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New Efficient Descent Direction of a Primal-dual Path-following Algorithm for Linear Programming

Billel Zaoui *, Djamel Benterki, Samia Khelladi

Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, University of Ferhat Abbas Setif-1, 19000, Algeria

Abstract We introduce a new primal-dual interior-point algorithm with a full-Newton step for solving linear optimization problems. The newly proposed approach is based on applying a new function on a simple equivalent form of the centering equation of the system which defines the central path. Thus, we get a new efficient search direction for the considered algorithm. Moreover, we prove that the method solves the studied problems in polynomial time and that the algorithm obtained has the best known complexity bound for linear optimization. Finally, a comparative numerical study is reported to show the efficiency of the proposed algorithm.

Keywords Linear programming, Interior-point methods, Primal-dual algorithm, Descent direction

AMS 2010 subject classifications 90C05, 90C51

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1. Introduction

Linear programming (LP) consists of optimizing a linear function subject to linear constraints on real variables. In this paper, we consider the LP in its standard form:

$$\begin{cases} \min c^T x \\ \text{subject to } Ax = b, x \geq 0, \end{cases} \tag{P}$$

and it's dual problem:

$$\begin{cases} \max b^T y \\ \text{subject to } A^T y + s = c, s \geq 0. \end{cases} \tag{D}$$

Where A is a $m \times n$ given matrix, $b, y \in \mathbb{R}^m$ and $c, x, s \in \mathbb{R}^n$.

LP is a classic topic in optimization with a large number of application areas. Among the important applications of the LP are in economics and industry.

There are many approaches to solving LP problems, the most important is the interior point method, which was first proposed by Karmarkar [10]. This method and its variants that were developed subsequently are now called interior-point methods IPMs. For a survey, we refer to [3, 8, 18, 19, 26]. Megiddo [16] and Sonnevend [21] were the first to recognize the relevance of the central path for LP. The authors in [20] investigated the first primal-dual path-following IPM for LP problems with full Newton step. This technique has been extensively extended to other optimization problems (e.g., [11, 15, 25]).

*Correspondence to: Billel Zaoui (Email: billel.zaoui@univ-setif.dz). Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, University of Ferhat Abbas Setif-1, 19000, Algeria.

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In IPMs, the determination of search directions plays a key role. In fact, Darvay [4] introduces a new method for finding search directions. He applied the square root function on both sides of an algebraic equivalent transformation (AET) on the centering equation of the system which defines the central path. Then he used the Newton method to the resulting system. This method is extended to other optimization problems such as: convex quadratic optimization (CQO)[1], semidefinite optimization (SDO)[22], second-order cone optimization (SOCO)[23] and symmetric optimization (SO)[24]. Moreover, Kheirfam and Nasrollahi in [13] extended this technique which is based on the square root function to the integer powers of this function. Furthermore, Based on the AET strategy, Darvay et al. in [5] considered a new function to present a new primal-dual IPM for LP. In the same way, Kheirfam and Haghighi in [12] presented a new primal-dual IPM for $P^*(k)$-linear complementarity. For more related papers, we refer to [2, 7, 17]

Currently, the AET technique has become a wide research interest, and the search for a new AET to describe a new primal-dual IPM has become an important motivation for researchers. In 2011, Zhang and Xu [27] proposed a specific search direction for LP. They considered the equivalent form $v^2 = v$ of the centering equation and transformed it into the form $xs = \mu v$. After that, they assumed that the variance vector is fixed and they applied Newton's method. Based on this new AET, Darvay and Takàcs [6] proposed another technique to obtain a new descent direction for solving LP. They applied the function $\psi(t) = t^2$ on both sides of the nonlinear equation $v^2 = v$. Next, they used Newton's method to get the new search direction. The authors proved the theoretical and numerical effectiveness of this new approach compared to other approaches. Furthermore, this technique has been extensively extended to other optimization problems (e.g., [9, 14]...).

Inspired by the papers mentioned earlier, our primary objective is to reevaluate the method proposed in [6] by incorporating a new function $\psi(t) = t^{\frac{3}{2}}$. This results in the introduction of a novel primal-dual interior point method for linear programming that offers improved computational efficiency. Furthermore, we develop some new results and prove that the complexity is $O\left(\log \frac{(n+\sqrt[3]{4})}{\epsilon}\right)$ iterations.

The paper is organized as follows. The concept of the central path is introduced in Section 2. Section 3 discusses the novel search direction and outlines the algorithm associated with it. The convergence of the algorithm towards an optimal solution is demonstrated in Section 4, along with an analysis of the maximum number of iterations required to meet optimality conditions. Section 5 presents comparative numerical experiments and accompanying remarks. The paper concludes with a summary and suggestions for future research in Section 6.

2. Position of the problem

Without loss of generality, throughout the paper, we assume that the pair (P) and (D) satisfy the conditions belows

- The matrix A is a full rank row, i.e., $Rank(A) = m$ $(m < n)$.
- There exists (x^0, y^0, s^0) such that:

$$Ax^0 = b, A^T y^0 + s^0 = c, x^0 > 0, s^0 > 0. \tag{1}$$

This last condition (1) is named the interior point condition (IPC).

