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Strong consistency of a deconvolution estimator of cumulative distribution function

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Abstract We study the strong consistency of a deconvolution estimator of cumulative distribution function when the distribution of error variable is assumed to be known exactly and ordinary smooth as well as supersmooth.

Keywords Cumulative distribution function, Deconvolution problem, Ordinary smooth error, Supersmooth error, Strong consistency.

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1. Introduction

Measurement error models have been attracted a lot of attentions in many scientific and engineering fields, see, e.g. Carroll et al. [2]. In this work, we consider the additive measurement error model

$$Y = X + \varepsilon, \tag{1}$$

where X is an unobservable random variable of interest distributed with an unknown density f_X, ε is an unobservable random error distributed with a known density f_ε, called error density, and Y is an observable random variable which can be viewed as a noisy version of X. Furthermore, X and ε are assumed to be independent. Let $Y_1, \dots, Y_n$ be a sample of independent and identically distributed (i.i.d.) observations from the distribution of Y. On the basis of the observations and the completely knowledge about f_ε, the problem of estimating the unknown cumulative distribution function F_X of X, i.e. $F_X(x) := \mathbb{P}(X \le x)$ for $x \in \mathbb{R}$, has been studied in some research, such as Gaffey [6], Fan [5], Dattner et al. [4], Dattner and Reiser [3] and Trong and Phuong [11]. More concretely, Gaffey [6] considered the asymptotic mean squared error for an estimator of F_X when ε has the normal distribution with zero mean and known variance σ^2. Fan [5] proposed an estimator of F_X derived by integrating the kernel deconvolution density estimator in Stefanski and Carroll [10] and then derived rates of convergence of his estimator with respect to the mean squared error when f_ε is ordinary smooth or supersmooth. Here a density f is said to be ordinary smooth (respectively, supersmooth) if the corresponding characteristic function decays at infinity with a polynomial rate (respectively, an exponential rate). Dattner et al. [4] (respectively, Dattner and Reiser [3]) showed the optimality of the estimator $\widehat{F}_{X;T}$ with $T > 0$, defined by

$$\widehat{F}_{X;T}(x) := \frac{1}{2} - \frac{1}{\pi}\int_0^T \frac{1}{t}\Im\left\{\frac{\frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\mathrm{i}t(Y_j - x)}}{\phi_\varepsilon(t)}\right\}\mathrm{d}t, \tag{2}$$

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in the case of ordinary smooth f_ε (respectively, supersmooth f_ε) under the mean squared error. It is emphasized that the error characteristic function ϕ_ε does not have any zeros on the real line $\mathbb{R}$ when f_ε is ordinary smooth or supersmooth. Recently, Trong and Phuong [11] studied the problem of estimating F_X in some cases of f_ε where ϕ_ε may have some isolated zeros on $\mathbb{R}$. The authors also considered some asymptotic properties of their estimator with respect to the mean squared error.

As mentioned, $\widehat{F}_{X;T}$ in (2) is an optimal estimator of F_X with respect to the mean squared error under the assumption that f_ε is ordinary smooth or supersmooth. A further question is whether $\widehat{F}_{X;T}$ is a strong consistent estimator of F_X under the same assumption on f_ε. Recall here that a deconvolution estimator $\widehat{F}_X(\cdot; Y_1, \ldots, Y_n)$ of $F_X(\cdot)$ is said to be strong consistency if $\widehat{F}_X(\cdot; Y_1, \ldots, Y_n)$ converges almost surely to $F_X(\cdot)$ as $n \to \infty$. Up to present, we have not found any research addressing that question, and our aim in this work is thus to give an answer for this question.

2. The main result

We first introduce some notations. The L^1-norm of $f \in L^1(\mathbb{R})$ is the quantity $\|f\|_1 := \int_{-\infty}^{\infty} |f(x)| \mathrm{d}x$. The characteristic function of a random variable U is defined by $\phi_U(t) := \mathbb{E}(\mathrm{e}^{\mathrm{i}tU})$, for $t \in \mathbb{R}$. The notation $\overset{a.s.}{\to}$ represents the almost sure convergence of a sequence of random variables. For a complex number z, the notations $\Re\{z\}$ and $\Im\{z\}$ stand respectively for the real and imaginary parts of z.

To study the strong consistency of $\widehat{F}_{X;T}$, we introduce the following assumptions on the error characteristic function ϕ_ε:

(A1) There exist constants $K, \xi, A > 0$ such that

$$|\phi_\varepsilon(t)| \geq 1 - K|t|^\xi, \quad \text{for all } \ t \in [-A, A].$$

(A2) There exist constants $c_1, c_2 > 0$ and $\alpha > 1$ such that

$$c_1(1 + |t|)^{-\alpha} \leq |\phi_\varepsilon(t)| \leq c_2(1 + |t|)^{-\alpha}, \quad \text{for all } \ t \in \mathbb{R}.$$

(A3) There exist constants $d_1, d_2, d, \beta > 0$ such that

$$d_1 \mathrm{e}^{-d|t|^\beta} \leq |\phi_\varepsilon(t)| \leq d_2 \mathrm{e}^{-d|t|^\beta}, \quad \text{for all } \ t \in \mathbb{R}.$$

The assumption (A1) describes local behavior of the function ϕ_ε around the point $t = 0$. It is satisfied if ϕ_ε is smooth at that point. For example, if ε has the normal distribution with zero mean and variance σ^2, then $\phi_\varepsilon(t) = \mathrm{e}^{-\sigma^2 t^2/2}$ for all $t \in \mathbb{R}$, so we apply the elementary inequality $\mathrm{e}^x \geq 1 + x$ for all $x \in \mathbb{R}$ to obtain that $|\phi_\varepsilon(t)| \geq 1 - \sigma^2 t^2/2$ for all $t \in \mathbb{R}$. Hence, ϕ_ε satisfies (A1) with $K \equiv \sigma^2/2$, $\xi \equiv 2$ and $A > 0$ arbitrarily. Another example is the case $f_\varepsilon(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$, i.e. ε has the Laplace distribution with location 0 and scale 1. In that case, $\phi_\varepsilon(t) = 1/(1 + t^2)$ for all $t \in \mathbb{R}$, so ϕ_ε satisfies (A1) with $K \equiv 1$, $\xi \equiv 2$ and $A > 0$ arbitrarily. The assumptions (A2) and (A3) are quite standard in the field of nonparametric deconvolution. The assumption (A2) is satisfied with many usual distributions, such as the Laplace and Gamma distributions. The normal and Cauchy distributions are typical examples for the assumption (A3). Under (A2), the error density f_ε is continuously differentiable on $\mathbb{R}$ up to order $\lfloor \alpha \rfloor$, the biggest integer number less than or equal to α, and so f_ε is called an ordinary smooth density. While, the density f_ε from (A3) is infinitely continuously differentiable on $\mathbb{R}$, so it is called a supersmooth density. The ordinary smooth and supersmooth terms were first introduced in Fan [5].

