

Rao-Robson-Nikulin Goodness-of-fit Test Statistic for Censored and Uncensored Real Data with Classical and Bayesian Estimation

Salwa L. AlKhayat^{1,2}, Haitham M. Yousof³, Hafida Goual⁴, Talhi Hamida⁴, Mohamed S. Hamed^{5,3}
Aiachi Hiba⁴ and Mohamed Ibrahim^{6,*}

¹*Department of Statistics, Faculty of Science, University of Jeddah, Kingdom of Saudi Arabia*

²*Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Kafr El-Sheikh University, Egypt*

³*Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Benha 13518, Egypt*

⁴*Laboratory of probability and statistics LaPS, University Badji Mokhtar, Annaba, Algeria*

⁵*Department of Business Administration, Gulf Colleges, KSA*

⁶*Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia*

Abstract In this work, we provide a new Pareto type-II extension for censored and uncensored real-life data. With an emphasis on the applied elements of the model, some mathematical properties of the new distribution are deduced without excess. A variety of traditional methods, including the Bayes method, are used to estimate the parameters of the new distribution. The censored case maximum likelihood technique is also inferred. Using Pitman's proximity criteria, the likelihood estimation and the Bayesian estimation are contrasted. Three loss functions such as the generalized quadratic, the Linex, and the entropy functions are used to derive the Bayesian estimators. All the estimation techniques provided have been evaluated through simulated studies. The BB algorithm is used to compare the censored maximum likelihood method to the Bayesian approach. With the aid of two applications and a simulation study, the construction of the Rao-Nikulin-Robson (RRN) statistic for the new model in the uncensored case is explained in detail. Additionally, the development of the Rao-Robson-Nikulin statistic for the novel model under the censored situation is shown using data from two censored applications and a simulation study.

Keywords Validation; Modeling; Simulation; Rao-Robson-Nikulin Censorship.

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1. Introduction and genesis

Lomax [68] used his continuous heavy-tail probability distribution called the Pareto type-II (PII) distribution to represent internet traffic, actuarial science, company failure, and other real phenomena. The PII distribution, which is another name for the Lomax model, is widely used. Lomax [68] initially proposed the heavy-tailed distribution known as the PII distribution in his study of lifetime data on business failure. It has a variety of applications in business, economics, and actuarial science. In survival analysis, the distribution basically a shifted Pareto distribution is commonly used and applied. There are particular attempts being made to widen the PII distribution and its relevant expansions in applied statistics and related fields like engineering, for instance, wealth inequality, income, medical, biological investigations, and dependability. The PII model is used to simulate real data on business sizes (see Corbellini et al. [29]), type-II progressive censored competing risks data analysis (see Cramer and Schemiedt [31]), and income and wealth statistics (see Harris [51], Asgharzadeh and Valiollahi [19]).

*Correspondence to: *Mohamed Ibrahim (Email:miahmed@kfu.edu.sa). Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia

Many authors have recently thought about the PII model's extensions, for example: two extended PII models with applications were studied by Gomes et al. [41]. For the first time ever, the PII-G (PII-G) family was designed and evaluated by Cordeiro et al. [30]. The new PII-G family was adaptable enough to define numerous more significant PII models with numerous helpful special situations. A novel practical PII log-location regression model including influence diagnostics, residual analysis, and various actual data applications was put out by Altun et al. [17]. The Zografos-Balakrishnan PII (ZBPII) distribution was explored by Altun et al. [18], who then created the corresponding regression model for prediction and demonstrated numerous real-world applications using the new model. Introducing a brand-new Weibull PII (W PII) distribution is Nasir et al. [78]. The odd -Lindley PII (OLPII) model was investigated by Korkmaz et al. [64] using Bayesian analysis, conventional inference, and some novel practical characterization results. The Poisson Topp-Leone-PII model (a zero-truncated version) was introduced by Yousof et al. [101], who also provided several new valuable characterizations. A novel PII lifetime model based on the Topp-Leone family including regression models, characterizations, and applications was proposed by Yousof et al. [114]. The PII-PII (PII-PII) model was examined by Gad et al. [40], who also gave it a statistical description and an application. A new generalisation of the Lomax model including properties, copulas, and real data applications was described by Elgohari and Yousof [33]. Introducing a unique generalised mixture class of probabilistic models, Chesneau and Yousof [28]. With four applications, Elsayed and Yousof [37] expanded the PII model and created the Poisson generalised PII (PG PII) distribution. The Poisson exponentiated exponential Lomax was introduced by Aboraya et al. [2]. It has statistical properties, applications, copulas, and a variety of estimation techniques, such as method of the maximum likelihood, ordinary (and weighted) least squares methods, Cramér-von-Mises, Anderson-Darling, and left tail Anderson-Darling estimation techniques.

We will be satisfied with the expansions of the PII distribution that have been discussed in the statistical literature to achieve this, and we will point the reader to those references for more information. However, we'll focus on how the new distribution was created and why it was so important for us to publish this work in the lines that follow this introduction. The cumulative distribution function (CDF) of the standard PII model can be written as

$$G_{\zeta}(z) = 1 - (1 + z)^{-\zeta} \mid z \geq 0, \quad (1)$$

The corresponding PDF is given by

$$g_{\zeta}(z) = \zeta (1 + z)^{-\zeta-1} \mid z > 0, \quad (2)$$

where $\zeta > 0$ is the shape parameter. Details and many mathematical properties of the PII model can be found in Burr ([24], [25] and [26]) and Burr and Cislak [27]. Consider the CDF of the Rayleigh-G (R-G) family of distributions proposed by Yousof et al. [100], and by inverting the CDF of the R-G family and then substituting the CDF of the PII model, then the CDF of our proposed model, called the the inverted Rayleigh Pareto type-II (IR-PII) model is expressed as

$$F_{\zeta}(z) = \exp \left\{ - \left[(1 + z)^{\zeta} - 1 \right]^{-2} \right\}, \quad (3)$$

the PDF of the IR-PII is given by

$$f_{\zeta}(z) = 2 [(1 + z) - 1]^{-3} \exp \left\{ - [(1 + z) - 1]^{-2} \right\}. \quad (4)$$

The hazard rate function of the IR-PII distribution can be derived directly from

$$h_{\zeta}(z) = f_{\zeta}(z) / [1 - F_{\zeta}(z)].$$

In this paper, we present a construction of RRN statistic for the IR-PII model. Classical estimation and Bayesian estimation under different loss functions are considered. Simulation studies for comparing the classical methods are performed. Uncensored and censored applications under the RRN statistics are presented. In addition to the Bayes' method, we estimate the parameters of the new distribution using a number of traditional techniques, such as the maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method, and the right-tail Anderson-Darling method. Furthermore, a thorough simulation is used to evaluate the maximum likelihood method

in the censored scenario. The likelihood estimation and the Bayesian estimation are contrasted using Pitman's closeness criterion. The Bayesian estimation is given under various loss functions. To generate the Bayesian estimators, we use three loss functions: the generalised quadratic, the Linex, and the entropy. Many helpful details are supplied in their respective sections below. Through simulation experiments with specific parameters and controls, all of the estimating methods given have been assessed. These simulation studies are all stated in the paper at the proper places.

Two real data applications are presented under the uncensored case, the first being the strengths of glass fibres data and the second being the construction of RRN statistic for the IR-PII model under the uncensored case (see Shehata et al. [44]). A simulation study is performed to evaluate the RRN statistics under the uncensored case. A simulation study is presented for evaluating the RRN statistics under the censored case, and two real data applications are investigated under the censored case: the first data is the capacitor data (reliability data), and the second data is the lung cancer data. In addition, the construction of the RRN statistic for the IR-PII model under the censored case is presented in detail (medical data).

2. Literature review

The field of statistical modeling has witnessed substantial advancements through the development of novel probability distributions, goodness-of-fit tests, and estimation methods, as demonstrated by a series of studies that have explored their applications in various domains. Ibrahim et al. [60] extended the Lindley distribution with characterizations and different estimation methods. Yadav et al. [97] proposed the Topp-Leone-Lomax model, employing a modified RRN test for survival analysis. Abouelmagd et al. [3] developed the zero truncated Poisson Burr X family of distributions for count data, and Mansour et al. [70] generalized the exponentiated Weibull model with copulas and mathematical properties, followed by Mansour et al. [71] proposing a two-parameter Burr XII distribution for acute bone cancer data. Mansour et al. [72] introduced a new parametric life distribution with a Bagdonavičius-Nikulin test for censored validation, and Mansour et al. [73], [74] and [75] expanded some new models with copulas and estimation methods. Salah et al. [86] expanded the Fréchet model with a modified Bagdonavičius-Nikulin test for censored data, and Ibrahim et al. [55] developed a modified goodness-of-fit test for censored validation under a new Burr type XII distribution. Yousof et al. [102] proposed a new lifetime distribution with a modified Chi-square test for right-censored data, and Ibrahim et al. [53] expanded the Nadarajah Haghighi model with copula-based methods for censored and uncensored validation, and Ibrahim et al. [59] also validated the double Burr XII model using a RRN test with Bayesian and non-Bayesian estimation (see also [54]). Yadav et al. [99] validated the xgamma exponential model via the RRN test under complete and censored samples, and Yousof et al. [106] explored accelerated failure time estimation for a novel exponential model with characterizations. For more related modifications, censored and uncensored applications see Goual et al. [45], [43], [38], [112], [104], [46], [67], [113], [87], [103], [48]. Finally, Teghri et al. [95] proposed a two-parameter Lindley-frailty model for censored and uncensored schemes under various baseline models, and Shehata et al. [92] explored censored and uncensored RRN validation with classical and Bayesian estimation methods.

3. Construction of RRN statistic for the IR-PII model

Several strategies can be used to check whether the mathematical model is acceptable for the data from the observations in the case of complete data. The most popular test is based on the Pearson Chi-square statistic. These techniques, however, are not always applicable, particularly when the model's parameters are unclear or the data is censored. Since the middle of the 20th century, scholars have started to suggest changes to the statistics that are now available. Unknown parameters must be taken into account on the one hand, while censorship must be taken into account on the other. Nikulin ([79], [80], [81]) and Rao and Robson [82], respectively, proposed the statistics known as RRN statistics for the whole data. The RRN statistical test is a natural modification of the Pearson statistic and follows the chi-square distribution.

