function is asymmetric and has the following form for λ

$$L(\delta) = \delta^q - q\ln(\delta) - 1; q \neq 0, \tag{15}$$

where $\delta = \frac{\hat{\lambda}}{\lambda}$ and $\hat{\lambda}$ is an estimator of λ (Varian [28]). The Bayes estimator of λ, under the general entropy loss function is

$$\hat{\lambda}_G = [E(\lambda^{-q})]^{-1/q}. \tag{16}$$

Under general entropy loss function, we obtain Bayes estimators of λ, $R(t)$ and $h(t)$ by combining (8) and (16), respectively, as follows:

$$\hat{\lambda}_G = \left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}, \tag{17}$$

$$\hat{R}_G = \left(\frac{T+b}{T+b-qt^\alpha)}\right)^{-\frac{m+a}{q}}, \tag{18}$$

$$\hat{h}_G = \alpha t^{\alpha-1}\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-\frac{1}{q}}(T+b)^{-1}. \tag{19}$$

3. E-Bayesian Estimators

In the reliability theory, the failure rate of a high reliability product is less than that of a low-quality product. For this reason and According to Han [17], a and b should be selected such that $g(\lambda)$ is a decreasing function of λ, in the E-Bayesian estimations. For more detail, we refer to Han [17]. The derivative of $g(\lambda)$ with respect to λ is

$$\frac{dg(\lambda)}{d(\lambda)} = \frac{b^a}{\Gamma(a)}\lambda^{a-2}e^{-b\lambda}[(a-1)-b\lambda].$$

Note that $a > 0$, $b > 0$, and $\lambda > 0$, it follows $0 < a < 1$, $b > 0$ due to $\frac{dg(\lambda)}{(\lambda)} < 0$; and therefore $g(\lambda)$ is a decreasing function of λ. Assuming that a and b are independent with bivariate density function

$$\pi(a,b) = \pi_1(a)\pi_2(b)$$

then, the E-Bayesian estimator of λ (expectation of the Bayesian estimator of λ) can be written as

$$\hat{\lambda}_{EB} = E(\hat{\lambda}|X) = \int\int \hat{\lambda}_B(a,b)\pi(a,b)dbda \tag{20}$$

where $\hat{\lambda}_B(a,b)$ is the Bayes estimator of λ given by (9), (10) and (17). In addition, to evaluate the precision of the E-Bayesian estimators, we have used the E-MSE that has been proposed by Han [15]:

$$\mathrm{E-MSE}\left(\hat{\lambda}_{EB}\right) = \int\int_D MSE[\hat{\lambda}_B(a,b)]\pi(a,b)dbda$$

To check the effect of different prior distributions of the hyperparameters a and b on the E-Bayesian estimations, we have derived the E-Bayesian estimations under three following different distributions. The following distributions of a and b may be used.

$$\begin{cases} \pi_1(a,b) = \frac{2(c-b)}{c^2}, & 0 < a < 1, \ 0 < b < c \\ \pi_2(a,b) = \frac{1}{c}, & 0 < a < 1, \ 0 < b < c \\ \pi_3(a,b) = \frac{2b}{c^2}, & 0 < a < 1, \ 0 < b < c. \end{cases} \tag{21}$$

3.1. E-Bayesian Estimators Under the Square Error Loss Function

The E-Bayesian estimations of λ is derived from (9) and (21) as

$$\begin{aligned} \hat{\lambda}_{EBS1} &= \int\int_D \hat{\lambda}_S(a,b)\pi_1(a,b)dbda = \int_0^1\int_0^c \frac{m+a}{T+b}\frac{2(c-b)}{c^2}dbda \\ &= \frac{2m+1}{c^2}\Big(c\ln(T+c) - c\ln T - T\ln(T+c) - T\ln T\Big), \end{aligned} \tag{22}$$

$$\hat{\lambda}_{EBS2} = \int\int_D \hat{\lambda}_S(a,b)\pi_2(a,b)dbda = \frac{\mathrm{m}+\frac{1}{2}}{\mathrm{c}}\ln(1+\frac{c}{T}), \tag{23}$$

and

$$\hat{\lambda}_{EBS3} = \int\int_D \hat{\lambda}_S(a,b)\pi_3(a,b)dbda = \frac{2m+1}{c^2}\left(c - T\ln(\frac{T+c}{T})\right). \tag{24}$$

Also, the E-Bayesian estimations of R from (11) and (21) can be obtained as follows

$$\hat{R}_{EBS1} = \int_0^1\int_0^c \left(\frac{T+b}{T+b+T^\star}\right)^{m+a}\frac{2(c-b)}{c^2}dbda, \tag{25}$$

$$\hat{R}_{EBS2} = \int_0^1\int_0^c \left(\frac{T+b}{T+b+T^\star}\right)^{m+a}\frac{1}{\mathrm{c}}dbda, \tag{26}$$

and

$$\hat{R}_{EBS3} = \int_0^1\int_0^c \left(\frac{T+b}{T+b+T^\star}\right)^{m+a}\frac{2b}{c^2}dbda. \tag{27}$$

Further, the E-Bayesian estimations of h from (12) and (21) can be given as follows

$$\hat{h}_{EBS1} = \frac{(\alpha t^{\alpha-1})(2m+1)}{c^2}\Big(\mathrm{c}\ln(T+c) - \mathrm{c}\ln\mathrm{T} - \mathrm{T}\ln(\mathrm{T}+\mathrm{c})\Big) \tag{28}$$

$$\hat{h}_{EBS2}=\frac{(\alpha t^{\alpha-1})(\mathrm{m}+\frac{1}{2})}{\mathrm{c}}\ln(1+\frac{c}{T}), \tag{29}$$

and

$$\hat{h}_{EBS3}=\frac{(\alpha t^{\alpha-1})(2m+1)}{c^2}\bigg(c-T\ln(\frac{T+c}{T})\bigg). \tag{30}$$

3.2. *E-Bayesian Estimators Under the General Entropy Loss Function*

Under the general entropy loss function, the E-Bayesian estimation of λ, R and h is computed from (17)-(19), respectively, and (21). The estimators are as follows:

$$\hat{\lambda}_{EBG1}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2(c-b)}{c^2}dbda, \tag{31}$$

$$\hat{\lambda}_{EBG2}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{1}{\mathrm{c}}dbda, \tag{32}$$

$$\hat{\lambda}_{EBG3}=\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2b}{c^2}dbda, \tag{33}$$

$$\hat{R}_{EBG1}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-qT^\star}\right)^{-\frac{m+a}{q}}\frac{2(c-b)}{c^2}dbda, \tag{34}$$

$$\hat{R}_{EBG2}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-qT^\star}\right)^{-\frac{m+a}{q}}\frac{1}{\mathrm{c}}dbda, \tag{35}$$

$$\hat{R}_{EBG3}=\int_0^1\int_0^c\left(\frac{T+b}{T+b-qT^\star}\right)^{-\frac{m+a}{q}}\frac{2b}{c^2}dbda, \tag{36}$$

$$\hat{h}_{EBG1}=(\alpha t^{\alpha-1})\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2(c-b)}{c^2}dbda, \tag{37}$$

$$\hat{h}_{EBG2}=(\alpha t^{\alpha-1})\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{1}{\mathrm{c}}dbda, \tag{38}$$

$$\hat{h}_{EBG3}=(\alpha t^{\alpha-1})\int_0^1\int_0^c\left(\frac{\Gamma(m+a-q)}{\Gamma(m+a)}\right)^{-1/q}(T+b)^{-1}\frac{2b}{c^2}dbda. \tag{39}$$