We note that under the previous assumptions, the optimal solution of the primal-dual pair (P) and (D) can be given with the following nonlinear system

$$\begin{cases} Ax = b, \ x \geq 0, \\ A^T y + s = c, \ s \geq 0, \\ xs = 0. \end{cases} \tag{2}$$

Where xs denotes the coordinatewise product of the vectors x and s, hence $xs = (x_1 s_1, x_2 s_2, ..., x_n s_n)^T \geq 0$.

The two first equations of the system (2) are named respectively primal feasibility and dual feasibility. The last one is called the complementarity condition.

The basic idea of the primal-dual interior-point algorithm is to replace the complementarity condition $xs = 0$ in (2), by the parameterized equation $xs = \mu e$, with $\mu > 0$ and e is the all-one vector of length n. Thusly, we consider the following system

$$\begin{cases} Ax = b, \ x > 0, \\ A^T y + s = c, \ s > 0, \\ xs = \mu e, \ \mu > 0. \end{cases} \tag{3}$$

Without a doubt, under our assumptions, the system (3) has a unique solution, for each $\mu > 0$ (see [21]). It is denoted as $(x(\mu), y(\mu), s(\mu))$, and we call $x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D). The set of μ-centers gives a homotopy, which is named the central path of (P) and (D). If μ goes to zero, then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for the pair (P) and (D).

3. New search direction

In this section, we reconsider the technique introduced by Darvay and Takàcs [6] with a new function $\psi(t) = t^{\frac{3}{2}}$ to get new efficient search direction for LP. Note that for $x, s > 0$ and $\mu > 0$, from the third equation of system (3), we deduce that

$$xs = \mu e \Leftrightarrow \frac{xs}{\mu} = e \Leftrightarrow \sqrt{\frac{xs}{\mu}} = e \Leftrightarrow \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}}.$$

Where, $\frac{xs}{\mu}$ denotes the coordinatewise product of the vectors x and s divided by $\mu > 0$, hence $\frac{xs}{\mu} = \left(\frac{x_1 s_1}{\mu}, \frac{x_2 s_2}{\mu}, ..., \frac{x_n s_n}{\mu}\right)^T > 0$ and $\sqrt{\frac{xs}{\mu}}$ is the vector obtained by taking the square roots of the components of $\frac{xs}{\mu}$.

Now, the perturbed central path can be equivalently stated as follows

$$\begin{cases} Ax = b, \\ A^T y + s = c, \\ \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}}. \end{cases} \tag{4}$$

Let us consider the function ψ defined and continuously differentiable on the interval $\left(k^2, \infty\right)$, where $0 \leq k < 1$, such that $2t\psi'\left(t^2\right) - \psi'\left(t\right) > 0, \forall t > k^2$. Using this, system (4) can be written in the following equivalent form

$$\begin{cases} Ax = b, \\ A^T y + s = c, \\ \psi\left(\frac{xs}{\mu}\right) = \psi\left(\sqrt{\frac{xs}{\mu}}\right). \end{cases} \tag{5}$$

This last system (5) can be written in the form $f(x, y, s) = 0$, where

$$f(x, y, s) = \begin{pmatrix} Ax - b \\ A^T y + s - Qx - c \\ \psi(\frac{xs}{\mu}) - \psi\left(\sqrt{\frac{xs}{\mu}}\right) \end{pmatrix} \tag{6}$$

Applying Newton's method to this system, we get: $x_+ = x + \Delta x,\ y_+ = y + \Delta y$ and $s_+ = s + \Delta s,$ where $(\Delta x, \Delta y, \Delta s)$ is the solution of the linear system

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s = 0, \\ \frac{1}{\mu}(s\Delta x + x\Delta s) = \dfrac{-\psi\left(\frac{xs}{\mu}\right) + \psi\left(\sqrt{\frac{xs}{\mu}}\right)}{\psi'\left(\frac{xs}{\mu}\right) - \frac{1}{2\sqrt{\frac{xs}{\mu}}}\psi'\left(\sqrt{\frac{xs}{\mu}}\right)}. \end{cases} \tag{7}$$

For the analysis of IPMs, it is convenient to associate any triple $(x, s, \mu) > 0$, and introduce the scaled vector v and the scaled search directions d_x and d_s according to

$$v = \sqrt{\frac{xs}{\mu}}, d_x = \frac{v\Delta x}{x} \text{ and } d_s = \frac{v\Delta s}{s}.$$

Hence, we obtain

$$\frac{1}{\mu}(s\Delta x + x\Delta s) = v(d_x + d_s), \tag{8}$$

and

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{9}$$

Obviously, with these notations, the scaled feasible Newton system of (7) can be expressed as

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T \Delta y + d_s = 0, \\ d_x + d_s = p_v. \end{cases} \tag{10}$$

Where

$$p_v = \frac{2\psi(v) - 2\psi(v^2)}{2v\psi'(v^2) - \psi'(v)} \text{ and } \overline{A} = \frac{1}{\mu} A diag\left(\frac{x}{v}\right).$$

Here, $diag(\frac{x}{v})$ is a diagonal matrix, which contains on its main diagonal the elements of the vector $\frac{x}{v}$ respectively in the original order.