The main result of the present work is the following theorem.

Theorem 1
Consider the estimator $\widehat{F}_{X;T}$ given by (2). Assume $\int_R^\infty t^{-1}|\phi_X(t)| \mathrm{d}t < \infty$, for some $R > 0$.

(a) Under (A1) and (A2), choosing $T = k\,n^a$ with $k > 0$, $0 < a < 1/(4\alpha - 1)$ gives $\widehat{F}_{X;T}(x) \overset{a.s.}{\to} F_X(x)$ as $n \to \infty$, for all $x \in \mathbb{R}$.

(b) Under (A1) and (A3), choosing $T = (b \ln n)^{1/\beta}$ with $0 < b < 1/(6d)$ gives $\widehat{F}_{X;T}(x) \overset{a.s.}{\to} F_X(x)$ as $n \to \infty$, for all $x \in \mathbb{R}$.

To prove Theorem 1, we need the following lemmas.

Lemma 1 (see Corollary C.1(ii) in Härdle et al. [8])
Let $U_1, \dots, U_n$ be i.i.d. random variables such that $\mathbb{E}(U_j) = 0$ for all j, and that there exists a constant $M > 0$ so that $|U_j| \le M$ almost surely, for all j. Then, for any $\rho \ge 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} U_j\right| \ge \rho\right) \le 2\exp\left\{-\frac{n\rho^2}{2\left(\mathbb{E}|U_1|^2 + \frac{M\rho}{3}\right)}\right\}.$$

Lemma 2 (see Proposition 7.2.3(a) in Athreya and Lahiri [1])
Let $\{A_n\}_{n\ge1}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\sum_{n=1}^{\infty} \mathbb{P}(|A_n| > \epsilon) < \infty$, for all $\epsilon > 0$, then

$$\mathbb{P}(\lim_{n\to\infty} A_n = 0) = 1.$$

Proof of Theorem 1
Fix $x \in \mathbb{R}$. We have

$$|\widehat{F}_{X;T}(x) - F_X(x)| \le |\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| + |\mathbb{E}\widehat{F}_{X;T}(x) - F_X(x)|. \tag{3}$$

By the Fubini theorem,

$$\begin{aligned}\mathbb{E}\widehat{F}_{X;T}(x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^T \frac{1}{t}\Im\left\{\frac{\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(\mathrm{e}^{\mathrm{i}tY_j})\mathrm{e}^{-\mathrm{i}tx}}{\phi_\varepsilon(t)}\right\}\mathrm{d}t \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^T \frac{1}{t}\Im\{\phi_X(t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t.\end{aligned}$$

Since X is a random variable of continuous type, one has (see, e.g. Gil-Pelaez [7])

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{t}\Im\{\phi_X(t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t.$$

Therefore,

$$|\mathbb{E}\widehat{F}_{X;T}(x) - F_X(x)| = \left|\frac{1}{\pi}\int_T^\infty \frac{1}{t}\Im\{\phi_X(t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t\right| \le \frac{1}{\pi}\int_T^\infty \frac{|\phi_X(t)|}{t}\mathrm{d}t.$$

Since $\lim_{n\to\infty} T = \infty$ and $\int_R^\infty t^{-1}|\phi_X(t)|\mathrm{d}t < \infty$, we apply the Lebesgue dominated convergence theorem to give $\lim_{n\to\infty}\int_T^\infty t^{-1}|\phi_X(t)|\mathrm{d}t = 0$, which implies $|\mathbb{E}\widehat{F}_{X;T}(x) - F_X(x)| \to 0$ as $n \to \infty$. From this result and the estimate (3), it remains to show that $|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| \overset{a.s.}{\to} 0$ as $n \to \infty$.

For that purpose, we write

$$\begin{aligned}|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| &= \left|\frac{1}{\pi}\int_0^T \frac{1}{t}\Im\left\{\frac{\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{\mathrm{i}t(Y_j - x)} - \phi_Y(t)\mathrm{e}^{-\mathrm{i}tx}}{\phi_\varepsilon(t)}\right\}\mathrm{d}t\right| \\ &= \left|\frac{g(T)}{n}\sum_{j=1}^{n} W_{j,T}(x)\right|,\end{aligned} \tag{4}$$

where

$$g(T) := \begin{cases} T^{\alpha} & \text{under (A2)}, \\ T^{-\beta} e^{d T^{\beta}} & \text{under (A3)}, \end{cases}$$

$$W_{j,T}(x) := \frac{1}{\pi g(T)} \int_0^T \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{(\mathrm{e}^{\mathrm{i}tY_j} - \phi_Y(t))\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t.$$

By the Fubibi theorem, we have for any $j \in \{1, \dots, n\}$ that

$$\mathbb{E}W_{j,T}(x) = \frac{1}{\pi g(T)} \int_0^T \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{(\mathbb{E}(\mathrm{e}^{\mathrm{i}tY_j}) - \phi_Y(t))\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t = 0. \tag{5}$$

Our next goal is to show that there is a constant $C > 0$ such that $|W_{j,T}(x)| \le C$, for all x, j. Indeed, we have

$$|W_{j,T}(x)| \le \frac{1}{\pi g(T)}(V_1 + V_2) \tag{6}$$

with

$$V_1 := \left| \int_0^{\varrho} \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{(\mathrm{e}^{\mathrm{i}tY_j} - \phi_Y(t))\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t \right|,$$

$$V_2 := \left| \int_{\varrho}^T \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{(\mathrm{e}^{\mathrm{i}tY_j} - \phi_Y(t))\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t \right|,$$

$$\varrho := \min\{A; (4K)^{-1/\xi}\}.$$

Now we estimate V_1. We first have

$$V_1 \le \left| \int_0^{\varrho} \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j - x)}\}\mathrm{d}t \right| + \left| \int_0^{\varrho} \frac{1}{t|\phi_\varepsilon(t)|^2} \Im\{\phi_Y(t)\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t \right| =: V_{1,1} + V_{1,2}. \tag{7}$$