If the data is censored and the parameters are unknown, the classical test is not sufficient to verify the hypothesis H_0 , according to which a series of parameters come from the parameter family $F_\zeta(z)$. Bagdonavičius and Nikulin ([21] and [22]) and Bagdonavičius et al. [20] proposed a modification of the RRN statistic taking into account random right censorship. These new statistics were used to adjust observations to the Burr XII inverse Rayleigh (BXXIR) model (see Goual and Yousof [42]), odd Lindley exponentiated exponential (OLEE) (Goual et al. [44]), Topp-Leone-Lomax (TLLx) model (see Yadav et al. [98]) and new reciprocal Rayleigh extension (see Yousof et al. [105]). In this Section, we construct a modified chi-square test for the IR-PII model case of complete and censored data. To test the hypothesis

$$H_0 : P\{z_i \leq z\} = F_\nabla(z), \quad z \in \mathbb{R}, \quad \nabla = (\nabla_1, \nabla_2, \dots, \nabla_s)^T,$$

wherein z_1, z_2, \dots, z_n , an n -sample belong to a parametric family $F(z; \nabla)$ where ∇ represents the vector of unknown parameters, Nikulin [79], [80], [81]) and Rao and Robson [82] proposed the RRN statistic Y^2 where

$$Y^2(\widehat{\nabla}_{n,Z}) = Y^2(\widehat{\nabla}_{n,Z}) + \frac{1}{n} \mathbf{L}^T(\widehat{\nabla}_{n,Z})(\mathbf{I}(\widehat{\nabla}_{n,Z}) - \mathbf{J}(\widehat{\nabla}_{n,Z}))^{-1} \mathbf{L}(\widehat{\nabla}_{n,Z}),$$

$$Y^2(\nabla_{n,Z}) = \left(\frac{\nu_1 - np_1(\nabla_{n,Z})}{[np_1(\nabla_{n,Z})]^{\frac{1}{2}}}, \frac{\nu_2 - np_2(\nabla_{n,Z})}{[np_2(\nabla_{n,Z})]^{\frac{1}{2}}}, \dots, \frac{\nu_b - np_b(\nabla_{n,Z})}{[np_b(\nabla_{n,Z})]^{\frac{1}{2}}} \right)^T,$$

and $\mathbf{J}(\nabla_{n,Z})$ is the information matrix for the grouped data with

$$B(\nabla_{n,Z}) = \left[\frac{1}{\sqrt{p_i}} \frac{\partial}{\partial \mu} p_i(\nabla_{n,Z}) \right]_{r \times s} \Big|_{(i=1,2,\dots,b \text{ and } k=1,\dots,s)},$$

then

$$\mathbf{L}(\nabla_{n,Z}) = (\mathbf{L}_1(\nabla_{n,Z}), \dots, \mathbf{L}_s(\nabla_{n,Z}))^z,$$

where

$$\mathbf{L}_k(\nabla_{n,Z}) = \sum_{i=1}^r \frac{\nu_i}{p_i} \frac{\partial}{\partial \nabla_k} p_i(\nabla_{n,Z}),$$

where $\mathbf{I}_n(\widehat{\nabla}_{n,Z})$ represents the estimated Fisher information matrix and $\widehat{\nabla}_n$ is the maximum likelihood estimator of the parameter vector. The Y^2 statistic follows a distribution of chi-square χ_{b-1}^2 with $(b-1)$ degrees of freedom. Consider the Observations z_1, z_2, \dots, z_n , they are grouped in b subintervals $\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_b$ mutually disjoint $\mathbf{I}_j =]a_{j-1}; a_j]$; where $j = \overline{1; b}$. The limits a_j of the intervals \mathbf{I}_j are obtained such that

$$p_j(\nabla_{n,Z}) = \int_{a_{j-1}}^{a_j} f_\nabla(z) dx \Big|_{(j=1,2,\dots,b)}, a_j = F^{-1} \left(\frac{j}{b} \right) \Big|_{(j=1,\dots,b-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \dots, \nu_b)^T$ is the vector of frequencies obtained by the grouping of data in these \mathbf{I}_j intervals

$$\nu_j = \sum_{i=1}^n 1_{\{z_i \in \mathbf{I}_j\}} \Big|_{(j=1,\dots,b)}.$$

In order to check whether the data used in this paper is distributed according to the IR-PII model, in the case of unknown parameters, we construct the chi-square goodness-of-fit test by fitting the RRN statistics developed previously. After calculating the maximum likelihood estimator $\widehat{\nabla}_n$ for the unknown parameters of the IR-PII distribution on the data set, we use $\mathbf{I}_n(\widehat{\nabla}_n)$ as the estimated Fisher information matrix to provide all the components of the Y^2 statistic of our IR-PII model.

4. Classical estimation

4.1. Maximum likelihood method

A statistical method known as maximum likelihood estimation (MLE) is used to estimate the parameters of a probability density function that has been assumed in light of some observed data. To achieve this, a likelihood function is optimized to increase the probability of the observed data under the presumptive statistical model. The parameter space position where the likelihood function is optimum is known as the maximum likelihood estimate. A common method for drawing statistical conclusions is maximum likelihood because of its adaptive and transparent justification. In the event that the likelihood function is differentiable, the derivative test for maxima location may be applied.

The ordinary least squares estimator for a linear regression model optimized likelihood when it was assumed that all observed outcomes had Normal distributions with the same variance. The likelihood function's first-order conditions can occasionally be solved analytically. When creating confidence intervals, maximum likelihood estimates (MLEs) can be employed since they have favourable properties. Let w_1, w_2, \dots, w_n be a random sample from this distribution. The log-likelihood function is given by

$$\ell(\zeta) = \log \left[\prod_{i=1}^n f_{\zeta}(z_m) \right]$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\zeta)$ by differentiation. The score vector, $U_{\zeta} = \left(\frac{\partial}{\partial \zeta} \ell(\zeta) \right)^T$, are easy to derive. Below, we aim to obtain the maximum likelihood estimator of inverted Burr X-II (IR-P-II) distribution under type II censored data. Consider the n -sample (z_1, z_2, \dots, z_n) and a fixed constant m , we assume that the m -sample (z_1, z_2, \dots, z_m) generated from the inverted Burr X-II (IR-P-II). The likelihood function of this sample is

$$L_{\zeta}(z) = N \prod_{i=1}^m f_{\zeta}(z_i) [1 - F_{\zeta}(z_m)]^{n-m},$$

where $N = \frac{n!}{(n-m)!}$, using (3) and (4) we get

$$L_{\zeta}(z) = N 2^m \zeta^m \prod_{i=1}^m (1 + z_i)^{-2\beta-1} q_{\zeta}(z_i)^{-3} \exp[b_{\zeta}(z_i)] [1 - b_{\zeta}(z_m)]^{n-m},$$

where

$$q_{\zeta}(z_i) = (1 - (1 + z_i)^{-\zeta}), b_{\zeta}(z_i) = -[(1 + z_i)^{\zeta} - 1]^{-2},$$

then, the logarithm of the likelihood function is

$$\begin{aligned} \ln L_{\zeta}(z) = l_{\zeta}(z) &= \ln N + m \ln 2 + m \ln \zeta - (2\beta + 1) \sum_{i=1}^m \ln(1 + z_i) \\ &\quad - 3 \sum_{i=1}^m \ln q_{\zeta}(z_i) + \sum_{i=1}^m b_{\zeta}(z_i) + (n - m) \ln[1 - b_{\zeta}(z_m)]. \end{aligned}$$

Where

$$\begin{aligned} \frac{\partial l_{\zeta}(z)}{\partial \zeta} &= \frac{m}{\zeta} - 2 \sum_{i=1}^m \ln(1 + z_i) - 3 \sum_{i=1}^m \frac{1}{q_{\zeta}(z_i)} \left[\frac{\partial q_{\zeta}(z_i)}{\partial \zeta} \right] \\ &\quad + \sum_{i=1}^m \frac{1}{b_{\zeta}(z_i)} \left[\frac{\partial b_{\zeta}(z_i)}{\partial \zeta} \right] - (n - m) \frac{1}{1 - b_{\zeta}(z_m)} \left[\frac{\partial b_{\zeta}(z_m)}{\partial \zeta} \right], \end{aligned}$$

$$\frac{\partial q_{\zeta}(z_i)}{\partial \zeta} = \ln(1 + z_i)(1 + z_i)^{-\zeta},$$

and

$$\frac{\partial b_{\zeta}(z_i)}{\partial \zeta} = 2(\ln(1 + z_i)(1 + z_i)^{\zeta}[(1 + z_i)^{\zeta} - 1]^{-3}.$$

The solution of the non-linear equation $\frac{\partial l_{\zeta}(z)}{\partial \zeta} = 0$ is the maximum likelihood estimator $\hat{\zeta}_{MLE}$ of the parameter ζ . There is no analytical solution for this system, thus so we use the "R" language to obtain the approximate values of maximum likelihood estimator $\hat{\zeta}_{MLE}$.

4.2. The Cramér-von Mises method

The Cramér-von Mises estimates (CVME) of the parameters $\hat{\zeta}$ are obtained via minimizing the following expression with respect to the parameter ζ respectively, where

$$\text{CVM}_{(\nabla)} = \frac{1}{12}n^{-1} + \sum_{i=1}^n [F_{\zeta}(z_{i,n}) - C_{(i,n)}]^2,$$

where $C_{(i,n)} = \frac{2i-1}{2n}$ and

$$\text{CVM}_{(\nabla)} = \sum_{i=1}^n [F_{\zeta}(z_{i,n}) - C_{(i,n)}]^2.$$

Then, CVME of the parameter ζ are obtained by solving the following non-linear equation

$$\sum_{i=1}^n [F_{\zeta}(z_{i,n}) - C_{(i,n)}] V_{(\zeta)}(z_{[i:n]}, \nabla) = 0,$$

where $V_{(\zeta)}(z_{[i:n]}, \nabla)$ are the first derivatives of the CDF of the new distribution with respect to ζ respectively.

4.3. The Anderson-Darling method

The Anderson-Darling estimate (ADE) of ζ are obtained by minimizing the function

$$\text{ADE}(\nabla) = -n - n^{-1} \sum_{i=1}^n (2i-1) \left\{ \log F_{\zeta}(z_{i,n}) + \log [1 - F_{\zeta}(z_{[-i+1+n:n]})] \right\}.$$

The parameter estimates of ζ follow by solving the nonlinear equation $\frac{\partial}{\partial \zeta} [\text{ADE}(\nabla)] = 0$.