It is worth noting that different choices of ψ will lead to different values of p_v and hence new search directions. The case $\psi(t) = t$ implies that $p_v = \frac{2v-2v^2}{2v-e}$, and we obtain that this direction is similar to the algorithm defined in [5]. The case where $\psi(t) = t^2$ yields $p_v = \frac{v-v^3}{2v^2-e}$ which is studied by Darvay and Takàcs in [6] for LP.

In this paper, we restrict our analysis to the case $\psi : \left(\frac{1}{\sqrt[3]{4}}, \infty\right) \to \mathbb{R}$, such that $\psi(t) = t^{\frac{3}{2}}$. This yields

$$p_v = \frac{4v - 4v^{\frac{5}{2}}}{6v^{\frac{3}{2}} - 3e}, \tag{11}$$

The condition $2t\psi'\left(t^2\right) - \psi'(t) > 0, \forall t > k^2$ is satisfied in this case, where $k^2 = \frac{1}{\sqrt[3]{4}}$. For the analysis of the algorithm, we define a norm-based proximity measure $\delta(xs, \mu)$ as follows:

$$\delta(v) = \delta(xs, \mu) = \frac{\|p_v\|}{2} = \frac{2}{3}\left\|\frac{v - v^{\frac{5}{2}}}{2v^{\frac{3}{2}} - e}\right\|, \tag{12}$$

where $\|.\|$ denotes the Euclidean norm.
Also, let us define $q_v = d_x - d_s$.
Then, using the above equation and the third equation of (10) we have

$$d_x = \frac{1}{2}(p_v + q_v) \text{ and } d_s = \frac{1}{2}(p_v - q_v).$$

This implies

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{13}$$

Since $d_x^T d_s = d_x^T\left(-\overline{A}^T \Delta y\right) = -\left(\overline{A}d_x\right)^T \Delta y = 0$, then

$$\|q_v\| = \|p_v\|. \tag{14}$$

Now, a framework of the algorithm is described in Figure 1 as follows
In the next section, we present some results related to algorithm complexity analysis.

Generic Primal-dual IPM for LP

Input:

a proximity parameter $0 < \tau < 1$ (default $\tau = \frac{1}{6}$);
an accuracy parameter $\varepsilon > 0$;
an update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{7\sqrt{n}}$);
a strictly feasible point (x^0, y^0, s^0); $\mu^0 = \frac{(x^0)^T s^0}{n}$; such that $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}} > \frac{1}{\sqrt[3]{4}} e$;

begin

$x = x^0$; $y = y^0$; $s = s^0$;
while $x^T s \geq \varepsilon$ **do**
  $\mu = (1 - \theta)\,\mu$;
  solve the system (10) via (7) to obtain $(\Delta x, \Delta y, \Delta s)$;
  $x = x + \Delta x$; $y = y + \Delta y$; $s = s + \Delta s$;
end.

Figure 1. Generic algorithm

4. Complexity Analysis

The following lemma shows the feasibility of the full-Newton step under the conditions $\delta(xs, \mu) < 1$ and $v > \frac{1}{\sqrt[3]{4}} e$.

Lemma 4.1
Suppose that $\delta(xs, \mu) < 1$ and $v > \frac{1}{\sqrt[3]{4}} e$. Then the full-Newton step is strictly feasible, hence $x_+ > 0$ and $s_+ > 0$.

Proof
For each $0 \leq \alpha \leq 1$ denote $x_+(\alpha) = x + \alpha \Delta x$ and $s_+(\alpha) = s + \alpha \Delta s$. Hence,

$$x_+(\alpha) s_+(\alpha) = xs + \alpha(s\Delta x + x\Delta s) + \alpha^2 \Delta x \Delta s.$$

Now, in view of (8) and (9) we have

$$\frac{1}{\mu} x_+(\alpha) s_+(\alpha) = \frac{xs}{\mu} + \alpha v(d_x + d_s) + \alpha^2 d_x d_s. \tag{15}$$

Also from (10) and (13), we can write

$$\frac{1}{\mu} x_+(\alpha) s_+(\alpha) = v^2 + \alpha v p_v + \alpha^2 \left(\frac{p_v^2 - q_v^2}{4} \right),$$

so

$$\frac{1}{\mu} x_+(\alpha) s_+(\alpha) = (1 - \alpha) v^2 + \alpha(v^2 + v p_v) + \alpha^2 \left(\frac{p_v^2 - q_v^2}{4} \right). \tag{16}$$

In addition, from (11) we obtain

$$v^2 + v p_v = \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e}. \tag{17}$$

Now, let's consider the function $f(x) = \frac{2x^{\frac{7}{2}} + x^2}{6x^{\frac{3}{2}} - 3}$, with $x > \frac{1}{\sqrt[3]{4}}$. We have $f(x) \geq f(1)$, so $f(x) \geq 1$.
Using this result, we get

$$v^2 + v p_v \geq e. \tag{18}$$

Then

$$\begin{aligned}
\frac{1}{\mu}x_+(\alpha)s_+(\alpha) &\geq (1-\alpha)v^2+\alpha e+\alpha^2\left(\frac{p_v^2}{4}-\frac{q_v^2}{4}\right)\\
&\geq (1-\alpha)v^2+\alpha e+\alpha^2\left(\frac{p_v^2}{4}-\frac{q_v^2}{4}\right)-\alpha\frac{p_v^2}{4}\\
&\geq (1-\alpha)v^2+\alpha e+\alpha(\alpha-1)\frac{p_v^2}{4}-\alpha^2\frac{q_v^2}{4},
\end{aligned}$$

so

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha)\geq(1-\alpha)v^2+\alpha\left(e-\left((1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right)\right). \tag{19}$$

