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Nondifferentiable Vector Optimization over Cones using Generalized E-convexity

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Abstract This paper discusses a novel approach to solve a nonconvex nondifferentiable vector optimization problem over cones making use of an operator $E : R^n \to R^n$ which renders differentiability to the considered problem. New classes of generalized convex functions are introduced and employed to obtain necessary and sufficient KKT-type E-optimality conditions. Further, a unified E-dual is associated with the given problem encapsulating both the Mond-Weir type and the Wolfe type duals in the same apparatus and duality results are proved.

Keywords E-differentiable function, Vector optimization over cones, Cone-generalized E-convexity, KKT optimality conditions, Unified Duality.

AMS 2010 subject classifications 90C26 . 90C30 . 90C46

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1. Introduction

Decision making has always been a central issue for mankind. Very often the decision maker is confronted with discordant objectives due to which he is unable to judge whether or not an alternative A is preferred to another alternative B. This motivates us to study multiobjective or vector optimization problems over partially ordered spaces. These problems deal with optimizing a vector-valued objective function with respect to a given partial order subject to certain constraints thus making incomparability of a pair of objects admissible. A host of problems in varied fields like engineering, finance, operations research, management science etc can be modelled using vector optimization (see, for example,[23, 19, 9, 10, 17, 22, 27, 28], and others). Frequently in vector optimization, the partial order on the underlying space is induced by pointed convex cones. The most common cone being the positive orthant of the Euclidean space. Extensive research has been done in the past to develop optimality conditions and duality results for both differentiable as well as nondifferentiable vector optimization problems over arbitrary cones [11, 20, 21, 29, 31, 30, 33, 32, 34].

To relax the ever so important notion of convexity in optimization theory, Youness [36] gave the concept of an E-convex set and an E-convex function. Further, Megahed et al. [24] defined the notion of an E-differentiable function which transforms a (not necessarily) differentiable function to a differentiable function via an operator $E : R^n \to R^n$. Owing to their importance and applicability, several researchers have generalised and studied the concepts of E-convexity and E-differentiability. Chen [12]introduced the concept of semi-E-convexity while Hu et. al [15] extended it to semilocal E-convexity. Iqbal et. al [18] explored geodesic E-convex sets and geodesic E-convex functions on Reimannian manifolds. Abdulaleem gave the concepts of E-invex functions [1], E-univex functions[4] and E-type-I functions [3] among many others. Antczak and Abdulaleem [8] proved relationships

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between vector variational-like inequalities and differentiable vector optimization problems under E-convexity and some of its generalizations. Recently, Mishra et. al [25] proved optimality and duality results for fractional programming problems under E-univexity. Ample of literature pertaining to E-convexity and E-differentiabity can be found in multiple arenas.See for instance, [3, 2, 7, 6, 5, 14, 16, 15, 24, 26, 36, 37] and the references therein.

To our knowledge, no research work so far attempts to explore E-convexity and E-differentiablilty in the framework of vector optimization over cones. In this paper, we consider an E-differentiable class of nonconvex vector optimization problems over cones. We introduce the notion of a cone-E-convex function and further present its generalizations with examples. We then employ the introduced classes of functions to obtain necessary and sufficient Karush-Kuhn-Tucker type E-optimality conditions for a vector optimization problem over general ordering cones. Furthermore, a unified E-dual is associated with the considered problem which enables us to study both the Mond-Weir and the Wolfe type duals in a unifying framework and weak and strong duality results are proved.

2. Definitions and Preliminaries

Youness [36] introduced the concept of an E-convex set and an E-convex function to weaken the notion of a convex set and a convex function. For the sake of convenience, let us recall these definitions.

Definition 1 [36] : A set $S \subseteq R^n$ is said to be an E-convex set if and only if there exists a map $E : R^n \to R^n$ such that

$$\lambda\, E(x) + (1 - \lambda\,)E(y) \in\; S,$$

for each $x, y \in\; S$ and $0 \leq \lambda \leq 1$.

It is apparent that every convex set is E-convex (with E as the identity map) and if $S \subseteq\; R^n$ is an E-convex set then $E(S) \subseteq S$.

