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Bayesian and Non-Bayesian Estimation for The Parameter of Inverted Topp-Leone Distribution Based on Progressive Type I Censoring

Hiba Z. Muhammed, Essam A. Muhammed *

Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt

Abstract In this paper, Bayesian and non-Bayesian estimations of the shape parameter of the Inverted Topp-Leone distribution are studied under a progressive Type I censoring scheme. The maximum likelihood estimator (MLE) and Bayes estimator (BE) of the unknown parameter under the squared error loss (SEL) function are obtained. Three types of confidence intervals are discussed for the unknown parameter. A simulation study is performed to compare the performances of the proposed methods, and two numerical examples have been analyzed for illustrative purposes.

Keywords Inverted Topp-Leone Distribution, Bayesian Estimation, Highest Posterior Density Interval, Bootstrap Confidence Interval, Progressive Type I Censoring

AMS 2010 subject classifications 62F10, 62N02

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1. Introduction

*T*he inverted distributions have a wide range of applications; in problems related to biological sciences, engineering sciences, econometrics, survey sampling, life testing problems, and medical research. In addition, it is employed in financial literature, environmental studies, and survival and reliability theory. Many researchers studied the inverted distributions and their applications; for example, [14] introduced the inverse Weibull, [4] reviewed the inverted Burr Type XII distribution, [2] discussed inverted Pareto Type I distribution, [5] described the inverted Pareto Type II distribution, [35] presented inverted exponential model, [6] studied the exponentiated inverted Weibull distribution and [1] introduced the inverted Kumaraswamy distribution. [38] introduced The Topp-Leone (TL) distribution as an alternative to beta distribution. The TL distribution indicates the j-shape form of the density function along with the bathtub shape of its hazard function. It helps model bounded lifetime phenomena, and different aspects of this distribution have been studied by [34] Several authors have studied the TL distribution; see, for example, [21], [39], [41], [29], [40], [33], [9], and [20]. The Inverted Topp-Leone (ITL) distribution has been introduced in different forms. [32] introduced the ITL distribution with support $x > 1$, and the probability density function (pdf) and the cumulative distribution function (cdf) of the ITL distribution, respectively, are given by:

$$g(x;\beta) = 2\beta x(x-1)x^{-2\beta-1}(2x-1)^{\beta-1}; 1 < x < \infty, \beta > 0$$

and

$$G(x;\beta) = 1 - x^{-2\beta}(2x-1)^{\beta-1}; 1 < x < \infty, \beta > 0$$

and [26] (also see [31]) introduced the ITL distribution, with support $x > 0$. the pdf and the cdf of the ITL distribution, respectively, are given by

*Correspondence to: Essam A. Muhammed (Email: essamabdelsalam16@gmail.com). Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt.

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$$f(x;\beta) = 2\beta x(1+2x)^{\beta-1}(1+x)^{-2\beta-1}; x, \beta > 0 \quad (1)$$

and

$$F(x;\beta) = 1-(1+2x)^{\beta}(1+x)^{-2\beta}; x \geq 0, \beta > 0 \quad (2)$$

where β is the shape parameter. (1) illustrates the behavior of the ITL distribution for various values of β.

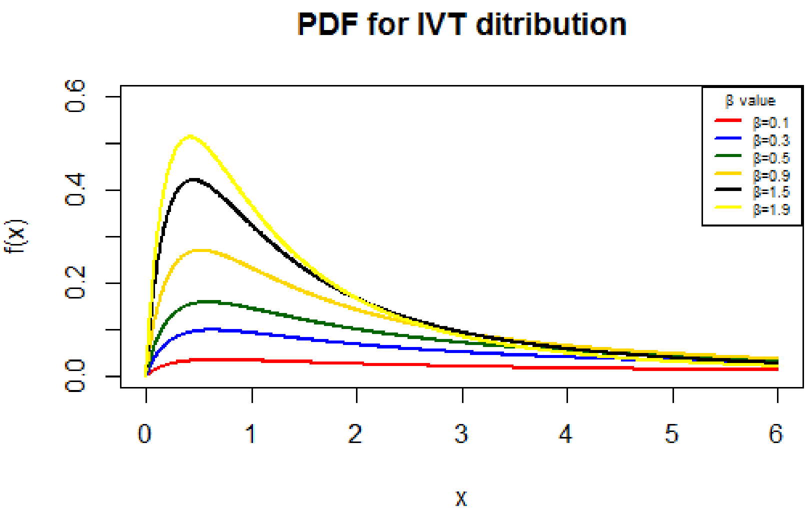

Figure 1. Density function of ITL distribution for some values of β

The survival function is given by

$$\overline{F}(x;\beta) = (1+2x)^{\beta}(1+x)^{-2\beta} \quad (3)$$

The hazard rate function of ITL distribution can be obtained as

$$h(x;\beta) = 2\beta x\left[(1+2x)(1+x)\right]^{-1} \quad (4)$$

(2) presents the hazard function of ITL distribution with different values of β. The ITL can be used to model many lifetime and reliability data and natural phenomena data; see, for example, [24], [22], [8], [7] and [25]. [3] introduced the Bayesian estimation of the shape parameter of ITL distribution. [31] discussed the Bayesian and non-Bayesian analysis of the shape parameter of ITL distribution under randomly censored data. [42] introduced the Bayes estimates for the ITLD parameters using MCMC method of system reliability under ranked set sampling. [27] studied the Bayesian estimation of ITLD under PT-IIC data based on CRs model. In life test experiments and reliability studies, it is not always possible to observe the failure times of all units on test. So, the experimenters depend on the censored data. In practice, censored data arise when the experiments, including the lifetimes of test items, have to be terminated before collecting complete observation. Type I and Type II censoring schemes are the two most common and popular censoring schemes. In Type I censoring, the life test stops at a pre-fixed time, and in Type II censoring, the life test stops after observing a pre-fixed number of units [30]. Sometimes the experimenter has to remove units intentionally because of restrictions (e.g., money, material resources, or time), or units are lost before failure (e.g., accidental breakage of a unit). The intentional withdrawal of objects from a running life test enables the release of items for other experiments. These scenarios can be modeled by progressive censoring. Two well-known censoring schemes are progressive Type I censoring and progressive Type II censoring.

[15] introduced the model of progressive Type I censoring as an extension of Type I censoring. Progressively Type I right censored samples are observed when a pre-fixed number of live test units are continuously removed

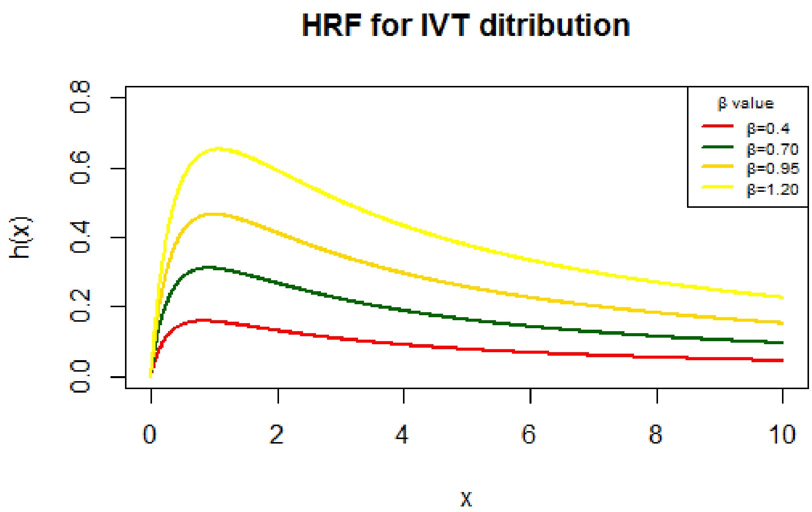

Figure 2. Hazard rate function of ITL distribution for some values of β

during the experiment at the end of each pre-specified time interval. For further information on progressive Type I censoring, we refer to the monograph by [11] and to the review paper by [10].

On the other hand, [28] introduced the model of progressive Type II censoring, named multiple censoring, and later on by [15] as 'Type II progressive censoring.' progressive Type II right censoring corresponds to the situation where a pre-fixed number of surviving units are continuously withdrawn from the experiment at each observed failure time until the pre-specified number of units have failed the life test.

In this article, our objective is to use statistical inference methods regarding a life test from which the available data are progressively Type-I censored. Under the assumption that the lifetimes of the test units independently have the ITL distribution. In Section 2, we obtain the approximate expression for the MLE. We discussed the Bayesian approach in Section 3. In Section 4, we introduce three types of confidence intervals. In Section 5, we provide a simulation study. Two numerical examples have been presented in Section 6. Finally, some conclusions are drawn in Section 7.

2. Model Description and Maximum Likelihood

To describe a life-testing procedure implemented with progressive Type-I censoring, we must first choose m censoring time points $T_1 < T_2 < \ldots < T_m$. Now, for $i = 1,\ 2, \ldots, m$, let v_i denote the number of units failed in time interval $]T_{i-1},\ T_i]$, $x_{i,l}$ denote the l^{th} ordered failure time among these v_i units for $l = 1,\ 2,\ \ldots,\ v_i$, and R_i denote the number of units randomly removed (i.e., censored) at time T_i. Furthermore, let N_i indicate the number of units operating and remaining on test at the beginning of the i^{th} time interval, i.e., $N_i = n - \sum_{j=1}^{i-1} v_j - \sum_{j=1}^{i-1} R_j$. Under this setup, a progressively Type-I censored life-testing experiment proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at time $T_0 \equiv 0$ and tested until time T_1, at which point R_1 live units are arbitrarily withdrawn from the test. In this first time interval, random v_1 failure times are also collected. The test is then continued on $N_2 = n - v_1 - R_1$ remaining units until time T_2, at which point R_2 live units are randomly withdrawn from the test, and so on. Finally, the number of failure times at time T_m is $r = \sum_{i=1}^{m} v_i$. Note that if $n > \sum_{i=1}^{m}(v_i + R_{i-1}))$ at time T_m then all the surviving units (R_m) are removed where $R_m = n - \sum_{i=1}^{m} v_i - \sum_{i=1}^{m} R_{i-1} = N_m - v_m$. The situation with no intermediate censoring (i.e., $r_1 = r_2 = \ldots = r_{m-1} = 0$) corresponds to a life-test under the conventional Type-I right censoring as a special case. (see [12]). The likelihood function, in this case, is given by [15]

$$\mathcal{L}(\theta) \propto \prod_{i=1}^{r} f\left(x_{(i)};\theta\right) \prod_{j=1}^{m} \left(1 - F\left(T_{(j)};\theta\right)\right)^{R_{(j)}} \tag{5}$$

Under the assumption that the lifetime of a test unit follows ITL distribution with pdf and cdf of the failure time of a test unit given in (1) and (2), respectively. Then the likelihood function under progressive Type I censored data is given by

$$\mathcal{L}(\beta) \propto (2\beta)^r \prod_{i=1}^{r} x_{(i)} \prod_{i=1}^{r} \left(1+x_{(i)}\right)^{-2\beta-1} \prod_{i=1}^{r} \left(1+2x_{(i)}\right)^{\beta-1}$$
$$\prod_{j=1}^{m} \left[\left(1+T_{(j)}\right)^{-2\beta} \left(1+T_{(j)}\right)^{\beta}\right]^{R_{(j)}} \tag{6}$$

Taking the logarithm of $\mathcal{L}(\beta)$ to obtain log-likelihood $l(\beta) = ln(\mathcal{L})$ as

$$l(\beta) \propto rln2 + rln\beta + \sum_{i=1}^{r} \ln x_{(i)} + (-2\beta-1)\sum_{i=1}^{r} \ln\left(1+x_{(i)}\right) + (\beta-1)\sum_{i=1}^{r} \ln\left(1+2x_{(i)}\right)$$
$$-2\beta \sum_{j=1}^{m} R_{(j)} \ln\left(1+T_{(j)}\right) + \beta \sum_{j=1}^{m} R_{(j)} \ln\left(1+T_{(j)}\right) \tag{7}$$

After differentiating the $l(\beta)$ and equating it to zero, the MLE for β can be expressed in closed form as follows.

$$\widehat{\beta} = \frac{r}{2\sum_{i=1}^{r} \ln\left(1+x_{(i)}\right) - \sum_{i=1}^{r} \ln\left(1+2x_{(i)}\right) + 2\sum_{j=1}^{m} R_{(j)} \ln\left(1+T_{(j)}\right) - \sum_{j=1}^{m} R_{(j)} \ln\left(1+T_{(j)}\right)}$$

3. Bayesian Estimation

In this section, we have discussed the Bayesian estimation procedure for the parameter of the ITL distribution, and we get a Bayesian estimate of the unknown parameter under the SEL function. We assume that the unknown parameter of the ITL distribution has gamma prior distribution and can be written with proportional as follows;

$$\pi\left(\beta|a_1,b_1\right) \propto \beta^{a_1-1} e^{-\beta b_1}, \qquad \beta>0,\ a_1,b_1>0 \tag{8}$$

Hyper-parameters determination: The hyper-parameters involved in priors (8) can be quickly evaluated if we consider that the prior mean and prior variance are known. The prior mean and prior variance will be obtained from the maximum likelihood estimate of (β) by equating the mean and variance of $(\widehat{\beta}^j)$ with the mean and variance of the considered priors (gamma prior), where $j = 1, 2, \ldots, k$ and k is the number of random samples generated from the model. Thus, by equating the mean and variance of $(\widehat{\beta}^j)$ with the mean and variance of gamma priors, we get [17]

$$\frac{1}{k}\sum_{j=1}^{k} \widehat{\beta}^j = \frac{a_1}{b_1} \qquad \& \qquad \frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2 = \frac{a_1}{{b_1}^2}\ .$$

Now, on solving the above two equations, the estimated hyper-parameters can be written as

$$a_1 = \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2} \qquad \& \qquad b_1 = \frac{\frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2}$$

Based on the likelihood function (6) and the gamma prior (8), the posterior density function of β, given the data, can be written as

$$(]\pi\left(\beta|\underline{x}\right) = \frac{\pi(\beta)L\left(\underline{x}|\beta\right)}{\int_0^\infty \pi(\beta)L\left(\underline{x}|\beta\right)d\beta}$$

.