3.3. *E-Bayesian Estimators Under the LINEX Loss Function*

The E-Bayesian estimations of λ under the LINEX loss function can be derived from (10) and (21) as follows

$$\hat{\lambda}_{EBL1}=\int_0^1\int_0^c\frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k}\right)\frac{2(c-b)}{c^2}dbda, \tag{40}$$

$$\hat{\lambda}_{EBL2}=\int_0^1\int_0^c\frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k}\right)\frac{1}{\mathrm{c}}dbda, \tag{41}$$

and

$$\hat{\lambda}_{EBL3}=\int_0^1\int_0^c\frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k}\right)\frac{2b}{c^2}dbda. \tag{42}$$

Also, the E-Bayesian estimations of R from (13) and (21) can be obtained as follows

$$\hat{R}_{EBL1}=\int_0^1\int_0^c \frac{-1}{k}\ln\left(\sum_{i=0}^{\infty}\frac{(-k)^i}{\Gamma(i)}(\frac{b+T}{b+T+iT^{\star}})^{m+a}\right)\frac{2(c-b)}{c^2}dbda, \tag{43}$$

$$\hat{R}_{EBL2}=\int_0^1\int_0^c \frac{-1}{k}\ln\left(\sum_{i=0}^{\infty}\frac{(-k)^i}{\Gamma(i)}(\frac{b+T}{b+T+iT^{\star}})^{m+a}\right)\frac{1}{\mathrm{c}}dbda, \tag{44}$$

and

$$\hat{R}_{EBL3}=\int_0^1\int_0^c \frac{-1}{k}\ln\left(\sum_{i=0}^{\infty}\frac{(-k)^i}{\Gamma(i)}(\frac{b+T}{b+T+iT^{\star}})^{m+a}\right)\frac{2b}{c^2}dbda. \tag{45}$$

Further, the E-Bayesian estimations of h from (14) and (21) can be given as follows

$$\hat{h}_{EBL1}=\int_0^1\int_0^c \frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k\alpha t^{\alpha-1}}\right)\frac{2(c-b)}{c^2}dbda, \tag{46}$$

$$\hat{h}_{EBL2}=\int_0^1\int_0^c \frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k\alpha t^{\alpha-1}}\right)\frac{1}{\mathrm{c}}dbda, \tag{47}$$

and

$$\hat{h}_{EBL3}=\int_0^1\int_0^c \frac{-(m+a)}{k}\ln\left(\frac{T+b}{T+b+k\alpha t^{\alpha-1}}\right)\frac{2b}{c^2}dbda. \tag{48}$$

4. Asymptotic Behaviours of the E-Bayesian Estimators

In this section, we discuss asymptotic behaviour of the E-Bayesian estimators. For the sake of brevity, we only present theorems for square error loss function. For general entropy and LINEX loss functions, the results can be easily derived by the corresponding relations.

4.1. Asymptotic Behaviours of $\hat{\lambda}_{EBSi}$ $(i = 1, 2, 3)$

Theorem 1
Let $0 < c < T$ and $\hat{\lambda}_{EBSi}$ $(i = 1, 2, 3)$ be given by (22)-(24), then:

1. $\hat{\lambda}_{EBS3} < \hat{\lambda}_{EBS2} < \hat{\lambda}_{EBS1}$.
2. $\lim_{T\to\infty}\hat{\lambda}_{EBS1} = \lim_{T\to\infty}\hat{\lambda}_{EBS2} = \lim_{T\to\infty}\hat{\lambda}_{EBS3}$.

Proof

1. It is easy to see that from (22)-(24), we have:

$$\hat{\lambda}_{EBS2} - \hat{\lambda}_{EBS3} = \hat{\lambda}_{EBS1} - \hat{\lambda}_{EBS2} = \frac{1}{c}\left(m+\frac{1}{2}\right)\left[\frac{c+2T}{c}\ln\left(\frac{c+T}{T}\right) - 2\right]. \tag{49}$$

Now, based on Mac laurin series, we have:

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}x^k/k, \qquad |x| < 1. \tag{50}$$

Thus, it suffices to replace x by c/T in (50). Also, note that when $0 < c < T$, then we have $0 < c/T < 1$, and:

$$\begin{aligned}\left[\frac{c+2T}{c}\ln\left(\frac{T+c}{T}\right)-2\right] &= \frac{c+2T}{c}\left((c/T)-\frac{1}{2}(c/T)^2+\frac{1}{3}(c/T)^3-\frac{1}{4}(c/T)^4+\frac{1}{5}(c/T)^5-\dots\right)-2\\ &= \left((c/T)-\frac{1}{2}(c/T)^2+\frac{1}{3}(c/T)^3-\frac{1}{4}(c/T)^4+\frac{1}{5}(c/T)^5-\dots\right)-2\\ &+\left(2-(c/T)+\frac{2}{3}(c/T)^2-\frac{2}{4}(c/T)^3+\frac{2}{5}(c/T)^4-\dots\right)\\ &= \left(c^2/6T^2-c^3/6T^3\right)+\left(3c^4/6T^4-2c^5/15T^5\right)+\dots\\ &= \frac{c^2}{6T^2}\left(1-c/T\right)+\frac{c^4}{60T^4}\left(9-8c/T\right)+\dots\ . \end{aligned} \tag{51}$$

Consequently, (49) and (51) yield:

$$\hat{\lambda}_{EBS2}-\hat{\lambda}_{EBS3}=\hat{\lambda}_{EBS1}-\hat{\lambda}_{EBS2}>0,$$

or

$$\hat{\lambda}_{EBS3}<\hat{\lambda}_{EBS2}<\hat{\lambda}_{EBS1}.$$

2. From (49) and (51), we obtain:

$$\lim_{T\to\infty}\left(\hat{\lambda}_{EBS2}-\hat{\lambda}_{EBS3}\right)=\lim_{T\to\infty}\left(\hat{\lambda}_{EBS1}-\hat{\lambda}_{EBS2}\right),$$

or

$$\lim_{T\to\infty}\hat{\lambda}_{EBS1}=\lim_{T\to\infty}\hat{\lambda}_{EBS2}=\lim_{T\to\infty}\hat{\lambda}_{EBS3}.\blacksquare$$

$\square$

The first part of Theorem 1 states that with different densities of prior distributions, (21), the E-Bayesian estimations $\hat{\lambda}_{EBSi}$ $(i=1,2,3)$ are also different and can be ordered. Part (2) shows that $\hat{\lambda}_{EBSi}$ $(i=1,2,3)$ are asymptotically equal when T is large.

