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Sushila-Geometric distribution, properties and applications

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Abstract In the present paper, we introduce a compound form of the Sushila distribution which offers a flexible model for lifetime data, the so-called Sushila-geometric (SG) distribution, and is obtained by compounding Sushila and geometric distributions. A three-parameter SG distribution is capable of modelling upside-down bathtub, bathtub-shaped, increasing and decreasing hazard rate functions which are widely used in engineering, economy and natural sciences. This new model contains some known distributions such as Lindley, Lindley-Geometric, and Sushila distributions in a special cases as sub-models. Several statistical properties of the SG distribution are derived. Simulation studies are conducted to investigate the performance of the maximum likelihood estimators derived through the EM algorithm. The flexibility of the new model is illustrated in the application of two real data sets.

Keywords EM algorithm; Maximum likelihood estimation; Geometric distribution; Sushila distribution

AMS 2010 subject classifications 60E05

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1. Introduction

The Sushila distribution is a lifetime distribution and was introduced by Shanker et al. [28]. This distribution, being a modified Lindley distribution, is a mixture of the exponential and gamma distributions. The Sushila distribution was discussed that its failure rate function and mean residual life function show flexibility over the Lindley and exponential distributions. Moreover, the moment generating function of Sushila distribution can be expressed in closed form. Therefore, we have been interested in using the Sushila distribution for creating a new alternative mixed model. This model, in spite of little attention in the statistical literature, is important for studying stress-strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the Sushila distribution, including also their properties. Several distributions have been proposed in the literature to model lifetime data. Lifetime distribution represents an attempt to describe, mathematically, the length of the life of a system or a device. Lifetime distributions are most frequently used in the fields like medicine, engineering etc.

The Poisson-Sushila distribution was introduced by Saratoon [25], which is a two-parameter discrete distribution. Various properties have been studied and shown that the Poisson-Sushila distribution is more flexible than Poisson distribution in real data. Elgarhy and Shawki [10], discussed exponentiated-Sushila distribution. Again Elgarhy and Shawki [27], obtained the various properties of kumaraswamy-Sushila distribution. Recently, Borah and Hazarika [6], discussed Poisson-Sushila distribution and its applications. Rather and Subramanian [23], have discussed the statistical properties and applications of length-biased Sushila distribution. Rather and Subramanian [24], introduced a new generalization of the Sushila distribution namely as weighted-Sushila distribution with three parameters. Pudprommarat [20], introduced the hurdle Poisson-Sushila distribution as an

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extension to the Poisson-Sushila distribution and indicated this distribution is a better fit than Poisson. Borah and Saikia [7], introduced discrete sushila distribution, de Oliveria et al. [9]used of a discrete sushila distribution in the analysis of right-censored lifetime data. Adetunji [4], introduced a three-parameter generalization of the Sushila distribution using the Quadratic transmuted technique and compared the performance of new distribution with the Sushila distribution.

Adamidis and Loukas [2], introduced the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus [12], introduced the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Marshall and Olkin [16], presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. [3], proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the EEG distribution can be monotone decreasing, increasing, or constant. Peng Xu [19], introduced the exponentiated-Lindley geometric Distribution (ELG). Bordbar and Nematollahi [8], introduced the modified exponential-geometric distribution with decreasing or increasing failure rate.

The Sushila distribution is one of the most commonly used lifetime distributions in modeling lifetime data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions with monotone failure rates but it is not useful for modeling the bathtub-shaped failure rates which are common in reliability and biological studies. In this paper, we introduce a new three-parameter lifetime distribution called Sushila-Geometric (SG) distribution by compounding Sushila and geometric distributions which generalizes the Lindely, Lindely-Geometric and Sushila distributions and study some of its properties. The hazard rate function of the SG distribution can be upside-down bathtub, bathtub-shaped, increasing and decreasing which makes the SG distribution to be superior to other lifetime distributions.

The paper is organized as follows. In section 2 we introduce the SG distribution. A comprehensive account of mathematical properties of the new distribution is provided in sections 3–12. The properties studied include; shapes of the probability density function and hazard rate function, stochastic orderings, quantile function, moments of the SG distribution, residual life and reserved residual, order statistics, asymptotic distribution of extreme values, Bonferroni and Lorenz curves, entropies, mean deviations, estimation of the parameters by maximum likelihood via an EM-algorithm and inference for a large sample, and simulation schemes. Finally, section (13) illustrates an application by using two real data sets and conclusions are provided in section (14).

2. Sushila-Geometric distribution

Consider the random variable Y having the Sushila distribution denoted $Su(\theta,\alpha)$ where its PDF and CDF are given by

$$f(y,\theta,\alpha) = \frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{y}{\alpha}\right)e^{-\frac{\theta}{\alpha}y};\quad y>0,\alpha>0,\theta>0 \tag{1}$$

$$F(y,\theta,\alpha) = 1-\left(1+\frac{\theta y}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}y};\quad y>0,\alpha>0,\theta>0 \tag{2}$$

Let $Y_1,\cdots,Y_N$ be iid random variables from Sushila distribution where N is a geometric random variable with the probability mass function $P(N=n)=(1-p)p^{n-1}$, $n=1,2,\cdots,$ $0<p<1$. Consider $X=min(Y_1,\cdots,Y_N)$, then the conditional CDF $X|N=n$ is given by

$$F_{X|N}(x|n) = 1-\left[(1+\frac{\theta x}{\alpha(\theta+1)})e^{-\frac{\theta}{\alpha}x}\right]^n$$

The Sushila-geometric (SG) distribution with three-parameter, denoted by $SG(\theta,\alpha,p)$, is defined by the marginal CDF of X;

$$F(x) = \frac{1-\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}{1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}} \tag{3}$$

The PDF of SG distribution for $\theta>0,\alpha>0$, and $0<p<1$ is given by

$$f(x) = \frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)(1-p)e^{-\frac{\theta}{\alpha}x}\left(1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)^{-2}, \quad x>0. \tag{4}$$

It should be noted that the PDF in (4) is still a well-defined density function when $p<0$ Thus, we can define the SG distribution to any $p<1$. The SG distribution includes several submodels. If $\alpha=1$, it becomes the Lindley-geometric (LG) distribution introduced by Zakerzadeh and Mahmoudi [30]. When $p=0$ it changes to the Sushila distribution due to Shanker et al. [28]. If $\alpha=1,p=0$ it turns out to be the Lindley distribution due to Lindley [15]. It converges a distribution degenerating at the point 0 when $p\to 1^-$. Using the series expansion

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j, \quad |z|<1, \quad k>0, \tag{5}$$

the density function (4) can be written as follows

$$f(x) = \frac{\theta^2}{\alpha(\theta+1)}(1-p)(1+\frac{x}{\alpha})e^{-\frac{\theta}{\alpha}x}\sum_{j=0}^{\infty}(j+1)p^j\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)^j e^{-\frac{\theta}{\alpha}jx} \tag{6}$$

Various mathematical properties of the SG distribution can be derived from (6).

Figure (1) shows different shapes of (4) for selected values of parameters. Since α is a scale parameter, when α increases, the kurtosis of (4) decreases. Also acorrding to Theorem 1 of Zakerzadeh and Mahmoudi [30], the density function of the SG distribution is (i) decreasing for $p>\frac{1-\theta^2}{1+\theta^2}$ and all values of θ and (ii) unimodal for $p\le\frac{1-\theta^2}{1+\theta^2}$ and all values for θ.