Then, the posterior density function can be written as

$$\pi\left(\beta|\underline{x}\right) = \frac{2^r\beta^{r+a_1}}{k(\beta)}\prod_{i=1}^{r}x_{(i)}\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-2\beta-1}\prod_{i=1}^{r}\left(1+2x_{(i)}\right)^{\beta-1}\prod_{j=1}^{m}\left[\left(1+T_{(j)}\right)^{-2\beta}\left(1+T_{(j)}\right)^{\beta}\right]^{R_{(j)}}e^{-\beta b_1} \tag{9}$$

Where,

$$k(\beta) = \int_0^\infty (2\beta)^r\beta^{a_1}\prod_{i=1}^{r}x_{(i)}\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-2\beta-1}\prod_{i=1}^{r}\left(1+2x_{(i)}\right)^{\beta-1}\prod_{j=1}^{m}\left(\left(1+T_{(j)}\right)^{-2\beta}\left(1+T_{(j)}\right)^{\beta}\right)^{R_{(j)}}e^{-\beta b_1}d\beta$$

Thus, the Bayes estimate (BE) of β based on the SEL function is given by

$$\widetilde{\beta} = E\left(\beta|\underline{x}\right)$$

$$\widetilde{\beta} = \frac{\int_0^\infty \beta\pi(\beta)L\left(\underline{x}|\beta\right) \quad d\beta}{\int_0^\infty \pi(\beta)L\left(\underline{x}|\beta\right) \quad d\beta} \tag{10}$$

It should be noted that the ratio of integral in (10) cannot be obtained in closed forms. So, we use the MCMC approximation method to generate samples from (9), calculate the BE of β, and construct associated highest posterior density (HPD) intervals.

Markov Chain Monte Carlo (MCMC) is a computer-driven sampling technique. It permits one to characterize a distribution without knowing its mathematical properties by random sampling values out of the distribution [36]. We use the Metropolis Hasting (M-H) method with normal proposal distribution to generate random numbers from (9). Thus, we perform the following steps of the M-H algorithm to draw samples from the posterior density (9) and, in turn, compute the BE of β and construct the corresponding HPD intervals [16]:

I. Set initial values $\beta^{(0)}$, M = burn-in.
II. For i = 1, . . . , N, repeat the following steps.

- Set $\beta = \beta^{(i-1)}$.
- Generate new candidate parameter values ω from $N_1\left(\log(\beta),\ S_\beta\right)$.
- Set $\beta^{'} = exp(\omega)$.
- Calculate $A = \min\left(1, \frac{\pi\left(\beta^{'}|\underline{x}\right)}{\pi(\beta|\underline{x})}\right)$
- Update $\beta^{(i)} = \beta^{'}$ with probability A; otherwise, set $\beta^{(i)} = \beta$.

The approximate BE of $\beta^{(i)}$, i =1,2,..., N concerning the SEL function is given by

$$\widetilde{\beta}_{BS} = \frac{1}{N-M} \sum_{i=M+1}^{N} \beta^{(i)},$$

Where $\widetilde{\beta}_{BS}$ is BE under the SEL function and M is the burn-in period (that is, several iterations before the stationary distribution is achieved)

4. Confidence Intervals

This section introduces three types of confidence intervals (CIs). One is based on the asymptotic distribution of β, two different bootstrap CIs, and finally, HPD intervals.

4.1. The Approximate Confidence Intervals

The second derivative for $\mathcal{L}(\beta)$ is trivially obtained as

$$\frac{d^2\mathcal{L}(\beta)}{d\beta^2} = -\frac{r}{\beta^2}.$$

The observed Fisher information matrix is given by

$$I\left(\widehat{\beta}\right) = -\left.\frac{d^2\mathcal{L}(\beta)}{d\beta^2}\right|_{\beta=\widehat{\beta}} = \frac{r}{\widehat{\beta}^2}.$$

The variance of $\widehat{\beta}$ is

$$V\left(\widehat{\beta}\right) = \frac{1}{I(\widehat{\beta})} = \frac{\widehat{\beta}^2}{r}$$

.

A standard normal distribution can approximate the sampling distribution of $\frac{\widehat{\beta}-\beta}{\sqrt{V(\widehat{\beta})}}$.

The large sample $(1-\alpha)100\%$ confidence interval for β is given by

$$(\widehat{\beta}_L, \widehat{\beta}_U) = \widehat{\beta} \mp Z_{\frac{\alpha}{2}} \sqrt{V\left(\widehat{\beta}\right)}.$$

Where $Z_{\frac{\alpha}{2}}$ is the standard normal random variable and $(1-\alpha)$ is the confidence coefficient.

4.2. Bootstrap Confidence Intervals

The bootstrap CIs are approximate CIs but, in general, are better approximate than standard intervals. A parametric bootstrap interval provides much more information about the population value of the quantity of interest than a point estimate. The parametric bootstrap methods are of two types:-

(i) The percentile bootstrap method (Boot-p) was proposed by [18].
(ii) The bootstrap-t method (Boot-t) was proposed by [23].

- **Percentile Bootstrap (Boot-P) Confidence Interval**

The boot-p method is relatively simple and constructs CIs directly from the estimated parameters' percentiles of the bootstrap distribution. The following steps give it:

I. A progressively Type I censored sample is generated from the original data $X = (x_1, x_2 \ldots x_n)$ based on the censoring times and the removed units that are shown in (1) and (2), respectively, and then computes the MLE $\widehat{\beta}$ of the parameter β.
II. Again, an independent bootstrap sample $X^* = (x_1{}^*, x_2{}^* \ldots x_n{}^*)$ is generated using $\widehat{\beta}$, then found in the progressive censoring data such as in the previous step.
III. Now, compute the bootstrap MLE $\widehat{\beta}^*$ of parameter β based on the progressive Type I censored data, as in step 1.
IV. Repeat steps 2-3, B times representing B bootstrap MLE's $\widehat{\beta}^{*\prime}s$ based on B different bootstrap samples, i=1, 2, … B.
V. Arrange all $\widehat{\beta}^{*\prime}s$ in an ascending order to obtain the bootstrap sample
VI. i.e., $\widehat{\beta}^*_{(1)} \leq \widehat{\beta}^*_{(2)} \leq \ldots \leq \widehat{\beta}^*_{(B)}$. An approximate $100(1-\omega)\%$ boot-p confidence interval for β is obtained by $\left(\widehat{\beta}^*_{[(\frac{\omega}{2})\times B]}, \widehat{\beta}^*_{[(1-\frac{\omega}{2})\times B]}\right)$.

Where $\frac{\omega}{2}$ is the quantity that helps to determine the bootstrap point.

- **Bootstrap-t (Boot-t) Confidence Intervals**

The following steps give the bootstrap-t confidence interval:

I. Steps 1 and 2 of the boot-p and boot-t methods are the same.
II. Compute the bootstrap-t statistic for β as follows

$$T^* = \frac{\widehat{\beta}^*_b - \widehat{\beta}}{\sqrt{v(\widehat{\beta}^*_b)}}, b = 1, 2, \ldots B.$$

III. To obtain a set of bootstrap statistics, $T^*{}_i; i = 1, 2, \ldots,\ B$ repeat steps 2-3, B times.
IV. Let $T^*{}_{(1)} \leq T^*{}_{(2)} \leq \ldots \leq T^*{}_{(B)}$ be the ordered values of $T^*{}_i; i = 1, 2, \ldots, B$.
V. Now, the approximate $100(1-\omega)\%$ boot-t confidence interval for parameter β is obtained by

$$\left(\widehat{\beta} - \widehat{T}^*_{[(1-\frac{\omega}{2})\times B]}\sqrt{v\left(\widehat{\beta}\right)}, \widehat{\beta} - \widehat{T}^*_{[(\frac{\omega}{2})\times B]}\sqrt{v\left(\widehat{\beta}\right)}\right)$$

4.3. Highest Posterior Density Intervals

The HPD intervals for the unknown parameters can be constructed by using the following algorithm, let $\beta_{(1)}, \beta_{(2)}, \ldots, \beta_{(n)}$ be the corresponding ordered MCMC sample from (9) to construct the HPD interval, let $\beta_{(j)}$ be the j$^{\text{th}}$ smallest of $\{\beta_{(i)}\}$ and denote $D_j(n) = \left(\beta_{(j)}, \beta_{(j+[(1-\varphi)n])}\right)$, where $0 < \varphi < 1$ For $j = 1, 2, \ldots, n - [(1-\varphi)n]$ be the HPD intervals then the best HPD interval that has smallest interval width from $D_j(n)$'s. So, we can say $D_{j^*}(n) = \left(\beta_{(j^*)}, \beta_{(j^*+[(1-\varphi)n])}\right)$, be HPD interval for the unknown parameters have the smallest interval width among all $D_{j^*}(n)^{'}s$. Where j^* is chosen so that

$$\beta_{(j^*+[(1-\varphi)n])} - \beta_{(j^*)} = \min_{1 \leq j \leq n-[(1-\varphi)n]} \left(\beta_{(j+[(1-\varphi)n])} - \beta_{(j)}\right)$$

Then, the HPD intervals for the unknown parameter β can be constructed as seen in Section 5.

5. Simulation Study

A simulation study was carried out to check the performance of the accuracy of point and interval estimates for several cases, for which estimate the one parameter of ITL distribution (β) for several replications ($k = 1000$) for different sample sizes (n) as $n = 25,\ 50,\ 100$ and different parameters values. All the computations are performed using statistical software R.

The following steps obtain the simulation results for maximum likelihood estimates (MLEs):

i. Specify true values for a parameter (β) as (0.5), (1) and (1.5).
ii. Specify the sample size n. as $n = 25,\ 50,\ 100$.
iii. Generate n standard uniform variates i.e. $U \sim Uniform\ (0,\ 1)$.
iv. Generated complete samples of size n from ITL (β) distribution by using the formula

$$x = \frac{-2\left((1-u)^{\frac{1}{\beta}}-1\right)+\sqrt{4\left((1-u)^{\frac{1}{\beta}}-1\right)^2-4(1-u)^{\frac{1}{\beta}}\left((1-u)^{\frac{1}{\beta}}-1\right)}}{2(1-u)^{\frac{1}{\beta}}}.$$

v. Determine the progressive Type I censoring data:

1. Determine the censoring times (CT) T_i based on the quantile function for the ITL distribution by using the provided expression (see [19]).

$$P\left(X \le T_i\right) = q_i$$

$$T_i = \frac{-2\left((1-q_i)^{\frac{1}{\beta}}-1\right)+\sqrt{4\left((1-q_i)^{\frac{1}{\beta}}-1\right)^2-4(1-q_i)^{\frac{1}{\beta}}\left((1-q_i)^{\frac{1}{\beta}}-1\right)}}{2(1-q_i)^{\frac{1}{\beta}}}\ i = 1, 2, \ldots, m.$$

Then the CT corresponding to the selected quantiles with $m = 3, 4, 5$ and the different β values are given in (3).