4.2. *Asymptotic Behaviours of* $\hat{R}_{EBSi}$ $(i=1,2,3)$

Theorem 2
Let $0<c<T$ and $\hat{R}_{EBSi}$ $(i=1,2,3)$ be given by (25)-(27), then:

$\lim_{T\to\infty}\hat{R}_{EBS1}<\lim_{T\to\infty}\hat{R}_{EBS2}<\lim_{T\to\infty}\hat{R}_{EBS3}.$

Proof
From (25)-(27), we have:

$$\lim_{T\to\infty}\left(\hat{R}_{EBS3}-\hat{R}_{EBS2}\right)=\lim_{T\to\infty}\left(\hat{R}_{EBS2}-\hat{R}_{EBS1}\right)=\frac{1}{c^2}\lim_{T\to\infty}\int_0^1\int_0^c(2b-c)\left(\frac{T+b}{T+b+T^*}\right)^{m+a}dbda.$$

Here, applying the monotone convergence theorem and equivalence rule:

$$\lim_{T\to\infty}\hat{R}_{EBS1}=\lim_{T\to\infty}\hat{R}_{EBS2}=\lim_{T\to\infty}\hat{R}_{EBS3}.\blacksquare$$

$\square$

The first part of Theorem 2 states that with different densities of prior distributions, (21), the E-Bayesian estimations $\hat{R}_{EBSi}$ $(i=1,2,3)$ are also different and can be ordered.

Remark 1
From (25)-(27), we have

$$\hat{R}_{EBS2} - \hat{R}_{EBS1} = \hat{R}_{EBS3} - \hat{R}_{EBS2} = \frac{1}{c^2} \int_0^1 \int_0^c (2b - c) \left(\frac{T + b}{T + b + T^*} \right)^{m+a} dbda. \tag{52}$$

The equation (52) has not a closed form. Therefore, numerical methods are used for calculating it, by R software. The results (Table 3) implied that the following relation

$$\hat{R}_{EBS1} < \hat{R}_{EBS2} < \hat{R}_{EBS1}.$$

4.3. *Asymptotic Behaviours of* $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$

Theorem 3
Let $0 < c < T$ and $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ be given by (28)-(30), then:

1. $\hat{h}_{EBS3} < \hat{h}_{EBS2} < \hat{h}_{EBS1}$.
2. $\lim_{T\to\infty} \hat{h}_{EBS1} = \lim_{T\to\infty} \hat{h}_{EBS2} = \lim_{T\to\infty} \hat{h}_{EBS3}$.

Proof

1. From (28)-(30), it is eay to see that:

$$\hat{h}_{EBS2} - \hat{h}_{EBS3} = \hat{h}_{EBS1} - \hat{h}_{EBS2} = \alpha t^{\alpha-1} \frac{1}{c} \left(m + \frac{1}{2} \right) \left[\frac{c + 2T}{c} \ln \left(\frac{c + T}{T} \right) - 2 \right]. \tag{53}$$

   Now, based on (53), (49) and (51), one can obtain:

$$\hat{h}_{EBS2} - \hat{h}_{EBS3} = \hat{h}_{EBS1} - \hat{h}_{EBS2} > 0,$$

   or:

$$\hat{h}_{EBS3} < \hat{h}_{EBS2} < \hat{h}_{EBS1}.$$

2. According to (53), (49) and (51), we have:

$$\lim_{T\to\infty} (\hat{h}_{EBS2} - \hat{h}_{EBS3}) = \lim_{T\to\infty} \left(\hat{h}_{EBS1} - \hat{h}_{EBS2} \right)$$

$$= \alpha t^{\alpha-1} \frac{1}{c} \left(m + \frac{1}{2} \right) \lim_{T\to\infty} \left\{ \frac{c^2}{6T^2} (1 - c/T) + \frac{c^4}{60T^4} (9 - 8c/T) + \dots \right\} = 0.$$

   Or,

$$\lim_{T\to\infty} \hat{h}_{EBS1} = \lim_{T\to\infty} \hat{h}_{EBS2} = \lim_{T\to\infty} \hat{h}_{EBS3} . \blacksquare$$

$\square$

The first part of Theorem 3 states that with different densities of prior distributions, (21), the E-Bayesian estimations $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ are also different and can be ordered. Part (2) shows that $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ are asymptotically equal when T is large.

5. Simulation Study

To generate a progressive type-II censored sample from a Weibull distribution, we use the algorithm of Balakrishnan et al. [7] as follows:

1. Simulate m independent exponential random variables $Z_1, Z_2, \dots, Z_m$. This can be done using inverse transformation $Z_i = -\ln(1-U_i)$ where U_i are independent uniform $(0,1)$ random variables.
2. Set
$$X_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \frac{Z_3}{n-R_1-R_2-2} + \cdots + \frac{Z_i}{n-R_1-R_2-\cdots-R_{i-1}-i+1}$$
for $i = 1, 2, \dots, m$. This is the standard exponential progressive type-II censored sample.
3. Finally, set $Y_i = F^{-1}(1-\exp(-X_i))$, for $i = 1, 2, \dots, m$, where $F^{-1}(.)$ is the inverse of cumulative distribution function of Weibull distribution. $Y_1, Y_2, , Y_m$ is the required progressive type-II censored sample from $F(.)$.
4. The E-Bayesian estimations of parameter λ and survival function $R(t)$ and hazard rate $h(t)$, can be computed, respectively, by (22)-(48).
5. The mentioned steps are repeated 2000 times and the E-MSEs of estimates are calculated as:
$$\mathrm{E-MSE}\left(\hat{Q}\right) = \frac{1}{2000}\sum \int \int MSE[\hat{Q}_B\,(a,b)]\pi(a,b)dbda,$$
where $\hat{Q}$ is an estimate of parameter Q.

To estimate the parameters via simulation, we use the following values as starting points in Table 1. The results are

Table 1. Initial values of the parameters for Monte Carlo simulation.

α	λ	k	q	a	b	c	t	Hazard rate	Reliability
1.9	1.247675	0.25	1.2	0.5060425	0.3919524	1.4	0.5	1.270364	0.7158345

displayed in Tables 2-4. Further, the corresponding figures are depicted in Figures 2-4. It is clear from Tables 2-4 and Figures 2-4 that the E-Bayesian estimators, are robust and satisfy the Theorems 1, 2, and 3.

Table 2. The value of $\hat{\lambda}$ and the corresponding E-MSE to estimate the parameter of the Weibull distribution, λ, based on Monte Carlo simulation.