Fix a $t \in (0, \varrho)$. Using the assumption (A1) gives $|\phi_\varepsilon(t)|^2 \ge (1 - Kt^\xi)^2 \ge 1 - 2Kt^\xi$, so $0 \le 1 - |\phi_\varepsilon(t)|^2 \le 2Kt^\xi \le 2K\varrho^\xi \le 1/2$. Hence, we have the Taylor expansion $|\phi_\varepsilon(t)|^{-2} = 1 + \sum_{\ell=1}^\infty (1 - |\phi_\varepsilon(t)|^2)^\ell$. From the expansion, the Fubini theorem, the estimate $\sup_{m>0} |\int_0^m \tau^{-1} \sin(\tau)\mathrm{d}\tau| < 2$ (see, e.g. Kawata [9, page 61]) and the equality $\sum_{\ell=1}^\infty \frac{x^\ell}{\ell} = -\ln(1 - x)$ for all $x \in (0, 1)$, we get

$$\begin{aligned} V_{1,1} &= \left| \int_0^{\varrho} \frac{1}{t} \left(1 + \sum_{\ell=1}^\infty (1 - |\phi_\varepsilon(t)|^2)^\ell \right) \Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j - x)}\}\mathrm{d}t \right| \\ &\le \left| \int_0^{\varrho} \frac{1}{t} \Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j - x)}\}\mathrm{d}t \right| + \int_0^{\varrho} \frac{1}{t} \sum_{\ell=1}^\infty (1 - |\phi_\varepsilon(t)|^2)^\ell \mathrm{d}t \\ &\le \left| \int_0^{\varrho} \frac{1}{t} \left(\int_{-\infty}^\infty f_\varepsilon(u) \sin(t(Y_j - x - u))\mathrm{d}u \right) \mathrm{d}t \right| + \int_0^{\varrho} \frac{1}{t} \sum_{\ell=1}^\infty (2Kt^\xi)^\ell \mathrm{d}t \\ &\le \int_{-\infty}^\infty f_\varepsilon(u) \left| \int_0^{\varrho} \frac{\sin(t(Y_j - x - u))}{t} \mathrm{d}t \right| \mathrm{d}u + \frac{1}{\xi} \sum_{\ell=1}^\infty \frac{(2K\varrho^\xi)^\ell}{\ell} \\ &\le 2 + \frac{1}{\xi} \ln\left(\frac{1}{1 - 2K\varrho^\xi} \right). \end{aligned}$$

By the same arguments as in the estimate of $V_{1,1}$, we also derive that

$$V_{1,2} \le 2 + \frac{1}{\xi} \ln\left(\frac{1}{1 - 2K\varrho^\xi} \right).$$

It follows from (7) and the estimates of $V_{1,1}$, $V_{1,2}$ that

$$V_1 \leq 4 + \frac{2}{\xi} \ln \left(\frac{1}{1 - 2K\varrho^{\xi}} \right). \tag{8}$$

We next estimate V_2. We have

$$V_2 \leq 2 \int_{\varrho}^{T} \frac{1}{t|\phi_\varepsilon(t)|} \mathrm{d}t.$$

Under (A2),

$$\begin{aligned} V_2 &\leq \frac{2}{c_1} \int_{\varrho}^{T} \frac{(1+t)^\alpha}{t} \mathrm{d}t = \frac{2}{c_1} \left(\int_{\varrho}^{1} \frac{(1+t)^\alpha}{t} \mathrm{d}t + \int_{1}^{T} \frac{(1+t)^\alpha}{t} \mathrm{d}t \right) \\ &\leq \frac{2}{c_1} \left(\int_{\varrho}^{1} \frac{(1+t)^\alpha}{t} \mathrm{d}t + 2^\alpha \int_{1}^{T} t^{\alpha-1} \mathrm{d}t \right) \leq C_1 T^\alpha, \end{aligned}$$

where C_1 is a positive constant depending on c_1, α, ϱ.

Under (A3),

$$V_2 \leq \frac{2}{d_1} \int_{\varrho}^{T} \frac{\mathrm{e}^{d\,t^\beta}}{t} \mathrm{d}t.$$

By the L'Hospital rule,

$$\lim_{T\to\infty} \frac{\int_{\varrho}^{T} t^{-1} \mathrm{e}^{d\,t^\beta} \mathrm{d}t}{T^{-\beta} \mathrm{e}^{d\,T^\beta}} = \lim_{T\to\infty} \frac{1}{d\beta - \beta T^{-\beta}} = \frac{1}{d\beta},$$

so, for large sufficiently T,

$$\int_{\varrho}^{T} \frac{\mathrm{e}^{d\,t^\beta}}{t} \mathrm{d}t \leq \left(1 + \frac{1}{d\beta}\right) T^{-\beta} \mathrm{e}^{d\,T^\beta}.$$

From there we derive $V_2 \leq C_2 T^{-\beta} \mathrm{e}^{d\,T^\beta}$, where C_2 is a positive constant depending on d_1, d, β, ϱ.

Hence, we have shown that

$$V_2 \leq \begin{cases} C_1 T^\alpha & \text{under (A2)}, \\ C_2 T^{-\beta} \mathrm{e}^{d\,T^\beta} & \text{under (A3)}. \end{cases} \tag{9}$$

From (6), (8) and (9), we conclude that

$$|W_{j,T}(x)| \leq \begin{cases} C_3 & \text{under (A2)}, \\ C_4 & \text{under (A3)}, \end{cases} \tag{10}$$

where $C_3 \equiv C_3(c_1, \alpha, \varrho, \xi, K) > 0$, $C_4 \equiv C_4(d_1, d, \beta, \varrho, \xi, K) > 0$ are constants.

Fix an arbitrary $\vartheta > 0$. From (4), (5), (10) and the fact that $W_{1,T}(x), \ldots, W_{n,T}(x)$ are i.i.d. random variables, we apply Lemma 1 to derive

$$\begin{aligned} \mathbb{P}(|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| > \vartheta) &= \mathbb{P}\left(\left| \frac{1}{n} \sum_{j=1}^{n} W_{j,T}(x) \right| > \frac{\vartheta}{g(T)} \right) \\ &\leq \begin{cases} 2\exp\left\{ -\frac{n\left(\frac{\vartheta}{g(T)}\right)^2}{2\left(\mathbb{E}|W_{j,T}(x)|^2 + \frac{C_3\vartheta}{3g(T)}\right)} \right\} & \text{under (A2)}, \\ 2\exp\left\{ -\frac{n\left(\frac{\vartheta}{g(T)}\right)^2}{2\left(\mathbb{E}|W_{j,T}(x)|^2 + \frac{C_4\vartheta}{3g(T)}\right)} \right\} & \text{under (A3)}. \end{cases} \end{aligned} \tag{11}$$