2. The removed units (R_i) are assumed at different sample sizes in (4).

vi. Obtain the MLEs.
vii. Obtain the mean, bias, mean squared error (MSE), approximate and bootstrap CIs for the unknown parameter, average interval lengths (AILs), and coverage probability (CP) for the different sample sizes.
viii. Repeat steps 1-5 1000 times.

The simulation results for the Bayesian estimate are obtained by the following steps:

i. Steps i, ii, iii, iv, and v of the MLE simulation are the same
ii. Using the M-H algorithm shown in Section 3 under the informative prior and the non-informative prior, repeat the chain N times (N=10000) to obtain MCMC samples.

- For informative prior, we compute the hyper parameters for all stages as shown in (9 : 11).
- For non-informative prior (P-II), we assume that hyper-parameter values are $a_1 = b_1 = 0$.

iii. Compute the approximate BE of β under the SEL function given by

$$\widetilde{\beta}_{SEL} = \frac{1}{N-M}\sum_{i=M+1}^{N} g_{SEL}\left(\beta^{(i)}\right)$$

, $i = 1, 2, \ldots, N$.

Where M (=2000) is the burn-in period (that is, several iterations before the stationary distribution is achieved).

Repeat steps i-iii 1000 times to obtain the mean, bias, mean squared error (MSE), HPD intervals for the unknown parameter β, average interval lengths (AILs), and coverage probability (CP) for different sample sizes.

From tabulated values in (12 : 20), it can be noticed that:

i. Depending on MSEs, higher values of n lead to better estimates.
ii. Depending on MSEs, the MLEs under Type I censoring data (the first removed cases in (2) are better than the MLEs under progressive Type I censoring data in all cases.
iii. It is noticed that the MLEs compete well with the Bayes estimates (BEs) under non-informative prior, and the performance of the BEs obtained under informative prior are better than the BEs under non-informative prior.
iv. It can also be noticed that the CPs under the approximate CIs are better than those of bootstrap (p, t), non-informative and informative priors, respectively.
v. It can also be noticed that under informative prior, the AILs of HPD intervals are better than those of non-informative priors, the approximate CIs and bootstrap (p, t) CIs, respectively.

6. Applications to Real Data Sets

In this section, the ITL distribution will be fitted to two real data sets to show how the ITL distribution can be applied in practice. Moreover, the ITL distribution will also be compared with other inverted distributions that are fitted to this data, such as inverse Weibull (IW), inverse exponential (IE), inverse Rayleigh (IR), and inverse Lindley (IL). For the data set, the unknown parameter of the previous distributions will be estimated by the maximum-likelihood method, and with this, the estimate (MLE), the values of the Kolmogorov-Smirnov (KS) statistic (the distance between the empirical CDFs and the fitted CDFs), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) will be calculated.

6.1. The First Data Set: Vinyl Chloride Data.

We consider vinyl chloride data obtained from clean upgradient monitoring wells in *mg/L*; see [13]. The data set is given as: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2. The results are summarized in the following Table:

Table 1. The Values of Goodness of Fit Test for Vinyl Chloride data Set to the ITL distribution.

Distribution		Measures						
		P-value	K-S	-2log L	AIC	BIC	AICc	HQIC
ITL	2.1172	0.92	0.09	-111.2	-109.2	-107.7	-111.1	-108.7
IW	0.6174	0.77	0.11	-117.3	-113.3	-110.2	-115.9	-112.2
	0.8805							
IE	0.5727	0.46	0.15	-118.4	-116.4	-114.8	-118.3	-115.9
IR	0.3389	0.00	0.49	-186.7	-187.7	-183.2	-186.6	-184.2
IL	0.8773	0.17	0.19	-123.6	-121.6	-120.1	-123.5	-121.1

From Table 1, we note that the vinyl chloride data can be modeled by the ITL distribution where the P-Value is 0.92 and MLE of β is $\widehat{\beta} = 2.1172$

To fit the given data set graphically, we plot the graph and the fitted pdf lines corresponding to the ITL, IW, IE, IR, and IL distributions; we also plot the empirical cdf and fitted cdfs corresponding to identical distributions. (3) shows the lines fitted for cdfs and pdfs for the vinyl chloride data set and the corresponding distributions. The numbers in (1) also indicate that the ITL distribution provides a better fit than other distributions for this data set.

We generate different progressive censoring Type I samples from the original vinyl chloride data. Removed items are considered as in the simulation study in (4) for n = 25. In addition, different stages of censoring are proposed at $m = 3$, $m = 4$ and $m = 5$ then the censoring times when $\beta = 1.5$ are given in (3).

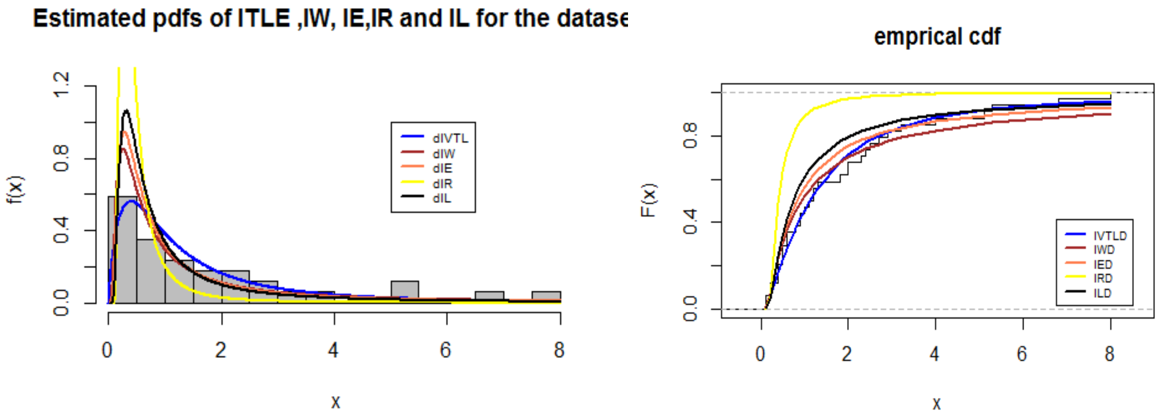

Figure 3. Estimated pdf and cdf for the vinyl chloride data set with corresponding distributions

In (2), we compute the MLE of the parameter β, their associated confidence interval estimates, and bootstrap CIs estimates (Boot-p and Boot-t). We also compute BEs and the Highest posterior density intervals using the M-H algorithm under informative and non-informative prior. For informative prior, we apply the same method shown in Section 3 to compute the hyper parameters. We use the MLE estimate for the parameter β and its variance under all stages with their different removed units shown in (5).

6.2. The Second Data Set: Annual Rainfall Data.

These data represent the total annual rainfall (in inches) during the month of January from 1880 to 1916, recorded at Los Angeles Civic Center introduced by [37]. These data show remission times (in a year) of a group of 37 observations. The data set is given as: (1.33, 1.43, 1.01, 1.62, 3.15, 1.05, 7.72, 0.20, 6.03, 0.25, 7.83, 0.25, 0.88, 6.29, 0.94, 5.84, 3.23, 3.70, 1.26, 2.64, 1.17, 2.49, 1.62, 2.10, 0.14, 2.57, 3.85, 7.02, 5.04, 7.27, 1.53, 6.70, 0.07, 2.01, 10.35, 5.42, 13.3). The results are summarized in the following (2):

Table 2. The Values of Goodness of Fit Test for Annual Rainfall Data to the ITL distribution.

Distribution	MLE	Measures						
		P-value	K-S	-2log L	AIC	BIC	AICc	HQIC
ITL	1.28	0.50	0.14	-176.1	354.2	355.8	352.3	354.8
IW	1.04	0.14	0.19	-186.9	377.7	380.9	375.0	378.8
	0.71							
IE	0.76	0.0	0.3	-198.6	399.1	400.7	397.2	399.7
IR	0.34	0.0	0.7	-339.3	680.6	682.2	678.7	681.1
IL	1.12	0.0	0.3	-210.4	422.8	424.4	420.9	423.3

From (2), we note the annual rainfall data can be modeled by the ITL distribution where P-Value is 0.50, and the MLE of β is $\widehat{\beta} = 1.28$.

We plot the graph and the fitted pdf lines corresponding to the ITL, IW, IE, IR, and IL distributions to fit the given dataset graphically. Also, we plot the empirical cdf and fitted cdfs corresponding to the same distributions. (4) shows the lines fitted for cdfs and pdfs for the annual rainfall data set and the corresponding distributions. The numbers in (2) also indicate that the ITL distribution provides a better fit than other distributions for this data set.

Table 3. The Censoring Times at the Different Stages with different β values.

Scheme	m	$q_i(\%)$	$TC(T_i)$ at $\beta = 0.5$	$TC(T_i)$ at $\beta = 1$	$TC(T_i)$ at $\beta = 1.5$
I	3	10,40,60	0.77,4.0,11	0.46,1.7,3.4	0.35,1.16,2.08
II	4	10,20,40,60	0.77,1.5,4.0,11	0.46,0.81,1.7,3.4	0.35,0.59,1.16,2.08
III	5	10,20,30,40,60	0.77,1.5,2.5,4.0,11	0.46,0.81,1.2,1.7,3.4	0.35,0.59,0.85,1.16,2.08

Table 4. The removed units at different stages during the test with different sample size.

m	Scheme	Sample Size		
		N=25	N=50	N=100
3	I	$(0^{(2)},Rm)$	$(0^{(2)},Rm)$	$(0^{(2)},Rm)$
	II	$(2^{(2)},Rm)$	$(6^{(2)},Rm)$	$(10^{(2)},Rm)$
	III	$(5,0, Rm)$	$(12,0, Rm)$	$(32,0, Rm)$
	IV	$(0,5, Rm)$	$(0,12, Rm)$	$(0,32, Rm)$
4	I	$(0^{(3)},Rm)$	$(0^{(3)},Rm)$	$(0^{(3)},Rm)$
	II	$(2^{(3)},Rm)$	$(6^{(3)},Rm)$	$(10^{(3)},Rm)$
	III	$(3, 0^{(2)}, Rm)$	$(12, 0^{(2)}, Rm)$	$(32, 0^{(2)}, Rm)$
	IV	$(0^{(2)}, 3, Rm)$	$(0^{(2)},12, Rm)$	$(0^{(2)}, 32, Rm)$
5	I	$(0^{(4)},Rm)$	$(0^{(4)},Rm)$	$(0^{(4)},Rm)$
	II	$(2^{(4)},Rm)$	$(5^{(4)},Rm)$	$(10^{(4)},Rm)$
	III	$(2, 0^{(3)}, Rm)$	$(12, 0^{(3)}, Rm)$	$(32, 0^{(3)}, Rm)$
	IV	$(0^{(3)}, 2, Rm)$	$(0^{(3)},12, Rm)$	$(0^{(3)}, 32, Rm)$

Table 5. The Hyper Parameters Values for Vinyl Chloride Data Set.

Hyper-parameters	initial values											
	m = 3				m = 4				m = 5			
	I	II	III	IV	I	II	III	IV	I	II	III	IV
a1	13.0	6.0	11.0	11.0	13.0	11.0	13.0	11.0	13.0	8.0	12.0	12.0
b1	10.1	4.4	7.7	8.9	10.1	8.2	8.4	9.3	10.1	8.1	9.3	9.5

Table 6. The Hyper Parameters Values for Annual Rainfall Data.

Hyper-parameters	initial values											
	m = 3				m = 4				m = 5			
	I	II	III	IV	I	II	III	IV	I	II	III	IV
a1	23.0	9.0	17.0	17.0	23.0	12.0	16.0	18.0	23.0	12.0	17.0	20.0
b1	23.0	9.5	15.5	19.2	23.0	15.2	14.8	18.5	23.0	13.1	15.6	18.5

Table 7. The MLEs, approximate CI, Bootstrap CT (P, T), BEs and HPD intervals of The Parameters From The Vinyl Chloride Data Set that Progressively Censored.