If $\left\|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right\|_\infty<1$, then the inequality $x_+(\alpha)s_+(\alpha)>0$ holds, where $\|.\|_\infty$ marks the Chebychev norm (or l_∞ norm). In this way

$$\begin{aligned}
\left\|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v^2\|_\infty}{4}+\alpha\frac{\|q_v^2\|_\infty}{4}\\
&\leq (1-\alpha)\frac{\|p_v\|^2}{4}+\alpha\frac{\|q_v\|^2}{4},
\end{aligned}$$

from (14), we get

$$\begin{aligned}
\left\|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v\|^2}{4}+\alpha\frac{\|p_v\|^2}{4}\\
&\leq \frac{\|p_v\|^2}{4}=\delta^2<1.
\end{aligned}$$

Hence, $x_+(\alpha)s_+(\alpha)>0$ for each $0\leq\alpha\leq1$, which means that the linear functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ do not change sign on the interval $[0,1]$ and for $\alpha=0$ we have $x_+(0)=x>0$ and $s_+(0)=s>0$. This leads to $x_+(1)=x_+>0$ and $s_+(1)=s_+>0$. This means that the full-Newton step is strictly feasible. □

We state the following lemma [5, Lemma 5.2] which will be useful in the next part of the analysis.

Lemma 4.2
Let $f:[d,1)\rightarrow(0,1)$ be a decreasing function with $d>0$. Furthermore, let us consider the positive vector v of length n such that $min(v)>d$. Then

$$\left\|f(v)\left(e-v^2\right)\right\|\leq f(\min(v))\left\|e-v^2\right\|\leq f(d)\left\|e-v^2\right\|.$$

The local quadratic convergence of the full-Newton step is proved in the following lemma.

Lemma 4.3
Let $\delta=\delta(xs,\mu)<\frac{1}{\sqrt[3]{4}}$ and $v>\frac{1}{\sqrt[3]{4}}e$. Then $v_+=\sqrt{\frac{x_+s_+}{\mu}}>\frac{1}{\sqrt[3]{4}}e$ and

$$\delta(x_+s_+,\mu)<8\delta^2,$$

which means local quadratic convergence of the full Newton step.

Proof
We know from Lemma 4.1 that $x_+>0$ and $s_+>0$, then $v_+=\sqrt{\frac{x_+s_+}{\mu}}$ is well defined.
Let $\alpha=1$. Then from (16) it follows that

$$v_+^2=v^2+vp_v+\frac{p_v^2}{4}-\frac{q_v^2}{4}. \tag{20}$$

Using (20) and the inequality (18), we obtain

$$v_+^2 \geq e + \frac{p_v^2}{4} - \frac{q_v^2}{4},$$

but $\frac{p_v^2}{4} \geq 0$, this imply that

$$v_+^2 \geq e - \frac{q_v^2}{4},$$

hence

$$\min(v_+^2) \geq 1 - \frac{\left\|q_v^2\right\|_\infty}{4} \geq 1 - \frac{\left\|q_v\right\|^2}{4} \geq 1 - \delta^2,$$

and this relation yields

$$\min(v_+) \geq \sqrt{1-\delta^2}. \tag{21}$$

We have $\delta < \frac{1}{\sqrt[3]{4}}$, then

$$\sqrt{1-\delta^2} > \sqrt{1 - \frac{1}{\sqrt[3]{16}}} > \frac{1}{\sqrt[3]{4}},$$

using this last inequality and (21) we get

$$v_+ > \frac{1}{\sqrt[3]{4}} e.$$

This completes the first part of the proof.

Now, from (20) and (17) we have

$$\begin{aligned}
\left\|e - v_+^2\right\| &= \left\|e - \left(\left(v^2 + v p_v\right) + \frac{p_v^2}{4} - \frac{q_v^2}{4}\right)\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|e - \left(v^2 + v p_v\right) - \frac{p_v^2}{4}\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|e - \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e} - \frac{p_v^2}{4}\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|\frac{6v^{\frac{3}{2}} - 3e - 2v^{\frac{7}{2}} - v^2}{6v^{\frac{3}{2}} - 3e} - \frac{p_v^2}{4}\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|\left[\frac{4 \times \left(6v^{\frac{3}{2}} - 3e - 2v^{\frac{7}{2}} - v^2\right)}{\left(6v^{\frac{3}{2}} - 3e\right) \times p_v^2} - e\right] \frac{p_v^2}{4}\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|\left[-\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2}\right] \frac{p_v^2}{4}\right\| \\
&\leq \left\|\frac{q_v^2}{4}\right\| + \left\|\left[\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2}\right] \frac{p_v^2}{4}\right\|.
\end{aligned}$$