4.4. The right-tail Anderson-Darling method

The right-tail Anderson-Darling estimate (ADERTE) ζ are obtained by minimizing

$$\text{ADERT}(\nabla) = \frac{1}{2}n - 2 \sum_{i=1}^n F_{\zeta}(z_{i,n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \left\{ \log [1 - F_{\zeta}(z_{[-i+1+n:n]})] \right\}.$$

The parameter estimates of ζ follow by solving the nonlinear equations and $\frac{\partial}{\partial \zeta} [\text{ADERT}(\nabla)] = 0$.

4.5. Kolmogorov method

The Kolmogorov estimate (KE) $\hat{\zeta}$ of ζ are obtained by minimizing the function

$$K=K(\zeta) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F_{\zeta}(z_{i,n}), F_{\zeta}(z_{i,n}) - \frac{i-1}{n} \right\}.$$

5. Bayesian estimation under different loss functions

5.1. Prior and posterior distributions

As prior distributions, we assume the parameters ζ follow a non informative distribution as a prior:

$$\pi(\zeta) = \frac{1}{\zeta}$$

Where K is the normalizing constant, the posterior distribution of ζ is given by

$$\pi(\zeta/z) = K 2^m \zeta^{m-1} \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m}$$

$$K = \int_0^{+\infty} 2^m \zeta^{m-1} \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta$$

Next, we use the three loss functions namely the generalised quadratic (GQ), the Linex and the entropy functions to obtain the Bayesian estimators, gamma, p , and r are integers.

5.2. Bayesian estimators and their posterior risks

The Bayesian estimator under the generalised quadratic loss function are

$$\zeta_{GQ} = \frac{I_0^{+\infty} [m-1+\gamma]}{I_0^{+\infty} [m-2+\gamma]},$$

where

$$I_0^{+\infty} [m-1+\gamma] = \int_0^{+\infty} 2^m \zeta^{m-1+\gamma} \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta$$

and

$$I_0^{+\infty} [m-2+\gamma] = \int_0^{+\infty} N 2^m \zeta^{m-2+\gamma} \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta.$$

The corresponding posterior risk is then

$$PR(\zeta_{GQ}) = E_\pi(\zeta^{\gamma+1}) - 2\hat{\beta}_{GQ} E_\pi(\zeta^{-\gamma}) + \zeta_{GQ}^2 E_\pi(\zeta^{\gamma-1}).$$

Under the entropy loss function, we obtain the following estimator

$$\zeta_E = [K I_0^{+\infty}(\zeta_E)]^{1/p},$$

where

$$I_0^{+\infty}(\zeta_E) = \int_0^{+\infty} 2^m \zeta^{m-1-p} \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta.$$

The corresponding posterior risk is then

$$PR(\zeta_E) = PE_\pi(\ln(\zeta) - \ln(\zeta_E)),$$

finally under the Linex loss function, the Bayesian estimator is

$$\zeta_L = \frac{-1}{a} K \ln [I_0^{+\infty}(\zeta_L)],$$

where

$$I_0^{+\infty}(\zeta_L) = \int_0^{+\infty} 2^m \zeta^{m-1} \exp(-r\beta) \prod_{i=1}^m (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta$$

and The corresponding posterior risk is then

$$PR(\zeta_L) = r(\zeta_{GQ} - \zeta_L).$$

Since it is unlikely possible to obtain all these estimators analytically, so we suggest the use of the MCMC procedures to evaluate them, which is the following section.

6. Simulation studies

6.1. Simulation studies for comparing the calssical methods

In order to assess and compare the performance of the proposed calssical methods, we perform three Monte Carlo simulation study through three carefully selected different scenarios. The results of these three scenarios in Table 1 ($\zeta = 1.5$), Table 2 ($\zeta = 0.8$) and Table 3 ($\zeta = 0.5$). All simulation studies are performed using $N = 1000$ samples with different sample sizes $n = 50, 100, 200$ and 300 . Specifically, Table 1 gives the mean squared errors (MSEs) under $\zeta = 1.5$. Table 2 lists the MSEs under $\zeta = 0.8$. Table 3 presents the MSEs under $\zeta = 0.5$. By looking closely at the three tables, we can find the following results:

- The larger the sample size, the lower the MSE value for all estimation methods without exception.
- Through the first scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00396$. Through the first scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00065$. Through the first scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00036$ (see Table 1).
- Through the second scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.01123$. Through the second scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00171$. Through the second scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00039$ (see Table 2).
- Through the third scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.01712$. Through the third scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00281$. Through the third scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where $\text{MLE}_{(\zeta)} = 0.00161$ (see Table 3).
- We cannot, however, ignore the fact that all of the possible methods for estimation produced highly acceptable results, and that there is no real distinction between them provided the estimates all meet the requisite standards for consistency and effectiveness.
- We will pay close attention to contrasting the maximum likelihood technique with Bayes's method in the next sections based on the previously trustworthy results, which were all in favored of the maximum likelihood method. As previously stated, this conclusion is not deliberate; rather, it is based on the outcomes of the three

prior scenarios (see Tables 1, 2, and 3) that were the subject of research and analysis.

Table 1: MSEs under $\zeta = 1.2$

$n \downarrow$ & Method \rightarrow	MLE	CVM	KE	ADE	RTADE
50	0.00396	0.00452	0.00523	0.00436	0.00544
100	0.00202	0.00241	0.00253	0.00212	0.00266
200	0.00092	0.00119	0.00124	0.00108	0.00132
300	0.00065	0.00079	0.00080	0.00072	0.00088
500	0.00036	0.00046	0.00048	0.00042	0.00053

Table 2: MSEs under $\zeta = 2$

$n \downarrow$ & Method \rightarrow	MLE	CVM	KE	ADE	RTADE
50	0.01123	0.01256	0.01354	0.01137	0.01451
100	0.00554	0.00640	0.00740	0.00612	0.00740
200	0.00275	0.00324	0.00346	0.00300	0.00378
300	0.00171	0.00207	0.00222	0.00199	0.00244
500	0.00039	0.00122	0.00133	0.00116	0.00147

Table 3: MSEs under $\zeta = 2.5$

$n \downarrow$ & Method \rightarrow	MLE	CVM	KE	ADE	RTADE
50	0.01712	0.02043	0.02115	0.01776	0.02268
100	0.00813	0.01006	0.01074	0.00914	0.01176
200	0.00429	0.00553	0.00548	0.00465	0.00564
300	0.00281	0.00341	0.00336	0.00295	0.00377
500	0.00161	0.00190	0.00199	0.001751	0.00227

6.2. Comparing the likelihood estimation and the Bayesian estimation using Pitman's closeness criterion

In order to compare the performance of the proposed Bayes estimators with the MLEs, we perform a Monte Carlo simulation study assuming that $\zeta = 1, 5$ using $N = 5000$ samples of the type II censored model with different sample sizes $n = 10, 50, 200$ while $m = 8, 40, 160$ respectively, we obtain the following results. Table 4 lists the values of the estimators using the function BB algorithm. We remark here that the estimated values of ζ are close to the true values of the parameter especially with the increase in sample size n . Table 5, 6 and 7 give the Bayesian estimators and PR (in brackets) under the generalized quadratic loss function, the entropy loss function and Linex loss function respectively. Table 8 shows the Bayesian estimators and PR (in brackets) under the three loss functions. In Table 5, the estimation under the GQ loss function, we remark that the value $\gamma = 1$ gives the best posterior risk. Also, we obtain the smallest suitable posterior risk when n is high. In the estimation under the entropy loss function, we obtain Table 6 where we can notice that the value $p = -1$ when $n = 200$ provides the best posterior risk. We can notice clearly that the value $r = 1$ provides the best PR. Summing up, making a small comparison between the three loss functions, it is clear that the best results are obtained by the quadratic loss function, Table 8 illustrate those results in details. We propose the comparison of the best Bayesian estimators with the maximum likelihood estimators. For this purpose, we use the Pitman closeness criterion (see Pitman [96], Fuller [39] and Jozani [61] for more details).

Table 4: MLEs under the quadratic error (in brackets).

$N = 5000$	$n = 10$	$n = 50$	$n = 200$
m	8	40	160
ζ	0.6235(0.0056)	0.8389(0.0044)	0.9675(0.00223)

Table 5: Bayes estimators and with PR (in brackets) under the loss function of GQ.

γ	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_2 = 200$
	m	8	40	160
-2	ζ	1,342(0.0031)	1.4632(0.0021)	1.4743(0.0032)
-1, 5	ζ	1.321(0.0534)	1.3839(0.0711)	1.6926(0.0032)
-1	ζ	1.3998(0.0031)	1.4213(0.0070)	1.3421(0.0018)
-0, 5	ζ	1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
0, 5	ζ	1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
1	ζ	1.4999(0.0025)	1.4705(0.0021)	1.5012(0.0012)
1, 5	ζ	1.6132(0.0012)	1.4308(0.0070)	1.3412(0.0021)
2	ζ	1.2732(0.1004)	1.5711(0.1231)	1.6903(0.0003)

Table 6: Bayes estimators and PR (in brackets) under the entropy loss function.

	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
$p \downarrow$	$m \rightarrow$	8	40	160
-2		11232(0.0042)	1.5632(0.0081)	1.6743(0.0098)
-1, 5		1.7510(0.0095)	2.1839(0.0020)	1.7926(0.0077)
-1		1.0994(0.0089)	1.0888(0.0033)	1.2138(0.0018)
-0, 5		1.4768(0.1241)	1.5158(0.0070))	1.4991(0.1181)
0, 5		1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
1		1.2999(0.0825)	1.2701(0.711)	1.6432(0.0016)
1, 5		1.7131(0.0012)	1.0888(0.0070)	1.6432(0.0016)
2		1.4768(0.1241)	1.6754(0.1181)	1.7903(0.0033)

Table 7: Bayes estimators and PR (in brackets) under Linex loss function.

r	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	$m \rightarrow$	8	40	160
-2		1.3188(0.0699)	1.2839(0.009)	0.7034(0.011)
-1, 5		1.4407(0.0611)	1.4077(0.0661)	1.7060(0.0012)
-1		1.4177(0.0072)	1.3633(0.0073)	0.7051(0.0003)
-0, 5		0.6493(0.0308)	07037(0.0009)	0.8755(0.319)
0, 5		1.8895(0.0729)	1.8998(0.0008)	1.9814(0.0733)
1		1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0001)
1, 5		1,6037(0.0009)	1.3055(0.319)	1.5491(0.0308)
2		1.4239(0.0199)	1.3881(0.0303)	1,7059(0.0003)

Table 8: Bayes estimators (PR in brackets) under the three loss functions.