Next we estimate $\mathbb{E}|W_{j,T}(x)|^2$. Thanks to the inequality $\mathrm{Var}U \le \mathbb{E}|U|^2$ and the inequality $(u+v)^2 \le 2(u^2 + v^2)$, for all $u, v \in \mathbb{R}$, we get

$$\begin{aligned}
\mathbb{E}|W_{j,T}(x)|^2 &= \frac{1}{\pi^2(g(T))^2}\mathbb{E}\left|\int_0^T \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{(\mathrm{e}^{\mathrm{i}tY_j} - \phi_Y(t))\phi_\varepsilon(-t)\mathrm{e}^{-\mathrm{i}tx}\}\mathrm{d}t\right|^2 \\
&= \frac{1}{\pi^2(g(T))^2}\mathrm{Var}\left(\int_0^T \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\mathrm{d}t\right) \\
&\le \frac{1}{\pi^2(g(T))^2}\mathbb{E}\left|\int_0^T \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\mathrm{d}t\right|^2 \\
&\le \frac{2}{\pi^2(g(T))^2}(Q_1 + Q_2), \qquad (12)
\end{aligned}$$

with

$$Q_1 := \mathbb{E}\left|\int_0^\varrho \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\mathrm{d}t\right|^2,$$

$$Q_2 := \mathbb{E}\left|\int_\varrho^T \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\mathrm{d}t\right|^2.$$

We have

$$\left|\int_0^\varrho \frac{1}{t|\phi_\varepsilon(t)|^2}\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\mathrm{d}t\right| \equiv V_{1,1} \le 2 + \frac{1}{\xi}\ln\left(\frac{1}{1-2K\varrho^\xi}\right),$$

so

$$Q_1 \le \left(2 + \frac{1}{\xi}\ln\left(\frac{1}{1-2K\varrho^\xi}\right)\right)^2. \qquad (13)$$

For estimating Q_2, we apply the Fubini theorem to give that

$$\begin{aligned}
Q_2 &= \left|\mathbb{E}\int_\varrho^T\int_\varrho^T \frac{\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(Y_j-x)}\}\Im\{\phi_\varepsilon(-s)\mathrm{e}^{\mathrm{i}s(Y_j-x)}\}}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s\right| \\
&= \left|\int_\varrho^T\int_\varrho^T \frac{A(t,s,x)}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s\right| \\
&\le \int_\varrho^T\int_\varrho^T \frac{|A(t,s,x)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s,
\end{aligned}$$

where

$$A(t,s,x) := \int_{-\infty}^\infty f_Y(y)\Im\{\phi_\varepsilon(-t)\mathrm{e}^{\mathrm{i}t(y-x)}\}\Im\{\phi_\varepsilon(-s)\mathrm{e}^{\mathrm{i}s(y-x)}\}\mathrm{d}y.$$

By the Fubini theorem and the formula $\sin(u)\sin(v) = [\cos(u-v) - \cos(u+v)]/2$, we get

$$\begin{aligned}
A(t,s,x) &= \int_{-\infty}^\infty f_Y(y)\left(\int_{-\infty}^\infty\int_{-\infty}^\infty f_\varepsilon(u)f_\varepsilon(v)\sin(t(y-x-u))\sin(s(y-x-v))\mathrm{d}u\mathrm{d}v\right)\mathrm{d}y \\
&= \int_{-\infty}^\infty\int_{-\infty}^\infty f_\varepsilon(u)f_\varepsilon(v)\left(\int_{-\infty}^\infty f_Y(y)\sin(t(y-x-u))\sin(s(y-x-v))\mathrm{d}y\right)\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty f_\varepsilon(u)f_\varepsilon(v)\Re\{\mathrm{e}^{\mathrm{i}[x(s-t)-ut+vs]}\phi_Y(t-s)\}\mathrm{d}u\mathrm{d}v \\
&\quad - \frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty f_\varepsilon(u)f_\varepsilon(v)\Re\{\mathrm{e}^{\mathrm{i}[-x(s+t)-ut-vs]}\phi_Y(t+s)\}\mathrm{d}u\mathrm{d}v,
\end{aligned}$$

and this yields

$$|A(t,s,x)| \le \frac{1}{2}|\phi_Y(t-s)| + \frac{1}{2}|\phi_Y(t+s)|.$$

Hence,

$$Q_2 \le \frac{1}{2}\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s + \frac{1}{2}\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t+s)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s.$$

Furthermore, we have by the Cauchy-Schwarz inequality and the Fubini theorem that

$$\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s \le \sqrt{\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t\mathrm{d}s \int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{s^2|\phi_\varepsilon(s)|^4}\mathrm{d}t\mathrm{d}s}.$$

Note that

$$\begin{aligned}\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t\mathrm{d}s &= \int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\left(\int_\varrho^T |\phi_Y(t-s)|\mathrm{d}s\right)\mathrm{d}t\\ &= \int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\left(\int_{t-T}^{t-\varrho} |\phi_Y(v)|\mathrm{d}v\right)\mathrm{d}t\\ &\le \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t,\end{aligned}$$

and by the same arguments, we also obtain

$$\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{s^2|\phi_\varepsilon(s)|^4}\mathrm{d}t\mathrm{d}s \le \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t.$$

Hence,

$$\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t-s)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s \le \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t.$$

Similarly,

$$\int_\varrho^T\int_\varrho^T \frac{|\phi_Y(t+s)|}{ts|\phi_\varepsilon(t)|^2|\phi_\varepsilon(s)|^2}\mathrm{d}t\mathrm{d}s \le \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t.$$

Thus we derive

$$Q_2 \le \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t. \tag{14}$$

Combining (12) with (13) and (14), we obtain that

$$\mathbb{E}|W_{j,T}(x)|^2 \le \frac{2}{\pi^2(g(T))^2}\left[\left(2+\frac{1}{\xi}\ln\left(\frac{1}{1-2K\varrho^\xi}\right)\right)^2 + \|\phi_\varepsilon\|_1\int_\varrho^T \frac{1}{t^2|\phi_\varepsilon(t)|^4}\mathrm{d}t\right]. \tag{15}$$

• Under (A2), the estimate (15) becomes

$$\mathbb{E}|W_{j,T}(x)|^2 \le \frac{2}{\pi^2T^{2\alpha}}\left[\left(2+\frac{1}{\xi}\ln\left(\frac{1}{1-2K\varrho^\xi}\right)\right)^2 + \frac{\|\phi_\varepsilon\|_1}{c_1^4}\int_\varrho^T \frac{(1+t)^{4\alpha}}{t^2}\mathrm{d}t\right].$$