Censor Scheme (n, m , R)	**Estimates E-MSE of E-Bayesian estimates**								
	EBS1	**EBS2**	**EBS3**	**EBG1**	**EBG2**	**EBG3**	**EBL1**	**EBL2**	**EBL3**
n=20, m=10; R=(5,5, 0*8)	1.339688	1.298638	1.257589	1.19909	1.162349	1.125608	1.316647	1.27708	1.237514
	0.1942	0.164863	0.140588	0.151155	0.137274	0.127449	0.176662	0.15159	0.131129
n=20, m=15; R=(5, 0*14)	1.316429	1.289729	1.263029	1.222897	1.198094	1.173291	1.301703	1.275622	1.249541
	0.125549	0.11259	0.101502	0.104877	0.098091	0.09292	0.118072	0.106533	0.096768
n=20, m=20; R=(0*20)	1.308804	1.288926	1.269048	1.238516	1.219705	1.200894	1.297926	1.278389	1.258851
	0.094842	0.087125	0.080387	0.081666	0.077277	0.073764	0.090457	0.083447	0.077379
n=30, m=15; R=(5*3, 0*12)	1.316237	1.289601	1.262964	1.222719	1.197975	1.17323	1.301547	1.275526	1.249505
	0.121252	0.108676	0.09795	0.101201	0.094735	0.089866	0.113992	0.102812	0.093385
n=30, m=20; R=(5,5, 0*18)	1.29502	1.275564	1.256108	1.225471	1.20706	1.188649	1.284376	1.265249	1.246122
	0.090184	0.083346	0.07744	0.079243	0.075587	0.072766	0.086291	0.080114	0.074833
n=30, m=30; R=(0*30)	1.297258	1.284235	1.271212	1.250445	1.237892	1.225339	1.290186	1.277307	1.264429
	0.05784	0.054484	0.051514	0.051465	0.049477	0.047847	0.055961	0.05286	0.050136
n=40, m=20; R=(5*4, 0*16)	1.309507	1.2896	1.269693	1.23918	1.220342	1.201504	1.298613	1.279047	1.259481
	0.095666	0.087952	0.081214	0.082316	0.077932	0.074423	0.091287	0.084274	0.078201
n=40, m=30; R=(5,5,0*28)	1.274577	1.262002	1.249427	1.228582	1.216461	1.20434	1.26775	1.255312	1.242874
	0.054003	0.051368	0.049093	0.049868	0.048511	0.047489	0.05252	0.050117	0.048066
n=40, m=40; R=(0*40)	1.273596	1.264158	1.254721	1.23899	1.229809	1.220627	1.26849	1.259129	1.249768
	0.042017	0.04038	0.038939	0.039204	0.038277	0.037536	0.041103	0.039589	0.038269
n=50, m=25; R=(5*5, 0*20)	1.292669	1.277154	1.261639	1.236868	1.222023	1.207178	1.284219	1.268911	1.253604
	0.069929	0.065485	0.061606	0.062286	0.059816	0.057864	0.067423	0.063361	0.059848
n=50, m=35; R=(5*3,0*32)	1.277341	1.266496	1.255651	1.237742	1.227233	1.216725	1.271464	1.260721	1.249977
	0.048625	0.046438	0.044517	0.044929	0.043689	0.042698	0.047401	0.045385	0.043629
n=50, m=50; R=(0*50)	1.276828	1.269244	1.261661	1.249006	1.241588	1.23417	1.272733	1.265199	1.257664
	0.031926	0.030799	0.029797	0.029738	0.029064	0.028508	0.0313	0.03025	0.029322

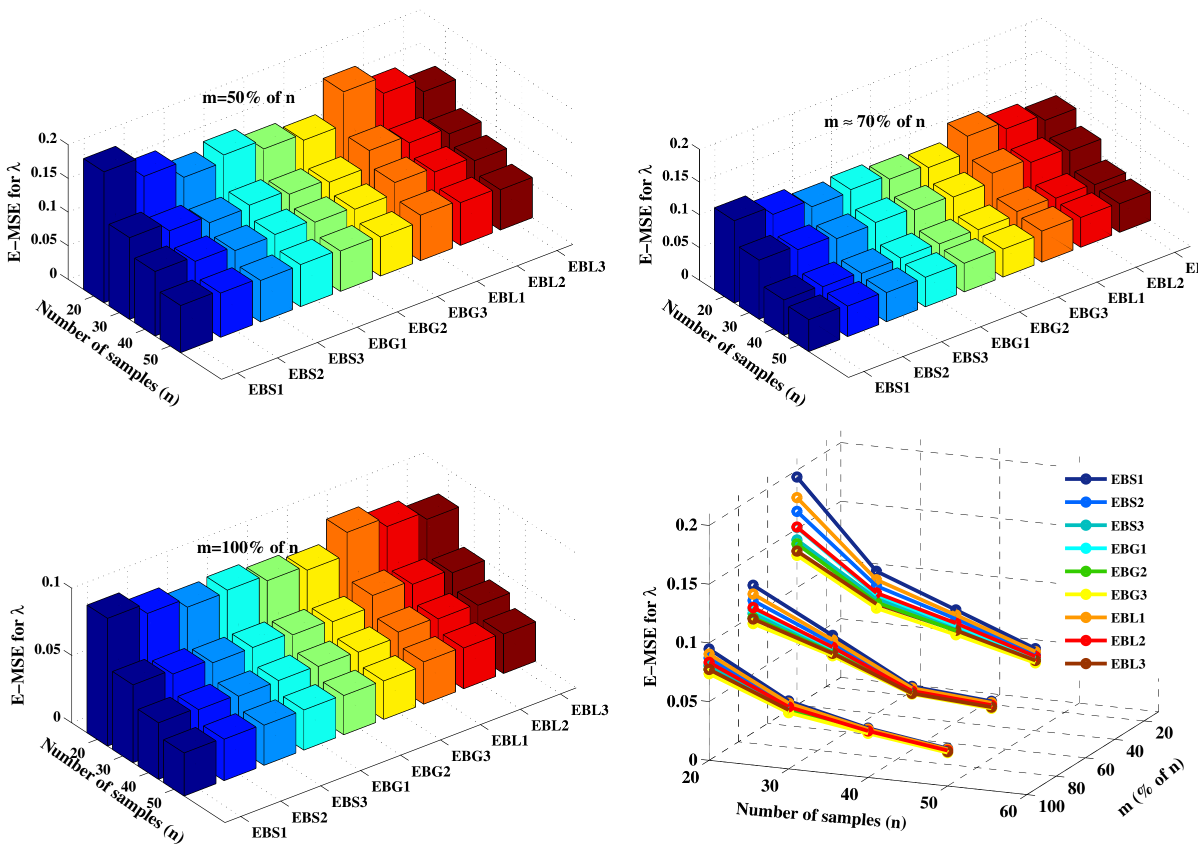

Figure 2. The values of E-MSEs to estimate parameter λ based on (50, 70 and 100)% of samples.

6. Real Data Analysis

In this section, we present a data analysis of the strength data reported by Kundu and Gupta [14]. The data represent the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20, and 50mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purpose, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes $n_1 = 69$ and $n_2 = 63$, respectively. The two real data sets are as follows.

Data Set I

1.312,1.314,1.479,1.552,1.700,1.803,1.861,1.865,1.944,1.958,1.966,1.997,2.006,2.021,2.027,2.055, 2.063,2.098,2.140,2.179,2.224,2.240,2.253,2.270,2.272,2.274,2.301,2.301,2.359,2.382,2.382,2.426, 2.434,2.435,2.478,2.490,2.511,2.514,2.535,2.554,2.566,2.570,2.586,2.629,2.633,2.642,2.648,2.684, 2.697,2.726,2.770,2.773,2.800,2.809,2.818,2.821,2.848,2.880,2.954,3.012,3.067,3.084,3.090,3.096, 3.128,3.233,3.433,3.585,3.585.

Data Set II

1.901,2.132,2.203,2.228,2.257,2.350,2.361,2.396,2.397,2.445,2.454,2.474,2.518,2.522,2.525,2.532, 2.575,2.614,2.616,2.618,2.624,2.659,2.675,2.738,2.740,2.856,2.917,2.928,2.937,2.937,2.977,2.996, 3.030,3.125,3.139,3.145,3.220,3.223,3.235,3.243,3.264,3.272,3.294,3.332,3.346,3.377,3.408,3.435, 3.493,3.501,3.537,3.554,3.562,3.628,3.852,3.871,3.886,3.971,4.024,4.027,4.225,4.395,5.020.