3. Survival and hazard rate functions

The survival function and hazard rate function of the SG distribution, are given respectively

$$S(x) = \frac{\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)(1-p)e^{-\frac{\theta}{\alpha}x}}{1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}$$

and

$$h(x) = \frac{\theta^2\left(1+\frac{x}{\alpha}\right)}{\alpha(\theta+1)+\theta x}\left(1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)^{-1} \tag{7}$$

for $x>0,\theta>0,\alpha>0,p<1$. It follows from (7) that

$$\frac{d\log h(x)}{dx} = \frac{\alpha e^{\frac{\theta}{\alpha}x}(1+\theta)-p(\alpha(1+\theta)+\theta x)\left(1+(1+\frac{x}{\alpha})^2\theta^2\right)}{(1+\frac{x}{\alpha})\left(\alpha(1+\theta)+\theta x\right)\left[\alpha e^{\frac{\theta}{\alpha}x}(1+\theta)-p(\alpha(1+\theta)+\theta x)\right]}. \tag{8}$$

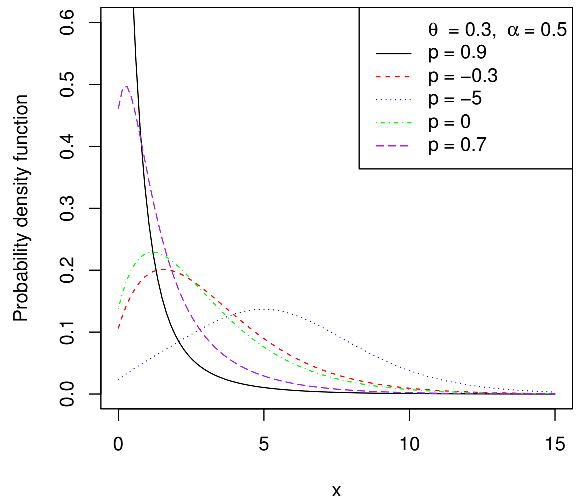

Figure 1. Plots of PDF of SG for selected values of parameters.

Note also that as the hazard rate function of the SG distribution in (7) tends to $\frac{\theta}{\alpha}$ and $\frac{\theta^2}{\alpha(\theta+1)(1-p)}$ when $x \to \infty$ and $x \to 0$, respectively. So, both the initial and ultimate hazard rates are constant. Figure 2 illustrates possible shapes of (7) for selected values of parameters. The shapes appear monotonically increasing for $p < 0$ and small p. For p values are close to 1 and small θ values and α values that are not small, the shape is upside down bathtub. When p values are close to 1, and θ and α values are almost large, the shape is decreasing, in this case, if the α values are not large, the hazard rate function is bathtub-shaped. But does not exhibit a constant hazard rate, which makes the SG distribution to be superior to other lifetime distributions, which exhibit only monotonically increasing/decreasing, or constant hazard rate.

4. Stochastic orderings

For positive continuous random variables, stochastic ordering is an important tool for judging the comparative behavior. A random variable X is said to be smaller than a random variable Y in the

- stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
- mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x.
- likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

From Shaked and Shanthikumar [26] we have the following implications:

$$\begin{array}{c} X \leq_{\text{lr}} Y \Rightarrow X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{mrl}} Y \\ \Downarrow \\ X \leq_{\text{st}} Y \end{array} \tag{9}$$

The following theorem shows that the SG distribution is ordered with respect to the strongest “likelihood ratio” ordering.

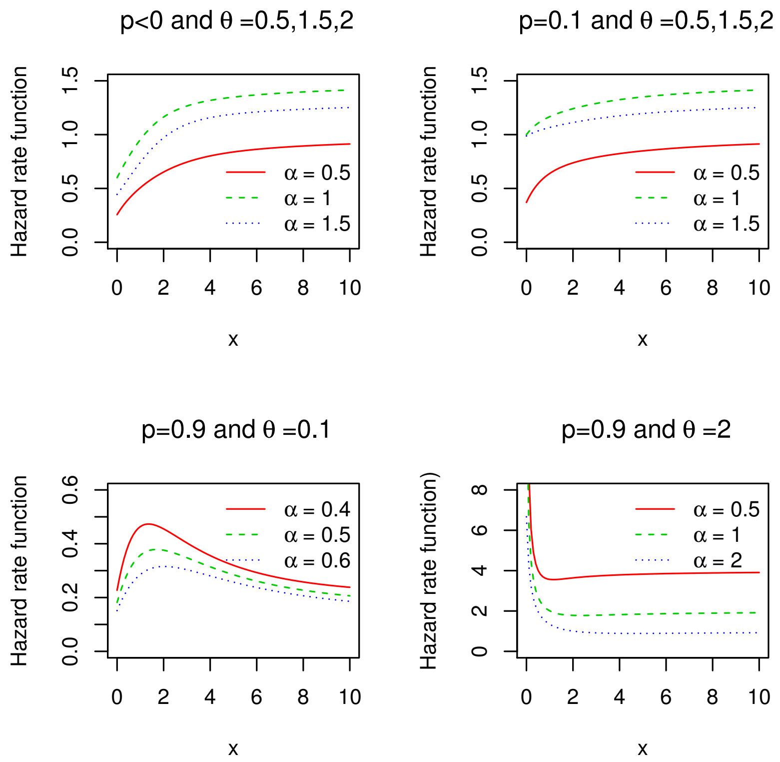

Figure 2. Plots of the hazard rate function for selected values of parameters

Theorem 4.1
Let $X \sim SG(\theta,\alpha,p_1)$ and $Y \sim SG(\theta,\alpha,p_2)$. If $p_1 > p_2$ then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof
First note that

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{1-p_1}{1-p_2}\right)\left(\frac{1-p_1\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}{1-p_2\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}\right)^{-2}.$$

Now

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left(\frac{1-p_1}{1-p_2}\right) - 2\log\left(\frac{1-p_1\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}{1-p_2\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}}\right).$$

Then

$$\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) = 2\left(\frac{p_1\left(\frac{\theta}{\alpha(\theta+1)} - \frac{\theta}{\alpha}(1+\frac{\theta x}{\alpha(\theta+1)})\right)e^{-\frac{\theta}{\alpha}x}}{1-p_1(1+\frac{\theta x}{\alpha(\theta+1)})e^{-\frac{\theta}{\alpha}x}} - \frac{p_2\left(\frac{\theta}{\alpha(\theta+1)} - \frac{\theta}{\alpha}(1+\frac{\theta x}{\alpha(\theta+1)})\right)e^{-\frac{\theta}{\alpha}x}}{1-p_2(1+\frac{\theta x}{\alpha(\theta+1)})e^{-\frac{\theta}{\alpha}x}}\right)$$

$\frac{f_X(x)}{f_Y(x)}$ is decreasing in x. That is $X \leq_{lr} Y$. The remaining statements follow from the implications in (9). □ □

5. Quantile Function

Theorem 5.1
Let $Y \sim Su(\theta, \alpha)$.The quantile function of Y is

$$F^{-1}(u) = -\alpha - \frac{\alpha}{\theta} - \frac{\alpha}{\theta} W_{-1}\left(-\frac{(\theta+1)}{e^{\theta+1}}(1-u)\right) \quad (10)$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof
For any fixed $\theta > 0, \alpha > 0$ and $0 < u < 1$, we solve $F_Y(y) = u$ with respect to y, for $y > 0$. From (2), we get

$$(\theta + 1 + \frac{\theta}{\alpha} y)e^{-\frac{\theta}{\alpha}y} = (1-u)(\theta+1).$$

Multiplying by $-\exp(-\theta-1)$ both sides of above Equation, we have

$$-(\theta + 1 + \frac{\theta}{\alpha} y)e^{-(\frac{\theta}{\alpha}y+\theta+1)} = -(1-u)(\theta+1)e^{-\theta-1} \quad (11)$$

From $W(z)exp(W(z)) = z$, (see Adler [5] in detail), we notice that $-(\theta + 1 + \frac{\theta}{\alpha}y)$ is the Lambert W function of the real argument $-(\theta+1)(1-u)\exp(-\theta-1)$. Then, we have

$$W\left(-\frac{(\theta+1)}{e^{\theta+1}}(1-u)\right) = -\theta - 1 - \frac{\theta}{\alpha} y. \quad (12)$$

Moreover, for any $\theta > 0, \alpha > 0$ and $y > 0$ it is immediate that $(\theta + 1 + \frac{\theta}{\alpha}y) > 1$ and it can also be checked that $(u-1)(\theta+1)\exp(-\theta-1) \in (-\frac{1}{e}, 0)$ since $0 < u < 1$. Therefore, by taking into account the properties of the negative branch of the Lambert W function Equation (12) becomes

$$W_{-1}\left(-\frac{(\theta+1)}{e^{\theta+1}}(1-u)\right) = -\theta - 1 - \frac{\theta}{\alpha} y \quad (13)$$

which in turn implies the result. □ □

Let X be a SG random variable with the CDF in (3). By inverting $F(x) = u$ for $0 < u < 1$, we obtain

$$\left(\frac{u - pu}{1 - pu}\right) = 1 - \frac{\alpha(\theta+1) + \theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x} \quad (14)$$