We see that

$$\|\phi_\varepsilon\|_1 = \int_{-\infty}^\infty |\phi_\varepsilon(t)|\mathrm{d}t \le c_2\int_{-\infty}^\infty (1+|t|)^{-\alpha}\mathrm{d}t < \infty,$$

$$\int_{\varrho}^{T} \frac{(1+t)^{4\alpha}}{t^2}\mathrm{d}t = \int_{\varrho}^{1} \frac{(1+t)^{4\alpha}}{t^2}\mathrm{d}t + \int_{1}^{T} \frac{(1+t)^{4\alpha}}{t^2}\mathrm{d}t \le \int_{\varrho}^{1} \frac{(1+t)^{4\alpha}}{t^2}\mathrm{d}t + \frac{2^{4\alpha}}{4\alpha-1}T^{4\alpha-1}.$$

Therefore, there is a positive constant C_5 depending on c_1, c_2, α, K, ξ and ϱ such that

$$\mathbb{E}|W_{j,T}(x)|^2 \le C_5 T^{2\alpha-1}.$$

Combining the latter estimate with (11), we obtain that

$$\mathbb{P}(|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| > \vartheta) \le 2\exp\left\{-\frac{n(\vartheta T^{-\alpha})^2}{2\left(C_5 T^{2\alpha-1} + C_3\vartheta T^{-\alpha}/3\right)}\right\}$$
$$\le 2\exp\left\{-\frac{\vartheta^2}{2(C_5 + C_3\vartheta/3)} n T^{1-4\alpha}\right\}.$$

For $T = k\,n^a$ with $k > 0$, $0 < a < 1/(4\alpha-1)$, we have

$$\mathbb{P}(|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| > \vartheta) \le 2\exp\left\{-\frac{\vartheta^2 k^{1-4\alpha}}{2(C_5 + C_3\vartheta/3)} n^{1+a(1-4\alpha)}\right\}.$$

Under the constraint $0 < a < 1/(4\alpha-1)$, we have

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{\vartheta^2 k^{1-4\alpha}}{2(C_5 + C_3\vartheta/3)} n^{1+a(1-4\alpha)}\right\} < \infty,$$

so

$$\sum_{n=1}^{\infty} \mathbb{P}(|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| > \vartheta) < \infty,$$

which together with Lemma 2 to imply $|\widehat{F}_{X;T}(x) - \mathbb{E}\widehat{F}_{X;T}(x)| \overset{a.s.}{\to} 0$ as $n \to \infty$.

- Under (A3), the estimate (15) becomes

$$\mathbb{E}|W_{j,T}(x)|^2 \le \frac{2}{\pi^2 T^{-2\beta} \mathrm{e}^{2dT^\beta}} \left[\left(2 + \frac{1}{\xi}\ln\left(\frac{1}{1-2K\varrho^\xi}\right)\right)^2 + \frac{\|\phi_\varepsilon\|_1}{d_1^4} \int_{\varrho}^{T} \frac{\mathrm{e}^{4d\,t^\beta}}{t^2}\mathrm{d}t\right].$$

We see that

$$\|\phi_\varepsilon\|_1 = \int_{-\infty}^{\infty} |\phi_\varepsilon(t)|\mathrm{d}t \le d_2 \int_{-\infty}^{\infty} \mathrm{e}^{-d|t|^\beta}\mathrm{d}t < \infty.$$

By the L'Hospital rule,

$$\lim_{T\to\infty} \frac{\int_{\varrho}^{T} t^{-2}\mathrm{e}^{4d\,t^\beta}\mathrm{d}t}{T^{-(\beta+1)}\mathrm{e}^{4d\,T^\beta}} = \lim_{T\to\infty} \frac{1}{4d\beta - (\beta+1)T^{-\beta}} = \frac{1}{4d\beta},$$

so, for large sufficiently T,

$$\int_{\varrho}^{T} \frac{\mathrm{e}^{4d\,t^\beta}}{t^2}\mathrm{d}t \le \left(1 + \frac{1}{4d\beta}\right) T^{-(\beta+1)}\mathrm{e}^{4d\,T^\beta}.$$

Therefore, we can find a positive constant C_6 depending on d_1, d_2, d, β, K, ξ, ϱ such that

$$\mathbb{E}|W_{j,T}(x)|^2 \le C_6 T^{\beta-1}\mathrm{e}^{4d\,T^\beta}.$$

Combining the latter estimate with (11), we obtain that

$$\mathbb{P}(|\widehat{F}_{X;T}(x)-\mathbb{E}\widehat{F}_{X;T}(x)|>\vartheta)\leq 2\exp\left\{-\frac{n(\vartheta T^{\beta}\mathrm{e}^{-dT^{\beta}})^2}{2\left(C_6T^{\beta-1}\mathrm{e}^{4dT^{\beta}}+C_4\vartheta T^{\beta}\mathrm{e}^{-dT^{\beta}}/3\right)}\right\}$$
$$\leq 2\exp\left\{-\frac{\vartheta^2}{2(C_6+C_4\vartheta/3)}n\mathrm{e}^{-6dT^{\beta}}\right\}.$$

For $T=(b\ln n)^{1/\beta}$ with $0<b<1/(6d)$, we infer that

$$\mathbb{P}(|\widehat{F}_{X;T}(x)-\mathbb{E}\widehat{F}_{X;T}(x)|>\vartheta)\leq 2\exp\left\{-\frac{\vartheta^2}{2(C_6+C_4\vartheta/3)}n^{1-6db}\right\}.$$

Under the constraint $0<b<1/(6d)$, we have

$$\sum_{n=1}^{\infty}\exp\left\{-\frac{\vartheta^2}{2(C_6+C_4\vartheta/3)}n^{1-6db}\right\}<\infty,$$

so

$$\sum_{n=1}^{\infty}\mathbb{P}(|\widehat{F}_{X;T}(x)-\mathbb{E}\widehat{F}_{X;T}(x)|>\vartheta)<\infty,$$

which together with Lemma 2 to give $|\widehat{F}_{X;T}(x)-\mathbb{E}\widehat{F}_{X;T}(x)|\overset{a.s.}{\to}0$ as $n\to\infty$.

The theorem is proved. □

3. Numerical example

We present in this section the two following numerical examples to illustrate the convergence of the estimator $\widehat{F}_{X;T}$ according to the sample size n. We use the R language for all numerical setups.