	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_3 = 200$
Loss function \downarrow	$m \rightarrow$	8	40	160
$GQ _{\gamma=1}$		1.4999(0.0534)	1.4705(0.711)	1.5012(0.0012)
$Entropy _{p=-1}$		1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$Linex _{r=1}$		1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0733)

Definition 1

An estimator θ_1 of a parameter θ dominates another estimator θ_2 in the sense of Pitman's closeness criterion if, for all $\theta \in \Theta$,

$$P_{\theta}[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5.$$

Table 9 presents the results of the probabilities of Pitman method which, for sure, helps us in comparing the Bayesian and MLE estimator under the three loss functions when for $\gamma = 1, p = 0.5$ and $r = 1.5$. According to definition 1, when the probability is greater than 0.5, the Bayesian estimators are better than the MLE estimators. Then we see that the Bayesian estimators of the parameters are superior to the MLE in terms of this criterion. Also the generalized quadratic loss function has the best values in comparison with the other two loss functions with $0.798|_{n=200, m=160}$.

Table 9: Pitman comparison of the estimators.

	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_3 = 200$
Loss function ↓	$m \rightarrow$	8	40	160
GQ $_{ \gamma=1}$		0.745	0.744	0.798
Entropy $_{ p=-1}$		0.656	0.582	0.567
Linex $_{ r=1}$		0.712	0.544	0.544

7. Uncensored distributional validation**7.1. Uncensored simulation study under the RRN statistics Y^2**

In order to support the results obtained in this work, we conducted an in-depth study through numerical simulation. Therefore, in order to test the null hypothesis H_0 that the sample belongs to the IR-PII model, we respectively calculated n statistical samples, which are the N statistics of 15000 simulated samples with sizes $n = 25, n = 50, n = 150, n = 400$ and $n = 700$. For different theoretical levels ($\epsilon = 0.01, 0.02, 0.05, 0.1$), we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \leq \chi_{\epsilon}^2(b-1)$. The corresponding empirical and theoretical levels are represented in Table 10. It can be seen that the calculated empirical level value is very close to its corresponding theoretical level value. Therefore, we conclude that the recommended test is very suitable for the IR-PII distribution.

Table 10: Empirical levels and corresponding theoretical levels under the artificial data.

$n \downarrow \& \epsilon \rightarrow$	$\epsilon_1 = 0.01$	$\epsilon_2 = 0.02$	$\epsilon_3 = 0.05$	$\epsilon_4 = 0.1$
$n_1 = 25$	0.9925	0.9838	0.9520	0.9031
$n_2 = 50$	0.9921	0.9829	0.9515	0.9026
$n_3 = 150$	0.9916	0.9821	0.9504	0.9019
$n_4 = 400$	0.9909	0.9810	0.9502	0.9005
$n_5 = 700$	0.9903	0.9804	0.9501	0.9002

7.2. Uncensored applications under the RRN statistics Y^2

7.2.1. Example 1: Heat exchanger tube crack data The crack data is taken from the book by Meeker and Escobar [76] and represents inspections performed at 8 selected times until cracks appeared in 167 identical turbine parts.

Time of inspection	186	606	902	1077	1209	1377	1592	1932
Number of fans found to have cracks	5	16	12	18	18	2	6	17

We use RRN Statistics previously obtained, to test the null hypothesis that these data are adjusted by our IR-PII distribution. Utilizing R programming and the BB algorithm (see Ravi [85]), we determine the MLE is

$$\hat{\nabla} = \hat{\zeta} = 0.8945.$$

At that point, the estimated Fisher information matrix is:

$$\mathbf{I}(\hat{\nabla}) = (5.60048).$$

Then, we derive the value of $Y^2 = 19.99007$. For significance level $\epsilon = 0.05$ and the critical value $\chi_{0.01}^2(12) = 21.02607$. Then

$$Y^2 = 19.99007 < \chi_{0.05}^2(12) = 19.99007.$$

The RRN statistic for this model (Y^2) is smaller than the critical value, which allows us to say that these data appropriately correspond to the IR-PII model.

7.2.2. Example 2: Strengths of glass fibers This data set consists of 100 carbon fiber fracture stresses (in Gba) given by Nichols and Padgett [77]. Assuming that our IR-PII model can fit the strength data of 1.5cm glass fiber, we can use the BB algorithm to find the MLE value of the parameter ∇ is

$$\hat{\nabla} = \hat{\zeta} = 2.89364.$$

Using the $\hat{\nabla}$ value, we can estimate and give the Fisher information matrix as follow:

$$\mathbf{I}(\hat{\nabla}) = (1.0661836).$$

After the calculation, we performed the RRN statistical test, and the critical values where

$$Y^2 = 12.000427 < \chi_{0.05}^2(6) = 12.59159.$$

What we can be sure of is that the 1.5 cm glass fiber data can be modeled satisfactorily with our IR-PII distribution.

8. Censored distributional validation

We apply the statistic type test based on a version of the RRN statistic given by Bagdonavičius and Nikulin ([21], [22]) and Bagdonavičius et al. [20] to confirm the sufficiency of the IR-PII model when the parameters are unknown and the data are censored. We adapt this test for a IR-PII model (the failure rate z_i follows an IR-PII distribution). Let us consider the composite hypothesis

$$H_0 : F(z) \in F_0 = \{F_0(z, \nabla), z \in R^1, \nabla \in \Theta \subset R^s\},$$

the survival function and the cumulative hazard function of the IR-PII distribution are:

$$S_{\text{IR-PII}}(z, \nabla) \exp \left\{ - \left[(1+z)^\zeta - 1 \right]^{-2} \right\}.$$

$$\Lambda_{\text{IR-PII}}(z, \nabla) = - \left[(1+z)^\zeta - 1 \right]^{-2}$$

Under such choice of intervals we have a constant value of $e_j(Z) = E_j(Z)/k$ for any j . There is no explicit form of the inverse hazard function of IR-PII distribution, so we can estimate intervals by iterative method. Let us dividing a finite time interval $[0, \tau]$ into $k > s$ smaller intervals $\mathbf{I}_j = (a_{j-1}, a_j]$, where τ is the maximum time of the study and $0 < a_0 < a_1 \dots < a_{k-1} < a_k = +\infty$. If Λ^{-1} is the inverse of cumulative hazard function Λ , $\hat{\nabla}$ is the maximum likelihood estimator of the parameter ∇ and $z_{(i)}$ is the i^{th} element in the ordered statistics $(z_{(1)}, \dots, z_{(n)})$,

we can give the estimated $\widehat{a_{j(Z)}}$ as:

$$\widehat{a_{j(Z)}} = \Lambda^{-1} \left((E_j(Z) - \sum_{l=1}^{i-1} \Lambda(z_{(l)}, \widehat{\nabla})) / (n - i + 1), \widehat{\theta} \right), \quad \widehat{a_k} = z_{(n)}|_{(j=1, \dots, k)},$$

where

$$E_j(Z) = \Lambda(\widehat{a_{j(Z)}}, \widehat{\nabla})(n - i + 1) + \sum_{l=1}^{i-1} \Lambda(z_{(l)}, \widehat{\nabla}) = \sum_{i: z_i > a_j} (\Lambda(a_j \wedge z_i, \widehat{\nabla}) - \Lambda(a_{j-1}, \widehat{\nabla})),$$

$$E_k(Z) = \sum_{i=1}^n \Lambda(z_i, \widehat{\nabla}).$$

and a_j are random data functions such as the k intervals chosen have equal expected numbers of failures $e_j(Z)$. For hypothesis H_0 , the test can be based on the statistic

$$Y_\epsilon^2(n, r) = \mathbf{Z}^T \widehat{\Sigma}^{-1} \mathbf{Z},$$

where $\mathbf{Z} = (Z_1, \dots, Z_k)^T$ and

$$\mathbf{Z}_{j,n} = \frac{1}{\sqrt{n}} (\mathbf{U}_j(Z) - e_j(Z))|_{(j=1, 2, \dots, k)}$$

and $\mathbf{U}_j(Z)$ represent the numbers of observed failures in these intervals. The test statistic of Bagdonavičius and Nikulin (2011a,b) and Bagdonavičius et al. (2013) can be written as:

$$Y_\epsilon^2(n, r) = \sum_{j=1}^k \frac{1}{\mathbf{U}_j(Z)} (\mathbf{U}_j(Z) - e_j(Z))^2 + \mathbf{Q},$$

where

$$\begin{aligned} \widehat{\Sigma}^{-1} &= \widehat{\mathbf{V}}^{-1} + \widehat{\mathbf{C}}^{-1} \widehat{\mathbf{V}}^z \widehat{\psi}^{-1} \widehat{\mathbf{C}} \widehat{\mathbf{V}}^{-1}, \\ \widehat{\psi} &= [\widehat{g}_{ll'}]_{s \times s} = \widehat{\mathbf{I}} - \widehat{\mathbf{C}} \widehat{\mathbf{V}}^{-1} \widehat{\mathbf{C}}^z, \\ \widehat{\mathbf{C}}_{lj} &= \frac{1}{n} \sum_{i: z_i \in \mathbf{I}_j} \rho_i \frac{\partial}{\partial \nabla} \ln [\lambda_i(z_i, \widehat{\nabla})], \\ \mathbf{U}_j(Z) &= \sum_{i: z_i \in \mathbf{I}_j} \rho_i, \\ \widehat{\mathbf{V}}_j &= n^{-1} \mathbf{U}_j(Z), \\ \mathbf{Q} &= \widehat{\mathbf{M}}^z \widehat{\psi}^{-1} \widehat{\mathbf{M}}, \widehat{\mathbf{M}}_l = \sum_{j=1}^k \widehat{\mathbf{C}}_{lj} \widehat{\mathbf{V}}_j^{-1} \mathbf{Z}_{j,n}, l, l' = 1, 2, \dots, s, \\ \widehat{i}_{ll'} &= n^{-1} \sum_{i=1}^n \rho_i \frac{\partial}{\partial \nabla_l} \ln [\lambda_i(z_i, \widehat{\nabla})] \frac{\partial}{\partial \nabla_{l'}} \ln [\lambda_i(z_i, \widehat{\nabla})] \end{aligned}$$

and

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^k \widehat{\mathbf{C}}_{lj} \widehat{\mathbf{C}}_{l'j} \widehat{A}_j^{-1}.$$

We calculate all the elements of the statistic Y^2 for the IR-PII model. The limit distribution of the statistic $Y_\epsilon^2(n, r)$ is chi-square and its degree of freedom is $df = \text{rank}(\Sigma) = \text{trace}(\Sigma^{-1} \Sigma)$. If ψ is non-degenerate, then $df = k$. If $Y^2 > \chi_\epsilon^2(df)$ (where $\chi_\epsilon^2(df)$ is the quantile of chi-square with df degrees of freedom), then the approximate

significance level ϵ is rejected Hypothesis. The principal element of the $Y_\epsilon^2(n, r)$ statistic test of the IR-P-II model is the matrix \widehat{C}_{lj} given as

$$\widehat{C}_{lj} = \frac{1}{n} \sum_{i: z_i \in I_j} \rho_i \frac{\partial}{\partial \nabla} \ln [\lambda(z_i, \widehat{\nabla})].$$

We expect that this test will catch the interest of scholars working in applied statistics across a range of disciplines. This test will enable researchers to determine whether the statistical distribution is appropriate for modelling particular right censored data, as mentioned in the research's main body. We advise people interested in statistical hypothesis tests to apply this test to different engineering, medical, agricultural, and actuarial data as well as to other probability distributions. We anticipate a significant advancement in statistical tests over the coming years, especially given the growing requirement for practical research that keeps up with current global alterations. Additionally, additional new tests based on this test may be developed, and these new tests may be simpler to use and include into statistical modeling.