On one hand, we have

$$\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \geq 0, \forall v > \frac{1}{\sqrt[3]{4}} e.$$

In the other hand, we have

$$\frac{16v^5+v^2-8v^{\frac{7}{2}}-36v^3+36v^{\frac{3}{2}}-9e}{4\left(v-v^{\frac{5}{2}}\right)^2}=5e+K(v),$$

where

$$K(v)=\frac{36v^{\frac{3}{2}}-36v^3-9e+32v^{\frac{7}{2}}-4v^5-19v^2}{4\left(v-v^{\frac{5}{2}}\right)^2},$$

since $K(v)\leq 0, \forall v>\frac{1}{\sqrt[3]{4}}e$, then, we conclude that

$$0\leq\frac{16v^5+v^2-8v^{\frac{7}{2}}-36v^3+36v^{\frac{3}{2}}-9e}{4\left(v-v^{\frac{5}{2}}\right)^2}\leq 5e,$$

which implies that

$$\|e-v_+^2\|\leq\left\|\frac{q_v^2}{4}\right\|+5\left\|\frac{p_v^2}{4}\right\|=6\delta^2. \tag{22}$$

Now, By the definition of δ we have

$$\delta(v_+)=\delta(x_+s_+,\mu)=\frac{\|p_{v_+}\|}{2}=\frac{2}{3}\left\|\frac{\left(v_+-v_+^{\frac{5}{2}}\right)}{\left(2v_+^{\frac{3}{2}}-e\right)(e-v_+^2)}(e-v_+^2)\right\|.$$

Let's consider the function: $f(t)=\frac{\left(t-t^{\frac{5}{2}}\right)}{\left(2t^{\frac{3}{2}}-1\right)(1-t^2)}$, for all $t>\frac{1}{\sqrt[3]{4}}$, $t\neq 1$. Since $f'(t)<0$, so f is decreasing. Hence, in view of Lemma 4.2, we obtain

$$\delta(x_+s_+,\mu)\leq\frac{2}{3}\frac{\left(\left(1-\delta^2\right)^{\frac{1}{2}}-\left(1-\delta^2\right)^{\frac{5}{4}}\right)}{\delta^2\left(2\left(1-\delta^2\right)^{\frac{3}{4}}-1\right)}\|e-v_+^2\|.$$

Then, from this last inequality and (22) we deduce

$$\delta(x_+s_+,\mu)\leq\frac{4\left(\left(1-\delta^2\right)^{\frac{1}{2}}-\left(1-\delta^2\right)^{\frac{5}{4}}\right)}{\delta^2\left(2\left(1-\delta^2\right)^{\frac{3}{4}}-1\right)}\delta^2. \tag{23}$$

Now, if we take $g(\delta)=\frac{4\left(\left(1-\delta^2\right)^{\frac{1}{2}}-\left(1-\delta^2\right)^{\frac{5}{4}}\right)}{\delta^2\left(2\left(1-\delta^2\right)^{\frac{3}{4}}-1\right)}$ for $\delta<\frac{1}{\sqrt[3]{4}}$, then we obtain that $g(\delta)\leq g(\frac{1}{\sqrt[3]{4}})<8$, and we conclude that

$$\delta(x_+s_+,\mu)<8\delta^2.$$

This completes the proof. □

The next lemma examines what is the effect of the full-Newton step on the duality gap.

Lemma 4.4
Let $\delta=\delta(xs,\mu)$. Then, the duality gap satisfies

$$(x_+)^Ts_+\leq\mu(n+4\delta^2).$$

Proof
From (17) we have

$$\begin{aligned} v^2 + vp_v &= \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e} \\ &= e + \frac{2v^{\frac{7}{2}} + v^2 - 6v^{\frac{3}{2}} + 3e}{6v^{\frac{3}{2}} - 3e} \\ &= e + \frac{4\left(2v^{\frac{7}{2}} + v^2 - 6v^{\frac{3}{2}} + 3e\right)}{\left(6v^{\frac{3}{2}} - 3e\right)p_v^2} \times \frac{p_v^2}{4} \\ &= e + \frac{12v^5 - 3v^2 - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \times \frac{p_v^2}{4} \\ &= e + \left[4e + \frac{36v^{\frac{3}{2}} - 36v^3 - 9e + 32v^{\frac{7}{2}} - 4v^5 - 19v^2}{4\left(v - v^{\frac{5}{2}}\right)^2}\right] \times \frac{p_v^2}{4} \\ &\leq e + 4\frac{p_v^2}{4}, \end{aligned}$$

because

$$\frac{36v^{\frac{3}{2}} - 36v^3 - 9e + 32v^{\frac{7}{2}} - 4v^5 - 19v^2}{4\left(v - v^{\frac{5}{2}}\right)^2} \leq 0, \forall v > \frac{1}{\sqrt[3]{4}}e.$$

Then $(x_+)^T(s_+) \leq \mu(n + 4\delta^2)$, which completes the proof. □

The following lemma investigates the effect on the proximity measure after a main iteration of the algorithm.