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			MLE approximate CI AILs	Boot.p CI AIL/CP	Boot.t CI AIL/CP	BE HPD intervals AILs	BE HPD intervals AILs
3	$\widehat{\beta}$	I	1.2933 (0.7119,2.1291) 1.4172	(0.8188,2.1165) 1.2977	(0.5819,2.0711) 1.4892	1.5920 (1.4987,1.6480) 0.1493	1.4372 (1.3825,1.5156) 0.1331
		II	1.3706 (0.5412,2.7773) 2.2361	(0.6438,2.3881) 1.7443	(0.4740,2.7830) 2.3090	1.4166 (1.3565,1.4832) 0.1267	1.5571 (1.4754,1.6140) 0.1387
		III	1.4315 (0.7437,2.4517) 1.7080	(0.8416,2.2296) 1.3879	(0.6947,2.5221) 1.8274	1.3971 (1.3234,1.5129) 0.1895	1.4529 (1.4021,1.5031) 0.1010
		IV	1.2291 (0.6386,2.1051) 1.4666	(0.6872,1.9199) 1.2326	(0.5854,2.2487) 1.6634	1.4901 (1.4104,1.5482) 0.1378	1.5452 (1.5000,1.5892) 0.0893
4	$\widehat{\beta}$	I	1.2933 (0.7119,2.1291) 1.4172	(0.8047,1.9494) 1.1447	(0.6777,2.1367) 1.4590	1.4951 (1.4417,1.5344) 0.0928	1.5094 (1.4451,1.5631) 0.1180
		II	1.3429 (0.6977,2.3000) 1.6023	(0.7910,2.2568) 1.4658	(0.5463,2.3401) 1.7938	1.5482 (1.4653,1.6733) 0.2080	1.5494 (1.4681,1.6843) 0.2162
		III	1.5421 (0.8488,2.5387) 1.6899	(0.8371,2.4469) 1.6098	(0.7740,2.7898) 2.0159	1.5025 (1.4269,1.5638) 0.1369	1.4912 (1.4431,1.5371) 0.0940
		IV	1.1775 (0.6118,2.0167) 1.4050	(0.7351,1.7999) 1.0648	(0.5745,1.9770) 1.4025	1.3816 (1.3068,1.4591) 0.1523	1.4607 (1.4320,1.5034) 0.0714
5	$\widehat{\beta}$	I	1.2933 (0.7119,2.1291) 1.4172	(0.8682,1.9516) 1.0834	(0.6880,1.9879) 1.2999	1.5623 (1.5197,1.6118) 0.0921	1.5970 (1.5509,1.6435) 0.0926
		II	0.9919 (0.4526,1.8465) 1.3940	(0.5776,1.5892) 1.0116	(0.3879,1.8634) 1.4755	1.4002 (1.3544,1.4565) 0.1021	1.4764 (1.4233,1.5100) 0.0867
		III	1.2957 (0.6943,2.1732) 1.4789	(0.6653,1.9745) 1.3091	(0.6640,2.5231) 1.8591	1.4276 (1.3450,1.4978) 0.1528	1.5487 (1.5104,1.5916) 0.0813
		IV	1.2586 (0.6744,2.1109) 1.4365	(0.8044,1.9101) 1.1057	(0.6251,2.0531) 1.4280	1.4851 (1.4374,1.5338) 0.0963	1.4237 (1.3797,1.4912) 0.1115

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 8. The MLEs, approximateCI, Bootstrap CT (P, T), BEs and HPD intervals of The Parameters From The Annual Rainfall Data Set that Progressively Censored.

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			MLE approximate CI AILs	Boot.p CI AIL/CP	Boot.t CI AIL/CP	BE HPD intervals AILs	BE HPD intervals AILs
3	$\widehat{\beta}$	I	1.0011 (0.6458,1.4678) 0.8220	(0.5898,1.4532) 0.8635	(0.6574,1.5649) 0.9075	0.9487 (0.8941,1.0105) 0.1163	0.8348 (0.8095,0.8877) 0.0782
		II	0.9478 (0.4553,1.7088) 1.2535	(0.3732,1.6201) 1.2469	(0.4751,1.9208) 1.4457	0.9280 (0.8928,0.9635) 0.0708	0.9612 (0.8853,1.0280) 0.1427
		III	1.0981 (0.6556,1.7060) 1.0504	(0.6609,1.7701) 1.1093	(0.6534,1.6825) 1.0291	0.8217 (0.7731,0.8912) 0.1181	0.9484 (0.9154,0.9876) 0.0722
		IV	0.8870 (0.5295,1.3780) 0.8485	(0.5256,1.3577) 0.8321	(0.5372,1.4099) 0.8728	0.8946 (0.8492,0.9317) 0.0824	0.8273 (0.7865,0.8615) 0.0750
4	$\widehat{\beta}$	I	1.0011 (0.6256,1.4326) 0.8070	(0.6741,1.5021) 0.8280	(0.6458,1.4678) 0.8220	0.9508 (0.9198,0.9866) 0.0669	0.8312 (0.7656,0.8728) 0.1072
		II	0.7877 (0.4186,1.2858) 0.8672	(0.4245,1.3892) 0.9647	(0.4221,1.3211) 0.8990	0.8284 (0.7894,0.8884) 0.0990	0.9718 (0.9122,1.0205) 0.1083
		III	1.0818 (0.6166,1.7311) 1.1144	(0.6282,1.7585) 1.1303	(0.6348,1.7019) 1.0671	0.9718 (0.9099,1.0548) 0.1448	0.8913 (0.8617,0.9402) 0.0785
		IV	0.9731 (0.5832,1.4127) 0.8294	(0.6444,1.5259) 0.8815	(0.5901,1.4944) 0.9043	1.0193 (0.9691,1.0830) 0.1139	0.8612 (0.7722,0.9372) 0.1651
5	$\widehat{\beta}$	I	1.0011 (0.6458,1.4678) 0.8220	(0.6315,1.4766) 0.8452	(0.6574,1.5048) 0.8474	0.9713 (0.9158,1.0308) 0.1150	0.8938 (0.8321,0.9374) 0.1053
		II	0.9133 (0.4894,1.5317) 1.0424	(0.4833,1.5356) 1.0523	(0.4756,1.6168) 1.1412	0.9239 (0.8903,0.9560) 0.0657	0.8550 (0.8059,0.9119) 0.1060
		III	1.0932 (0.6526,1.6983) 1.0457	(0.6302,1.8552) 1.2250	(0.6060,1.7393) 1.1333	0.8545 (0.8050,0.9356) 0.1306	0.8103 (0.7694,0.8427) 0.0733
		IV	1.0791 (0.6728,1.6237) 0.9509	(0.6161,1.6064) 0.9903	(0.7076,1.7073) 0.9997	0.8697 (0.8281,0.9120) 0.0839	0.9581 (0.9223,1.0055) 0.0832

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 9. The Hyper Parameters Values under simulation data at m=3.

Hyper-parameters		initial values											
		$\beta_0 = \mathbf{0.5}$				$\beta_0 = \mathbf{1}$				$\beta_0 = \mathbf{1.5}$			
		I	II	III	IV	I	II	III	IV	I	II	III	IV
25	a1	14.7	11.5	29.1	21.9	14.6	12.8	12.1	13.2	14.7	12.9	12.0	13.1
	b1	13.0	13.2	25.2	25.4	14.6	12.8	11.5	12.6	9.7	8.5	7.8	8.5
50	a1	29.7	26.3	58.9	52.0	29.4	28.2	27.0	28.1	29.4	28.2	27.4	28.2
	b1	28.1	28.3	55.2	55.2	29.7	27.5	26.7	27.7	19.7	18.5	17.6	18.5
100	a1	59.8	58.1	56.0	58.0	59.7	58.0	56.9	58.6	59.9	57.7	56.9	58.8
	b1	118.9	114.9	113.2	115.6	59.5	57.7	56.7	57.4	39.7	38.4	37.7	38.3

Table 10. The Hyper Parameters Values under simulation data at m=4.

Hyper-parameters		initial values											
		$\beta_0 = \mathbf{0.5}$				$\beta_0 = \mathbf{1}$				$\beta_0 = \mathbf{1.5}$			
		I	II	III	IV	I	II	III	IV	I	II	III	IV
25	a1	15.0	12.1	13.3	13.8	14.7	12.2	13.2	13.7	14.9	12.2	13.3	13.5
	b1	28.5	23.5	25.7	27.1	14.4	11.5	12.8	13.4	9.6	7.7	8.4	9.1
50	a1	29.6	21.3	23.1	25.6	29.7	21.5	23.0	13.9	29.6	21.4	22.8	25.7
	b1	59.0	42.5	45.7	50.8	29.5	20.8	23.0	13.5	19.8	14.0	15.3	16.9
100	a1	60.2	46.2	42.2	49.0	59.8	45.8	42.3	49.0	60.3	45.9	42.1	48.5
	b1	119.2	90.8	83.2	97.5	59.3	45.6	41.7	48.8	39.4	30.3	27.6	32.8

Table 11. The Hyper Parameters Values under simulation data at m=5.

Hyper-parameters		initial values											
		$\beta_0 = \mathbf{0.5}$				$\beta_0 = \mathbf{1}$				$\beta_0 = \mathbf{1.5}$			
		I	II	III	IV	I	II	III	IV	I	II	III	IV
25	a1	14.7	11.2	13.9	14.1	14.7	12.1	13.7	14.5	14.8	12.1	13.9	14.2
	b1	28.9	21.4	26.1	27.7	14.5	11.7	13.3	13.5	9.6	7.8	8.9	9.2
50	a1	29.6	20.6	23.3	25.4	29.4	20.8	23.2	25.5	29.7	20.7	23.0	25.5
	b1	59.0	40.8	45.7	51.0	29.6	20.1	22.6	25.5	19.7	13.5	15.2	17.0
100	a1	60.1	41.7	41.8	48.9	59.4	41.8	41.9	49.0	59.6	41.5	42.0	49.1
	b1	118.5	82.2	83.7	97.9	59.9	41.1	41.8	48.7	39.8	27.4	27.7	32.5

Table 12. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=3, $\beta_0 = \mathbf{0.5}$ and CT=(0.77,4.0,11.0).

m	Par	Remove	MLE Mean MSE/Bias Asymptotic CI AILs/CP	MLE Boot.p CI AIL/CP	MLE Boot.t CI AIL/CP	Bayesian P-I Mean MSE/Bias HPD intervals AILs/CP	Bayesian P-II Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	0.5033 0.0177/0.0033 (0.2910,0.8012) 0.5102/97.2	(0.2826,0.8065) 0.5239/92.4	(0.2905,0.8243) 0.5338/91.8	0.5004 0.0011/0.0004 (0.4716,0.5300) 0.0585/79.6	0.5900 0.0117/0.0900 (0.5162,0.6647) 0.1485/90.6
		II	0.5162 0.0200/0.0162 (0.2880,0.8427) 0.5547/96.4	(0.2791,0.8569) 0.5778/90.8	(0.2857,0.8685) 0.5828/89.2	0.5052 0.0013/0.0052 (0.4755,0.5345) 0.0590/80.8	0.5975 0.0129/0.0975 (0.5234,0.6714) 0.1480/89.2
		III	0.5260 0.0276/0.0260 (0.2830,0.8811) 0.5982/97	(0.2779,0.9046) 0.6267/92.7	(0.2788,0.8970) 0.6182/92.7	0.5020 0.0012/0.0020 (0.4723,0.5316) 0.0593/79.3	0.5980 0.0133/0.0980 (0.5229,0.6729) 0.1499/90.7
		IV	0.5224 0.0202/0.0224 (0.2931,0.8494) 0.5564/96	(0.2851,0.8696) 0.5845/91.7	(0.2900,0.8730) 0.5830/91.3	0.5008 0.0012/0.0008 (0.4723,0.5302) 0.0579/79.3	0.5995 0.0139/0.0995 (0.5219,0.6740) 0.1521/89.7
4	$\widehat{\beta}$	I	0.5054 0.0078/0.0054 (0.3460,0.7079) 0.3619/98.2	(0.3411,0.7069) 0.3657/93.2	(0.3481,0.7186) 0.3705/91.8	0.5030 0.0012/0.0030 (0.4750,0.5322) 0.0572/78.2	0.5596 0.0070/0.0596 (0.4955,0.6234) 0.1279/86.8
		II	0.5089 0.0099/0.0089 (0.3445,0.7189) 0.3744/97.6	(0.3386,0.7188) 0.3802/94	(0.3468,0.7312) 0.3844/92	0.5035 0.0013/0.0035 (0.4739,0.5336) 0.0597/79.6	0.5656 0.0077/0.0656 (0.5004,0.6304) 0.1300/87.8
		III	0.5058 0.0111/0.0058 (0.3380,0.7219) 0.3838/97.3	(0.3339,0.7245) 0.3906/91.7	(0.3384,0.7320) 0.3936/91	0.4965 0.0011/0.0035 (0.4683,0.5258) 0.0575/79.7	0.5704 0.0083/0.0704 (0.5071,0.6339) 0.1268/84.3
		IV	0.5128 0.0099/0.0128 (0.3477,0.7236) 0.3759/97	(0.3435,0.7257) 0.3822/92	(0.3492,0.7335) 0.3843/91	0.5005 0.0012/0.0005 (0.4707,0.5312) 0.0605/81.3	0.5657 0.0075/0.0657 (0.5002,0.6309) 0.1307/87
5	$\widehat{\beta}$	I	0.5030 0.0038/0.0030 (0.3865,0.6409) 0.2544/98	(0.3836,0.6396) 0.2560/94.2	(0.3883,0.6463) 0.2580/93.4	0.5008 0.0011/0.0008 (0.4722,0.5299) 0.0577/80	0.5032 0.0033/0.0032 (0.4480,0.5587) 0.1107/83.4
		II	0.5057 0.0050/0.0057 (0.3871,0.6463) 0.2592/96.8	(0.3844,0.6453) 0.2609/89.2	(0.3890,0.6513) 0.2623/88.6	0.5027 0.0014/0.0027 (0.4733,0.5326) 0.0593/79.4	0.5047 0.0042/0.0047 (0.4505,0.5609) 0.1103/80
		III	0.4950 0.0045/0.0050 (0.3770,0.6356) 0.2586/96	(0.3757,0.6351) 0.2594/89.3	(0.3781,0.6394) 0.2613/89.3	0.4999 0.0013/0.0001 (0.4709,0.5300) 0.0591/79.3	0.5081 0.0039/0.0081 (0.4547,0.5623) 0.1076/80.7
		IV	0.5016 0.0037/0.0016 (0.3837,0.6414) 0.2577/97.3	(0.3816,0.6416) 0.2599/92.3	(0.3845,0.6461) 0.2616/91.7	0.5053 0.0013/0.0053 (0.4765,0.5349) 0.0584/80.7	0.5103 0.0040/0.0103 (0.4544,0.5657) 0.1113/81