It follows from (13) that the quantile function of the SG distribution is given by

$$F^{-1}(u) = -\alpha - \frac{\alpha}{\theta} - \frac{\alpha}{\theta} W_{-1}\left(-\frac{\theta+1}{e^{\theta+1}}\left[1 - \left(\frac{u-pu}{1-pu}\right)\right]\right) \quad (15)$$

note that $-\frac{1}{e} < \left(-\frac{\theta+1}{e^{\theta+1}}\left[1 - \left(\frac{u-pu}{1-pu}\right)\right]\right) < 0$ so the W_{-1} is unique, which implies that $F^{-1}(u)$ is also unique. Thus, one can use (15) for generating random data from the SG distribution. In addition, the qth quantile x_q of $SG(\theta, \alpha, p)$ is given by

$$x_q = -\alpha - \frac{\alpha}{\theta} - \frac{\alpha}{\theta} W_{-1}\left(-\frac{\theta+1}{e^{\theta+1}}\left[1 - \left(\frac{q-pq}{1-pq}\right)\right]\right); \quad 0 < q < 1. \quad (16)$$

In particular, we obtain Median by putting $q = 0.5$ in (16). Table 1 displays the median of SG distribution for different values of parameters.

6. Moments of the SG distribution

Let $X \sim SG(\theta, \alpha, p)$. Using (6) and applying the binomial expansion for $(1+\frac{\theta x}{\alpha(\theta+1)})^j$, the rth moment of X is given by

$$E(X^r) = \frac{\theta^2(1-p)}{\alpha(\theta+1)} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \binom{j}{i}(j+1)p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^i \frac{\Gamma(r+i+1)}{\left(\frac{\theta}{\alpha}(j+1)\right)^{r+i+1}} \left[1+\frac{r+i+1}{\theta(j+1)}\right].$$

Then

$$E(X) = \frac{\theta^2(1-p)}{\alpha(\theta+1)} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(j+1)!}{(j-i)!}p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^i \frac{(i+1)}{\left(\frac{\theta}{\alpha}(j+1)\right)^{i+2}} \left[1+\frac{i+2}{\theta(j+1)}\right], \tag{17}$$

and

$$E(X^2) = \frac{\theta^2(1-p)}{\alpha(\theta+1)} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(j+1)!}{(j-i)!}p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^i \frac{(i+1)(i+2)}{\left(\frac{\theta}{\alpha}(j+1)\right)^{i+3}} \left[1+\frac{i+3}{\theta(j+1)}\right].$$

For selected values of parameters, the mean and variance of X are presented in Table 1.

Table 1. Median, Mean and variance of SG distribution for different values of parameters

θ	α	p	$Median$	E(X)	Var(X)
1.5	0.2	−0.1	0.1479	0.1952	0.0304
1.5	0.2	−0.2	0.1564	0.2031	0.0316
0.5	0.2	0.2	0.4644	0.6041	0.2754
0.5	0.2	0.1	0.4987	0.6367	0.2895
0.3	0.5	0.9	0.5131	0.9368	1.5763
0.3	0.5	0.7	1.1437	1.6917	3.0827
0.3	0.5	−0.3	2.7560	3.2811	5.9623

7. Residual life time and reversed residual life time of the SG distribution

Given that a component of a system survives up to time $t > 0$, the residual life will be the period beyond t until the time of failure occurs in the system and is thus defined by the conditional random variable $X - t|X > t$. The mean residual life plays an important role in survival analysis and reliability of characterizing lifetime, because it can be used to determine a unique corresponding lifetime distribution. The rth moment of the residual life of the SG distribution can be obtained by the general formula.

$$\mu_r(t) = E[(X-t)^r|X>t] = \frac{1}{S(t)}\int_t^{\infty}(x-t)^r f(x)dx,$$

where $S(t)$ is the survival function. Using the binominal expansion to $(x-t)^r$ gives

$$\begin{aligned}\mu_r(t) = {} & \frac{\theta^2(1-p)}{\alpha(\theta+1)s(t)} \sum_{i=0}^{r}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^i\binom{j}{k}t^j p^j(j+1)\left(\frac{\theta}{\alpha(\theta+1)}\right)^k \left[\left(\frac{\alpha}{\theta(j+1)}\right)^{r+k-i+2}\right. \\ & \left.\times\left(\Gamma\left(r+k-i+2;\frac{\theta}{\alpha}t(j+1)\right)+\frac{\theta}{\alpha}(j+1)\left(\Gamma(r+k-i+1;\frac{\theta}{\alpha}t(j+1)\right)\right)\right],\end{aligned}$$

where $\Gamma(s;t)=\int_t^\infty x^{s-1}e^{-x}dx$ shows the upper incomplete gamma function. The mean residual life time of the SG distribution is given by

$$\begin{aligned}\mu(t) &= \frac{\theta^2(1-p)}{\alpha(\theta+1)s(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}(j+1)p^j\left(\frac{\theta}{\alpha(\theta+1)}\right)^k\left[\left(\frac{\alpha}{\theta(j+1)}\right)^{k+3}\right.\\ &\left.\times\Gamma\left(k+3;\frac{\theta}{\alpha}t(j+1)\right)+\frac{\theta}{\alpha}(j+1)\Gamma\left(k+2;\frac{\theta}{\alpha}t(j+1)\right)\right]-t.\end{aligned}$$

In particular we have

$$\begin{aligned}\mu(0)=E(X) &= \frac{\theta^2(1-p)}{(\theta+1)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{(j+1)!}{(j-i)!}p^j\left(\frac{\theta}{\alpha(\theta+1)}\right)^i\\ &\times\left(\frac{\alpha}{\theta(j+1)}\right)^{i+2}(i+1)\left(1+\frac{(i+2)}{\theta(j+1)}\right).\end{aligned}$$

The variance of the residual life time of the SG distribution can be obtained easily by using $\mu_2(t)$ and $\mu(t)$.

The reversed residual life time can be defined as the conditional random variable $X-t|X<t$ which denotes the time elapsed from the failure of a component given that its life time is less than or equal to t. This random variable may also be called the inactivity time (or time since failure); (for more details see Kundu and Nanda [11]; Nanda et al [18]). The rth-order moment of the reversed residual life time can be obtained by

$$m_r(t) = E[(t-X)^r|X<t]=\frac{1}{F(t)}\int_0^t(t-x)^rf(x)dx$$

hence

$$\begin{aligned}m_r(t) &= \frac{\theta^2(1-p)}{\alpha(\theta+1)F(t)}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{r-i}\binom{j}{k}t^jp^j(j+1)\left(\frac{\theta}{\alpha(\theta+1)}\right)^k\left[\left(\frac{\alpha}{\theta(j+1)}\right)^{r+k-i+2}\right.\\ &\times\left.\left(\gamma\left(r+k-i+2;\frac{\theta}{\alpha}t(j+1)\right)+\frac{\theta}{\alpha}(j+1)\gamma\left(r+k-i+1;\frac{\theta}{\alpha}t(j+1)\right)\right)\right]\end{aligned}$$

where $\gamma(s;t)=\int_0^t x^{s-1}e^{-x}dx$ shows the lower incomplete gamma function. The reserved residual life of the SG distribution is given by

$$\begin{aligned}m(t) &= t-\frac{\theta^2(1-p)}{\alpha(\theta+1)F(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}(j+1)p^j\left(\frac{\theta}{\alpha(\theta+1)}\right)^k\left[\left(\frac{\alpha}{\theta(j+1)}\right)^{k+3}\right.\\ &\left.\times\,\gamma\left(k+3;\frac{\theta}{\alpha}t(j+1)\right)+\frac{\theta}{\alpha}(j+1)\gamma\left(k+2;\frac{\theta}{\alpha}t(j+1)\right)\right].\end{aligned}\tag{18}$$

Using $m(t)$ and $m_2(t)$ the variance and the coefficient of variation of the reversed residual life time of the SG distribution can be obtained.