(E1) $X\sim 0.5\mathcal{N}(0,1/4)+0.5\mathcal{N}(2,1/4)$. In this example, we consider two cases of ε:

- $\varepsilon\sim\mathcal{L}(0,\sqrt{5/8})$. In that case, ϕ_ε satisfies (A2) with $\alpha=2$.
- $\varepsilon\sim\mathcal{N}(0,5/4)$. In that case, ϕ_ε satisfies (A3) with $d=5/8$, $\beta=2$.

(E2) $X\sim 0.5\mathcal{G}(4,1/2)+0.5\mathcal{G}(3,1/\sqrt{3})$. In this example, we consider two cases of ε:

- $\varepsilon\sim\mathcal{L}(0,0.7134)$. In that case, ϕ_ε satisfies (A2) with $\alpha=2$.
- $\varepsilon\sim\mathcal{N}(0,1.0179)$. In that case, ϕ_ε satisfies (A3) with $d=0.5089$, $\beta=2$.

Here $\mathcal{N}(\mu,v)$ denotes the normal distribution with mean $\mu\in\mathbb{R}$ and variance $v>0$, $\mathcal{G}(m,s)$ stands for the Gamma distribution with shape parameter $m\in\mathbb{R}$ and scale parameter $s>0$, and $\mathcal{L}(p,q)$ represents the Laplace distribution with location parameter $p\in\mathbb{R}$ and scale parameter $q>0$.

To set up $\widehat{F}_{X;T}$, in each example of X and in each case of ε, we generate randomly a sample $X_1,\dots,X_n$ from f_X and a sample $\varepsilon_1,\dots,\varepsilon_n$ from f_ε. After that, based on the model (1), we obtain the sample $Y_1,\dots,Y_n$, with $Y_j=X_j+\varepsilon_j$, $j=1,\dots,n$. Besides the sample $Y_1,\dots,Y_n$, we have to choose the parameter T concretely. In view of Theorem 1, for each example, we choose $T=1.25n^{1/8}$ for the case of the Laplace error, and $T=(b\log n)^{1/2}$ with $b=17/(120d)$ for the case of the normal error.

A competitor of $\widehat{F}_{X;T}$ is the estimator $\widehat{F}_{X;\mathrm{Ker}}$ defined by

$$\widehat{F}_{X;\mathrm{Ker}}(x):=\int_{-\infty}^{x}\widehat{f}_{X;\mathrm{Ker}}(u)\mathrm{d}u,\quad x\in\mathbb{R},$$

where

$$\widehat{f}_{X;\mathrm{Ker}}(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi_K(ht)\frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}tY_j}}{\phi_\varepsilon(t)} \mathrm{e}^{-\mathrm{i}tu}\mathrm{d}t.$$

Here K is a symmetric kernel with compactly supported Fourier transform ϕ_K ($\phi_K(t) := \int_{-\infty}^{\infty} K(x)\mathrm{e}^{\mathrm{i}tx}\mathrm{d}x, t \in \mathbb{R}$), and $h > 0$ is a bandwidth parameter depending on n. The quantity $\widehat{f}_{X;\mathrm{Ker}}$ is known as the deconvolution kernel density estimator of f_X (see Stefanski and Carroll [10]). Therefore, $\widehat{F}_{X;\mathrm{Ker}}$ is called the deconvolution kernel estimator of F_X. Computing directly gives

$$\widehat{F}_{X;\mathrm{Ker}}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{1}{t}\Im\left\{\frac{\phi_K(ht)\frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}tY_j}}{\phi_\varepsilon(t)} \mathrm{e}^{-\mathrm{i}tx}\right\}\mathrm{d}t.$$

To compute $\widehat{F}_{X;\mathrm{Ker}}$, we have to choose h and K. Concerning h, based on Theorems 1 and 2 in Fan [5], we take $h = cn^{-1/(2\alpha+5)}$ for some constant $c > 0$ under (A2), and $h = (4d)^{1/\beta}(\ln n)^{-1/\beta}$ under (A3). Precisely, if $\varepsilon \sim \mathcal{L}(0, s)$ with $s > 0$, we take $c = (5s^4)^{1/9}$. Regarding K, we choose K in the form

$$K(x) := \frac{48\cos(x)}{\pi x^4}\left(1 - \frac{15}{x^2}\right) - \frac{144\sin(x)}{\pi x^5}\left(2 - \frac{5}{x^2}\right).$$

This is a symmetric kernel of order 2 with the Fourier transform $\phi_K(t) = (1 - t^2)^3\mathbb{I}_{[-1,1]}(t)$, in which $\mathbb{I}_{[-1,1]}$ is the indicator function on $[-1, 1]$. This kernel K has been popularly used in the deconvolution literature. For this kernel, $\widehat{F}_{X;\mathrm{Ker}}(x)$ becomes

$$\widehat{F}_{X;\mathrm{Ker}}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{1/h} \frac{1}{t}\Im\left\{\frac{(1 - (ht)^2)^3\frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}tY_j}}{\phi_\varepsilon(t)} \mathrm{e}^{-\mathrm{i}tx}\right\}\mathrm{d}t. \quad (16)$$

Some performances of $\widehat{F}_{X;\mathrm{Ker}}$ at (16) will be compared with those of $\widehat{F}_{X;T}$.

In Figures 1 and 2, the left sub-figures are respect to the Laplace error, whereas the right sub-figures are respect to the normal error. In each sub-figure, we plot five curves with different colors, in which the black curve is the graph of the target function F_X, and the curves with green, violet, red and blue colors are the graphs of $\widehat{F}_{X;T}$ with respect to $n = 50$, 200, 800 and 3200. It is obvious that the curves become closer when n is increased, and this confirms surely the convergence of the estimator $\widehat{F}_{X;T}$. Observe that the smoothness of the error variable ε, characterized by the decaying rate of the characteristic function ϕ_ε, also affects to the convergence of $\widehat{F}_{X;T}$. In fact, in Figures 1 and 2, the convergence trend of the curves in the left sub-figures seems to be more obvious than the one in the right sub-figures. This can be explained by the fact that the Laplace error is less smoother than the normal error.

To compare performances of $\widehat{F}_{X;T}$ and $\widehat{F}_{X;\mathrm{Ker}}$, we show some their curves with respect to $n = 50$ and $n = 200$ in Figures 3, 4, 5 and 6. In each sub-figure of these figures, the curves with black, red and blue colors are respectively the graphs of F_X, $\widehat{F}_{X;T}$ and $\widehat{F}_{X;\mathrm{Ker}}$. The left sub-figures are respect to the Laplace error and the right sub-figures are respect to the normal error. In both examples of X, we see that the performances of $\widehat{F}_{X;T}$ are better than those of $\widehat{F}_{X;\mathrm{Ker}}$ in the two cases of ε. Note that we present here the results only for $n = 50$ and $n = 200$, but our conclusions extend to simulation results with another sample sizes n, namely, $n = 800$ and $n = 3200$.