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 13. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=4, $\beta_0 = \mathbf{0.5}$ and CT=(0.77,1.5,4.0,11.0).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	0.5033 0.0177/0.0033 (0.2910,0.8012) 0.5102/97.2	(0.2826,0.8065) 0.5239/92.4	(0.2905,0.8243) 0.5338/91.8	0.5004 0.0011/0.0004 (0.4716,0.5300) 0.0585/79.6	0.5900 0.0117/0.0900 (0.5162,0.6647) 0.1485/90.6
		II	0.5162 0.0200/0.0162 (0.2880,0.8427) 0.5547/96.4	(0.2791,0.8569) 0.5778/90.8	(0.2857,0.8685) 0.5828/89.2	0.5052 0.0013/0.0052 (0.4755,0.5345) 0.0590/80.8	0.5975 0.0129/0.0975 (0.5234,0.6714) 0.1480/89.2
		III	0.5260 0.0276/0.0260 (0.2830,0.8811) 0.5982/97	(0.2779,0.9046) 0.6267/92.7	(0.2788,0.8970) 0.6182/92.7	0.5020 0.0012/0.0020 (0.4723,0.5316) 0.0593/79.3	0.5980 0.0133/0.0980 (0.5229,0.6729) 0.1499/90.7
		IV	0.5224 0.0202/0.0224 (0.2931,0.8494) 0.5564/96	(0.2851,0.8696) 0.5845/91.7	(0.2900,0.8730) 0.5830/91.3	0.5008 0.0012/0.0008 (0.4723,0.5302) 0.0579/79.3	0.5995 0.0139/0.0995 (0.5219,0.6740) 0.1521/89.7
4	$\widehat{\beta}$	I	0.5054 0.0078/0.0054 (0.3460,0.7079) 0.3619/98.2	(0.3411,0.7069) 0.3657/93.2	(0.3481,0.7186) 0.3705/91.8	0.5030 0.0012/0.0030 (0.4750,0.5322) 0.0572/78.2	0.5596 0.0070/0.0596 (0.4955,0.6234) 0.1279/86.8
		II	0.5089 0.0099/0.0089 (0.3445,0.7189) 0.3744/97.6	(0.3386,0.7188) 0.3802/94	(0.3468,0.7312) 0.3844/92	0.5035 0.0013/0.0035 (0.4739,0.5336) 0.0597/79.6	0.5656 0.0077/0.0656 (0.5004,0.6304) 0.1300/87.8
		III	0.5058 0.0111/0.0058 (0.3380,0.7219) 0.3838/97.3	(0.3339,0.7245) 0.3906/91.7	(0.3384,0.7320) 0.3936/91	0.4965 0.0011/0.0035 (0.4683,0.5258) 0.0575/79.7	0.5704 0.0083/0.0704 (0.5071,0.6339) 0.1268/84.3
		IV	0.5128 0.0099/0.0128 (0.3477,0.7236) 0.3759/97	(0.3435,0.7257) 0.3822/92	(0.3492,0.7335) 0.3843/91	0.5005 0.0012/0.0005 (0.4707,0.5312) 0.0605/81.3	0.5657 0.0075/0.0657 (0.5002,0.6309) 0.1307/87
5	$\widehat{\beta}$	I	0.5030 0.0038/0.0030 (0.3865,0.6409) 0.2544/98	(0.3836,0.6396) 0.2560/94.2	(0.3883,0.6463) 0.2580/93.4	0.5008 0.0011/0.0008 (0.4722,0.5299) 0.0577/80	0.5032 0.0033/0.0032 (0.4480,0.5587) 0.1107/83.4
		II	0.5057 0.0050/0.0057 (0.3871,0.6463) 0.2592/96.8	(0.3844,0.6453) 0.2609/89.2	(0.3890,0.6513) 0.2623/88.6	0.5027 0.0014/0.0027 (0.4733,0.5326) 0.0593/79.4	0.5047 0.0042/0.0047 (0.4505,0.5609) 0.1103/80
		III	0.4950 0.0045/0.0050 (0.3770,0.6356) 0.2586/96	(0.3757,0.6351) 0.2594/89.3	(0.3781,0.6394) 0.2613/89.3	0.4999 0.0013/0.0001 (0.4709,0.5300) 0.0591/79.3	0.5081 0.0039/0.0081 (0.4547,0.5623) 0.1076/80.7
		IV	0.5016 0.0037/0.0016 (0.3837,0.6414) 0.2577/97.3	(0.3816,0.6416) 0.2599/92.3	(0.3845,0.6461) 0.2616/91.7	0.5053 0.0013/0.0053 (0.4765,0.5349) 0.0584/80.7	0.5103 0.0040/0.0103 (0.4544,0.5657) 0.1113/81

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 14. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=5, $\beta_0 = \mathbf{0.5}$ and CT=(0.77,1.5,2.5,4.0,11.0).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	0.5095 0.0179/0.0095 (0.2950,0.8100) 0.5150/97.0	(0.2856,0.8176) 0.5319/93.7	(0.2937,0.8349) 0.5413/92.3	0.4998 0.0012/0.0002 (0.4721,0.5284) 0.0564/78.0	0.4985 0.0052/0.0015 (0.4401,0.5571) 0.117/81.3
		II	0.5208 0.0237/0.0208 (0.2767,0.8797) 0.6029/97.7	(0.2795,0.9071) 0.6276/92.3	(0.2714,0.8820) 0.6106/93.0	0.5024 0.0011/0.0024 (0.4735,0.5332) 0.0598/80.3	0.5014 0.0047/0.0014 (0.4435,0.5575) 0.114/81.3
		III	0.5346 0.0228/0.0346 (0.3048,0.8595) 0.5547/95.7	(0.2951,0.8772) 0.5821/92.0	(0.3028,0.8819) 0.5791/90.7	0.5038 0.0012/0.0038 (0.4743,0.5339) 0.0597/79.7	0.4979 0.0047/0.0021 (0.4400,0.5557) 0.116/80.3
		IV	0.5101 0.0151/0.0101 (0.2911,0.8192) 0.5281/98.7	(0.2880,0.8305) 0.5425/91.0	(0.2891,0.8333) 0.5442/90.7	0.5020 0.0013/0.0020 (0.4724,0.5313) 0.0589/79.3	0.4964 0.0045/0.0036 (0.4393,0.5555) 0.116/79.3
4	$\widehat{\beta}$	I	0.5024 0.0086/0.0024 (0.3438,0.7039) 0.3601/98.3	(0.3410,0.7042) 0.3632/88.3	(0.3450,0.7116) 0.3666/87.7	0.5003 0.0012/0.0003 (0.4724,0.5291) 0.0567/77.7	0.4898 0.0047/0.0102 (0.4317,0.5494) 0.118/79.0
		II	0.5052 0.0120/0.0052 (0.3193,0.7525) 0.4332/96.3	(0.3164,0.7618) 0.4454/89.0	(0.3183,0.7645) 0.4462/89.0	0.5016 0.0014/0.0016 (0.4717,0.5318) 0.0601/75.3	0.4890 0.0045/0.0110 (0.4332,0.5463) 0.113/80.3
		III	0.5092 0.0105/0.0092 (0.3311,0.7425) 0.4115/98.0	(0.3250,0.7463) 0.4213/93.3	(0.3321,0.7565) 0.4244/92.3	0.5040 0.0012/0.0040 (0.4746,0.5336) 0.0591/79.0	0.5001 0.0040/0.0001 (0.4413,0.5587) 0.117/84.7
		IV	0.4985 0.0101/0.0015 (0.3306,0.7159) 0.3852/96.3	(0.3299,0.7203) 0.3904/90.3	(0.3299,0.7226) 0.3926/90.3	0.5062 0.0013/0.0062 (0.4767,0.5357) 0.0591/79.7	0.4930 0.0049/0.0070 (0.4367,0.5512) 0.114/79.3
5	$\widehat{\beta}$	I	0.5071 0.0042/0.0071 (0.3899,0.6457) 0.2558/96.7	(0.3872,0.6438) 0.2566/89.3	(0.3926,0.6504) 0.2579/88.7	0.5007 0.0012/0.0007 (0.4718,0.5310) 0.0592/80.0	0.4959 0.0041/0.0041 (0.4427,0.5494) 0.107/77.7
		II	0.5077 0.0072/0.0077 (0.3697,0.6764) 0.3067/96.3	(0.3663,0.6774) 0.3111/91.0	(0.3716,0.6842) 0.3126/90.7	0.5009 0.0011/0.0009 (0.4732,0.5294) 0.0562/80.7	0.4949 0.0040/0.0051 (0.4378,0.5528) 0.115/79.7
		III	0.4997 0.0052/0.0003 (0.3639,0.6660) 0.3021/98.3	(0.3621,0.6667) 0.3046/91.7	(0.3647,0.6712) 0.3065/91.7	0.5015 0.0011/0.0015 (0.4731,0.5304) 0.0573/81.7	0.4973 0.0040/0.0027 (0.4429,0.5516) 0.109/81.7
		IV	0.4994 0.0048/0.0006 (0.3727,0.6519) 0.2792/97.7	(0.3710,0.6527) 0.2817/93.3	(0.3738,0.6573) 0.2835/93.0	0.5028 0.0011/0.0028 (0.4734,0.5326) 0.0592/82.0	0.4880 0.0039/0.0120 (0.4346,0.5421) 0.107/79.0