8.1. Censored simulation study under the RRN statistics Y^2

In order to test the sample belongs to the null hypothesis H_0 of the IR-P-II model, it is assumed that the generated sample ($N = 15000$) is censored at 25% and $df = 5$ grouping intervals. For different theoretical levels ($\epsilon = 0.01, 0.02, 0.05, 0.1$), when $Y_\epsilon^2(n, r) \leq \chi_\epsilon^2(r - 1)$, we calculate the average value of the non-rejection numbers of the null hypothesis. Table 11 displays the relevant theoretical and empirical levels. The computed empirical level value is quite similar to the matching theoretical level value, as can be seen in Table 11. As a result, we draw the conclusion that the customised test is ideal for the IR-P-II model.

Table 11: The empirical significance levels and corresponding significance theoretical levels.

$n \downarrow \& \epsilon \longrightarrow$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.1$
$n_1 = 25$	0.9933	0.9769	0.9530	0.9041
$n_2 = 50$	0.9924	0.9780	0.9523	0.9031
$n_3 = 150$	0.9915	0.9796	0.9513	0.9017
$n_4 = 400$	0.9912	0.9798	0.9506	0.9009
$n_5 = 700$	0.9906	0.9802	0.9502	0.9004

Based on these results, we find that the empirical significance level of the Y^2 statistics corresponds to the level of the theoretical level of the chi-square distribution on df degrees of freedom. For that reason, it can be said that the proposed test can rightly fit the censored data from the IR-P-II distribution.

8.2. Censored applications under the RRN statistics Y^2

8.2.1. Capacitor data Reliability data set A set of data for basic reliability analyses, drawn from Meeker and Escobar's book (see Escobar [76]). . data on glass capacitor longevity as a function of voltage and operating temperature from a factorial experiment. Each combination of temperature and voltage had 8 capacitors. Testing was stopped after the fourth failure for each combination ($n=64$ and censored items=32). This data is available in the Sirvival package of R. Assuming that these data are distributed according to the IR-P-II distribution, the maximum likelihood estimator $\widehat{\nabla}$ of the parameter vector ∇ is $\widehat{\nabla} = \widehat{\zeta} = 2.07514$. We choose $df = 8$ a number of classes. The element of the statistic test Y^2 are presented as:

$\widehat{a_j(Z)}$	263.68	342.91	443.99	560.76	613.25	949.80	1091.22	1110.83
$\widehat{U_j(Z)}$	6	5	9	11	6	8	9	10
$e_j(Z)$	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088

The estimated matrix \widehat{C}_{lj} and fisher's estimated matrix $\widehat{I}_{(1 \times 1)}$ are:

$$\widehat{C}_{lj} = \begin{pmatrix} -0.604587 & -0.47777 & 0.703498 & 0.39953 & 0.42076 & -0.71194 & 0.95151 & 0.312846 \end{pmatrix}$$

and

$$\widehat{I}_{(1 \times 1)} = (2.8467512).$$

Then, we evaluate the value of the statistical test $Y_{0.05}^2(64, 8) = 14.00387$. The critical value is $\chi_{0.05}^2(8) = 15.50731 > Y_{0.05}^2(64, 8)$. We can come to the conclusion that the life data of glass capacitors are adjusted with the IR-PII model.

8.2.2. Lung cancer data set The lung cancer data given by Loprinzi et al. [69] from the North Central cancer treatment group, study the survival in patients ($n = 228$ and number of the censored items = 63) with advanced lung cancer and their Performance scores rate how well the patient can perform usual daily activities. We can estimate the vector parameter $\widehat{\nabla}$ by using the maximum likelihood estimation method as: $\widehat{\nabla} = (\widehat{\zeta}) = 1.428136$, if we suppose that this data are distributed according to IR-PII distribution. We use $df = 8$ as a number of classes. The test statistic Y^2 elements are presented as following:

$\widehat{a_j(Z)}$	60.612	109.546	168.197	201.662	267.207	374.183	651.111	1023.4391
$\widehat{U_j(Z)}$	17	19	22	28	31	48	43	20
$e_j(Z)$	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094

The estimated matrix \widehat{C}_{lj} and fisher's estimated matrix $\widehat{I}_{(1 \times 1)}$ are

$$\widehat{C}_{lj} = \begin{pmatrix} 0.377451 & -0.400044 & -0.94775 & 0.699978 & -0.39765 & 0.269199 & 0.738495 & 0.367344 \end{pmatrix}$$

and

$$\widehat{I}_{(1 \times 1)} = (4.558107).$$

The critical value of the chi-squared test is $\chi_{0.05}^2(8) = 15.50731$. Using the previous results, we find that the calculated statistic of the proposed test is $Y_{0.05}^2(228, 8) = 14.81071$. Since the tabulated value of the $Y_{0.05}^2(228, 8)$ statistic is greater than the calculated value, then we can say that our hypothesis H_0 is accepted. Which leads us to conclude that the Lung cancer data can follow the IR-PII distribution with a 5% risk of error.

9. Conclusions

In this paper, we introduce a new extension of the Pareto type-II distribution called the inverted Rayleigh Pareto type-II model. By demonstrating an emphasis on the applicable features of the model, some mathematical properties of the new distribution are determined without excess. There are three alternative methods presented for characterizing the IR-PII distribution: utilizing two truncated moments, using the hazard function, and using the conditional expectation of a random variable function. The maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method, the right-tail Anderson-Darling method, and the Bayes' method are only a few of the traditional techniques used to estimate the parameters of the new distribution. Furthermore, the censored case maximum likelihood method is deduced in detail and evaluated using a thorough simulation. The likelihood estimation and the Bayesian estimation are compared using Pitman's proximity criterion. Under different loss functions, the Bayesian estimation is provided. Three loss functions including the extended quadratic, the Linex, and the entropy are used to produce the Bayesian estimators, and many useful details are provided. All of the provided estimate methods have been evaluated using simulation experiments with particular settings and controls. Every one of these simulation studies is mentioned in the study in the appropriate areas.

The BB algorithm for process estimation under censored samples is used to compare the Bayesian technique and the censored maximum likelihood method. It is shown in detail how the RRN statistic is created for the IR-P-II model in the uncensored situation. Two real data applications are shown under the uncensored scenario: the first data is the strengths of fibreglass, and the second data is the heat exchanger tube crack data. A simulation research is carried out to assess the RRN statistic under the uncensored situation. Additionally, a simulation study for evaluating the RRN statistic for the new model under the censored case is presented, along with two real data applications that are investigated under the censored case. The first data is the capacitor data (reliability data), and the second data is the lung cancer data (medical data).

In the context of the distributional validity and statistical hypothesis tests for the uncensored data, the RRN statistic, which is based on the uncensored maximum likelihood estimators on initial non-grouped data, is considered under the IR-P-II model. The RRN statistic is assessed under two uncensored data sets and the following results can be highlighted:

- For the uncensored heat exchanger tube crack data, $Y^2 = 19.99007 < \chi_{0.05}^2(12) = 19.99007$. Hence, we can come to the conclusion that the life data of the uncensored heat exchanger tube crack data are adjusted with the IR-P-II model. In other words, we can accept the null hypothesis that the right censored capacitor data set follows the IR-P-II distribution.
- For the uncensored strengths of glass fibers data set, $Y^2 = 12.000427 < \chi_{0.05}^2(6) = 12.59159$. Hence, we can come to the conclusion that the life data of the uncensored strengths of glass fibers data set are adjusted with the IR-P-II model. In other words, we can accept the null hypothesis that the right censored lung cancer set follows the IR-P-II distribution.

A modified RRN statistic, which is based on the censored maximum likelihood estimators on initial non-grouped data, is taken into consideration under the IR-P-II model in the context of the distributional validity and statistical hypothesis testing for the censored data. The updated RRN statistic is evaluated using two data sets with right censoring, and the following findings stand out:

- For the right censored capacitor data set, $\chi_{0.05}^2(8) = 15.50731 > Y_{0.05}^2(64, 8) = 14.00387$. Hence, we can come to the conclusion that the life data of capacitor data set are adjusted with the IR-P-II model. In other words, we can accept the null hypothesis that the right censored capacitor data set follows the IR-P-II distribution.
- For the right censored lung cancer data set, $\chi_{0.05}^2(8) = 15.50731 > Y_{0.05}^2(228, 8) = 14.81071$. Hence, we can come to the conclusion that the life data of right censored lung cancer data set are adjusted with the IR-P-II model. In other words, we can accept the null hypothesis that the right censored lung cancer set follows the IR-P-II distribution.