Lemma 4.5
Let $\delta = \delta(xs, \mu) < \frac{1}{\sqrt[3]{4}}$, $v > \frac{1}{\sqrt[3]{4}}e$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$. In addition, let $v_{++} = \sqrt{\frac{x_+s_+}{\mu_+}}$. Then $v_{++} > \frac{1}{2^{\frac{4}{7}}}e$, and

$$\delta(x_+s_+, \mu_+) < \frac{2}{3\sqrt{1-\theta}}\left[\frac{\left((1-\delta^2)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}\right)\left(6\delta^2 + \theta\sqrt{n}\right)}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}}\right].$$

Moreover, if $\delta < \frac{1}{6}$ and $\theta = \frac{1}{7\sqrt{n}}$, then $\delta(x_+s_+, \mu_+) < \frac{1}{6}$.

Proof
Using Lemma 4.3 we have $v_+ > \frac{1}{\sqrt[3]{4}}e$. From $v_{++} = \sqrt{\frac{x_+s_+}{\mu_+}}$ it follows that

$$v_{++} = \sqrt{\frac{x_+s_+}{\mu_+}} = \sqrt{\frac{x_+s_+}{(1-\theta)\mu}} = \frac{1}{\sqrt{1-\theta}}v_+ > \frac{1}{\sqrt[3]{4}}e. \tag{24}$$

This last inequality follows from $0 < \theta < 1 \Rightarrow \frac{1}{\sqrt{1-\theta}} > 1$.

Now, from the definition of δ, we write

$$\delta(v_+) = \delta(x_+s_+, \mu_+) = \frac{\|p_{v_{++}}\|}{2} = \frac{2}{3}\left\|\frac{\left(v_{++} - v_{++}^{\frac{5}{2}}\right)}{\left(2v_{++}^{\frac{3}{2}} - e\right)\left(e - v_{++}^2\right)}(e - v_{++}^2)\right\|.$$

Let us compute the three expressions of the previous norm. From (24) we obtain

$$v_{++} - v_{++}^{\frac{5}{2}} = \frac{1}{\sqrt{1-\theta}} v_+ - \left(\frac{1}{\sqrt{1-\theta}} v_+\right)^{\frac{5}{2}} = \frac{1}{(1-\theta)^{\frac{5}{4}}} \left[(1-\theta)^{\frac{3}{4}} v_+ - v_+^{\frac{5}{2}}\right],$$

and

$$2v_{++}^{\frac{3}{2}} - e = \frac{2v_+^{\frac{3}{2}}}{(1-\theta)^{\frac{3}{4}}} - e,$$

also

$$e - v_{++}^2 = \frac{1}{(1-\theta)} \left[(1-\theta)e - v_+^2\right].$$

Then

$$\left(2v_{++}^{\frac{3}{2}} - e\right)\left(e - v_{++}^2\right) = \frac{\left[2\left(1-\theta\right) v_+^{\frac{3}{2}} - 2v_+^{\frac{7}{2}} + (1-\theta)^{\frac{3}{4}} v_+^2 - (1-\theta)^{\frac{7}{4}}\right]}{(1-\theta)^{\frac{7}{4}}}. \tag{25}$$

And

$$\left(v_{++} - v_{++}^{\frac{5}{2}}\right)\left(e - v_{++}^2\right) = \frac{(1-\theta)^{\frac{3}{4}} v_+ - v_+^{\frac{5}{2}}}{(1-\theta)^{\frac{9}{4}}} \left[(1-\theta)e - v_+^2\right]. \tag{26}$$

These two last equalities give

$$\delta(v_+) = \frac{2}{3} \left\| \frac{\left((1-\theta)^{\frac{3}{4}} v_+ - v_+^{\frac{5}{2}}\right)\left[(1-\theta)e - v_+^2\right]}{2\left(1-\theta\right)^{\frac{3}{2}} v_+^{\frac{3}{2}} - 2(1-\theta)^{\frac{1}{2}} v_+^{\frac{7}{2}} + (1-\theta)^{\frac{5}{4}} v_+^2 - (1-\theta)^{\frac{9}{4}}} \right\|. \tag{27}$$

Let us consider the function

$$f(t) = \frac{(1-\theta)^{\frac{3}{4}} t - t^{\frac{5}{2}}}{2\left(1-\theta\right)^{\frac{3}{2}} t^{\frac{3}{2}} - 2(1-\theta)^{\frac{1}{2}} t^{\frac{7}{2}} + (1-\theta)^{\frac{5}{4}} t^2 - (1-\theta)^{\frac{9}{4}}}, \forall t > \frac{1}{\sqrt[3]{4}}.$$

After some calculation we obtain $f'(t) < 0$ for all $n \in \mathbb{N}^*$and $t > \frac{1}{\sqrt[3]{4}}$, then the function f is decreasing. From lemma 4.2 and (27) we deduce that

$$\delta(x_+ s_+, \mu_+) < \frac{2\left[(1-\theta)^{\frac{3}{4}}\left(1-\delta^2\right)^{\frac{1}{2}} - \left(1-\delta^2\right)^{\frac{5}{4}}\right] \left\|\left[(1-\theta)e - v_+^2\right]\right\|}{3\sqrt{1-\theta}\left[2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}\right]}. \tag{28}$$

According to (22), we get

$$\|(1-\theta)e - v_+^2\| \leq \|e - v_+^2\| + \|\theta e\| \leq 6\delta^2 + \theta\sqrt{n}.$$

Hence, by using this last inequality in (28) we get

$$\delta(x_+ s_+, \mu_+) < \frac{2}{3\sqrt{1-\theta}} \left[\frac{\left(\left(1-\delta^2\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}\right)\left(6\delta^2 + \theta\sqrt{n}\right)}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}}\right].$$

which proves the first part of the lemma.