8. Order statistics

Order statistics deal with the properties and applications of ordered random variables and their functions. In the study of many natural problems related to flood, longevity, breaking strength, atmospheric temperature, atmospheric pressure, wind etc., the future possibilities in the recurrence of extreme situations are of much importance and accordingly the problem of interest in these cases reduces to that of the extreme observations. In this section we get the probability density function and the cumulative distribution function of the kth order statistic of the SG distribution. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from (4). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. The PDF and CDF of the kth order statistic, say $Y = X_{k:n}$, are given by

$$\begin{aligned} f_Y(y) &= \frac{n!\theta^2(1-p)}{\alpha(\theta+1)(k-1)!(n-k)!}\left(1+\frac{y}{\alpha}\right)e^{-\frac{\theta}{\alpha}y} \\ &\times \sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(-1)^l\left[1-\left(1+\frac{\theta y}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}y}\right]^{k+l-1}}{\left[1-p\left(1+\frac{\theta y}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}y}\right]^{k+l+1}} \end{aligned}$$

and

$$\begin{aligned} F_Y(y) &= \sum_{j=k}^{n}\binom{n}{j}F^j(y)\left[1-F(y)\right]^{n-j} \\ &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[\frac{1-\left(1+\frac{\theta y}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}y}}{1-p\left(1+\frac{\theta y}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}y}}\right]^{j+l}. \end{aligned}$$

If $X_1, \cdots, X_n$ is a random sample from (4) and $\bar{X} = \frac{X_1+X_2+\cdots+X_n}{n}$ denotes the sample mean, by the central limit theorem as $n \to \infty$ then $\frac{\sqrt{n}(\bar{X}-E(X))}{\sqrt{Var(X)}}$ approaches the standard normal distribution. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \cdots, X_n)$ and $m_n = \min(X_1, \cdots, X_n)$. For the CDF in (3), by using L'Hospital's rule, we have

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)} &= \lim_{t\to\infty}\frac{f(t+x)}{f(t)} \\ &= \lim_{t\to\infty}\frac{1-\left(1-\frac{\alpha(\theta+1)+\theta(t+x)}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}(t+x)}\right)}{1-\left(1-\frac{\alpha(\theta+1)+\theta t}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}t}\right)} \\ &= e^{-\frac{\theta}{\alpha}x}. \end{aligned}$$

In addition, by using L'Hospital's rule, it can be easily shown that

$$\begin{aligned} \lim_{t\to 0}\frac{F(tx)}{F(t)} &= \lim_{t\to 0}\frac{xf(tx)}{f(t)} \\ &= \lim_{t\to 0}\frac{\left(1-\frac{\alpha(\theta+1)+\theta tx}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}tx}\right)}{1-\left(1-\frac{\alpha(\theta+1)+\theta t}{\alpha(\theta+1)}e^{\frac{-\theta}{\alpha}t}\right)} \\ &= x. \end{aligned}$$

By following Theorem 1.6.2 in Leadbetter et al. [13], we observe that there must be some normalizing constants $a_n > 0, b_n, c_n > 0$ and d_n, such that

$$Pr[a_n(M_n - b_n) \le x] \;\to\; exp\left(-e^{-\frac{\theta}{\alpha}x}\right)$$

$$Pr[c_n(m_n - d_n) \le x] \;\to\; 1 - exp(-x)$$

as $n \to \infty$. The form of the normalizing constants can be determined by using Corollary 1.6.3 in Leadbetter et al. [13]. As an illustration, one can see that $a_n = 1$ and $b_n = F^{-1}(1-\frac{1}{n})$ where$F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$.

9. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have many practical applications not only in economics and poverty, but also in other fields like reliability, life time testing, insurance, and medicine. For a random variable X with CDF (3), the Bonferroni curve is defined by

$$B_F[F(x)] = \frac{1}{\mu F(x)} \int_0^x u f(u) du,$$

where $\int_0^x u f(u) du$ is incomplete moment of X, that for SG distribution is given by

$$\begin{aligned}\int_0^x u f(u) du &= \frac{\theta^2(1-p)}{\alpha(\theta+1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{k} (j+1) p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^k \left[\left(\frac{\alpha}{\theta(j+1)}\right)^{k+3}\right. \\ &\quad \left. \times \gamma\left(k+3; \frac{\theta}{\alpha} x(j+1)\right) + \frac{\theta}{\alpha}(j+1)\gamma\left(k+2; \frac{\theta}{\alpha} x(j+1)\right)\right].\end{aligned}$$

where $\gamma(\cdot;\cdot)$ is the lower incomplete gamma function. Hence, the Bonferroni curve of the SG distribution is given by

$$\begin{aligned}B_F[F(x)] &= \frac{\theta^2(1-p)\left[1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right) e^{\frac{-\theta}{\alpha} x}\right]}{\mu\alpha(\theta+1)\left[1-\left(1+\frac{\theta x}{\alpha(\theta+1)}\right) e^{\frac{-\theta}{\alpha} x}\right]} \\ &\quad \times \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{k} (j+1) p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^k \left[\left(\frac{\alpha}{\theta(j+1)}\right)^{k+3}\right. \\ &\quad \left. \times \gamma\left(k+3; \frac{\theta}{\alpha} x(j+1)\right) + \frac{\theta}{\alpha}(j+1)\gamma\left(k+2; \frac{\theta}{\alpha} x(j+1)\right)\right]\end{aligned}$$

where μ is given in (17).

The Lorenz curve of F that follows the SG distribution can be obtained via the expression $L_F[F(y)] = F(y)B_F[F(y)]$. The scaled total time and cumulative total time on test transform of a distribution function F (Pundir et al., [21]) respectively are defined by

$$\begin{aligned}S_F[F(t)] &= \frac{1}{\mu} \int_0^t S(u) du \\ C_F &= \int_0^1 S_F[F(t)] f(t) dt,\end{aligned}$$

where $S(\cdot)$ and $F(\cdot)$ denotes the survival function and CDF of X. Then, for SG distribution we get

$$S_F[F(t)] = \frac{1-p}{\mu} \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} \binom{j+1}{k} p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^k \left(\frac{\alpha}{\theta(j+1)}\right)^{k+1} \gamma\left(k+1; \frac{\theta}{\alpha} t(j+1)\right).$$

The Gini index can be derived from the $G = 1 - C_F$.

10. Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. The Rényi entropy is defined as

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log\left[\int f^{\gamma}(x) dx\right], \quad \gamma > 0, \gamma \neq 1.$$

Suppose $X \sim SG(\theta, \alpha, p)$, Then, one can calculate

$$
\begin{aligned}
\int f^{\gamma}(x)dx &= \frac{\theta^{2\gamma}}{(\alpha(\theta+1))^{\gamma}}(1-p)^{\gamma}\int_0^{\infty}(1+\frac{x}{\alpha})^{\gamma}e^{-\frac{\theta}{\alpha}\gamma x}\left(1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta x}{\alpha}}\right)^{-2\gamma}dx \\
&= \frac{\theta^{2\gamma}}{(\alpha(\theta+1))^{\gamma}}(1-p)^{\gamma}\sum_{j=0}^{\infty}\frac{\Gamma(2\gamma+j)}{\Gamma(2\gamma)j!}p^j \\
&\quad \times \int_0^{\infty}(1+\frac{x}{\alpha})^{\gamma}e^{-\frac{\theta}{\alpha}\gamma x}\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)^j e^{-\frac{\theta}{\alpha}xj}dx \\
&= \frac{\theta^{2\gamma}}{(\alpha(\theta+1))^{\gamma}}(1-p)^{\gamma} \\
&\quad \times \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}\frac{\Gamma(2\gamma+j)}{\Gamma(2\gamma)j!}(\frac{p}{\theta+1})^j\frac{\alpha e^{\theta(\gamma+j)}}{\theta^{\gamma}(\gamma+j)^{\gamma+k}}\int_{\theta(\gamma+j)}^{\infty}u^{k+\gamma}e^{-u}du \\
&= \frac{\theta^{2\gamma}}{(\alpha(\theta+1))^{\gamma}}(1-p)^{\gamma} \\
&\quad \times \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}\frac{\Gamma(2\gamma+j)}{\Gamma(2\gamma)j!}(\frac{p}{\theta+1})^j\frac{\alpha e^{\theta(\gamma+j)}}{\theta^{\gamma}(\gamma+j)^{\gamma+k}}\Gamma(\gamma+k+1;\theta(\gamma+j))
\end{aligned}
$$

where$\Gamma(\cdot;\cdot)$ is the upper incomplete gamma function. So, one obtains the Rényi entropy of SG distribution as

$$
\begin{aligned}
\mathcal{J}_R(\gamma) &= \frac{\gamma}{1-\gamma}\log\left(\frac{\theta^2(1-p)}{\alpha(\theta+1)}\right)+\frac{\alpha}{1-\gamma} \\
&\quad \times\log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}\frac{\Gamma(2\gamma+j)}{\Gamma(2\gamma)j!}(\frac{p}{\theta+1})^j\frac{\alpha e^{\theta(\gamma+j)}}{\theta^{\gamma}(\gamma+j)^{\gamma+k}}\Gamma\left(\gamma+k+1;\theta(\gamma+j)\right)\right]. \qquad (19)
\end{aligned}
$$