Table 3. The value of $\hat{R}$ and the corresponding E-MSE to estimate the reliability function of the Weibull distribution based on Monte Carlo simulation.

Censor Scheme	Estimates								
	E-MSE of E-Bayesian estimates								
(n, m , R)	**EBS1**	**EBS2**	**EBS3**	**EBG1**	**EBG2**	**EBG3**	**EBL1**	**EBL2**	**EBL3**
n=20, m=10; R=(5,5, 0*8)	0.707295	0.714362	0.721428	0.697555	0.705103	0.71265	0.767281	0.775364	0.783447
	0.005433	0.004865	0.004422	0.006452	0.005647	0.004986	0.009657	0.009923	0.01035
n=20, m=15; R=(5, 0*14)	0.708623	0.713376	0.718128	0.702266	0.707229	0.712192	0.766451	0.771967	0.777483
	0.003827	0.00354	0.003306	0.004305	0.003926	0.003605	0.00779	0.008053	0.008388
n=20, m=20; R=(0*20)	0.708608	0.712201	0.715794	0.703872	0.707583	0.711294	0.768926	0.773025	0.777125
	0.002985	0.002804	0.002652	0.003274	0.003044	0.002846	0.006437	0.00672	0.00704
n=30, m=15; R=(5*3, 0*12)	0.708554	0.713302	0.718051	0.702202	0.707161	0.71212	0.769842	0.775249	0.780656
	0.003703	0.003424	0.003198	0.004169	0.003799	0.003487	0.007908	0.008215	0.00859
n=30, m=20; R=(5,5, 0*18)	0.711113	0.714646	0.71818	0.706458	0.710106	0.713754	0.767676	0.771808	0.775939
	0.002901	0.002742	0.002612	0.003158	0.002953	0.002779	0.006434	0.006703	0.00701
n=30, m=30; R=(0*30)	0.709192	0.711592	0.713992	0.706075	0.708526	0.710978	0.771065	0.773786	0.776506
	0.001922	0.001835	0.001761	0.002055	0.001949	0.001856	0.005515	0.005746	0.005992
n=40, m=20; R=(5*4, 0*16)	0.708504	0.712102	0.715699	0.703762	0.707478	0.711194	0.770491	0.774554	0.778618
	0.003039	0.002855	0.0027	0.003331	0.003098	0.002898	0.006568	0.006863	0.007196
n=40, m=30; R=(5,5,0*28)	0.713413	0.715745	0.718077	0.710385	0.712766	0.715148	0.768771	0.771527	0.774283
	0.001834	0.001769	0.001717	0.001936	0.001854	0.001785	0.005269	0.005491	0.00573
n=40, m=40; R=(0*40)	0.712957	0.714718	0.716479	0.710686	0.712475	0.714264	0.770654	0.772708	0.774762
	0.001443	0.001401	0.001366	0.001507	0.001455	0.001411	0.00498	0.005163	0.005354
n=50, m=25; R=(5*5, 0*20)	0.710644	0.71349	0.716335	0.706927	0.709846	0.712765	0.769921	0.773205	0.776489
	0.002302	0.002192	0.002101	0.002474	0.002336	0.002218	0.00619	0.006431	0.006697
n=50, m=35; R=(5*3,0*32)	0.712551	0.714567	0.716582	0.709944	0.711996	0.714049	0.770974	0.773315	0.775655
	0.001638	0.001583	0.001537	0.001722	0.001655	0.001596	0.005335	0.005536	0.005749
n=50, m=50; R=(0*50)	0.711881	0.713302	0.714723	0.710056	0.711495	0.712934	0.773383	0.775003	0.776624
	0.001104	0.001074	0.001048	0.00115	0.001113	0.001081	0.00488	0.00504	0.005205

Table 4. The value of $\hat{h}$ and the corresponding E-MSE to estimate the hazard rate function of the Weibull distribution based on Monte Carlo simulation.

Censor Scheme	Estimates E-MSE of E-Bayesian estimates								
(n, m , R)	EBS1	EBS2	EBS3	EBG1	EBG2	EBG3	EBL1	EBL2	EBL3
n=20, m=10; R=(5,5, 0*8)	1.364049	1.322254	1.280458	1.220896	1.183486	1.146077	1.340174	1.299915	1.259655
	0.201327	0.170913	0.145748	0.156703	0.142312	0.132127	0.182843	0.156926	0.135781
n=20, m=15; R=(5, 0*14)	1.340368	1.313183	1.285997	1.245135	1.219881	1.194627	1.325106	1.298562	1.272018
	0.130157	0.116722	0.105227	0.108726	0.101691	0.096331	0.122273	0.110336	0.100237
n=20, m=20; R=(0*20)	1.332605	1.312365	1.292125	1.261038	1.241885	1.222732	1.32133	1.301443	1.281556
	0.098323	0.090322	0.083337	0.084664	0.080113	0.076471	0.093699	0.086444	0.080165
n=30, m=15; R=(5*3, 0*12)	1.340173	1.313052	1.285931	1.244954	1.219759	1.194565	1.324948	1.298465	1.271981
	0.125702	0.112665	0.101545	0.104915	0.098212	0.093164	0.118048	0.106482	0.096732
n=30, m=20; R=(5,5, 0*18)	1.31857	1.29876	1.27895	1.247756	1.22901	1.210264	1.307537	1.288069	1.2686
	0.093494	0.086405	0.080282	0.082152	0.078362	0.075436	0.089388	0.082996	0.077534
n=30, m=30; R=(0*30)	1.320848	1.307588	1.294329	1.273184	1.260403	1.247621	1.313518	1.300408	1.287298
	0.059963	0.056484	0.053404	0.053354	0.051292	0.049603	0.057981	0.054771	0.051951
n=40, m=20; R=(5*4, 0*16)	1.33332	1.313051	1.292782	1.261714	1.242534	1.223353	1.322029	1.302113	1.282198
	0.099177	0.09118	0.084195	0.085337	0.080792	0.077154	0.094558	0.087301	0.081017
n=40, m=30; R=(5,5,0*28)	1.297755	1.284951	1.272148	1.250924	1.238582	1.226241	1.290678	1.278017	1.265355
	0.055985	0.053253	0.050895	0.051698	0.050292	0.049232	0.05442	0.051934	0.049811
n=40, m=40; R=(0*40)	1.296756	1.287147	1.277537	1.261521	1.252173	1.242824	1.291463	1.281933	1.272403
	0.043559	0.041861	0.040368	0.040643	0.039682	0.038914	0.042595	0.041028	0.039662
n=50, m=25; R=(5*5, 0*20)	1.316175	1.300378	1.284581	1.25936	1.244245	1.22913	1.307417	1.291835	1.276253
	0.072496	0.067889	0.063867	0.064572	0.062012	0.059987	0.069852	0.065649	0.062013
n=50, m=35; R=(5*3,0*32)	1.300569	1.289527	1.278485	1.260251	1.24955	1.23885	1.294478	1.283541	1.272603
	0.050409	0.048143	0.046151	0.046578	0.045292	0.044265	0.049119	0.047032	0.045214
n=50, m=50; R=(0*50)	1.300047	1.292325	1.284604	1.271719	1.264166	1.256613	1.295802	1.288132	1.280461
	0.033097	0.03193	0.03089	0.030829	0.03013	0.029554	0.032438	0.03135	0.030389