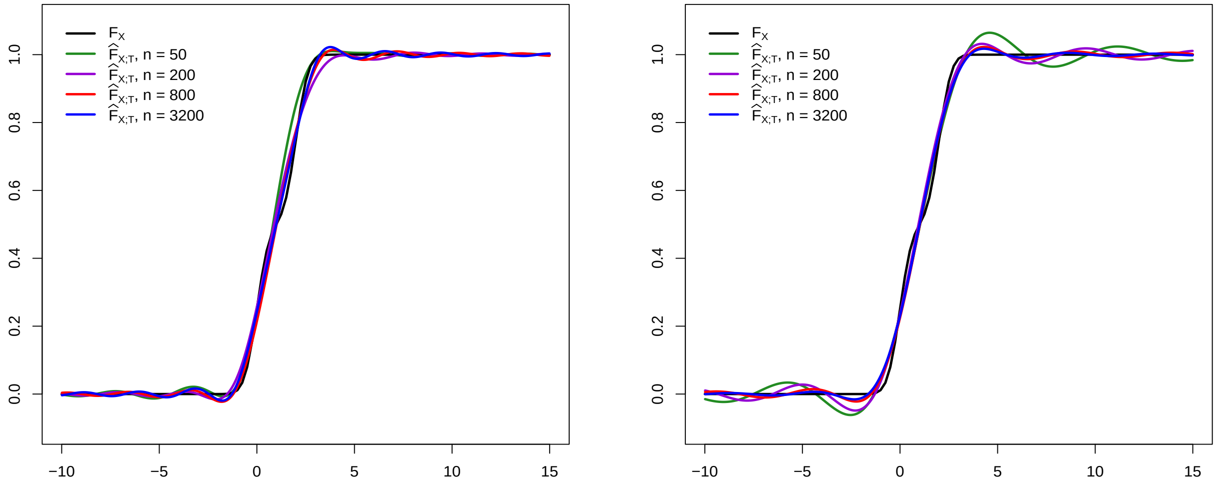

Figure 1. Graphs of the estimator $\widehat{F}_{X;T}$ with respect to $n = 50, 200, 800, 3200$ and of the target function F_X in (E1).

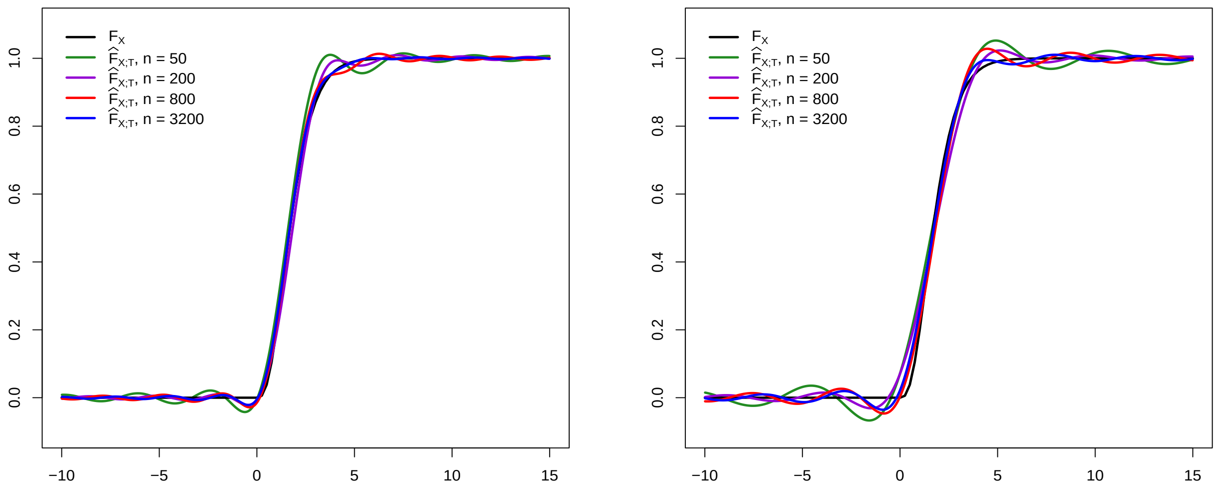

Figure 2. Graphs of the estimator $\widehat{F}_{X;T}$ with respect to $n = 50, 200, 800, 3200$ and of the target function F_X in (E2).

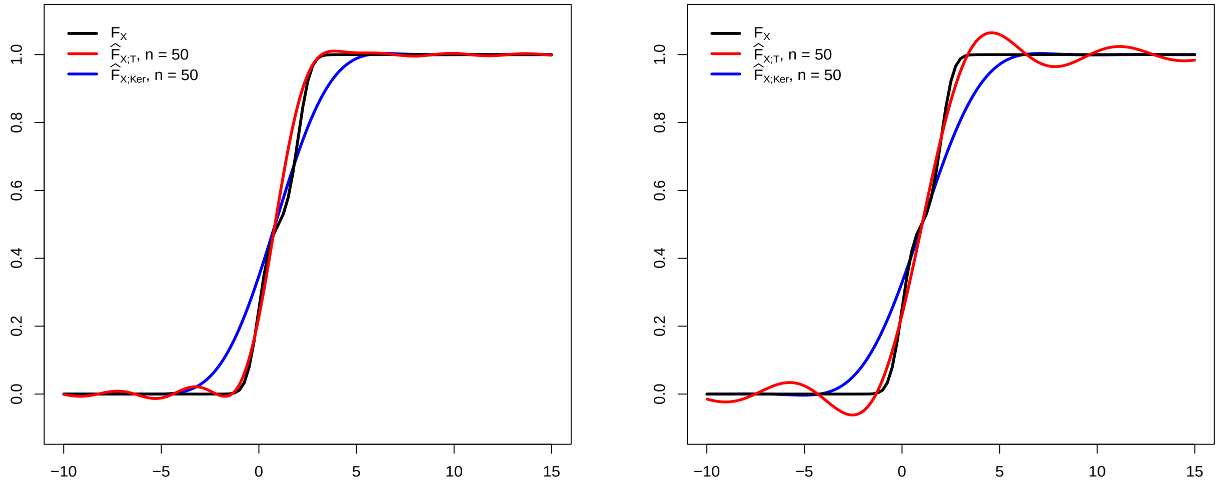

Figure 3. Graphs of $\widehat{F}_{X;T}$, $\widehat{F}_{X;\mathrm{Ker}}$ with respect to $n = 50$ and of the target function F_X in (E1).