Shannon entropy defined by $-E[\log f(X)]$ is the particular case of (19) for $\gamma\uparrow 1$. Limiting $\gamma \uparrow 1$ in (19) and using L'Hospital's rule, after considerable algebraic manipulation we get

$$
\begin{aligned}
E[-\log f(X)] &= -\log\left(\frac{\theta^2}{\theta+1}\right)+\frac{\theta}{\alpha}E(X)+\sum_{k=1}^{\infty}\frac{(-1)^k}{k\alpha^k}E(X^K) \\
&\quad -\frac{2\theta^2}{\theta+1}(1-p)\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{k}{l}\binom{j+1}{k}\left(\frac{\theta}{\alpha(\theta+1)}\right)^{l+i}p^{j+k} \\
&\quad \times\frac{\Gamma(l+i+1)}{\left(\frac{\theta}{\alpha}(k+j+1)\right)^{l+i+1}}\left(1+\frac{l+i+1}{\theta(k+j+1)}\right). \qquad (20)
\end{aligned}
$$

Finally, consider the cumulative residual entropy (Rao [22]) defined by

$$\mathcal{J}_C = -\int Pr(X>x)\log Pr(X>x)dx.$$

Let $V(x) = 1-\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}$, and using the series expansion, $\log(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}$ and (5), one can calculate

$$
\begin{aligned}
\mathcal{J}_C &= (1-p)\sum_{i=1}^{\infty}\frac{1}{i}\int_0^{\infty}V(x)(1-V(x))^i(1-pV(x))^{-(i+1)}dx \\
&= \frac{\alpha(1-p)}{\theta}\sum_{i=1}^{\infty}\sum_{l=0}^{i}\sum_{j=0}^{\infty}\sum_{k=0}^{l+j}\sum_{m=0}^{k}(-1)^{l+k}\binom{i}{l}\binom{l+j}{k}\binom{k}{m} \\
&\quad \times\frac{p^j}{i(k(\theta+1))^{m+1}}\times\frac{\Gamma(i+j+1)m!}{\Gamma(i+1)j!}. \qquad (21)
\end{aligned}
$$

11. Mean deviations

The amount of scatter in a population can be measured by the totality of deviations from the mean and median. For a random variable X, with $\mu = E(X)$ and $M = Median(X)$, the mean deviation about the mean and the mean deviation about the median, are defined respectively by

$$\begin{aligned}\delta_1 &= \int_0^\infty |x-\mu| f(x)dx = 2\mu F(\mu) - 2I(\mu),\\ \delta_2 &= \int_0^\infty |x-M| f(x)dx = \mu - 2I(M),\end{aligned}$$

where $I(b) = \int_0^b x f(x)dx$. For the SG distribution we have

$$\begin{aligned}I(b) &= \frac{\theta^2(1-p)}{(\theta+1)} \sum_{j=0}^{\infty}\sum_{k=0}^{j} \binom{j}{k} (j+1)p^j \left(\frac{\theta}{\alpha(\theta+1)}\right)^k \left[\left(\frac{\alpha}{\theta(j+1)}\right)^{k+3}\right.\\ &\quad \times \left.\left(\gamma(k+3;\frac{\theta}{\alpha}b(j+1)) + \frac{\theta}{\alpha}(j+1)\gamma(k+2;\frac{\theta}{\alpha}b(j+1))\right)\right]. \qquad (22)\end{aligned}$$

The Mean deviations of the SG distribution respectively are given by

$$\delta_1 = 2\mu \left[\frac{1-\left(1+\frac{\theta\mu}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}\mu}}{1-p\left(1+\frac{\theta\mu}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}\mu}}\right] - 2I(\mu),$$

and

$$\delta_2 = \mu - 2I(M),$$

where μ and M are defined in (17) and (16) respectively.

12. Maximum Likelihood Estimation

It is well known that the MLE is often used to estimate the unknown parameters of a distribution because of its attractive properties, such as consistency and asymptotic normality. In this section the MLEs of the parameters θ ,α and p are derived. Let$X_1, \cdots, X_n$ be a random sample from the SG distribution with unknown vector of parameters $\boldsymbol{\theta} = (\theta, \alpha, p)'$. Then the log-likelihood function is given by

$$\begin{aligned}\log f(x;\boldsymbol{\theta}) &= 2n\log(\theta) - n\log(1+\theta) + n\log(1-p) - n\log(\alpha) + \sum_{i=1}^{n}\log\left(1+\frac{x_i}{\alpha}\right)\\ &\quad -\frac{\theta}{\alpha}\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\log\left(1-p\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}\right). \qquad (23)\end{aligned}$$

The MLEs of the unknown parameters can be obtained by taking the first partial derivatives of (23) with respect to θ, α and p. We get the following likelihood equations

$$\begin{aligned}\frac{\partial \log f(x;\boldsymbol{\theta})}{\partial p} &= \frac{-n}{1-p} + 2\sum_{i=1}^{n}\frac{\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}}{1-p\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}} = 0\\ \frac{\partial \log f(x;\boldsymbol{\theta})}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{1+\theta} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i - 2p\sum_{i=1}^{n}\frac{\left(\frac{1}{\alpha}+\frac{\theta x_i}{\alpha^2(\theta+1)}-\frac{1}{\alpha(\theta+1)^2}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{1-p\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \log f(x;\boldsymbol{\theta})}{\partial \alpha} &= \frac{-n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha+x_i}\right) + \frac{\theta}{\alpha^2}\sum_{i=1}^{n} x_i \\ &\quad -2p\sum_{i=1}^{n}\frac{\left(\frac{\theta}{\alpha^2(\theta+1)} - \frac{\theta}{\alpha^2} - \frac{\theta^2 x_i}{\alpha^3(\theta+1)}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{1-p\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}} = 0.\end{aligned}$$

The solutions of these nonlinear equations do not have a closed form, so numerical methods can be employed to get the MLEs.

12.1. Expectation-Maximization Algorithm

An EM algorithm is used to estimate the parameters when some observations are treated as incomplete data. Suppose that $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ and $\boldsymbol{Z} = (Z_1, Z_2, \cdots, Z_n)$ represent the observed and hypothetical data, respectively. Here, the hypothetical data can be thought of as missing data because $Z_1, Z_2, \cdots, Z_n$ are not observable. We formulate the problem of finding the MLEs as an incomplete data problem, and thus, the EM algorithm is applicable to determine the MLEs of the SG distribution. Let $\boldsymbol{W} = (\boldsymbol{X}, \boldsymbol{Z})$ denote the complete data. To start this algorithm, define the PDF of each (X_i, Z_i) for $i = 1, ..., n$ as

$$f(x,z;\boldsymbol{\theta}) = (1-p)\frac{\theta^2 z}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\left(p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)^{z-1}$$

where $\boldsymbol{\theta} = (\theta, \alpha, p), \quad x > 0, z \in N.$

Under the formulation, the E-step of an EM cycle requires the expectation of $(Z|X;\boldsymbol{\theta}^r)$ where $\boldsymbol{\theta}^{(r)} = (\theta^{(r)}, \alpha^{(r)}, p^{(r)})$, is the current estimate (in the rth iteration) of $\boldsymbol{\theta}$.