Definition 2 [36] : A function $f : S \to\; R$ is said to be E-convex on S if and only if there is a map $E : R^n \to\; R^n$ such that S is an E-convex set and

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)),$$

for each $x, y \in S$ and $0 \leq \lambda \leq 1$.
If E is considered to be the identity map, the notion of E-convexity reduces to convexity.

In an attempt to solve vector optimization problems wherein the multiple objectives are evaluated on the basis of partial order relations induced by arbitrary cones in the Eudidean spaces, we introduce the notion of a vector-valued cone-E-convex function and its generalizations.

Let $K \subseteq R^m$ be a closed convex pointed cone with nonempty interior and let *intK* denote the interior of K. The positive dual cone K^* of K, is defined as

$$K^* = \{y^* \in R^m : \langle y, y^* \rangle \geq 0 \;\text{ for all }\; y \in K\}$$

Consider a vector-valued function $f : S \subseteq R^n \to R^m$.

Definition 3 : The function f is said to be K-E-convex at $\bar{x} \in S$ on S if there exists an operator $E : R^n \to R^n$ such that S is an E-convex set and

$$\lambda f(E(x)) + (1 - \lambda)f(E(\bar{x})) - f(\lambda E(x) + (1 - \lambda)E(\bar{x})) \in K$$

for each $x \in S$ and $0 \leq \lambda \leq 1$.

Note that, since S is E-convex, therefore $E(S) \subseteq S$ and hence the above relation is well defined.
If f is K-E-convex at each $\bar{x} \in S$ then f is said to be K-E-convex on S.

Remark 1 : (a) If E is the identity map, K-E-convexity reduces to K-convexity defined by Craven [13]

(b) If f is a real-valued function and $K = R_+$, then the above definition reduces to the definition of E-convexity given by Youness [36]

In [24], Megahed et al. introduced the notion of an *E*-differentiable function which transforms a non-differentiable real-valued function to a differentiable function under an operator $E : R^n \to R^n$.

Definition 4 [24] : Let $E : R^n \to R^n$, S be an open *E*-convex set and $f : S \subseteq R^n \to R$ be a not necessarily differentiable real-valued function. The function f is said to be *E*-differentiable at $\bar{x} \in S$ if and only if (*foE*) is differentiable at $\bar{x}$ (in the usual sense), that is,

$(foE)(x) = (foE)(\bar{x}) + \bigtriangledown(foE)(\bar{x})(x - \bar{x}) + \alpha(\bar{x}, x - \bar{x}) \parallel x - \bar{x} \parallel$, where $\alpha(\bar{x}, x - \bar{x}) \to 0$ as $x \to \bar{x}$.

Here, $\bigtriangledown(foE)(\bar{x})$ denotes the unique gradient vector of (*foE*) at $\bar{x}$ given by

$$\bigtriangledown(foE)(\bar{x}) = \left(\frac{\partial(foE)(\bar{x})}{\partial x_1}, ..., \frac{\partial(foE)(\bar{x})}{\partial x_n}\right)^T$$

We now state the concept of a vector-valued *E*-differentiable function.

Definition 5 : Let $E : R^n \to R^n$, S be an open *E*-convex set and $f = (f_1, f_2..., f_m) : S \subseteq R^n \to R^m$ be a not necessarily differentiable vector-valued function. We say that f is *E*-differentiable at $\bar{x} \in S$ if and only if $(f_i oE)$ is *E*-differentiable at $\bar{x}$, for each $i = 1, 2, ..., m$, and in this case

$$\bigtriangledown(foE)(\bar{x}) = (\bigtriangledown(f_1 oE)(\bar{x}), ..., \bigtriangledown(f_m oE)(\bar{x}))^T$$

is the $m \times n$ Jacobian matrix of (foE) at $\bar{x}$.

Next, we provide a necessary condition for a vector-valued *E*-differentiable cone-*E*-convex function.