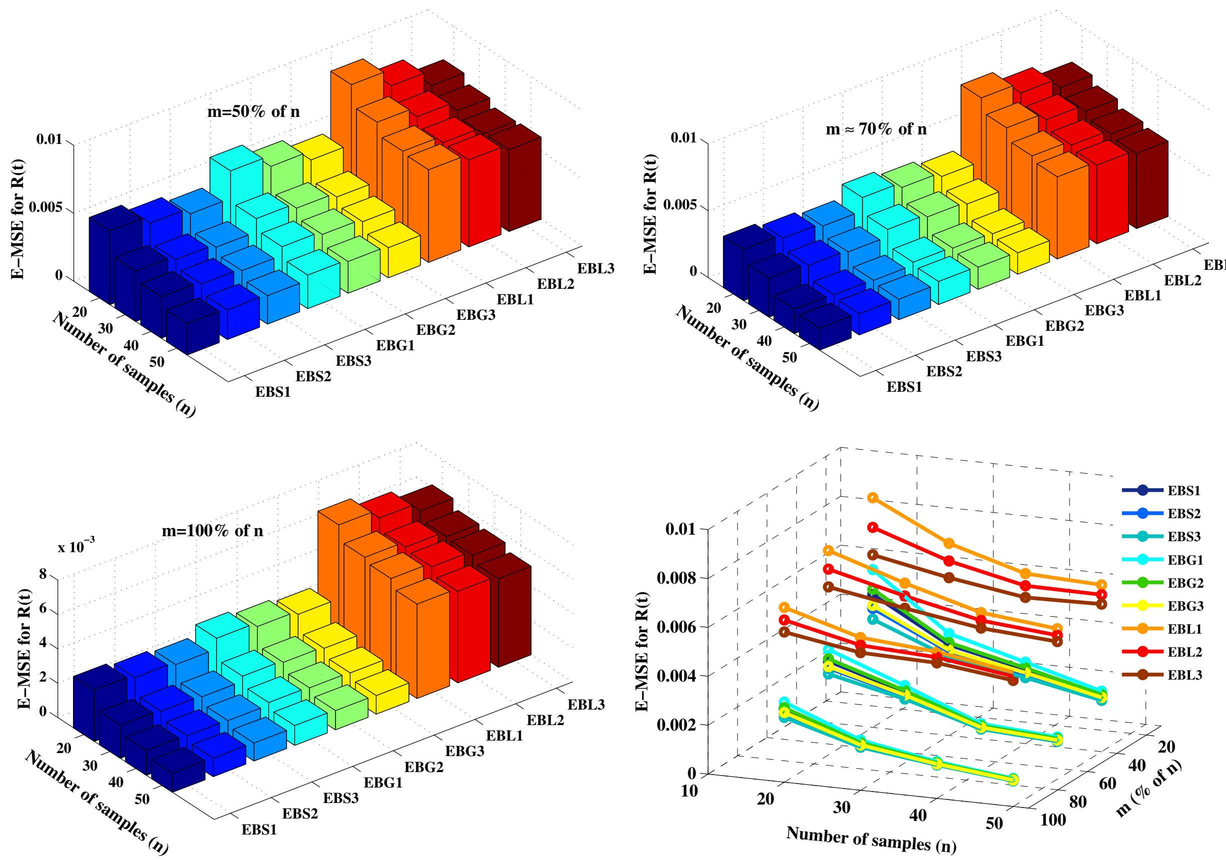

Figure 3. The values of E-MSEs to estimate $R(t)$ based on (50, 70 and 100)% of samples.

Kundu and Gupta [14] have shown that the Weibull distribution is acceptable for these data. Using equations (22)-(48), different estimates are computed and reported in Table 6. The first column is the censor scheme. For example, $R = (8 * 3, 2, 0 * 30)$ means $(8, 8, 8, 8, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. We use the following initial values (Table 5) to estimate the parameters.

Table 5. Initial values for two real data sets.

c	k	q	t	α(Data set 1)	α(Data set 2)
1.4	0.25	1.2	0.5	3.8428	3.9090

The results of Table 6 indicates that E-Bayesian estimates are robust and satisfy the Theorems 1, 2 and 3.

7. Conclusions and Concluding Remark

The two-parameter Weibull distribution is one of the most important statistical lifetime distributions in reliability engineering due to its simplicity and flexibility. On the other hand, E-Bayesian estimators cover the uncertainty in parameters more than other estimators such as maximum likelihood and Bayesian estimators, and in fact, they are robust to parameters uncertainties.

We have obtained the E-Bayesian estimation of the parameter, the reliability, and the hazard functions of Weibull distribution when the progressive type-II censored data are available. Our interest in this stems from the

Table 6. Estimates based on real data set.

Censor Scheme (n, m , R)	EBS1	EBS2	EBS3	EBG3	EBG2	EBG1	EBL1	EBL2	EBL3
				Estimates E-MSE of E-Bayesian estimates					
				DATA SET I					
				Lambda=0.08838451					
n=69, m=35; R=(8*4, 2, 0*30)	0.014144	0.014142	0.014141	0.013705	0.013704	0.013703	0.014143	0.014142	0.014140
n=69, m=49; R=(5*4, 0*45)	0.019071	0.019069	0.019067	0.018647	0.018645	0.018644	0.019070	0.019068	0.019067
n=69, m=69; R=(0*69)	0.026273	0.026271	0.026269	0.025858	0.025855	0.025853	0.026272	0.026270	0.026267
				Hazard rate=0.04734301					
n=69, m=35; R=(8*4, 2, 0*30)	0.007576	0.007575	0.007574	0.007341	0.007340	0.007340	0.007576	0.007575	0.007574
n=69, m=49; R=(5*4, 0*45)	0.010215	0.010214	0.010213	0.009988	0.009987	0.009986	0.010215	0.010214	0.010213
n=69, m=69; R=(0*69)	0.014073	0.014072	0.014071	0.013850	0.013849	0.013848	0.014073	0.014072	0.014070
				Survival=0.993859					
n=69, m=35; R=(8*4, 2, 0*30)	0.99902	0.99901	0.99901	0.99902	0.99901	0.99901	0.86847	0.86847	0.86845
n=69, m=49; R=(5*4, 0*45)	0.99868	0.99867	0.99867	0.99867	0.99867	0.99867	0.86823	0.868224	0.86822
n=69, m=69; R=(0*69)	0.99817	0.99817	0.99817	0.99817	0.99817	0.99816	0.86786	0.86785	0.86785
				DATA SET II					
				Lambda=0.02594377					
n=63, m=32; R=(8*3, 7, 0*28)	0.0054363	0.0054361	0.0054357	0.0052522	0.0052520	0.0052518	0.0054362	0.0054359	0.0054357
n=63, m=42; R=(5*4, 1, 0*37)	0.006920	0.006920	0.006920	0.006741	0.006741	0.006741	0.006920	0.006920	0.006920
n=63, m=63; R=(0*63)	0.0101612	0.0101608	0.0101604	0.0099851	0.0099847	0.0099843	0.0101610	0.0101606	0.0101602
				Hazard=0.01350214					
n=63, m=32; R=(8*3, 7, 0*28)	0.0028293	0.0028292	0.0028291	0.0027335	0.0027334	0.0027332	0.0028292	0.0028291	0.0028290
n=63, m=42; R=(5*4,1, 0*37)	0.0036019	0.0036017	0.0036016	0.0035086	0.0035085	0.0035083	0.0036018	0.0036017	0.0036016
n=63, m=63; R=(0*63)	0.0052882	0.0052880	0.0052878	0.0051966	0.005196457 0.0051962	0.0052882	0.0052880	0.0052878	
				Survival= 0.9982744					
n=63, m=32; R=(8*3, 7, 0*28)	0.99964	0.99963	0.99965	0.99963	0.99963	0.99963	0.86893	0.86893	0.86893
n=63, m=42; R=(5*4,1, 0*37)	0.99953	0.99952	0.99951	0.99953	0.9995394 0.99953	0.86887	0.86886	0.86886	
n=63, m=63; R=(0*63)	0.99932	0.99932	0.99931	0.99932	0.99932	0.99932	0.86870	0.86870	0.86870