The PDF of Z given X is given by

$$f(z|x) = z\left(p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)^{z-1}\left(1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)^2.$$

Therefore, its expected value is given by

$$E(Z|X;\boldsymbol{\theta}) = \frac{\left(1+p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)}{\left(1-p\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x}\right)}.$$

The EM cycle is completed with the M-step by using the maximum likelihood estimation over $\boldsymbol{\theta}$, with the missing $Z's$ replaced by their conditional expectations given above. The log-likelihood for the complete-data is

$$\begin{aligned}\ln^*(x,z,\boldsymbol{\theta}) &= \sum_{i=1}^{n}\log z_i + 2n\log(\theta) - n\log(1+\theta) + n\log(1-p) - n\log(\alpha) \\ &\quad + \sum_{i=1}^{n}\log(1+\frac{x_i}{\alpha}) - \frac{\theta}{\alpha}\sum_{i=1}^{n} x_i \\ &\quad - \sum_{i=1}^{n}(z_i-1)\log\left(p\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}\right).\end{aligned}$$

The components of the score function are given by

$$\frac{\partial l}{\partial p} = \sum_{i=1}^{n}\frac{z_i-1}{p} - \frac{n}{1-p}$$

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{1+\theta} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}(z_i-1)\frac{\left(\frac{-1}{\alpha}+\frac{\theta x_i}{\alpha^2(\theta+1)}+\frac{1}{\alpha(\theta+1)^2}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}}\\
\frac{\partial l}{\partial \alpha} &= \frac{-n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha+x_i}\right) + \frac{\theta}{\alpha^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}(1-z_i)\frac{\left(-\frac{\theta}{\alpha^2(\theta+1)}+\frac{\theta}{\alpha^2}+\frac{\theta^2 x_i}{\alpha^3(\theta+1)}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}}.
\end{aligned}$$

From a nonlinear system of equations $U^*(\boldsymbol{\theta}) = 0$, we obtain the interative procedure of the EM algorithm as

$$\begin{aligned}
\hat{p}^{(r+1)} &= 1 - \frac{n}{\sum_{i=1}^{n} z_i^{(r)}}\\
\frac{2n}{\hat{\theta}^{(r+1)}} &- \frac{n}{1+\hat{\theta}^{(r+1)}} - \frac{1}{\hat{\alpha}^{(r+1)}}\sum_{i=1}^{n} x_i\\
&- \sum_{i=1}^{n}(z_i^{(r)}-1)\frac{\left(\frac{-1}{\hat{\alpha}^{(r+1)}}+\frac{\hat{\theta}^{(r+1)}x_i}{\hat{\alpha}^{2(r+1)}(\hat{\theta}^{(r+1)}+1)}+\frac{1}{\hat{\alpha}^{(r+1)}(\hat{\theta}^{(r+1)}+1)^2}\right)x_i e^{-\frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{(r+1)}}x_i}}{\left(1+\frac{\hat{\theta}^{(r+1)}x_i}{\hat{\alpha}^{(r+1)}(\hat{\theta}^{(r+1)}+1)}\right)e^{-\frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{(r+1)}}x_i}}\\
\frac{-n}{\hat{\alpha}^{(r+1)}} &- \frac{1}{\hat{\alpha}^{(r+1)}}\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\alpha}^{(r+1)}+x_i}\right) + \frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{2(r+1)}}\sum_{i=1}^{n} x_i\\
&- \sum_{i=1}^{n}(1-z_i^{(r)})\frac{\left(-\frac{\hat{\theta}^{(r+1)}}{\alpha^{2(r+1)}(\hat{\theta}^{(r+1)}+1)}+\frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{2(r+1)}}+\frac{\hat{\theta}^{2(r+1)}x_i}{\hat{\alpha}^{3(r+1)}(\hat{\theta}^{(r+1)}+1)}\right)x_i e^{-\frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{(r+1)}}x_i}}{\left(1+\frac{\hat{\theta}^{(r+1)}x_i}{\hat{\alpha}^{(r+1)}(\hat{\theta}^{(r+1)}+1)}\right)e^{-\frac{\hat{\theta}^{(r+1)}}{\hat{\alpha}^{(r+1)}}x_i}}
\end{aligned}$$

where $\hat{\theta}^{(r+1)}$ and $\hat{\alpha}^{(r+1)}$ are found numerically. Hence for $i = 1, ..., n$ we have

$$z_i^{(r)} = \frac{\left(1+\hat{p}^{(r)}\left(1+\frac{\hat{\theta}^{(r)}x_i}{\hat{\alpha}^{(r)}(\hat{\theta}^{(r)}+1)}\right)e^{-\frac{\hat{\theta}^{(r)}}{\hat{\alpha}^{(r)}}x_i}\right)}{\left(1-\hat{p}^{(r)}\left(1+\frac{\hat{\theta}^{(r)}x_i}{\hat{\alpha}^{(r)}(\hat{\theta}^{(r)}+1)}\right)e^{-\frac{\hat{\theta}^{(r)}}{\hat{\alpha}^{(r)}}x_i}\right)}.$$

13. Simulation study

In this section, the performance of the MLEs is assessed via simulation in terms of the sample size n. Also, the empirical bias and empirical mean square error (MSE), for different values of parameters are calculated. Let $\boldsymbol{\theta} = (\theta, \alpha, p)'$ be the parameters vector. Given n and $\boldsymbol{\theta}$, the following algorithm to calculate the biases and MSEs, for $i = 1, 2, 3$ is applied.

Algorithm 1

(i) Generate the values $x_1, x_2, \cdots, x_n$ from the $SG(\theta, \alpha, p)$ using (15).

(ii) Compute $\widehat{\boldsymbol{\theta}} = (\hat{\theta}, \hat{\alpha}, \hat{p})'$ for $x_1, \cdots, x_n$.

(iii) Repeat the steps (i) and (ii) for 10000 times.

(iv) Obtain the bias using the formulas bias($\hat{\theta}_i$)=$\frac{1}{10000}\sum_{j=1}^{10000}\hat{\theta}_{ij} - \theta_i$ and MSE($\hat{\theta}_i$)=$\frac{1}{10000}\sum_{j=1}^{10000}(\hat{\theta}_{ij} - \theta_i)^2$ where $\hat{\theta}_{ij}$ denote the MLE of θ_i in the jth replication, for $i = 1, 2, 3$ and $j = 1, ..., 10000$. Table (2) gives the biases and the MSE of $\hat{\theta}, \hat{\alpha}, \hat{p}$. It can be concluded from the table that the efficiency of the MLE estimation method increased with the increase of the sample size where the bias and the MSE got smaller.

Table 2. The biase and MSE (in parentheses) of MLEs

(α, p, θ)	n	$\hat{\alpha}$	$\hat{p}$	$\hat{\theta}$
$(1, 0.5, 0.5)$	20	$-0.4087(0.3428)$	$-0.0104(0.1097)$	$-0.1857(0.1085)$
	50	$-0.3276(0.2474)$	$0.05(0.0742)$	$-0.1469(0.0726)$
	100	$-0.296(0.2063)$	$0.0572(0.0509)$	$-0.1397(0.0543)$
	200	$-0.2484(0.1535)$	$0.0645(0.0297)$	$-0.1163(0.0367)$
	500	$-0.2095(0.1114)$	$0.0554(0.016)$	$-0.0995(0.0262)$
	1000	$-0.1677(0.0746)$	$0.0441(0.0101)$	$-0.0803(0.0174)$
$(0.5, 0.5, 0.8)$	20	$-0.1345(0.1102)$	$-0.0648(0.1238)$	$-0.2503(0.236)$
	50	$-0.0911(0.0787)$	$0.0209(0.0869)$	$-0.1657(0.1647)$
	100	$-0.0609(0.0579)$	$0.014(0.0561)$	$-0.1007(0.1136)$
	200	$-0.0386(0.0432)$	$-0.0024(0.0384)$	$-0.0731(0.0856)$
	500	$-0.0253(0.028)$	$0.0096(0.0199)$	$-0.0421(0.0559)$
	1000	$-0.008(0.0191)$	$0.0056(0.011)$	$-0.0132(0.0386)$
$(1, -0.5, 1.5)$	20	$-0.6398(0.5599)$	$0.6861(0.5462)$	$-0.8589(1.2468)$
	50	$-0.6134(0.4812)$	$0.645(0.4691)$	$-0.8323(1.0426)$
	100	$-0.5725(0.4084)$	$0.5782(0.3583)$	$-0.7886(0.8819)$
	200	$-0.5498(0.3579)$	$0.5508(0.3171)$	$-0.7541(0.7522)$
	500	$-0.5429(0.3152)$	$0.5215(0.2764)$	$-0.7607(0.6428)$
	1000	$-0.5423(0.3039)$	$0.5085(0.2598)$	$-0.7643(0.6152)$
$(0.8, 0.2, 1.5)$	20	$-0.3237(0.2778)$	$0.1377(0.1288)$	$-0.5418(0.972)$
	50	$-0.2306(0.195)$	$0.1063(0.099)$	$-0.3949(0.6862)$
	100	$-0.1813(0.1604)$	$0.0985(0.0776)$	$-0.3148(0.5583)$
	200	$-0.1275(0.1216)$	$0.0775(0.0559)$	$-0.214(0.418)$
	500	$-0.0484(0.0883)$	$0.035(0.0352)$	$-0.085(0.2923)$
	1000	$-0.0253(0.0693)$	$0.0201(0.0244)$	$-0.0466(0.222)$