We hope that this new IR-P-II model (as flexible extension of P-II model) will attract the attention of applied statistical researchers in various fields such as the risk analysis under insurance and financial data set (see, for more applications and details, Ahmed et al. [6], Alizadeh et al. [12], Alizadeh et al. [13], Aljadani et al. [15], Elbatal et al. [32], Hamed et al. [47], Hashempour et al. [52], Ibrahim et al. [58], Khedr et al. [63], Korkmaz et al. [65], Rasekhi et al. [83], Shrahili et al. [94], Yousof et al. [107], Yousof et al. [109], Yousof et al. [115], Yousof et al. [111] and Yousof et al. [116]). As mentioned in the body of the paper, this test will help researchers determine whether the statistical distribution is suitable for modeling specific right censored data. We suggest to those interested in statistical hypothesis tests to apply this test to other probability distributions, as well as to various engineering, medical, agricultural and actuarial data. On the other hand, we hope that this test will receive more attention for application in new topics, including reliability applications, continuous families and discrete families (see Yousof et al. [108], Shehata et al. [88], Alkhayyat et al. [16], Ahmed et al. [7], Aidi et al. [8], Shehata et al. ([90], [91], [93], [89]), Elgohari and Yousof ([34],[35],[36]), Alizadeh et al. [14], Korkmaz et al. [66], Yousof et al. [110], Rasekhi et al. ([84]), Aboraya et al. [1] and Hamedani et al. [49], [50], [57], [56], [5], [62], [9], [11], [23], [4] and [10]). We believe that the next few years may witness a major development in statistical tests, especially with the increasing demand for applied studies that keep pace with recent global changes. Moreover, more new tests based on this test

can be introduced and the new tests may be easier to apply and statistical modeling. Finally, if researchers succeed in providing more tests, it will be necessary to provide new statistical packages ready to fit the new statistical tests.

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REFERENCES

1. Aboraya, M., Ali, M. M., Yousof, H. M. and Ibrahim, M. (2022). A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data. *Pakistan Journal of Statistics and Operation Research*, 18(2), 437-463. <https://doi.org/10.18187/pjsor.v18i2.3938>
2. Aboraya, M., Ali, M. M., Yousof, H. M. and Ibrahim, M. (2022). A novel Lomax extension with statistical properties, copulas, different estimation methods and applications. *Bulletin of the Malaysian Mathematical Sciences Society*, (2022) <https://doi.org/10.1007/s40840-022-01250-y>
3. Abouelmagd, T. H. M., Hamed, M. S., Hamedani, G. G., Ali, M. M., Goual, H., Korkmaz, M. C., & Yousof, H. M. (2019). The zero truncated Poisson Burr X family of distributions with properties, characterizations, applications, and validation test. *Journal of Nonlinear Sciences and Applications*, 12(5), 314-336.
4. Abouelmagd, T. H. M., Hamed, M. S., Almamy, J. A., Ali, M. M., Yousof, H. M., & Korkmaz, M. C. (2019). Extended Weibull log-logistic distribution. *Journal of Nonlinear Sciences and Applications*, 12(8), 523-534.
5. Abonongo, J., Abonongo, A. I. L., Aljadani, A., Mansour, M. M., & Yousof, H. M. (2025). Accelerated failure model with empirical analysis and application to colon cancer data: Testing and validation. *Alexandria Engineering Journal*, 113, 391-408.
6. Ahmed, B., Ali, M. M. and Yousof, H. M. (2022). A Novel G Family for Single Acceptance Sampling Plan with Application in Quality and Risk Decisions, *Annals of Data Science*, 10.1007/s40745-022-00451-3
7. Ahmed, B., Chesneau, C. Ali, M. M. and Yousof, H. M. (2022). Amputated Life Testing for Weibull Reciprocal Weibull Percentiles: Single, Double and Multiple Group Sampling Inspection Plans with Applications, *Pakistan Journal of Statistics and Operation Research*, 18(4), 995-1013.
8. Aidi, K., Butt, N. S., Ali, M. M., Ibrahim, M., Yousof, H. M. and Shehata, W. A. M. (2021). A Modified Chi-square Type Test Statistic for the Double Burr X Model with Applications to Right Censored Medical and Reliability Data. *Pakistan Journal of Statistics and Operation Research*, 17(3), 615-623.
9. Ali, M. M., Ali, I., Yousof, H. M., & Ahmed, M. I. M. (Eds.). (2023). *G Families of Probability Distributions: Theory and Practices*. CRC Press, Taylor & Francis.
10. Ali, M. M., Imon, R., Ali, I. and Yousof, H. M. (2025). *Statistical Outliers and Related Topics*, CRC Press, Taylor & Francis.
11. Ali, M. M., Yousof, H. M., & Ibrahim, M. (2021). A new Lomax type distribution: Properties, copulas, applications, Bayesian and non-Bayesian estimation methods. *Int. J. Stat. Sci*, 21, 61-104.
12. Alizadeh, M., Afshari, M., Contreras-Reyes, J. E., Mazarei, D., & Yousof, H. M. (2024). The Extended Gompertz Model: Applications, Mean of Order P Assessment and Statistical Threshold Risk Analysis Based on Extreme Stresses Data. *IEEE Transactions on Reliability*, doi: 10.1109/TR.2024.3425278.
13. Alizadeh, M., Afshari, M., Ranjbar, V., Merovci, F. and Yousof, H. M. (2023). A novel XGamma extension: applications and actuarial risk analysis under the reinsurance data. *São Paulo Journal of Mathematical Sciences*, 1-31.
14. Alizadeh, M., Lak, F., Rasekhi, M., Ramires, T. G., Yousof, H. M., & Altun, E. (2018). The odd log-logistic Topp–Leone G family of distributions: heteroscedastic regression models and applications. *Computational Statistics*, 33(3), 1217-1244.
15. Aljadani, A., Mansour, M. M., & Yousof, H. M. (2024). A Novel Model for Finance and Reliability Applications: Theory, Practices and Financial Peaks Over a Random Threshold Value-at-Risk Analysis. *Pakistan Journal of Statistics and Operation Research*, 20(3), 489-515. <https://doi.org/10.18187/pjsor.v20i3.4439>
16. Alkhayyat, S. L., Mohamed, H. S., Butt, N. S., Yousof, H. M., & Ali, E. I. (2023). Modeling the Asymmetric Reinsurance Revenues Data using the Partially Autoregressive Time Series Model: Statistical Forecasting and Residuals Analysis. *Pakistan Journal of Statistics and Operation Research*, 425-446.
17. Altun, E., Yousof, H. M. and Hamedani, G. G. (2018a). A new log-location regression model with influence diagnostics and residual analysis. *Facta Universitatis, Series: Mathematics and Informatics*, 33, 417-449.
18. Altun, E., Yousof, H. M., Chakraborty, S. and Handique, L. (2018b). Zografos-Balakrishnan Lomax distribution: regression modeling and applications. *International Journal of Mathematics and Statistics*, 19(3), 46-70.
19. Asgharzadeh, A. and Valiollahi, R. (2011). Estimation of the scale parameter of the Lomax distribution under progressive censoring, *International Journal for Business and Economics* 6, 37-48.
20. Bagdonavičius, V., Levulienė, R., J., and Nikulin, M. (2013). Chi-squared goodness-of-fit tests for parametric accelerated failure time models. *Communications in Statistics-Theory and Methods*, 42, 15, 2768-2785.
21. Bagdonavičius, V. and Nikulin, M. (2011a). Chi-squared goodness-of-fit test for right censored data. *International Journal of Applied Mathematics and Statistics*, 24, 30-50.
22. Bagdonavičius, V. and Nikulin, M. (2011b). Chi-squared tests for general composite hypotheses from censored samples. *Comptes Rendus Mathématique*, 349(3-4), 219-223.
23. Bhatti, F. A., Hamedani, G. G., Korkmaz, M. Ç., Yousof, H. M., & Ahmad, M. (2023). On the new modified Burr XII distribution: Development, properties, characterizations and applications. *Pakistan Journal of Statistics and Operation Research*, 327-348.
24. Burr, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13, 215-232.