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 15. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=3, $\beta_0 = \mathbf{1}$ and CT=(0.46,1.7,3.4).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.0056 0.0690/0.0056 (0.5811,1.6010) 1.0199/96.7	(0.5686,1.6230) 1.0544/93.3	(0.5759,1.6410) 1.0651/92.7	1.0012 0.0040/0.0012 (0.9449,1.0591) 0.1142/81.7	1.0505 0.0065/0.0505 (0.9884,1.1123) 0.1240/85
		II	1.0002 0.0837/0.0002 (0.5554,1.6386) 1.0832/95.0	(0.5461,1.6603) 1.1143/92	(0.5504,1.6774) 1.1270/92	1.0005 0.0043/0.0005 (0.9414,1.0599) 0.1185/81.3	1.0530 0.0068/0.0530 (0.9896,1.1178) 0.1282/86
		III	1.0520 0.1045/0.0520 (0.5749,1.7430) 1.1681/96.0	(0.5595,1.7827) 1.2232/91	(0.5698,1.7849) 1.2151/91	0.9999 0.0051/0.0001 (0.9410,1.0598) 0.1187/80	1.0493 0.0070/0.0493 (0.9838,1.1128) 0.1290/84
		IV	1.0457 0.0940/0.0457 (0.5864,1.7010) 1.1146/95.7	(0.5733,1.7403) 1.1670/91.3	(0.5794,1.7418) 1.1624/91.3	1.0071 0.0043/0.0071 (0.9491,1.0668) 0.1178/78.3	1.0627 0.0083/0.0627 (0.9965,1.1267) 0.1302/82.3
4	$\widehat{\beta}$	I	0.9918 0.0362/0.0082 (0.6780,1.3907) 0.7128/97.0	(0.6716,1.3917) 0.7201/92	(0.6795,1.4084) 0.7289/90.7	1.0095 0.0045/0.0095 (0.9528,1.0680) 0.1153/79.7	1.0288 0.0048/0.0288 (0.9709,1.0880) 0.1171/81.7
		II	1.0261 0.0382/0.0261 (0.6955,1.4484) 0.7528/96.7	(0.6845,1.4558) 0.7713/92	(0.6969,1.4713) 0.7744/91.7	1.0045 0.0051/0.0045 (0.9453,1.0635) 0.1182/79.3	1.0414 0.0061/0.0414 (0.9799,1.1022) 0.1223/83.7
		III	1.0146 0.0384/0.0146 (0.6815,1.4423) 0.7608/97.3	(0.6695,1.4545) 0.7850/93.7	(0.6789,1.4686) 0.7897/92	1.0014 0.0046/0.0014 (0.9449,1.0595) 0.1147/79.3	1.0396 0.0056/0.0396 (0.9786,1.1011) 0.1224/81.7
		IV	1.0115 0.0375/0.0115 (0.6847,1.4293) 0.7447/96.7	(0.6765,1.4321) 0.7556/90.7	(0.6872,1.4499) 0.7627/89.7	1.0018 0.0052/0.0018 (0.9428,1.0624) 0.1196/79	1.0343 0.0055/0.0343 (0.9756,1.0927) 0.1170/80.7
5	$\widehat{\beta}$	I	1.0031 0.0185/0.0031 (0.7707,1.2782) 0.5075/96.7	(0.7676,1.2763) 0.5087/90.7	(0.7738,1.2850) 0.5112/90.7	1.0012 0.0047/0.0012 (0.9404,1.0631) 0.1227/83.3	1.0018 0.0045/0.0018 (0.9433,1.0578) 0.1144/81.3
		II	1.0057 0.0163/0.0057 (0.7696,1.2860) 0.5164/97.3	(0.7669,1.2831) 0.5163/93.7	(0.7739,1.2924) 0.5186/91.7	1.0023 0.0046/0.0023 (0.9442,1.0619) 0.1177/81	1.0071 0.0044/0.0071 (0.9499,1.0652) 0.1152/79.3
		III	1.0038 0.0165/0.0038 (0.7661,1.2863) 0.5203/98.7	(0.7618,1.2847) 0.5229/89.7	(0.7689,1.2952) 0.5263/88.7	1.0047 0.0043/0.0047 (0.9467,1.0634) 0.1166/80	1.0016 0.0046/0.0016 (0.9437,1.0585) 0.1147/79
		IV	1.0217 0.0197/0.0217 (0.7831,1.3046) 0.5215/98.0	(0.7781,1.3050) 0.5270/93	(0.7859,1.3131) 0.5272/92	1.0018 0.0046/0.0018 (0.9445,1.0600) 0.1155/79.3	1.0074 0.0044/0.0074 (0.9487,1.0659) 0.1172/81

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 16. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=4, $\beta_0 = \mathbf{1}$ and CT=(0.46,0.81,1.7,3.4).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.0213 0.0730/0.0213 (0.5907,1.6251) 1.0344/97.0	(0.5749,1.6465) 1.0716/91.0	(0.5849,1.6691) 1.0843/90.3	0.9948 0.0041/0.0052 (0.9362,1.0517) 0.1154/81.7	1.0576 0.0082/0.0576 (0.9917,1.1231) 0.1315/80.0
		II	1.0607 0.0948/0.0607 (0.5801,1.7560) 1.1760/97.0	(0.5723,1.8128) 1.2404/91.7	(0.5693,1.7833) 1.2140/91.7	1.0000 0.0047/0.0000 (0.9411,1.0595) 0.1184/80.7	1.0575 0.0078/0.0575 (0.9935,1.1197) 0.1262/83.3
		III	1.0304 0.0764/0.0304 (0.5770,1.6778) 1.1008/98.0	(0.5639,1.7073) 1.1433/92.7	(0.5698,1.7171) 1.1473/92.0	1.0104 0.0048/0.0104 (0.9510,1.0702) 0.1191/78.3	1.0590 0.0073/0.0590 (0.9963,1.1233) 0.1270/84.0
		IV	1.0204 0.0840/0.0204 (0.5788,1.6464) 1.0675/94.3	(0.5768,1.6728) 1.0961/91.3	(0.5739,1.6646) 1.0907/92.0	1.0011 0.0039/0.0011 (0.9446,1.0563) 0.1117/79.3	0.9967 0.0046/0.0033 (0.9375,1.0546) 0.1170/80.7
4	$\widehat{\beta}$	I	1.0064 0.0345/0.0064 (0.6888,1.4097) 0.7209/97.3	(0.6837,1.4107) 0.7269/91.3	(0.6911,1.4239) 0.7328/90.7	0.9988 0.0043/0.0012 (0.9431,1.0578) 0.1147/79.3	1.1009 0.0140/0.1009 (1.0208,1.1808) 0.1600/88.7
		II	1.0338 0.0512/0.0338 (0.6607,1.5273) 0.8665/96.0	(0.6581,1.5479) 0.8898/90.3	(0.6585,1.5412) 0.8827/90.3	1.0157 0.0048/0.0157 (0.9568,1.0756) 0.1188/80.3	1.1126 0.0168/0.1126 (1.0318,1.1919) 0.1600/87.7
		III	1.0008 0.0383/0.0008 (0.6490,1.4623) 0.8134/98.0	(0.6457,1.4681) 0.8223/94.0	(0.6496,1.4768) 0.8271/93.3	1.0000 0.0045/0.0000 (0.9438,1.0569) 0.1131/79.7	1.1048 0.0149/0.1048 (1.0262,1.1834) 0.1572/88.7
		IV	1.0284 0.0652/0.0284 (0.5844,1.6570) 1.0725/97.7	(0.5746,1.6866) 1.1120/93.0	(0.5795,1.6897) 1.1102/93.0	1.0025 0.0050/0.0025 (0.9444,1.0626) 0.1183/78.3	1.0549 0.0073/0.0549 (0.9901,1.1202) 0.1302/82.7
5	$\widehat{\beta}$	I	1.0089 0.0172/0.0089 (0.7752,1.2855) 0.5102/96.0	(0.7708,1.2844) 0.5136/88.7	(0.7779,1.2939) 0.5159/88.7	1.0024 0.0045/0.0024 (0.9445,1.0619) 0.1174/80.0	1.1881 0.0388/0.1881 (1.0716,1.3003) 0.2287/98.3
		II	1.0054 0.0223/0.0054 (0.7431,1.3234) 0.5803/97.3	(0.7372,1.3237) 0.5865/92.0	(0.7466,1.3366) 0.5900/91.7	1.0088 0.0056/0.0088 (0.9499,1.0693) 0.1194/79.3	1.2045 0.0458/0.2045 (1.0773,1.3288) 0.2515/97.3
		III	1.0138 0.0229/0.0138 (0.7396,1.3490) 0.6094/98.0	(0.7348,1.3535) 0.6188/91.3	(0.7403,1.3594) 0.6191/91.0	1.0058 0.0054/0.0058 (0.9443,1.0671) 0.1228/80.0	1.2085 0.0473/0.2085 (1.0791,1.3314) 0.2522/98.0
		IV	1.0049 0.0212/0.0049 (0.7506,1.3112) 0.5606/97.7	(0.7467,1.3126) 0.5659/91.3	(0.7535,1.3222) 0.5688/90.0	0.9951 0.0055/0.0049 (0.9358,1.0551) 0.1193/81.0	1.2066 0.0462/0.2066 (1.0775,1.3308) 0.2533/98.3

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 17. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=5, $\beta_0 = \mathbf{1}$ and CT=(0.46,0.81,1.2,1.7,3.4).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.0185 0.0659/0.0185 (0.5892,1.6201) 1.0309/97.3	(0.5784,1.6353) 1.0570/91.7	(0.5868,1.6541) 1.0673/91.7	1.0033 0.0046/0.0033 (0.9456,1.0634) 0.1178/79.7	0.9961 0.0036/0.0039 (0.9380,1.0543) 0.1163/83.7
		II	1.0321 0.0892/0.0321 (0.5623,1.7134) 1.1511/97.0	(0.5627,1.7612) 1.1986/90.7	(0.5515,1.7280) 1.1765/91.0	1.0067 0.0045/0.0067 (0.9476,1.0668) 0.1192/80.3	0.9967 0.0050/0.0033 (0.9370,1.0526) 0.1156/79.3
		III	1.0298 0.0808/0.0298 (0.5836,1.6628) 1.0792/97.0	(0.5676,1.6918) 1.1242/90.3	(0.5765,1.7063) 1.1298/90.0	1.0032 0.0056/0.0032 (0.9452,1.0624) 0.1172/78.0	0.9959 0.0049/0.0041 (0.9381,1.0532) 0.1151/80.0
		IV	1.0730 0.0877/0.0730 (0.6186,1.7113) 1.0928/96.3	(0.6064,1.7478) 1.1415/87.3	(0.6153,1.7387) 1.1234/86.3	1.0016 0.0054/0.0016 (0.9437,1.0608) 0.1172/75.3	0.9972 0.0042/0.0028 (0.9414,1.0528) 0.1115/80.3
4	$\widehat{\beta}$	I	0.9926 0.0338/0.0074 (0.6780,1.3925) 0.7145/97.0	(0.6707,1.3924) 0.7217/94.3	(0.6790,1.4116) 0.7326/94.0	1.0057 0.0047/0.0057 (0.9489,1.0637) 0.1148/78.0	0.9941 0.0046/0.0059 (0.9338,1.0540) 0.1202/80.0
		II	1.0356 0.0593/0.0356 (0.6571,1.5383) 0.8811/98.0	(0.6512,1.5594) 0.9082/91.0	(0.6566,1.5578) 0.9012/89.3	0.9990 0.0045/0.0010 (0.9415,1.0579) 0.1165/82.3	0.9996 0.0046/0.0004 (0.9423,1.0575) 0.1152/82.7
		III	1.0280 0.0521/0.0280 (0.6688,1.4982) 0.8293/95.7	(0.6627,1.5127) 0.8500/93.0	(0.6671,1.5142) 0.8471/93.0	1.0028 0.0043/0.0028 (0.9452,1.0619) 0.1167/78.7	0.9996 0.0047/0.0004 (0.9399,1.0576) 0.1177/80.7
		IV	1.0000 0.0380/0.0000 (0.6636,1.4350) 0.7714/97.3	(0.6590,1.4376) 0.7786/93.3	(0.6666,1.4535) 0.7869/92.7	0.9980 0.0044/0.0020 (0.9437,1.0535) 0.1098/80.0	1.0030 0.0046/0.0030 (0.9429,1.0609) 0.1180/81.0
5	$\widehat{\beta}$	I	0.9923 0.0138/0.0077 (0.7617,1.2655) 0.5039/98.0	(0.7570,1.2612) 0.5043/92.0	(0.7657,1.2757) 0.5100/91.7	1.0003 0.0043/0.0003 (0.9444,1.0572) 0.1128/79.7	1.0009 0.0046/0.0009 (0.9443,1.0567) 0.1123/80.3
		II	1.0167 0.0241/0.0167 (0.7404,1.3549) 0.6145/97.3	(0.7368,1.3583) 0.6215/95.0	(0.7434,1.3654) 0.6219/94.7	0.9985 0.0045/0.0015 (0.9424,1.0546) 0.1121/80.3	0.9980 0.0056/0.0020 (0.9375,1.0572) 0.1197/78.7
		III	1.0020 0.0243/0.0020 (0.7299,1.3349) 0.6051/97.0	(0.7241,1.3334) 0.6094/89.0	(0.7336,1.3486) 0.6150/87.3	1.0058 0.0049/0.0058 (0.9475,1.0657) 0.1182/78.0	0.9987 0.0043/0.0013 (0.9406,1.0541) 0.1135/82.0
		IV	1.0067 0.0219/0.0067 (0.7519,1.3137) 0.561/96.3	(0.7476,1.3148) 0.5672/91.0	(0.7552,1.3251) 0.5699/90.7	1.0044 0.0049/0.0044 (0.9455,1.0637) 0.1181/78.0	1.0055 0.0050/0.0055 (0.9466,1.0654) 0.1188/79.0