14. Application to Real-Data

In this section, the comparison of the SG with some continuous life time distributions is considerd. For this purpose, we consider two real data set to analyze. In the first data set, we show that the SG is a good competitor for four important life time distributions, Gamma, Weibull , Weibull-Geometric(WG) and Lindley-Poisson(LP). In the second set, it is observed that the SG distribution performs well comparing to its special submodels. In order to identify the shape of the hazard rate function of the data, we consider a graphical method based on the Total Time on Test (TTT) plot. As we know, the empirical TTT plot is given by

$$G\left(\frac{r}{n}\right) = \frac{\left(\sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}\right)}{\sum_{i=1}^{n} X_{i:n}}$$

where $X_{i:n}$ denotes the ith order statistic of the sample. If the empirical TTT transform is convex, concave, convex then concave and concave then convex, the shape of the corresponding hazard rate function is, respectively, decreasing, increasing, bathtub-shaped and upside-down bathtub, see Aarset [1]. To compare the SG with the four distributions (i) $Gamma(\alpha, \beta)$, (ii) $Weibull(\alpha, \beta)$, (iii) $Weibull - Geometric(\alpha, \beta, p)$, (iv) $Lindley - Poisson(\theta, \lambda)$ with PDF listed

$$\begin{aligned} f_1(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}; \quad \alpha, \beta > 0 \\ f_2(x) &= \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}; \quad \alpha, \beta > 0 \\ f_3(x) &= \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}} \left[1 - pe^{-(\beta x)^{\alpha}}\right]^{-2}; \quad \alpha, \beta > 0, \ p < 1 \end{aligned}$$

$$f_4(x) = \frac{\lambda\theta^2(x+1)e^{\frac{\lambda e^{-\theta x}(\theta x+\theta+1)}{(\theta+1)}-\theta x}}{(\theta+1)(e^{\lambda}-1)}; \quad \theta,\lambda>0$$

the values of the log-likelihood($-\log L$), Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the AIC with a correction (AICc) for the real data set are calculated. The better distribution corresponds to smaller -logL, AIC, BIC, and AICc. In addition, we apply the Kolmogrov-Smirnov statistic (and associated p-value) to verify which distribution fits better to data.

The first data set is about the remission time (in months) of a random sample of 128 bladder cancer patients. This data set was studied by [14] in fitting the extended Lomax distribution and [29] for the modified Weibull geometric distribution. The first data set is given by as follows:
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14 ,79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.
From Figure 3, we can notice that, the hazard rate of the data set is upside down bathtub. We observe from Table (3) and Table (4) that the SG distribution provide an improved fit over other distributions that are fitting lifetime data. The fitted density, the empirical CDF plot and p-p plot of the SG distribution model are presented in Figure 5. The Figure indicates a desirable fit of the SG distribution.

The second data set studied by Maguire et al. [17], represent the time intervals between two deadly accidents in the mines of the Division no.5, of Great Britain National Cole Board in 1950. The second data set is given by as follows:
21, 2, 15, 1, 5, 1, 9, 1, 0, 17, 0, 1, 24, 14, 4, 9, 20, 14, 1, 1, 44, 4, 5, 1, 13, 6, 9, 3.
We intend to illustrate the applicability of the new distribution, hence we fit SG distribution for the data. We compare the SG distribution with two continuous life time distributions that are sepecial submodels of SG distribution (i) $Lindley(\theta)$, (ii) $Lindley-Geometric(\theta,p)$ with PDF listed

$$f_1(x) = \frac{\theta^2}{\theta+1}(1+x)\,e^{-\frac{\theta}{x}}; \quad \theta>0$$
$$f_2(x) = \frac{\theta^2}{\theta+1}(1+x)(1-p)e^{-\theta x}\left(1-p\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)^{-2}; \quad \theta>0, 0<p<1$$

From Figure 4, we can notice that, the hazard rate of the data set is decreasing. We observe from Table (5) that the SG distribution provides an improved fit over other distributions that are commonly used for fitting lifetime data.

To test the null hypothesis H_0: Lindley-Geometric versus H_1: SG or equivalently $H_0:\alpha=1$ versus $H_1:\alpha\neq 1$, we use the likelihood ratio test statistic whose value is 4.4696 (p-value=0.0345). As result, the null model LG is rejected in favore of the alternative model SG at any level>0.0345. To test the null hypothesis H_0: Lindley versus H_1: SG or equivalently $H_0:\alpha=1, p=0$ versus $H_1:\alpha\neq 1, p\neq 0$, we use the likelihood ratio test statistic whose value is 11.077 (p-value=0.0039). So, the null model Lindley is rejected in favore of the alternative model SG at any level>0.0039, the results are shown in Table (6). The fitted density and the empirical cdf plot of the SG distribution model are presented in Figure 6. The figure represent a desirable fit of the SG distribution.

15. Conclusion

A new three-parameter distribution named the SG distribution is proposed. This new model contains Lindley, Lindley-Geometric, and Sushila distributions as submodels. The hazard rate function of the SG distribution can be an upside-down bathtub, bathtub-shaped, increasing, and decreasing which are widely used in applications.

Some mathematical properties such as density expansion, hazard rate function, quantile function, moments, mean deviations, order statistics, and the Shannon and Rényi entropies in the closed forms in terms of some well-known mathematical functions are obtained. The log-likelihood equations were obtained and the EM algorithm was presented to calculate the MLEs of parameters. The performance of the MLEs was investigated for different values of sample sizes and different values of the parameters using simulation. The simulation results show that the MLEs perform well in terms of bias and MSE criteria. Finally, we fitted the SG model to two real data sets to show the potential of the new proposed distribution.

Table 3. MLEs of the parameters, standard errors (in parentheses) and corresponding criteria for data set 1.

$Model$	$parameters$	$-\log L$	AIC	BIC	$AICc$
$Gamma$	$\hat{\alpha} = 1.1726(0.131)$	413.3678	830.7356	836.4396	830.8316
	$\hat{\beta} = 0.1252(0.017)$				
$Weibull$	$\hat{\alpha} = 9.5607(0.853)$	414.0869	832.1738	837.8778	832.2698
	$\hat{\beta} = 1.0478(0.068)$				
WG	$\hat{\alpha} = 1.6042(0.159)$	410.0921	826.1842	834.7403	826.3777
	$\hat{\beta} = 0.0286(0.012)$				
	$\hat{p} = 0.9362(0.059)$				
LP	$\hat{\theta} = 0.1103(0.019)$	411.3845	826.7691	833.4732	826.8651
	$\hat{\lambda} = 3.1743(0.996)$				
SG	$\hat{\theta} = 0.0469(0.039)$	409.3992	824.7984	833.3045	824.9919
	$\hat{\alpha} = 0.6444(0.462)$				
	$\hat{p} = 0.9023(0.086)$				

Table 4. K-S test for data set 1.

$Model$	$K-S$	$p-value$
$Gamma$	0.073209	0.4989
$Weibull$	0.070037	0.5566
WG	0.032507	0.9993
LP	0.059978	0.7465
SG	0.31345	0.9996

Table 5. MLEs of the parameters, standard errors (in parentheses) and corresponding criteria for data set 2.