25. Burr, I. W. (1968). On a general system of distributions, III. The simpler range. *Journal of the American Statistical Association*, 63, 636-643.
26. Burr, I. W. (1973). Parameters for a general system of distributions to match a grid of 3 and 4. *Communications in Statistics*, 2, 1-21.
27. Burr, I. W. and Cislak, P. J. (1968). On a general system of distributions: I. Its curve-shaped characteristics; II. The sample median. *Journal of the American Statistical Association*, 63, 627-635.
28. Chesneau, C. and Yousof, H. M. (2021). On a special generalized mixture class of probabilistic models. *Journal of Nonlinear Modeling and Analysis*, 3, 71-92.
29. Corbellini, A., Crosato, L., Ganugi, P. and Mazzoli, M. (2010). Fitting Pareto II distributions on firm size: Statistical methodology and economic puzzles. In *Advances in Data Analysis* (pp. 321-328). Birkhäuser Boston.
30. Cordeiro, G. M., Yousof, H. M., Ramires, T. G. and Ortega, E. M. (2018). The Burr XII system of densities: properties, regression model and applications. *Journal of Statistical Computation and Simulation*, 88(3), 432-456.
31. Cramer, E. and Schemiedt, A.B. (2011). Progressively type-II censored competing risks data from Lomax distribution, *Computational Statistics and Data Analysis* 55, 1285-1303.
32. Elbatal, I., Diab, L. S., Ghorbal, A. B., Yousof, H. M., Elgarhy, M. and Ali, E. I. (2024). A new losses (revenues) probability model with entropy analysis, applications and case studies for value-at-risk modeling and mean of order-P analysis. *AIMS Mathematics*, 9(3), 7169-7211.
33. Elgohari, H. and Yousof, H. M. (2020). A generalization of Lomax distribution with properties, copula and real data applications. *Pakistan Journal of Statistics and Operation Research*, 16, 697-711.
34. Elgohari, H. and Yousof, H. M. (2021b). A New Extreme Value Model with Different Copula, Statistical Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 17(4), 1015-1035. <https://doi.org/10.18187/pjsor.v17i4.3471>
35. Elgohari, H. and Yousof, H. M. (2020c). New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. *Statistics, Optimization & Information Computing*, 8(4), 972-993. <https://doi.org/10.19139/soic-2310-5070-1036>
36. Elgohari, H., Ibrahim, M. and Yousof, H. M. (2021). A New Probability Distribution for Modeling Failure and Service Times: Properties, Copulas and Various Estimation Methods. *Statistics, Optimization & Information Computing*, 8(3), 555-586.
37. Elsayed, H. A. and Yousof, H. M. (2021). Extended poisson generalized Burr XII distribution. *Journal of Applied Probability and Statistics*, 16(1), 01-30.
38. Emam, W.; Tashkandy, Y.; Goual, H.; Hamida, T.; Hiba, A.; Ali, M.M.; Yousof, H.M.; Ibrahim, M. A New One-Parameter Distribution for Right Censored Bayesian and Non-Bayesian Distributional Validation under Various Estimation Methods. *Mathematics* 2023, 11, 897. <https://doi.org/10.3390/math11040897>
39. Fuller, W.A. (1982). Closeness of estimators. *Encyclopedia of statistical sciences*, Vol 2, Wiley.
40. Gad, A. M., Hamedani, G. G., Salehabadi, S. M. and Yousof, H. M. (2019). The Burr XII-Burr XII distribution: mathematical properties and characterizations. *Pakistan Journal of Statistics*, 35(3), 229-248.
41. Gomes, A. E., da-Silva, C. Q. and Cordeiro, G. M. (2015). Two extended Burr models: Theory and practice. *Commun. Stat. Theory - Methods* 44, 1706-1734.
42. Goual, H., and Yousof, H. M. (2020). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. *Journal of Applied Statistics*, 47(3), 393-423.
43. Goual, H., Hamida, T., Hiba, A., Hamedani, G.G., Ibrahim, M. and Yousof, H. M. (2022). Bayesian and Non-Bayesian Distributional Validations under Censored and Uncensored Schemes with Characterizations and Applications
44. Goual, H., Yousof, H. M., and Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. *Pakistan Journal of Statistics and Operation Research*, 745-771.
45. Goual, H., Yousof, H. M., & Ali, M. M. (2020). Lomax inverse Weibull model: properties, applications, and a modified Chi-squared goodness-of-fit test for validation. *Journal of Nonlinear Sciences & Applications (JNSA)*, 13(6), 330-353.
46. Hashem, A. F., Alotaibi, N., Alyami, S. A., Abdelkawy, M. A., Elgawad, M. A. A., Yousof, H. M., & Abdel-Hamid, A. H. (2024). Utilizing Bayesian inference in accelerated testing models under constant stress via ordered ranked set sampling and hybrid censoring with practical validation. *Scientific Reports*, 14(1), 14406.
47. Hamed, M. S., Cordeiro, G. M. and Yousof, H. M. (2022). A New Compound Lomax Model: Properties, Copulas, Modeling and Risk Analysis Utilizing the Negatively Skewed Insurance Claims Data. *Pakistan Journal of Statistics and Operation Research*, 18(3), 601-631. <https://doi.org/10.18187/pjsor.v18i3.3652>
48. Hamedani, G. G., Goual, H., Emam, W., Tashkandy, Y., Ahmad Bhatti, F., Ibrahim, M., & Yousof, H. M. (2023). A new right-skewed one-parameter distribution with mathematical characterizations, distributional validation, and actuarial risk analysis, with applications. *Symmetry*, 15(7), 1297.
49. Hamedani, G. G., Korkmaz, M. C., Butt, N. S. and Yousof, H. M. (2021). The Type I Quasi Lambert Family: Properties, Characterizations and Different Estimation Methods. *Pakistan Journal of Statistics and Operation Research*, 17(3), 545-558.
50. Hamedani, G., Korkmaz, M. C., Butt, N. S. and Yousof, H. M. (2022). The Type II Quasi Lambert G Family of Probability Distributions. *Pakistan Journal of Statistics and Operation Research*, 18(4), 963-983. <https://doi.org/10.18187/pjsor.v18i4.3907>
51. Harris, C.M. (1968). The Pareto distribution as a queue service discipline, *Operations Research* 16, 307-313.
52. Hashempour, M., Alizadeh, M., & Yousof, H. M. (2024). A new Lindley extension: estimation, risk assessment and analysis under bimodal right skewed precipitation data. *Annals of Data Science*, 11(6), 1919-1958.
53. Ibrahim, M., Aidi, K., Ali, M. M. and Yousof, H. M. (2021). The Exponential Generalized Log-Logistic Model: Bagdonavičius-Nikulin test for Validation and Non-Bayesian Estimation Methods. *Communications for Statistical Applications and Methods*, 29(1), 681-705.
54. Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. (2022). The Double Burr Type XII Model: Censored and Uncensored Validation Using a New Nikulin-Rao-Robson Goodness-of-Fit Test with Bayesian and Non-Bayesian Estimation Methods. *Pakistan Journal of Statistics and Operation Research*, 18(4), 901-927. <https://doi.org/10.18187/pjsor.v18i4.3600>
55. Ibrahim, M., Altun, E., Goual, H., and Yousof, H. M. (2020). Modified goodness-of-fit type test for censored validation under a new Burr type XII distribution with different methods of estimation and regression modeling. *Eurasian Bulletin of Mathematics*, 3(3), 162-182.

56. Ibrahim, M., Ansari, S. I., Al-Nefaie, A. H., & Yousof, H. M. (2025). A New Version of the Inverse Weibull Model with Properties, Applications and Different Methods of Estimation. *Statistics, Optimization & Information Computing*, 13(3), 1120-1143.
57. Ibrahim, M., Butt, N. S., Al-Nefaie, A. H., Hamedani, G. G., Yousof, H. M., & Mahmoud, A. S. (2025). An Extended Discrete Model for Actuarial Data and Value at Risk Analysis: Properties, Applications and Risk Analysis under Financial Automobile Claims Data. *Statistics, Optimization & Information Computing*, 13(1), 27-46.
58. Ibrahim, M.; Emam, W.; Tashkandy, Y.; Ali, M.M.; Yousof, H.M. (2023). Bayesian and Non-Bayesian Risk Analysis and Assessment under Left-Skewed Insurance Data and a Novel Compound Reciprocal Rayleigh Extension. *Mathematics* 2023, 11, 1593. <https://doi.org/10.3390/math11071593>
59. Ibrahim, M., Hamedani, G. G., Butt, N. S. and Yousof, H. M. (2022). Expanding the Nadarajah Haghighi Model: Copula, Censored and Uncensored Validation, Characterizations and Applications. *Pakistan Journal of Statistics and Operation Research*, 18(3), 537-553. <https://doi.org/10.18187/pjsor.v18i3.3420>
60. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H., & Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. *Communications for Statistical Applications and Methods*, 26(5), 473-495.
61. Jozani, M. J., Davies, K. F. and Balakrishnan, N. (2012). Pitman closeness results concerning ranked set sampling. *Statistics & Probability Letters*, 82(12), 2260-2269.
62. Khan, M. I., Aljadani, A., Mansour, M. M., Abd Elrazik, E. M., Hamedani, G. G., Yousof, H. M., & Shehata, W. A. (2024). A New Heavy-Tailed Lomax Model With Characterizations, Applications, Peaks Over Random Threshold Value-at-Risk, and the Mean-of-Order-P Analysis. *Journal of Mathematics*, 2024(1), 5329529.
63. Khedr, A. M., Nofal, Z. M., El Gebaly, Y. M. and Yousof, H. M. (2023). A Novel Family of Compound Probability Distributions: Properties, Copulas, Risk Analysis and Assessment under a Reinsurance Revenues Data Set. *Thailand Statistician*, forthcoming.
64. Korkmaz, M. Ç., Altun, E., Yousof, H. M., & Hamedani, G. G. (2019). The odd power Lindley generator of probability distributions: properties, characterizations and regression modeling. *International Journal of Statistics and Probability*. 8(2), 70-89.
65. Korkmaz, M. Ç., Altun, E., Yousof, H. M., Afify, A. Z. and Nadarajah, S. (2018). The Burr X Pareto Distribution: Properties, Applications and VaR Estimation. *Journal of Risk and Financial Management*, 11(1), 1.
66. Korkmaz, M. Ç., Yousof, H. M., & Ali, M. M. (2017). Some theoretical and computational aspects of the odd Lindley Fréchet distribution. *İstatistikçiler Dergisi: İstatistik ve Aktüerya*, 10(2), 129-140.
67. Loubna, H., Goual, H., Alghamdi, F. M., Mustafa, M. S., Tekle Mekiso, G., Ali, M. M., ... & Yousof, H. M. (2025). The quasi-xgamma frailty model with survival analysis under heterogeneity problem, validation testing, and risk analysis for emergency care data. *Scientific Reports*, 14(1), 8973.
68. Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data, *Journal of the American Statistical Association*, 49, 847-852.
69. Loprinzi, C. L., Laurie, J. A., Wieand, H. S., Krook, J. E., Novotny, P. J., Kugler, J. W. and Klatt, N. E. (1994). Prospective evaluation of prognostic variables from patient-completed questionnaires. North Central Cancer Treatment Group. *Journal of Clinical Oncology*, 12(3), 601-607.
70. Mansour, M. M., Ibrahim, M., Aidi, K., Butt, N. S., Ali, M. M., Yousof, H. M., & Hamed, M. S. (2020). A New Log-Logistic Lifetime Model with Mathematical Properties, Copula, Modified Goodness-of-Fit Test for Validation and Real Data Modeling. *Mathematics*, 8(9), 1508.
71. Mansour, M. M., Butt, N. S., Ansari, S. I., Yousof, H. M., Ali, M. M., & Ibrahim, M. (2020). A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications. *Contributions to Mathematics*, 1 (2020) 57-66. DOI: 10.47443/cm.2020.0018
72. Mansour, M., Korkmaz, M. Ç., Ali, M. M., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020). A generalization of the exponentiated Weibull model with properties, Copula and application. *Eurasian Bulletin of Mathematics*, 3(2), 84-102.
73. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M., & Elrazik, E. A. (2020). A New Parametric Life Distribution with Modified Bagdonavičius-Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. *Entropy*, 22(5), 592.
74. Mansour, M., Yousof, H. M., Shehata, W. A. M., & Ibrahim, M. (2020). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. *Journal of Nonlinear Science and Applications*, 13(5), 223-238.
75. Mansour, M. M., Butt, N. S., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020). A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets. *Pakistan Journal of Statistics and Operation Research*, 16(2), 373-386.
76. Meeker, W. Q., Escobar, L. A., & Lu, C. J. (1998). Accelerated degradation tests: modeling and analysis. *Technometrics*, 40(2), 89-99.
77. Nichols, M. D., & Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. *Quality and reliability engineering international*, 22(2), 141-151.
78. Nasir, A., Korkmaz, M. C., Jamal, F., Tahir, M. H. and Yousof, H. M. (2018). A New Weibull Lomax Distributions for Lifetime Data. *Sohag J. Math.*, 5(2), 1-10.
79. Nikulin, M. S., (1973a). Chi-square Test For Continuous Distributions with Shift and Scale Parameters. *teor. Veroyatn. Primen.*, 18 : 3, 559-568.
80. Nikulin, M. S., (1973b). Chi-squared test for continuous distributions with shift and scal parameters. *Theory of Probability and its Applications*, 18, 559-568.
81. Nikulin, M. S., (1973c). On a Chi-squared test for continuous distributions. *Theory of Probability and its Applications*. 19, 638-639.
82. Rao, K. C., Robson, D. S. (1974). A Chi-square statistic for goodness-of-fit tests within the exponential family. *Communication in Statistics*, 3, 1139-1153.
83. Rasekhi, M., Altun, E., Alizadeh, M. and Yousof, H. M. (2022). The Odd Log-Logistic Weibull-G Family of Distributions with Regression and Financial Risk Models. *Journal of the Operations Research Society of China*, 10(1), 133-158.