Proposition 1 : Let $E : R^n \to R^n$, S be an open *E*-convex subset of R^n, and $f : S \to R^m$ be *K*-*E*-convex at $\bar{x} \in S$ on *S*. If f is *E*-differentiable at $\bar{x}$, then for each $x \in S$

$$f(E(x)) - f(E(\bar{x})) - \bigtriangledown f(E(\bar{x}))(E(x)) - E(\bar{x})) \in K \tag{1}$$

Proof : Since f is *K*-*E*-convex at $\bar{x} \in S$ on *S*, therefore for each $x \in S$, we have

$$\lambda f(E(x)) + (1 - \lambda)f(E(\bar{x})) - f(\lambda E(x) + (1 - \lambda)E(\bar{x})) \in K, \quad 0 \leq \lambda \leq 1$$

Since *K* is a closed cone, this implies

$$f(E(x)) - f(E(\bar{x})) - \frac{[f(\lambda E(x) + (1 - \lambda)E(\bar{x})) - f(E(\bar{x}))]}{\lambda} \in K, \quad 0 \leq \lambda \leq 1$$

Taking limit as $\lambda \to 0$, we obtain

$$f(E(x)) - f(E(\bar{x})) - \bigtriangledown f(E(\bar{x}))(E(x) - E(\bar{x})) \in K.$$

□

Note that if f is *K*-*E*-convex on *S*, then relation (1) holds for all $x, \bar{x}, \in S$.

To justify the introduction of a cone-*E*-convex function, we now give an example of a function which is cone-*E*-convex but not *E*-convex.

Example 1 : Let $S = \{(x, y) : x^2 + y^2 < 1\}$ and $K = \{(x, y) : x \geq 0, x \geq -2y\}$

Consider the nonsmooth function $f : S \to R^2$ defined as

$$f(x, y) = \left(\sqrt[3]{x^2} + \sqrt[3]{y^2}, \sqrt[3]{xy}\right)$$

Let $E : R^2 \to R^2$ be an operator defined as $E(x, y) = (x^3, y^3)$.

The function $(foE)(x, y) = (x^2 + y^2, xy)$ is differentiable at $(\bar{x}, \bar{y}) = (0, 0)$ so that f is *E*-differentiable at $(\bar{x}, \bar{y})$

We have,

$$(foE)(x, y) - (foE)(\bar{x}, \bar{y}) - \bigtriangledown(foE)(\bar{x}, \bar{y})\{E(x, y) - E(\bar{x}, \bar{y})\} = (x^2 + y^2, xy) \in K$$

for each $(x, y) \in S$. Therefore, f is *K*-*E*-convex at $(\bar{x}, \bar{y})$, while it is easy to check that f is not *E*-convex at $(\bar{x}, \bar{y})$, since $(foE)(x, y) - (foE)(\bar{x}, \bar{y}) - \bigtriangledown(foE)(\bar{x}, \bar{y})\{E(x, y) - E(\bar{x}, \bar{y})\} \ngeq 0$, for all $(x, y) \in S$.

Let $E : R^n \to R^n$ be an operator and S be an open E-convex subset of R^n. We now define various generalizations of an E-differentiable cone-E-convex function. Consider a vector-valued function
$f : S \subseteq R^n \to R^m$.

Definition 6 : The function f is said to be *K*-*E*-pseudoconvex at $\bar{x} \in S$ on *S* if, for every $x \in S$

$$-\bigtriangledown(foE)(\bar{x})(E(x) - E(\bar{x})) \notin intK \Rightarrow -\{(foE)(x) - (foE)(\bar{x})\} \notin intK$$

It is clear that every *K*-*E*-convex function is *K*-*E*-pseudoconvex. However the converse may not be true as shown by the following example.

Example 2 : Let $S = (-5, 5) \subseteq R$ and $K = \{(x, y) : y \leq 0, y \leq x\}$
Consider the function $f : S \to R^2$ defined as

$$f(x) = \left(-\sqrt[3]{x^2}, 1 - e^{\sqrt[3]{x}}\right)$$

and an operator $E : R \to R$ defined as $E(x) = x^3$

The function $(foE)(x) = (-x^2, 1 - e^x)$ is differentiable at $\bar{x} = 0$ and therefore f is *E*-differentiable at $\bar{x}$.

We have,

$$\bigtriangledown(foE)(\bar{x})\{E(x) - E(\bar{x})\} = (0, -x^3)$$

Consequently, for $x \in S$

$$\{(foE)(x) - (foE)(\bar{x})\} \in -intK \Rightarrow x < 0 \Rightarrow \bigtriangledown(foE)(\bar{x})\{E(x) - E(\bar{x})\} \in -intK.$$

Thus, f is *K*-*E*-pseudoconvex at $\bar{x}$ on *S*.