$Model$	$parameters$	$-\log L$	AIC	BIC	$AICc$
$Lindley$	$\hat{\theta} = 0.2088(2.188)$	94.31056	190.6211	191.9533	190.775
LG	$\hat{\theta} = 0.1022(0.051)$ $\hat{p} = 0.8677(0.867)$	91.0067	186.0134	188.6778	186.4934
SG	$\hat{\theta} = 0.2921(0.312)$ $\hat{\alpha} = 8.4412(6.438)$ $\hat{p} = 0.9555(0.154)$	88.77188	181.5438	184.2082	182.0238

Table 6. K-S test for data set 2.

$Model$	$K-S$	$p-$value
$Lindely$	0.27316	0.03064
LG	0.22807	0.1086
SG	0.16569	0.4356

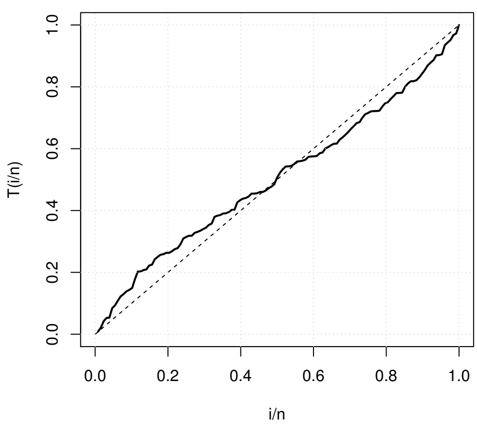

Figure 3. The empirical TTT plot of the data set 1.

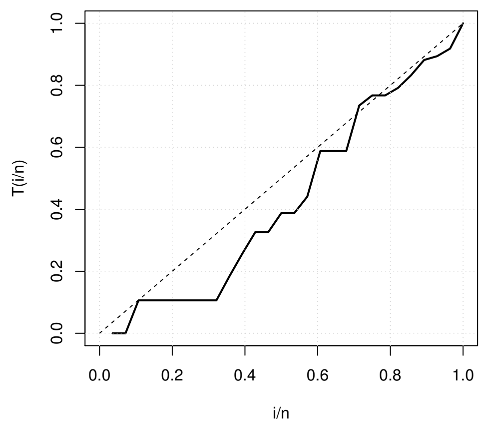

Figure 4. The empirical TTT plot of the data set 2.

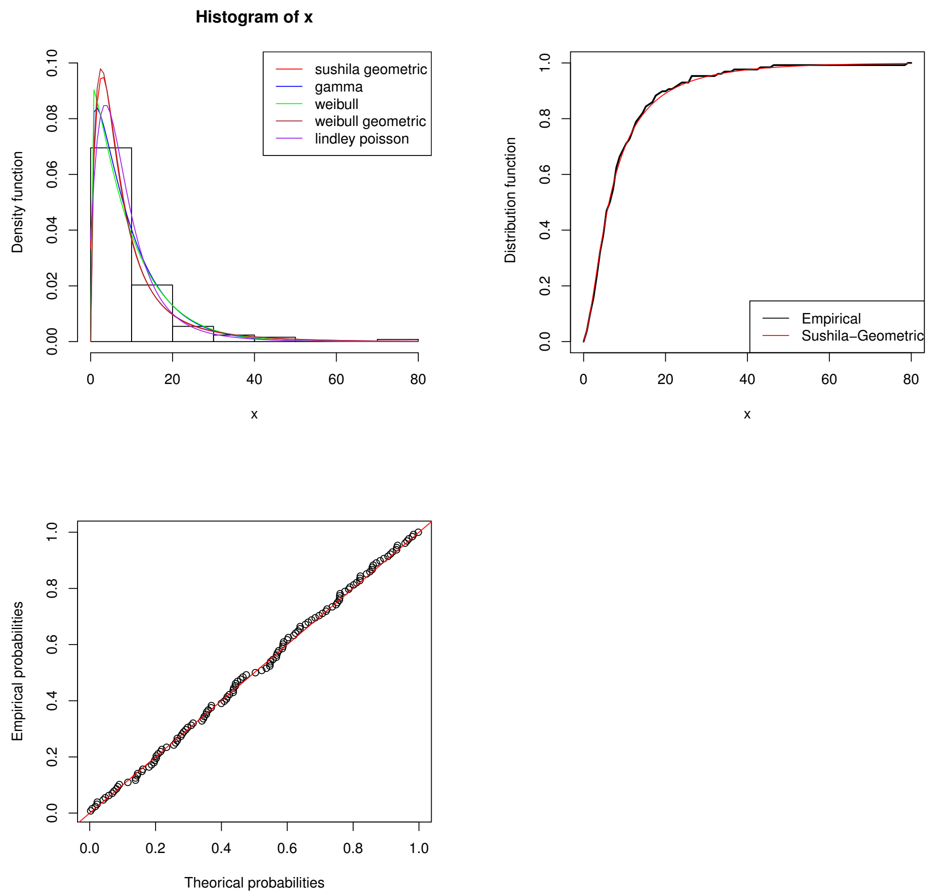

Figure 5. Plots of the estimated pdf and cdf and p-p plot of the SG model for data set1

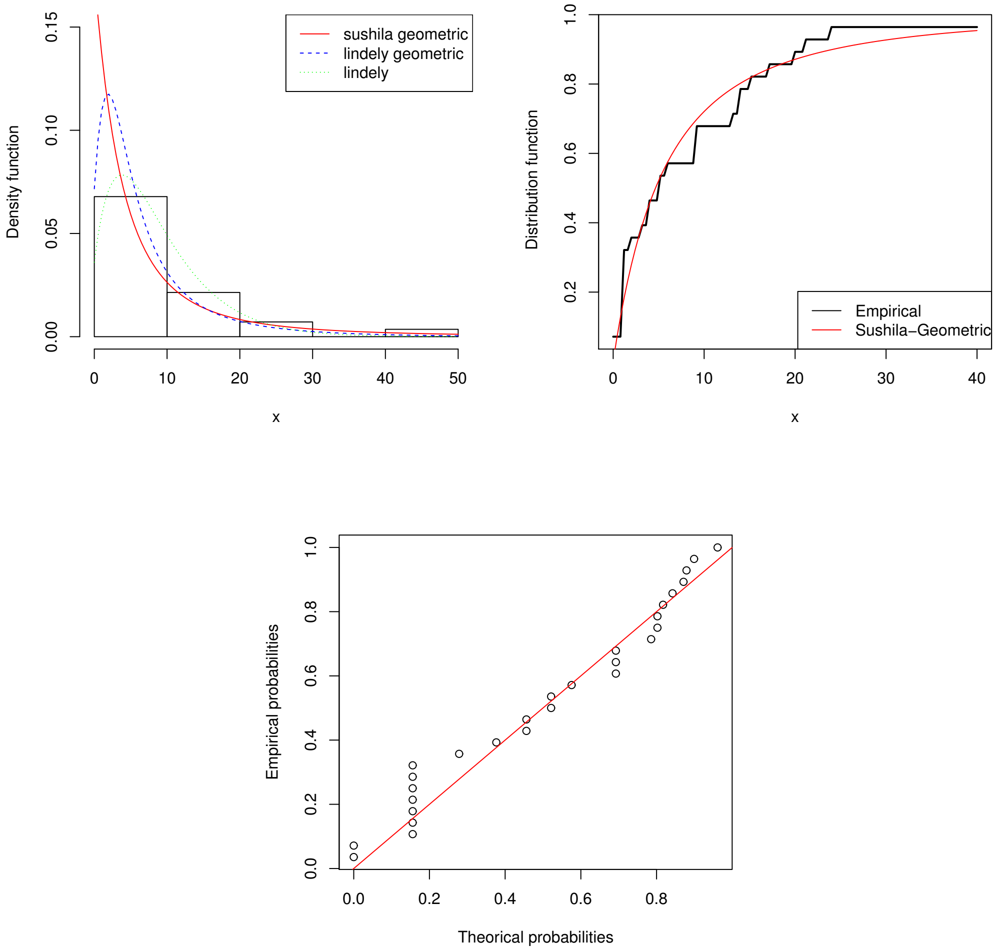

Figure 6. Plots of the estimated pdf and cdf and p-p plot of the SG model for data set2

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