Now, suppose that $\delta < \frac{1}{6}$ and $\theta = \frac{1}{7\sqrt{n}}$. Let's consider the function

$$f(\delta) = \frac{\left(1-\delta^2\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}},$$

we obtain $f'(\delta) > 0$, so f is increasing for each $\delta < \frac{1}{6}$, then

$$f(\delta) \leq f\left(\frac{1}{6}\right), \tag{29}$$

where

$$f\left(\frac{1}{6}\right) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2(1-\theta)\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2\left(\frac{35}{36}\right)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}\left(\frac{35}{36}\right) - (1-\theta)^{\frac{7}{4}}}.$$

Also we have for all $n \in N^*$, $3\sqrt{1-\theta} > 0$ and $2\left(6\delta^2 + \theta\sqrt{n}\right) = 2\left(6\delta^2 + \frac{1}{7}\right) < 2\left(\frac{1}{6} + \frac{1}{7}\right)$, then

$$\frac{2\left(6\delta^2 + \theta\sqrt{n}\right)}{3\sqrt{1-\theta}} < \frac{2\left(\frac{1}{6} + \frac{1}{7}\right)}{3\sqrt{1-\theta}} = \frac{13}{63\sqrt{1-\theta}}. \tag{30}$$

According to (29) and (30) we obtain

$$\delta(x_+ s_+, \mu_+) < \frac{13}{63} g(\theta), \tag{31}$$

with

$$g(\theta) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2(1-\theta)^{\frac{3}{2}}\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2(1-\theta)^{\frac{1}{2}}\left(\frac{35}{36}\right)^{\frac{7}{4}} + (1-\theta)^{\frac{5}{4}}\left(\frac{35}{36}\right) - (1-\theta)^{\frac{9}{4}}},$$

if $n \geq 1$ then $0 < \theta \leq \frac{1}{7}$. The function g is continuous and monotonic decreasing on $0 < \theta \leq \frac{1}{7}$, consequently

$$g(\theta) < g(0) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2\left(\frac{35}{36}\right)^{\frac{7}{4}} + \left(\frac{35}{36}\right) - 1}, \tag{32}$$

Finally, using (31) and (32), we get

$$\delta(x_+ s_+, \mu_+) < \frac{13}{63} \times \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2\left(\frac{35}{36}\right)^{\frac{7}{4}} + \left(\frac{35}{36}\right) - 1} = 0.1598 < \frac{1}{6}.$$

Which completes the second part of the proof. □

The next result yields an upper bound on the duality gap after a full-Newton step.

Lemma 4.6
Suppose that the pair $\left(x^0, s^0\right)$ is strictly feasible, $\mu^0 = \frac{\left(x^0\right)^T s^0}{n}$ and $\delta(x^0 s^0, \mu^0) < \frac{1}{\sqrt[3]{4}}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. Then the inequality $\left(x^k\right)^T s^k < \varepsilon$ is satisfied when

$$k \geq \frac{1}{\theta} \log\left[\frac{\mu^0(n + \sqrt[3]{4})}{\varepsilon}\right].$$

Proof
After k iterations, we have $\mu^k = (1-\theta)^k \mu^0$. From Lemma 4.4 and $\delta(xs, \mu) < \frac{1}{\sqrt[3]{4}}$, we get

$$(x^k)^T s^k < \mu^k \left(n + 4\left(\frac{1}{\sqrt[3]{4}}\right)^2\right) = \mu^0(1-\theta)^k \left(n + \sqrt[3]{4}\right).$$

Hence, the inequality $\left(x^k\right)^T s^k < \varepsilon$ holds if

$$\begin{aligned} & \mu^0(1-\theta)^k \left(n + \sqrt[3]{4}\right) \leq \varepsilon \\ \Longleftrightarrow \quad & \log(1-\theta)^k + \log \mu^0 \left(n + \sqrt[3]{4}\right) \leq \log \varepsilon \\ \Longleftrightarrow \quad & -k \log(1-\theta) \geq \log \frac{\mu^0\left(n + \sqrt[3]{4}\right)}{\varepsilon}. \end{aligned}$$

As $\theta \leq -\log(1-\theta)$, then the last inequality is valid only if

$$k \geq \frac{1}{\theta} \log \frac{\mu^0 \left(n + \sqrt[3]{4}\right)}{\varepsilon}.$$

This completes the proof. □

Theorem 4.1
Suppose that $x^0 = s^0 = e$. If we consider the default values for θ and τ, we obtain that the algorithm represented in Figure 1 requires no more than

$$7\sqrt{n} \log \frac{\left(n + \sqrt[3]{4}\right)}{\varepsilon}$$

iterations. The resulting vectors satisfy $\left(x^k\right)^T s^k < \varepsilon$.