However, it is not true that for each $x \in S$

$$(foE)(x) - (foE)(\bar{x}) - \bigtriangledown(foE)(\bar{x})\{E(x) - E(\bar{x})\} = (-x^2, 1 - e^x + x^3) \in K$$

Hence, f is not *K*-*E*-convex at $\bar{x}$ on *S*.

Definition 7 : The function f is said to be strictly *K*-*E*-pseudoconvex at $\bar{x} \in S$ on *S* if, for every $x \in S$ such that $E(x) \neq E(\bar{x})$,

$$-\bigtriangledown(foE)(\bar{x})(E(x) - E(\bar{x})) \notin intK \Rightarrow -\{(foE)(x) - (foE)(\bar{x})\} \notin K \setminus \{0\}$$

. **Definition 8 :** The function f is said to be *K*-*E*-quasiconvex at $\bar{x} \in S$ on *S* if, for every $x \in S$

$$\{(foE)(x) - (foE)(\bar{x})\} \notin intK \Rightarrow -\bigtriangledown(foE)(\bar{x})(E(x) - E(\bar{x})) \in K$$

. **Remark 2 :** (a) If f is *K*-*E*-pseudoconvex (strictly *K*-*E*-pseudoconvex, *K*-*E*-quasiconvex) at every $\bar{x} \in S$ then f is said to be *K*-*E*-pseudoconvex (strictly *K*-*E*-pseudoconvex, *K*-*E*-quasiconvex) on *S*.

(b) If f is a real-valued function and $K = R_+$, the above definitions take the form of the analogous definitions introduced in [5].

(c) If *E* is the identity map, the above definitions correspond to the definitions given by Cambini [11].

3. Vector Optimization over Cones with *E*-differentiable functions

Consider the following vector optimization problem (VOP) over arbitrary ordering cones. This problem generalizes the most commonly encountered and studied multiobjective programming problem with coordinate wise ordering induced by the positive orthant as ordering cone.

$$(VOP) \qquad K\text{-minimize} \quad f(x)$$

$$\text{subject to} \qquad -g(x) \in Q$$

where $f = (f_1, ..., f_m) : S \to R^m$, $g : (g_1, ..., g_p) : S \to R^p$ are vector-valued (not necessarily) differentiable functions and S is a nonempty open subset of R^n, K and Q are closed convex pointed cones with nonempty interiors in R^m and R^p respectively.

Let $S_o = \{x \in S : -g(x) \in Q\}$ be the feasible set of (VOP).

We shall investigate the following optimality notions for the problem (VOP) in this paper.

Definition 9 : A vector $\bar{x} \in S_o$ is said to be

(i) a weak minimum of (VOP) if, for every $x \in S_o$

$$f(x) - f(\bar{x}) \notin -intK$$

(ii) a minimum of (VOP) if, for every $x \in S_o$

$$f(x) - f(\bar{x}) \notin -K \setminus \{0\}$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we assume that the functions f and g are E-differentiable at any feasible solution.We associate the considered problem (VOP) with the following differentiable vector optimization problem:

$$(VOP)_E \qquad K\text{-minimize} \quad (foE)(x)$$

$$\text{subject to} \qquad -(goE)(x) \in Q.$$

Let $S_E = \{x \in S : -(goE)(x) \in Q\}$ be the set of all feasible solutions of $(VOP)_E$. Since the functions f and g constituting the problem (VOP) are assumed to be E-differentiable at any feasible solution of (VOP) therefore, the functions (foE) and (goE) constituting the problem $(VOP)_E$ are differentiable at any feasible solution of $(VOP)_E$.

Since the functions f and g constituting the problem (VOP) are assumed to be E-differentiable at any feasible solution of (VOP) therefore, the functions (foE) and (goE) constituting the problem $(VOP)_E$ are differentiable at any feasible solution of $(VOP)_E$.

We now establish the equivalence between (VOP) and $(VOP)_E$ in terms of their (weak) minimum solutions. We begin with the following result.