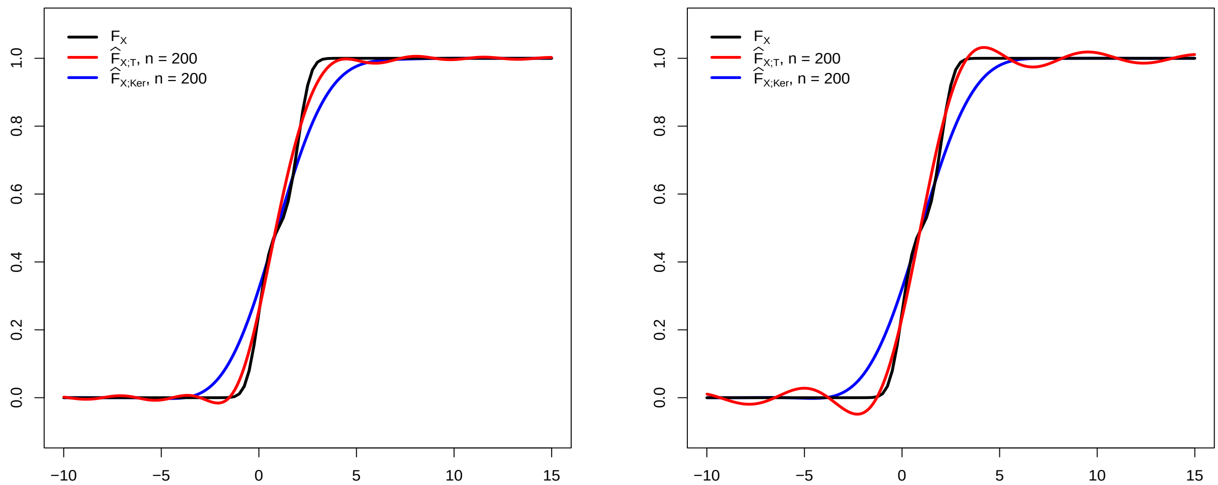

Figure 4. Graphs of $\widehat{F}_{X;T}$, $\widehat{F}_{X;\mathrm{Ker}}$ with respect to $n = 200$ and of the target function F_X in (E1).

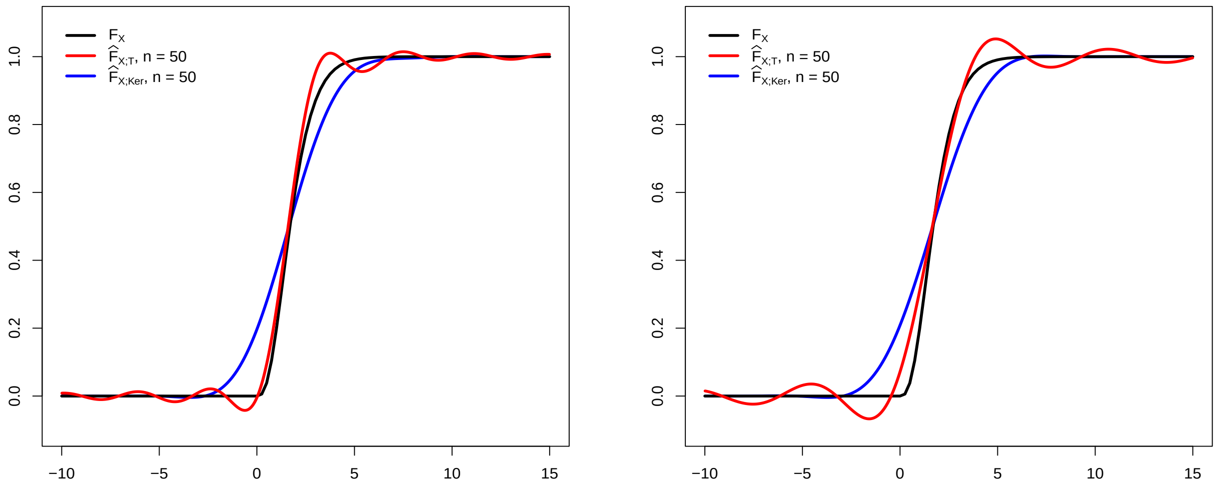

Figure 5. Graphs of $\widehat{F}_{X;T}$, $\widehat{F}_{X;\mathrm{Ker}}$ with respect to $n = 50$ and of the target function F_X in (E2).

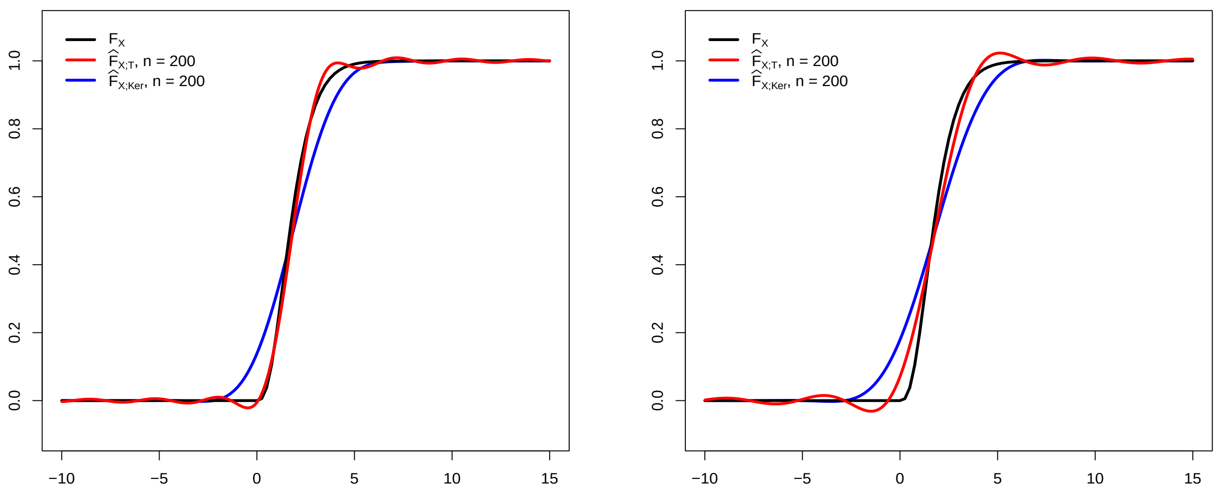

Figure 6. Graphs of $\widehat{F}_{X;T}$, $\widehat{F}_{X;\mathrm{Ker}}$ with respect to $n = 200$ and of the target function F_X in (E2).

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