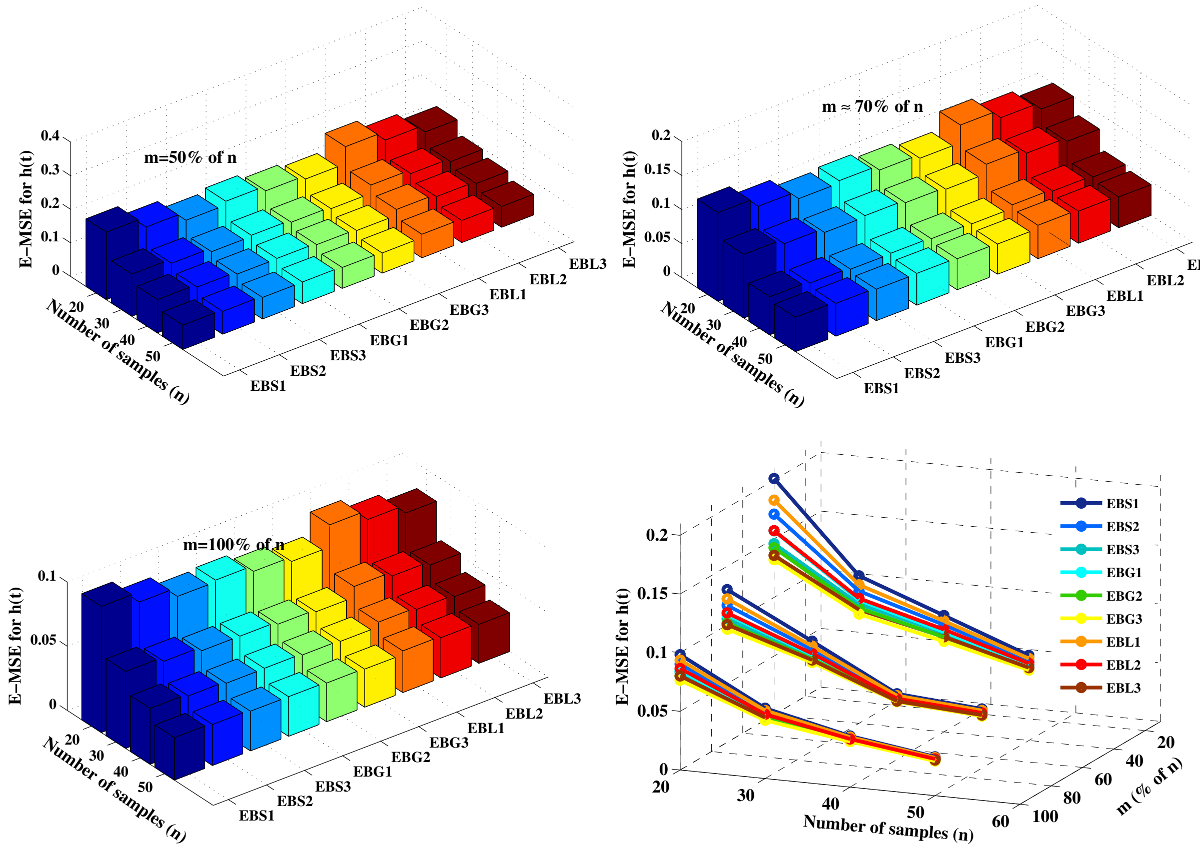

Figure 4. The values of E-MSEs to estimate $h(t)$ based on (50, 70 and 100)% of samples.

fact that we have predicted the different estimators to be differently affected by the availability of the progressive type-II censored samples in a known statistical distribution such as Weibull distribution. To compare the E-Bayesian estimators, we have used the E-MSE of the resulting estimators, which is a new measure to compare the E-Bayesian estimators. The E-MSE of the E-Bayesian estimators have obtained using Monte Carlo simulation studies with different censor schemes to compare the E-MSE of the resulting estimators. Asymptotic behaviours of the E-Bayesian have shown that the E-Bayesian estimations can be ordered and asymptotically equal when T is large. In contrast to our anticipation, we found that:

- The obtained E-Bayesian estimations from various prior parameters are more close to each other when the sample size increases. By various priors, the E-Bayesian estimators are robust, and also satisfy the corresponding theorems.
- The E-MSEs of the E-Bayesian estimators decrease when the sample size n increases.
- This paper states that the E-Bayesian estimation in all scenarios, as well as the simplicity of calculations, it also has high efficiency.
- From Table 2 and Figure 2, the E-MSEs of estimates decreases as sample size increases. The E-MSEs of E-Bayesian estimators are ordered as follows:

  a. $\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBS3}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBS2}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBS1}})$,
  b. $\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBG3}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBG2}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBG1}})$,
  c. $\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBL3}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBL2}})<\mathrm{E-MSE}(\hat{\lambda}_{\mathrm{EBL1}})$.

From Table 3 and Figure 3, the E-MSEs of estimates decrease as sample size increases. The E-MSEs of E-Bayesian estimators are ordered as follows:

a. $\mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBS3}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBS2}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBS1}})$,
b. $\mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBG3}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBG2}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBG1}})$,
c. $\mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBL3}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBL2}}) < \mathrm{E-MSE}(\hat{\mathrm{R}}_{\mathrm{EBL1}})$.

- From Table 4 and Figure 4, the E-MSEs of estimates decrease as sample size increases. The E-MSEs of E-Bayesian estimators are ordered as follows:

a. $\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS3}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS2}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBS1}})$,
b. $\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG3}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG2}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBG1}})$,
c. $\mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBL3}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBL2}}) < \mathrm{E-MSE}(\hat{\mathrm{h}}_{\mathrm{EBL1}})$.

- The E-Bayesian estimates, based on general entropy loss function have smaller E-MSEs than the others.

Declarations

Funding: Not applicable.
Conflicts of interest: The authors declare that they have no conflict of interest.