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 18. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=3, $\beta_0 = \mathbf{1.5}$ and CT=(0.35,1.16,2.08).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.5197 0.1591/0.0197 (0.8790,2.4177) 1.5387/98.7	(0.8592,2.4360) 1.5769/92	(0.8764,2.4774) 1.6010/92	1.5071 0.0108/0.0071 (1.4202,1.5964) 0.1762/81.3	1.5335 0.0054/0.0335 (1.4717,1.5947) 0.1230/82.3
		II	1.5180 0.1772/0.0180 (0.8455,2.4814) 1.6359/96.3	(0.8296,2.5263) 1.6967/91	(0.8352,2.5383) 1.7031/90.7	1.5090 0.0100/0.0090 (1.4238,1.5971) 0.1733/79.7	1.5374 0.0050/0.0374 (1.4774,1.5992) 0.1218/83
		III	1.5485 0.2220/0.0485 (0.8446,2.5694) 1.7248/96.7	(0.8257,2.6250) 1.7993/91.7	(0.8333,2.6278) 1.7945/91	1.5036 0.0095/0.0036 (1.4184,1.5899) 0.1715/80.7	1.5335 0.0062/0.0335 (1.4711,1.5961) 0.1251/82.7
		IV	1.5426 0.1795/0.0426 (0.8633,2.5126) 1.6492/97.7	(0.8473,2.5717) 1.7244/93.7	(0.8519,2.5705) 1.7186/93.7	1.4921 0.0107/0.0079 (1.4072,1.5806) 0.1734/80.3	1.5353 0.0057/0.0353 (1.4761,1.5940) 0.1180/82.7
4	$\widehat{\beta}$	I	1.4913 0.0761/0.0087 (1.0191,2.0915) 1.0724/97.7	(1.0110,2.0898) 1.0788/88.3	(1.0220,2.1162) 1.0942/87.7	1.5010 0.0107/0.0010 (1.4164,1.5881) 0.1717/79.7	1.5223 0.0050/0.0223 (1.4618,1.5822) 0.1203/81.3
		II	1.5254 0.0733/0.0254 (1.0333,2.1542) 1.1209/97.7	(1.0202,2.1626) 1.1424/93	(1.0356,2.1831) 1.1475/92.3	1.5114 0.0112/0.0114 (1.4221,1.6035) 0.1814/79	1.5263 0.0054/0.0263 (1.4681,1.5850) 0.1169/79
		III	1.5596 0.1026/0.0596 (1.0511,2.2114) 1.1603/96.0	(1.0402,2.2235) 1.1833/90.7	(1.0543,2.2327) 1.1784/89.3	1.5100 0.0099/0.0100 (1.4235,1.5951) 0.1716/81.3	1.5277 0.0053/0.0277 (1.4686,1.5859) 0.1173/81.3
		IV	1.5242 0.0866/0.0242 (1.0327,2.1520) 1.1193/97.3	(1.0217,2.1625) 1.1408/91.7	(1.0340,2.1794) 1.1454/91	1.5124 0.0122/0.0124 (1.4250,1.6007) 0.1757/77.3	1.5238 0.0049/0.0238 (1.4646,1.5828) 0.1182/82.7
5	$\widehat{\beta}$	I	1.5102 0.0365/0.0102 (1.1607,1.9239) 0.7632/97.0	(1.1558,1.9224) 0.7667/93	(1.1649,1.9342) 0.7693/92	1.5111 0.0116/0.0111 (1.4236,1.6010) 0.1775/79	1.4975 0.0047/0.0025 (1.4411,1.5545) 0.1135/79.7
		II	1.5041 0.0442/0.0041 (1.1503,1.9241) 0.7738/95.7	(1.1458,1.9216) 0.7758/91.3	(1.1547,1.9355) 0.7808/91	1.5056 0.0094/0.0056 (1.4211,1.5894) 0.1683/82	1.5049 0.0035/0.0049 (1.4479,1.5623) 0.1144/85.3
		III	1.5091 0.0384/0.0091 (1.1518,1.9338) 0.7820/98.0	(1.1461,1.9334) 0.7873/91	(1.1549,1.9462) 0.7912/90.7	1.5177 0.0118/0.0177 (1.4282,1.6116) 0.1834/77.3	1.5072 0.0047/0.0072 (1.4475,1.5662) 0.1187/80.3
		IV	1.5367 0.0434/0.0367 (1.1782,1.9617) 0.7835/95.7	(1.1680,1.9618) 0.7939/90.7	(1.1832,1.9774) 0.7942/88.7	1.5054 0.0108/0.0054 (1.4191,1.5933) 0.1741/78.3	1.5091 0.0049/0.0091 (1.4504,1.5666) 0.1162/78.7

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 19. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=4, $\beta_0 = 1.5$ and CT=(0.35,0.59,1.16,2.08).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.5527 0.1613/0.0527 (0.9012,2.4643) 1.5631/97.7	(0.8772,2.4999) 1.6228/93.3	(0.8957,2.5268) 1.6311/92.7	1.5049 0.0103/0.0049 (1.4200,1.5915) 0.1715/80.3	1.5366 0.0059/0.0366 (1.4775,1.5976) 0.1201/80.7
		II	1.5846 0.2073/0.0846 (0.8665,2.6240) 1.7575/96.7	(0.8504,2.7093) 1.8589/92.3	(0.8486,2.6746) 1.8260/92.3	1.5103 0.0097/0.0103 (1.4242,1.5980) 0.1738/80.7	1.5420 0.0061/0.0420 (1.4804,1.6039) 0.1235/80.7
		III	1.5756 0.2265/0.0756 (0.8865,2.5574) 1.6709/95.7	(0.8613,2.6208) 1.7595/92.3	(0.8754,2.6203) 1.7449/92.0	1.5026 0.0112/0.0026 (1.4169,1.5906) 0.1737/78.0	1.5349 0.0053/0.0349 (1.4762,1.5942) 0.1180/81.0
		IV	1.4908 0.1589/0.0092 (0.8408,2.4154) 1.5746/96.3	(0.8256,2.4297) 1.6042/90.7	(0.8395,2.4772) 1.6378/90.0	1.5100 0.0102/0.0100 (1.4254,1.5985) 0.1731/80.7	1.5271 0.0049/0.0271 (1.4661,1.5891) 0.1230/81.0
4	$\widehat{\beta}$	I	1.4962 0.0762/0.0038 (1.0233,2.0969) 1.0735/97.7	(1.0176,2.0985) 1.0810/92.0	(1.0264,2.1168) 1.0905/92.0	1.5094 0.0117/0.0094 (1.4212,1.5978) 0.1766/81.7	1.5638 0.0083/0.0638 (1.4963,1.6291) 0.1329/85.7
		II	1.5287 0.1160/0.0287 (0.9753,2.2615) 1.2862/98.0	(0.9626,2.2803) 1.3177/91.7	(0.9753,2.3024) 1.3271/91.3	1.4978 0.0095/0.0022 (1.4141,1.5829) 0.1688/82.3	1.5799 0.0108/0.0799 (1.5087,1.6515) 0.1427/85.3
		III	1.4935 0.0874/0.0065 (0.9664,2.1856) 1.2193/97.0	(0.9584,2.1877) 1.2294/92.0	(0.9688,2.2179) 1.2491/91.3	1.5105 0.0103/0.0105 (1.4237,1.5980) 0.1743/80.0	1.5761 0.0102/0.0761 (1.5058,1.6467) 0.1409/87.3
		IV	1.5200 0.1066/0.0200 (1.0107,2.1779) 1.1672/96.7	(0.9997,2.1857) 1.1860/90.3	(1.0139,2.2097) 1.1959/89.7	1.5131 0.0108/0.0131 (1.4235,1.6027) 0.1792/83.0	1.5732 0.0104/0.0732 (1.5041,1.6443) 0.1403/83.7
5	$\widehat{\beta}$	I	1.5326 0.0431/0.0326 (1.1792,1.9506) 0.7714/96.7	(1.1709,1.9485) 0.7775/89.7	(1.1853,1.9641) 0.7789/89.3	1.5144 0.0113/0.0144 (1.4250,1.6051) 0.1801/81.3	1.6419 0.0239/0.1419 (1.5478,1.7356) 0.1877/93.7
		II	1.5149 0.0435/0.0149 (1.1197,1.9940) 0.8743/99.3	(1.1144,1.9974) 0.8830/92.7	(1.1227,2.0091) 0.8864/92.0	1.5118 0.0110/0.0118 (1.4243,1.5991) 0.1748/79.0	1.6510 0.0271/0.1510 (1.5477,1.7549) 0.2072/95.3
		III	1.5258 0.0579/0.0258 (1.1125,2.0311) 0.9186/97.0	(1.1064,2.0370) 0.9306/91.7	(1.1140,2.0460) 0.9320/91.3	1.5041 0.0104/0.0041 (1.4167,1.5907) 0.1740/81.7	1.6576 0.0286/0.1576 (1.5532,1.7566) 0.2034/94.7
		IV	1.4809 0.0508/0.0191 (1.1045,1.9348) 0.8303/97.3	(1.1012,1.9341) 0.8329/91.0	(1.1087,1.9493) 0.8405/91.0	1.5072 0.0118/0.0072 (1.4192,1.5970) 0.1778/79.7	1.6531 0.0277/0.1531 (1.5488,1.7542) 0.2054/95.0

Note: par- parameter, AIL-average interval length, CT-censoring time

Table 20. Average estimated values, MSEs, bias, approximateand bootstrap (t-p) CI intervals of MLEs and BEs of ITL distribution parameters under progressive Type I censoring at m=5, $\beta_0 = \mathbf{1.5}$ and CT=(0.35,0.59,0.85,1.16,2.08).

m	Par	Remove	MLE			Bayesian P-I	Bayesian P-II
			Mean MSE/Bias Asymptotic CI AILs/CP	Boot.p CI AIL/CP	Boot.t CI AIL/CP	Mean MSE/Bias HPD intervals AILs/CP	Mean MSE/Bias HPD intervals AILs/CP
3	$\widehat{\beta}$	I	1.5466 0.1652/0.0466 (0.8973,2.4554) 1.5581/95.7	(0.8734,2.4866) 1.6132/92.3	(0.8923,2.5188) 1.6266/91.7	1.5131 0.0129/0.0131 (1.4244,1.6022) 0.1778/78.0	1.4966 0.0047/0.0034 (1.4396,1.5538) 0.1142/79.3
		II	1.5474 0.1922/0.0474 (0.8438,2.5673) 1.7235/96.3	(0.8423,2.6307) 1.7884/91.3	(0.8310,2.5951) 1.7640/91.7	1.5050 0.0120/0.0050 (1.4176,1.5945) 0.1769/80.7	1.4945 0.0050/0.0055 (1.4349,1.5527) 0.1178/80.3
		III	1.5577 0.1454/0.0577 (0.8840,2.5122) 1.6282/98.3	(0.8620,2.5621) 1.7001/94.7	(0.8725,2.5717) 1.6992/94.3	1.5122 0.0116/0.0122 (1.4238,1.6004) 0.1766/81.0	1.4919 0.0048/0.0081 (1.4347,1.5499) 0.1153/80.0
		IV	1.5438 0.1617/0.0438 (0.8831,2.4760) 1.5929/96.7	(0.8703,2.5155) 1.6452/92.7	(0.8768,2.5213) 1.6444/92.3	1.5050 0.0106/0.0050 (1.4182,1.5915) 0.1734/81.0	1.4996 0.0047/0.0004 (1.4434,1.5564) 0.1130/79.7
4	$\widehat{\beta}$	I	1.5061 0.0685/0.0061 (1.0306,2.1099) 1.0793/99.0	(1.0220,2.1135) 1.0915/91.7	(1.0328,2.1334) 1.1006/91.3	1.5106 0.0142/0.0106 (1.4205,1.6019) 0.1814/77.3	1.4911 0.0048/0.0089 (1.4332,1.5496) 0.1164/79.3
		II	1.5305 0.1084/0.0305 (0.9685,2.2780) 1.3095/97.7	(0.9575,2.3028) 1.3453/92.0	(0.9685,2.3147) 1.346291.7	1.5165 0.0109/0.0165 (1.4305,1.6038) 0.1734/79.0	1.5012 0.0055/0.0012 (1.4402,1.5615) 0.1213/77.7
		III	1.5128 0.0945/0.0128 (0.9811,2.2099) 1.2288/97.7	(0.9719,2.2113) 1.2394/91.0	(0.9876,2.2391) 1.2515/90.0	1.5122 0.0116/0.0122 (1.4219,1.6026) 0.1807/79.3	1.4913 0.0051/0.0087 (1.4348,1.5477) 0.1129/80.3
		IV	1.5017 0.1087/0.0017 (0.9974,2.1537) 1.1563/96.3	(0.9895,2.1702) 1.1807/89.3	(0.9950,2.1820) 1.1870/89.0	1.5089 0.0119/0.0089 (1.4198,1.6009) 0.1812/80.7	1.5032 0.0045/0.0032 (1.4457,1.5605) 0.1148/80.0
5	$\widehat{\beta}$	I	1.4974 0.0373/0.0026 (1.1500,1.9087) 0.7587/97.3	(1.1461,1.9072) 0.7611/92.7	(1.1533,1.9184) 0.7650/92.3	1.5152 0.0112/0.0152 (1.4252,1.6075) 0.1823/81.3	1.5004 0.0044/0.0004 (1.4424,1.5577) 0.1153/81.0
		II	1.5153 0.0534/0.0153 (1.1023,2.0209) 0.9186/97.7	(1.0952,2.0224) 0.9272/94.3	(1.1075,2.0408) 0.9333/93.3	1.5117 0.0106/0.0117 (1.4228,1.6047) 0.1820/78.7	1.4960 0.0045/0.0040 (1.4393,1.5550) 0.1157/81.3
		III	1.5177 0.0600/0.0177 (1.1063,2.0207) 0.9143/96.0	(1.0997,2.0170) 0.9173/88.7	(1.1138,2.0369) 0.9231/87.7	1.5053 0.0112/0.0053 (1.4216,1.5910) 0.1695/77.3	1.4963 0.0045/0.0037 (1.4386,1.5538) 0.1152/78.3
		IV	1.5117 0.0485/0.0117 (1.1295,1.9721) 0.8425/97.0	(1.1230,1.9754) 0.8524/90.0	(1.1332,1.9890) 0.8558/89.7	1.5027 0.0105/0.0027 (1.4144,1.5911) 0.1767/79.3	1.4997 0.0043/0.0003 (1.4424,1.5566) 0.1142/79.7