84. Rasekhi, M., Saber, M. M., & Yousof, H. M. (2020). Bayesian and classical inference of reliability in multicomponent stress-strength under the generalized logistic model. *Communications in Statistics-Theory and Methods*, 50(21), 5114-5125.
85. Ravi, V., Gilbert, P. D. (2009). BB : An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. *J. Statist. Software*, 32, 1-26.
86. Salah, M. M., El-Morshedy, M., Eliwa, M. S. and Yousof, H. M. (2020). Expanded Fréchet Model: Mathematical Properties, Copula, Different Estimation Methods, Applications and Validation Testing. *Mathematics*, 8(11), 1949.
87. Salem, M., Emam, W., Tashkandy, Y., Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. M. (2023). A new lomax extension: Properties, risk analysis, censored and complete goodness-of-fit validation testing under left-skewed insurance, reliability and medical data. *Symmetry*, 15(7), 1356.
88. Shehata, W. A. M., Aljadani, A., Mansour, M. M., Alrweili, H., Hamed, M. S., & Yousof, H. M. (2024). A Novel Reciprocal-Weibull Model for Extreme Reliability Data: Statistical Properties, Reliability Applications, Reliability PORT-VaR and Mean of Order P Risk Analysis. *Pakistan Journal of Statistics and Operation Research*, 20(4), 693-718. <https://doi.org/10.18187/pjsor.v20i4.4302>
89. Shehata, W. A. M., Butt, N. S., Yousof, H., & Aboraya, M. (2022). A New Lifetime Parametric Model for the Survival and Relief Times with Copulas and Properties. *Pakistan Journal of Statistics and Operation Research*, 18(1), 249-272.
90. Shehata, W. A. M. and Yousof, H. M. (2022). A novel two-parameter Nadarajah-Haghighi extension: properties, copulas, modeling real data and different estimation methods. *Statistics, Optimization & Information Computing*, 10(3), 725-749.
91. Shehata, W. A. M. and Yousof, H. M. (2021). The four-parameter exponentiated Weibull model with Copula, properties and real data modeling. *Pakistan Journal of Statistics and Operation Research*, 17(3), 649-667.
92. Shehata, W. A., Goual, H., Hamida, T., Hiba, A., Hamedani, G. G., Al-Nefaie, A. H., Ibrahim, M., Butt, N. S., Osman, R. M. A., and Yousof, H. M. (2024). Censored and Uncensored Nikulin-Rao-Robson Distributional Validation: Characterizations, Classical and Bayesian estimation with Censored and Uncensored Applications. *Pakistan Journal of Statistics and Operation Research*, 20(1), 11-35.
93. Shehata, W. A. M., Yousof, H. M., & Aboraya, M. (2021). A Novel Generator of Continuous Probability Distributions for the Asymmetric Left-skewed Bimodal Real-life Data with Properties and Copulas. *Pakistan Journal of Statistics and Operation Research*, 17(4), 943-961. <https://doi.org/10.18187/pjsor.v17i4.3903>
94. Shrahili, M.; Elbatal, I. and Yousof, H. M. Asymmetric Density for Risk Claim-Size Data: Prediction and Bimodal Data Applications. *Symmetry* 2021, 13, 2357. <https://doi.org/10.3390/sym13122357>
95. Teghri, S., Goual, H., Loubna, H., Butt, N. S., Khedr, A. M., Yousof, H. M., Ibrahim, M. & Salem, M. (2024). A New Two-Parameters Lindley-Frailty Model: Censored and Uncensored Schemes under Different Baseline Models: Applications, Assessments, Censored and Uncensored Validation Testing. *Pakistan Journal of Statistics and Operation Research*, 109-138.
96. Pitman, E. (1937). The closest estimates of statistical parameters. *Mathematical proceeding of the Cambridge philosophical society*, 33(2).
97. Yadav, A. S., Altun, E., & Yousof, H. M. (2021). Burr–Hatke Exponential Distribution: A Decreasing Failure Rate Model, Statistical Inference and Applications. *Annals of Data Science*, 8(2), 241–260.
98. Yadav, A. S., Goual, H., Alotaibi, R. M., Ali, M. M. and Yousof, H. M. (2019). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. *Symmetry*, 12(1), 57.
99. Yadav, A. S., Shukla, S., Goual, H., Saha, M. and Yousof, H. M. (2022). Validation of xgamma exponential model via Nikulin-Rao-Robson goodness-of-fit test under complete and censored sample with different methods of estimation. *Statistics, Optimization & Information Computing*, 10(2), 457-483.
100. Yousof, H. M., Afify, A. Z., Hamedani, G. G. and Aryal, G. (2017). The Burr X generator of distributions for lifetime data. *Journal of Statistical Theory and Applications*, 16(3), 288-305.
101. Yousof, H. M., Ahsanullah, M., and Khalil, M. G. (2019a). A new zero-truncated version of the Poisson Burr XII distribution: Characterizations and properties. *Journal of Statistical Theory and Applications*, 18(1), 1-11.
102. Yousof, H. M., Aidi, K., Hamedani, G. G. and Ibrahim, M. (2021). A new parametric lifetime distribution with modified Chi-square type test for right censored validation, characterizations and different estimation methods. *Pakistan Journal of Statistics and Operation Research*, 17(2), 399-425.
103. Yousof, H. M., Aidi, K., Hamedani, G. G., & Ibrahim, M. (2023). Chi-squared type test for distributional censored and uncensored validity with numerical assessments and real data applications. *Japanese Journal of Statistics and Data Science*, 6(2), 729-758.
104. Yousof, H. M., Ali, M. M., Aidi, K., Ibrahim, M. (2023). The modified Bagdonavičius-Nikulin goodness-of-fit test statistic for the right censored distributional validation with applications in medicine and reliability. *Statistics in Transition New Series*, 24(4), 1-18.
105. Yousof, H. M., Ali, M. M., Goual, H. and Ibrahim, M. (2021). A new reciprocal Rayleigh extension: properties, copulas, different methods of estimation and modified right censored test for validation, *Statistics in Transition New Series*, 23(3), 1-23.
106. Yousof, H. M., Ali, M. M., Hamedani, G. G., Aidi, K. & Ibrahim, M. (2022). A new lifetime distribution with properties, characterizations, validation testing, different estimation methods. *Statistics, Optimization & Information Computing*, 10(2), 519-547.
107. Yousof, H. M., Aljadani, A., Mansour, M. M., & Abd Elrazik, E. M. (2024). A New Pareto Model: Risk Application, Reliability MOOP and PORT Value-at-Risk Analysis. *Pakistan Journal of Statistics and Operation Research*, 20(3), 383-407. <https://doi.org/10.18187/pjsor.v20i3.4151>
108. Yousof, H. M., Al-Nefaie, A. H., Butt, N. S., Hamedani, G., Alrweili, H., Aljadani, A., Mansour, M. M., Hamed, M. S., & Ibrahim, M. (2024). A New Discrete Generator with Mathematical Characterization, Properties, Count Statistical Modeling and Inference with Applications to Reliability, Medicine, Agriculture, and Biology Data. *Pakistan Journal of Statistics and Operation Research*, 20(4), 745-770. <https://doi.org/10.18187/pjsor.v20i4.4616>
109. Yousof, H. M., Ansari, S. I., Tashkandy, Y., Emam, W., Ali, M. M., Ibrahim, M., Alkhayyat, S. L. (2023). Risk Analysis and Estimation of a Bimodal Heavy-Tailed Burr XII Model in Insurance Data: Exploring Multiple Methods and Applications. *Mathematics*. 2023; 11(9):2179. <https://doi.org/10.3390/math11092179>
110. Yousof, H. M., Altun, E., & Hamedani, G. G. (2018). A new extension of Fréchet distribution with regression models, residual analysis and characterizations. *Journal of Data Science*, 16(4), 743-770.

111. Yousof, H.M.; Emam, W.; Tashkandy, Y.; Ali, M.M.; Minkah, R.; Ibrahim, M. (2023). A Novel Model for Quantitative Risk Assessment under Claim-Size Data with Bimodal and Symmetric Data Modeling. *Mathematics* 2023, 11, 1284. <https://doi.org/10.3390/math11061284>
112. Yousof, H. M., Goual, H., Emam, W., Tashkandy, Y., Alizadeh, M., Ali, M. M., & Ibrahim, M. (2023). An Alternative Model for Describing the Reliability Data: Applications, Assessment, and Goodness-of-Fit Validation Testing. *Mathematics*, 11(6), 1308.
113. Yousof, H. M., Goual, H., Khaoula, M. K., Hamedani, G. G., Al-Aefaie, A. H., Ibrahim, M., ... & Salem, M. (2023). A novel accelerated failure time model: Characterizations, validation testing, different estimation methods and applications in engineering and medicine. *Pakistan Journal of Statistics and Operation Research*, 19(4), 691-717.
114. Yousof, H. M., Rasekhi, M., Altun, E., Alizadeh, M. Hamedani G. G. and Ali M. M. (2019b). A new lifetime model with regression models, characterizations and applications. *Communications in Statistics-Simulation and Computation*, 48(1), 264-286.
115. Yousof, H. M., Saber, M. M., Al-Nefaie, A. H., Butt, N. S., Ibrahim, M. and Alkhayyat, S. L. (2024). A discrete claims-model for the inflated and over-dispersed automobile claims frequencies data: Applications and actuarial risk analysis. *Pakistan Journal of Statistics and Operation Research*, 261-284.
116. Yousof, H.M.; Tashkandy, Y.; Emam, W.; Ali, M.M.; Ibrahim, M. (2023). A New Reciprocal Weibull Extension for Modeling Extreme Values with Risk Analysis under Insurance Data. *Mathematics* 2023, 11, 966. <https://doi.org/10.3390/math11040966>