Lemma 1 : Let $E : R^n \to R^n$ be a one-to-one and onto operator. Then $E(S_E) = S_o$.

Proof : Let $x \in E(S_E)$. Since E is a one-to-one and onto operator, there exists $z \in S_E$ such that $x = E(z)$. Suppose $x \notin S_o$, then $-g(x) \notin Q$, which means $-g(E(z)) = -(goE)(z) \notin Q$. But this contradicts $z \in S_E$. Hence, $E(S_E) \subset S_o$.

On the other hand, let $x \in S_o$. Suppose $x \notin E(S_E)$, then $E^{-1}(x) \notin S_E$, so that, $-(goE)(E^{-1}(x)) \notin Q$. This means, $-g(x) \notin Q$ and $x \notin S_o$ which is not true. □

Lemma 2 : Let $\bar{x} \in S_o$ be a (weak) minimum of (VOP). Then there exists $\bar{z} \in S_E$ such that $\bar{x} = E(\bar{z})$ and $\bar{z}$ is a (weak) minimum of $(VOP)_E$.

Proof : Let $\bar{x} \in S_o$ be a (weak) minimum of (VOP). By virtue of Lemma 1, there exists $\bar{z} \in S_E$ such that $\bar{x} = E(\bar{z})$. We prove that, $\bar{z}$ is a weak minimum of $(VOP)_E$. Suppose not, then there exists $\tilde{z} \in S_E$ such that

$$(foE)(\tilde{z}) - (foE)(\bar{z}) \in -intK$$

or,

$$f(E(\tilde{z}) - (f(E(\bar{z})) \in -intK$$

Again, by Lemma 1, there exists $\tilde{x} \in S_o$ such that $\tilde{x} = E(\tilde{z})$, so that the above relation gives

$$f(\tilde{x}) - f(\bar{x}) \in -intK.$$

This contradicts the fact that $\bar{x}$ is a weak minimum of (VOP). Analogous proof can be given if $\bar{x} \in S_o$ is a minimum solution of (VOP). □

Lemma 3: Let $\bar{z} \in S_E$ be a (weak) minimum of $(VOP)_E$. Then $E(\bar{z})$ is a (weak) minimum of (VOP).

Proof : Let $\bar{z} \in S_E$ be a (weak) minimum of $(VOP)_E$. By Lemma 1, $E(\bar{z}) \in S_o$. Suppose $E(\bar{z})$ is not a weak minimum of (VOP), then there exists $\tilde{x} \in S_o$ such that

$$f(\tilde{x}) - f(E(\bar{z})) \in -intK.$$

Again, using Lemma 1, there exists $\tilde{z} \in S_E$ such that $\tilde{x} = E(\tilde{z})$, so that the above relation gives

$$f(E(\tilde{z})) - f(E(\bar{z})) \in -intK,$$

which contradicts the fact that $\bar{z}$ is a weak minimum of $(VOP)_E$.

Similar proof holds for minimum solutions of (VOP) and $(VOP)_E$. □

Thus far, we have proved that the (not necessarily) differentiable problem (VOP) and the differentiable problem $(VOP)_E$ are equivalent in terms of their (weak) minimum solutions. Next, we derive Fritz John type necessary optimality conditions for $\bar{x} \in S_E$ to be a weak minimum of $(VOP)_E$ and hence, by virtue of Lemma 3, we obtain Fritz John type necessary optimality conditions for $E(\bar{x})$ to be a weak minimum of (VOP).

Theorem 1 : Fritz John type cone-*E*-optimality conditions

Let $E : R^n \to R^n$ be a one-to-one and onto operator. Let $\bar{x} \in S_E$ be a weak minimum of $(VOP)_E$ (and thus, $E(\bar{x})$ be a weak minimum of the considered problem (VOP)). Also assume that the objective function f and the constraint function g constituting the original problem (VOP) are *E*-differentiable at $\bar{x}$, then there exist $\bar{\lambda} \in K^*$, $\bar{\mu} \in Q^*$, not both zero, such that

$$[\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x})](x - \bar{x}) \geq 0, \forall x \in R^n \quad (2)$$

$$\bar{\mu}^T (goE)(\bar{x}) = 0 \quad (3)$$

Proof : Let $\bar{x} \in S_E$ be a weak minimum of $(VOP)_E$, then by Lemma 3, $E(\bar{x})$ is a weak minimum of (VOP). We shall show that the system

$$-F(x) \in int(K \times Q) \quad \text{has no solution} \quad x \in S,$$

where $F(x) = (\bigtriangledown(foE)(\bar{x})(x - \bar{x}), \bigtriangledown(goE)(\bar{x})(x - \bar{x}) + (goE)(\bar{x}))$