Note: par- parameter, AIL-average interval length, CT-censoring time

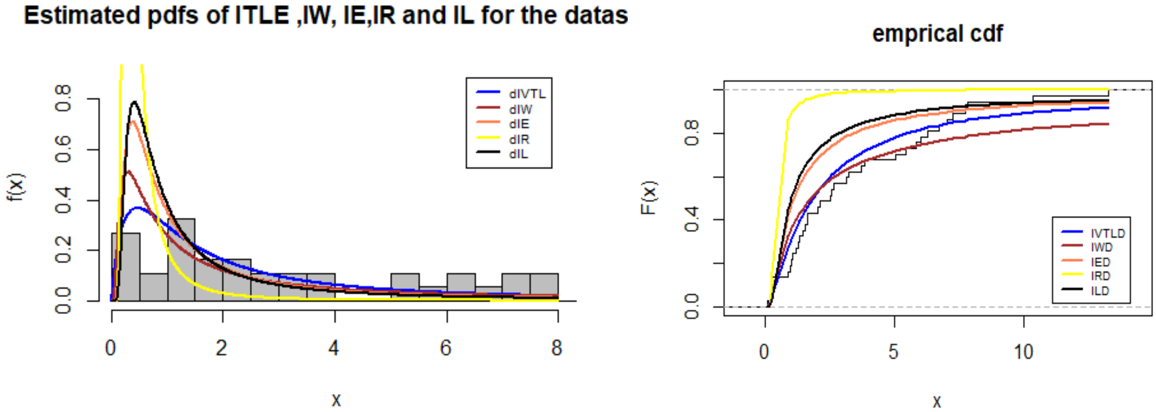

Figure 4. Estimated pdf and cdf for the annual rainfall data set with corresponding distributions.

We generate different progressive Type I censoring samples from the original annual rainfall data. Removed items are considered as in the simulation study in (4) for n = 50. In addition, different stages of censoring are proposed at $m = 3$, $m = 4$, and $m = 5$, then the censoring times when $\beta = 1$ are given in (3).

In (8), we compute the MLE of the parameter β, their associated confidence interval estimates, and bootstrap CIs estimates (Boot-p and Boot-t). We also compute BEs and the Highest posterior density intervals by using the M-H algorithm under informative prior and non-informative prior. For informative prior, we apply the same method shown in Section 3 to compute the hyper parameters. We use the MLE estimate for the parameter β and its variance under all stages with their different removed units shown in (9).

7. Conclusion

In this paper, we have obtained the MLEs and BEs under non-informative and informative priors for the unknown parameters of the ITL distribution based on progressive Type I censoring data; the approximate CIs, HPD intervals, and bootstrap (p-t) intervals are also obtained. We perform some simulations to see the performances of the MLEs and BEs in progressive Type I censoring samples. Two real data sets have been re-analyzed based on progressive Type I censoring data. The simulation results indicate that the performance of estimates under informative prior is better than the non-informative prior and MLEs; on the other hand, the MLEs are better than the BEs under non-informative prior. The values of hyper-parameters in Bayesian estimation were determined for simulation and real data based on the MLE and its variance in all statuses when informative prior is considered.

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REFERENCES

1. A. M. Abd AL-Fattah, A. A. El-Helbawy, and G. R. Al-Dayian, *inverted Kumaraswamy distribution: properties and estimation*, Pakistan Journal of Statistics, vol. 33, no. 1, 2017,
2. R.I. Abd EL-Kader, G.R. AL-Dayian, and S.A. AL-Gendy, *Inverted Pareto Type I distribution: properties and estimation*, Journal of Faculty of Commerce AL-Azhar University, Girls' Branch, 21, pp. 19-40, 2003.

3. A. Aijaz, S. Q. Ain, and R. Tripathi, *Bayesian analysis of inverse Topp-Leone distribution under different loss functions*, transformation, 1, 1, 2020.
4. G.R. AL-Dayian, *Burr Type III distribution: Properties and estimation*, The Egyptian Statistical Journal, 43, PP.102-116, 1999.
5. G.R. Al-Dayian, *Inverted Pareto Type II distribution: Properties and estimation*, Journal of Faculty of Commerce AL-Azhar University Girls' Branch, 22, 1-18, 2004.
6. A. Aljuaid, *Estimating the parameters of an exponentiated inverted Weibull distribution under Type II censoring*, Applied Mathematical Sciences, 7(35), 1721-1736, 2013.
7. E.M. Almetwally, R. Alharbi, D. Alnagar, and E.H. Hafez, *A new inverted top-leone distribution: applications to the COVID-19 mortality rate in two different countries*, Axioms, 10(1), 25, 2021.
8. Z.A. Al-Saiary and R.A. Bakoban, *The Topp-Leone generalized inverted exponential distribution with real data applications*, Entropy, 22(10), 1144, 2020.
9. B. Al-Zahrani, *Goodness-of-fit for the Topp-Leone distribution with unknown parameters*, Applied Mathematical Sciences, 6(128), 2012.
10. N. Balakrishnan, *Progressive censoring methodology: an appraisal*, Test, 16(2), 211-259, 2007.
11. N. Balakrishnan and E. Cramer, *The art of progressive censoring*, Statistics for industry and technology, 2014.
12. N. Balakrishnan, D. Han, and G. Iliopoulos, *Exact inference for progressively Type-I censored exponential failure data*, Metrika, 73(3), 335-358, 2011.
13. D. K. Bhaumik, K. Kapur, and R. D. Gibbons, *Testing parameters of a gamma distribution for small samples*, Technometrics, 51(3), 326-334, 2009.
14. R. Calabria and G. Pulcini, *On the maximum likelihood and least squares estimation in the inverse Weibull distribution*, Journal of Statistica Applicate, 2(1), 1990.
15. A. C. Cohen, *Progressively censored samples in life testing*, Technometrics, 5(3), 327-339, 1963.
16. S. Dey and B. Pradhan, *Generalized inverted exponential distribution under hybrid censoring*, Statistical Methodology, 18, 101-114, 2014.
17. S. Dey, S. Singh, Y.M. Tripathi, and A. Asgharzadeh, *Estimation and prediction for a progressively censored generalized inverted exponential distribution*, Statistical Methodology, 132, 185-202, 2016.
18. B. Efron and R. J. Tibshirani, *An Introduction to the Bootstrap*, New York: Chapman and Hall, 1993.
19. I. Elbatal, N. Alotaibi, S. A. Alyami, M. Elgarhy, and A. R. El-Saeed, *Bayesian and non-Bayesian estimation of the Nadarajah–Haghighi distribution: using progressive Type-1 censoring scheme*, Mathematics, 10(5), 760, 2022.
20. A. I. Genç, *Moments of order statistics of Topp-Leone distribution*, Statistical Papers, 53, 117-131, 2012.
21. M. E. Ghitany, S. Kotz, and M. Xie, *On some reliability measures and their stochastic orderings for the Topp-Leone distribution*, Journal of Applied Statistics, 32, 715-722, 2005.
22. R. C. Gupta and J. Huang, *The Weibull–Conway–Maxwell–Poisson distribution to analyze survival data*, Journal of Computational and Applied Mathematics, 311, 171-182, 2017.
23. P. Hall, *Theoretical comparison of bootstrap confidence intervals*, The Annals of Statistics, 927-953, 1988.
24. A. S. Hassan and S. M. Assar, *The exponentiated Weibull power function distribution*, Journal of Data Science, 16(2), 589-614, 2017.
25. A. S. Hassan, E. M. Almetwally, and G. M. Ibrahim, *Kumaraswamy inverted Topp–Leone distribution with applications to COVID-19 data*, Computers, Materials, and Continua, 337-358, 2021.
26. A. S. Hassan, M. Elgarhy, and R. Ragab, *Statistical properties and estimation of inverted Topp-Leone distribution*, J. Stat. Appl. Probab, 9(2), 319-331, 2020.
27. A. S. Hassan, R. M. Mousa, and M. H. Abu-Moussa, *Analysis of Progressive Type-II Competing Risks Data, with Applications*, Lobachevskii Journal of Mathematics, 43(9), 2479-2492., 2022.
28. G. R. Herd, *Estimation of the parameters of a population from a multi-censored sample*, Iowa State University, 1956.
29. S. Kotz and E. Seier, *Kurtosis of the Topp-Leone distributions*, Interstate, 1, 1-15, 2007.
30. N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, *Methods for statistical analysis of reliability and life data*, John Wiley and Sons, Inc., New York, 1974.
31. H. Z. Muhammed and E. A. E. Muhammed, *Inverted Topp-Leone Distribution: Contribution to a Family of J-Shaped Frequency Functions in Presence of Random Censoring*, Journal of Reliability and Statistical Studies, 615-650, 2021.
32. H. Z. Muhammed, *On The Inverted Topp-Leone Distribution*, international journal of reliability and applications, 20, 17–28, 2019.
33. S. Nadarajah, *Bathtub-shaped failure rate functions*, Quality and Quantity, 43, 855-863, 2009.
34. S. Nadarajah and S. Kotz, *Moments of Some J-Shaped Distributions*, Journal of Applied Statistics, 30, 311-317, 2003.
35. G. Prakash, *Inverted exponential distribution under a Bayesian view point*, Journal of Modern Applied Statistical Methods, 11(1), 190-202, 2012.
36. D. van Ravenzwaaij, P. Cassey, and S.D. Brown, *A simple introduction to Markov Chain Monte-Carlo sampling*, Psychonomic Bulletin Review, 25, 143-154, 2018.
37. M. A. Selim, *Estimation and prediction for Nadarajah-Haghighi distribution based on record values*, Pak. J. Statist, 34(1), 77-90, 2018.
38. C. W. Topp and F. C. Leone, *A family of J-shaped frequency functions*, Journal of the American Statistical Association, 50, 209–219, 1955.
39. J. R. Van Dorp and S. Kotz, *Modeling income distributions using elevated distributions on a bounded domain*, In Distribution models theory, 1-25, 2006.
40. D. Vicaria, J. R. Van Dorp, and S. Kotz, *Two-sided generalized Topp and Leone (TS-GTL) distributions*, Journal of Applied Statistics, 35(10), 1115-1129, 2008.
41. M. Zhou, D. W. Yang, Y. Wang, and S. Nadarajah, *Some J-shaped distributions: Sums, products, and ratios*, RAMS '06. Annual Reliability and Maintainability Symposium, 175-181, 2006.

42. M. M. Yousef, A. S. Hassan, A. H. Al-Nefaie, E. M. Almetwally and H. M. Almongy, *Bayesian estimation using MCMC method of system reliability for inverted Topp–Leone distribution based on ranked set sampling.*, Mathematics, 10(17), 3122, 2022.