REFERENCES

1. A.A. Abdel-Ghaly, and A.F. Attia, *On a new density function*, Microelectronics and Reliability, vol. 33, no. 5, pp. 671–679, 1993.
2. B. Al-Zahrani, and M. Ali, *Recurrence relations for moments of multiply type-II censored order statistics from Lindley distribution with applications to inference*, Statistics, Optimization and Information Computing, vol. 2, pp. 147–160, 2014.
3. E. M. Almetwally, M. A. Sabry, R. Alharbi, D. Alnagar, S. A. Mubarak and E. H. Hafez, Marshall–olkin alpha power Weibull distribution: different methods of estimation based on type-I and type-II censoring, Complexity, 2021.
4. A. Asgharzadeh, R. Valiollahi, and M. Z. Raqab, *Estimation of $Pr(Y < X)$ for the two-parameter generalized exponential records*, Communications in Statistics-Simulation and Computation, vol. 46, no. 1, pp. 379–394, 2020.
5. R. Athirakrishnan, and E. Abdul-Sathar, *E-Bayesian and hierarchical Bayesian estimation of inverse Rayleigh distribution*, American Journal of Mathematical and Management Sciences, pp. 1–22, 2021.
6. N. Balakrishnan, *Progressive censoring methodology:an appraisal (with discussions)*, Test, vol. 16, no. 2, pp. 211–259, 2007. .
7. N. Balakrishnan, N. Balakrishnan, and R. Aggarwala, *Progressive censoring: theory, methods, and applications*, Springer Science & Business Media, 2000.
8. N. Balakrishnan, and E. Cramer, *The art of progressive censoring*, Statistics for industry and technology, 2014.
9. S.K. Bhattacharya, and R.K. Tyagi, *Bayesian survival analysis based on the Rayleigh model*, Trabajos de Estadistica, vol. 5, no. 1, pp. 81–92, 1990.
10. D. Demiray, and F. Kızılaslan, *Stress–strength reliability estimation of a consecutive k-out-of-n system based on proportional hazard rate family*, Journal of Statistical Computation and Simulation, pp. 1–32, 2021.
11. A. Diamoutene, F. Noureddine, B. Kamsu-Foguem, and D. Barro, *Reliability Analysis with Proportional Hazard Model in Aeronautics*, International Journal of Aeronautical and Space Sciences, pp. 1–13, 2021.
12. F.J., Zeinhum and M.O., Hassan *E-Bayesian estimation for the Burr type XII model based on type-2 censoring*, Applied Mathematical Modelling, vol. 35, no. 10, pp. 4730–4737, 2011.
13. N. Feroze, M. Aslam, I. H. Khan, and M.H. Khan, *Bayesian reliability estimation for the Topp-Leone distribution under progressively type-II censored samples*, Soft Computing, vol. 25, no. 3, pp. 2131–2152, 2021.
14. D., Kundu and R. D. Gupta, *Estimation of $P[Y < X]$ for Weibull distributions*, IEEE Trans. Reliab., vol. 55, no. 2, pp. 270–280, 2006.
15. M. Han, *E-Bayesian estimation and its E-MSE under the scaled squared error loss function, for exponential distribution as example*, Communications in Statistics-Simulation and Computation, vol. 48, no. 6, pp. 1880–1890, 2019.
16. M. Han, *The structure of hierarchical prior distribution and its applications*, Chinese Operations Research and Management Science, vol. 6, no. 3, pp. 31–40, 1997.
17. M. Han, *E-Bayesian estimation of failure probability and its application*, Mathematical and Computer Modelling, vol. 45, no. 9-10, pp. 1272–1279, 2007.
18. N. Jana and S. Bera, *Interval estimation of multicomponent stress-strength reliability based on inverse Weibull distribution*, Mathematics and Computers in Simulation, vol. 191, pp. 95–119, 2022.
19. R. Jiang, *A novel parameter estimation method for the Weibull distribution on heavily censored data*, Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, vol. 236, no. 2, pp. 307–316, 2022.
20. W. Jianhua, L. Dan and C. Difang, *E-Bayesian Estimation and Hierarchical Bayesian Estimation of the System Reliability Parameter*, Systems Engineering Procedia, vol. 3, pp. 282–289, 2012.
21. T. Kayal, Y.M. Tripathi, D. Kundu, and M. Rastogi, *Statistical inference of Chen distribution based on type I progressive hybrid censored Samples*, Statistics, Optimization and Information Computing, 2019.

22. J. g. Ma, L. Wang, Y.M. Tripathi, and M.K. Rastogi, *Reliability inference for stress-strength model based on inverted exponential Rayleigh distribution under progressive Type-II censored data*, Communications in Statistics-Simulation and Computation, pp. 1–25, 2021.
23. A. B. Mastor, O. Ngesa, J. Mung'atu, N. M. Alfaer and A. Z. Afify, *The Extended Exponential Weibull Distribution: Properties, Inference, and Applications to Real-Life Data*, Complexity, 2022.
24. H. Z. Muhammed and E. M. Almetwally, *Bayesian and non-Bayesian estimation for the bivariate inverse weibull distribution under progressive type-II censoring*, Annals of Data Science, pp. 1–32, 2020.
25. H. Okasha, Y. Lio, and M. Al-bassam, *On Reliability Estimation of Lomax Distribution under Adaptive Type-I Progressive Hybrid Censoring Scheme*, Journal of Mathematics, vol. 9, no. 22, pp. 2903, 2021.
26. J. Ren, and W. Gui, *Inference and optimal censoring scheme for progressively Type-II censored competing risks model for generalized Rayleigh distribution*, Computational Statistics, vol. 36, no. 1, pp. 479–513, 2021.
27. R. Valiollahi, A. Asgharzadeh, and M.Z. Raqab, *Estimation of $P(Y < X)$ for Weibull distribution under progressive Type-II censoring*, Communications in Statistics-Theory and Methods, vol. 42, no. 24, pp. 4476–4498, 2013.
28. H.R. Varian, *A Bayesian approach to real estate assessment*, Studies in Bayesian econometric and statistics in Honor of Leonard J. Savage, pp. 195–208, 1975.
29. H. Panahi, *Reliability analysis for stress-strength model from inverted exponentiated Rayleigh based on the hybrid censored sample*, International Journal of Quality & Reliability Management, 2022.
30. A. Pak, G.A. Parham and M. Saraj, *Inference for the Weibull distribution based on fuzzy data*, Revista Colombiana de Estadistica, vol. 36, no. 2, pp. 337–356, 2013.
31. O. Shojaee, H. Piriaei and M.Babanezhad, *E-Bayesian Estimations and Its E-MSE For Compound Rayleigh Progressive Type-II Censored Data*, Statistics, Optimization & Information Computing, vol. 10, no. 4, pp. 1056–1071, 2022.
32. O. Shojaee, R. Azimi and M.Babanezhad, *Empirical Bayes estimators of parameter and reliability function for compound Rayleigh distribution under record data*, American Journal of Theoretical and Applied Statistics, vol. 1, no. 1, pp. 12–15, 2012.
33. R. Vila and M. Niyazi Çankaya, *A bimodal Weibull distribution: properties and inference*, Journal of Applied Statistics, vol. 49, no. 12, pp. 3044–3062, 2022.