Proof
Since $x^0 = s^0 = e$, we replace $\mu^0 = \frac{\left(x^0\right)^T s^0}{n}$ by 1 and θ by $\frac{1}{7\sqrt{n}}$ in Lemma 4.6, the result holds. □

5. Numerical experiments

To prove the effectiveness of our new function and evaluate its effect on the behavior of the algorithm, we offer a comparative study based essentially on the two following functions.

- The first function, given by Darvay and Takàcs [6] defined by $\psi(t) = t^2$.
- Our new function defined by $\psi(t) = t^{\frac{3}{2}}$.

Let us consider the linear problem (P) and its dual (D), where:

$$n = 2m, \quad A(i,j) = \begin{cases} 1 \text{ If } i = j \text{ or } j = i + m \\ 0 \text{ If } i \neq j \text{ or } j \neq i + m \end{cases},$$

$$c(i) = -1, \quad c(i+m) = 0, \quad b(i) = 2, \text{ for } i = 1, .., m.$$

The starting points are:
$x^0(i) = x^0(i+m) = 1, y^0(i) = -2, z^0(i) = 1, z^0(i+m) = 2$ for $i = 1, ..., m$.
The algorithm was implemented using MATLAB. We considered the following different values for the update parameter $\theta : 0.1, 0.2, 0.4, 0.5, 0.7, 0.9$. We set the value for the accuracy parameter $\varepsilon = 10^{-4}$.
In the table of results, (m, n) represent respectively the number of constraints and the number of variables, $Iter$ represents the number of iterations necessary for optimality and $T(s)$ is the execution time in seconds.
For different values of θ, we summarize the obtained results in Tables 1, 2, 3, 4, 5 and 6.

Comments
The numerical results show that the number of iterations and the execution time necessary for the optimality of the algorithm depends on the values of the parameter θ. It is quite surprising that $\theta = 0.9$ gives the smallest number of iterations and minimal time. Moreover, the realized numerical experiments prove the effectiveness of our new function on all tested instances. We note that when the dimension of the problem becomes large and $\theta = 0.9$, the difference between our new function and that of Darvay and Takàcs [6] becomes large in terms of the number of iterations. These numerical results consolidate and confirm our theoretical results.

Table 1. comparative results for $\theta = 0.1$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	129	0.3204	129	0.3087
$(50, 100)$	136	1.4265	136	1.4346
$(100, 200)$	142	8.0714	142	7.8226
$(250, 500)$	151	97.0597	151	89.6265
$(500, 1000)$	157	593.3854	157	585.5760
$(750, 1500)$	161	2234.7482	161	1933.9494

Table 2. comparative results for $\theta = 0.2$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	61	0.1756	61	0.1743
$(50, 100)$	65	0.6439	64	0.6325
$(100, 200)$	68	4.0990	67	4.0664
$(250, 500)$	72	42.0901	72	41.7091
$(500, 1000)$	75	282.4912	75	280.5850
$(750, 1500)$	77	1162.5452	77	933.3365

Table 3. comparative results for $\theta = 0.4$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	28	0.0959	27	0.0984
$(50, 100)$	29	0.3025	29	0.2783
$(100, 200)$	31	1.9525	30	1.8170
$(250, 500)$	33	21.5579	32	19.7897
$(500, 1000)$	34	139.1897	33	124.8724
$(750, 1500)$	35	448.7335	34	417.4751

6. Conclusion

We have reconsidered the technique presented by Darvay et al. in [6] by using a new function. With this technique, we proposed a new primal-dual IPM for solving linear optimization problems and proved that the obtained algorithm solves the problem in polynomial time and has the best-known complexity bound for LP. Moreover, we provided some numerical experiments to show the efficiency of the proposed algorithm. Future research would be to extend this technique to other important optimization problems such as convex quadratic optimization, semidefinite optimization, second-order cone optimization and symmetric optimization.

Table 4. comparative results for $\theta = 0.5$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	23	0.0861	21	0.0854
$(50, 100)$	24	0.3034	22	0.2271
$(100, 200)$	25	1.4499	23	1.3457
$(250, 500)$	26	16.4222	24	14.6800
$(500, 1000)$	27	101.6608	25	94.8041
$(750, 1500)$	28	345.8178	26	307.0373

Table 5. comparative results for $\theta = 0.7$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	21	0.0876	14	0.0701
$(50, 100)$	22	0.2788	15	0.1845
$(100, 200)$	23	1.5355	16	0.9504
$(250, 500)$	24	14.4541	17	10.7135
$(500, 1000)$	25	93.3130	17	64.9954
$(750, 1500)$	25	297.3467	18	224.1331

Table 6. comparative results for $\theta = 0.9$.

	$\psi(t) = t^2$		$\psi(t) = t^{\frac{3}{2}}$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	20	0.0743	13	0.0708
$(50, 100)$	21	0.2286	14	0.2143
$(100, 200)$	22	1.3844	14	0.8667
$(250, 500)$	23	14.5164	15	9.7474
$(500, 1000)$	24	90.6246	16	64.7198
$(750, 1500)$	25	294.2109	16	193.1574

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