Let if possible, there be a solution $x \in S$, then

$-\bigtriangledown (foE)(\bar{x})(x - \bar{x}) \in intK$ and $-(\bigtriangledown(goE)(\bar{x})(x - \bar{x}) + (goE)(\bar{x})) \in intQ$

Now, for $0 < \alpha < 1$, we have

$$(goE)(\bar{x} + \alpha(x - \bar{x})) = (goE)(\bar{x}) + \bigtriangledown(goE)(\bar{x})\alpha(x - \bar{x}) + o(\alpha)$$

$$= \alpha[(goE)(\bar{x}) + \bigtriangledown(goE)(\bar{x})(x - \bar{x})] + (1 - \alpha)(goE)(\bar{x}) + o(\alpha) \in -Q$$

where

$$\lim_{\alpha \to 0^+} o(\alpha) = 0$$

Also, for $0 < \alpha < 1$

$$(foE)(\bar{x} + \alpha(x - \bar{x})) = (foE)(\bar{x}) + \bigtriangledown(foE)(\bar{x})\alpha(x - \bar{x}) + o(\alpha)$$

where

$$\lim_{\alpha \to 0^+} o(\alpha) = 0$$

This implies,

$$(foE)(\bar{x} + \alpha(x - \bar{x})) - (foE)(\bar{x}) = \bigtriangledown(foE)(\bar{x})\alpha(x - \bar{x}) + o(\alpha) \in -intK$$

This contradicts the fact that $E(\bar{x})$ is a weak minimum of (VOP). Hence, the system $-F(x) \in int(K \times Q)$ has no solution $x \in S$.

Thus, by the Alternative Theorem given by Jeyakumar [20], there exist $\bar{\lambda} \in K^*$ and $\bar{\mu} \in Q^*$, not both zero, such that

$$\bar{\lambda}^T \bigtriangledown (foE)(\bar{x})(x-\bar{x}) + \bar{\mu}^T \{\bigtriangledown(goE)(\bar{x})(x-\bar{x}) + (goE)(\bar{x})\} \geq 0, \forall x \in S$$

For $x = \bar{x}$, the above relation gives

$$\bar{\mu}^T (goE)(\bar{x}) \geq 0.$$

Since, $-(goE)(\bar{x}) \in Q$, $\bar{\mu} \in Q^*$, therefore

$$\bar{\mu}^T (goE)(\bar{x}) \leq 0$$

Thus, $\bar{\mu}^T (goE)(\bar{x}) = 0$.

Hence, we obtain

$$[\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x})](x-\bar{x}) \geq 0, \forall x \in R^n$$

and

$$\bar{\mu}^T (goE)(\bar{x}) = 0.$$

□

For deriving Karush-Kuhn-Tucker type necessary optimality conditions for the differentiable vector optimization problem $(VOP)_E$ and hence for the (not necessarily) differentiable problem (VOP), we need a certain cone-constraint qualification.

Definition 10 : The problem $(VOP)_E$ is said to satisfy **cone-*E*-constraint qualification** at $\bar{x} \in S_E$ if there exists $\hat{x} \in R^n$ such that

$$(goE)(\bar{x}) + \bigtriangledown(goE)(\bar{x})(\hat{x}-\bar{x}) \in -intQ.$$

Theorem 2 : KKT type cone-*E*-optimality conditions

Let $E : R^n \to R^n$ be a one-to-one and onto operator. Let $\bar{x} \in S_E$ be weak minimum of $(VOP)_E$ (and thus, $E(\bar{x})$ be a weak minimum of the considered problem (VOP)) at which cone-E-constraint qualification holds. Further, assume that the objective function f and the constraint function g are E-differentiable at $\bar{x}$, then there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that (2) and (3) hold.

Proof : Since all the conditions of Theorem 1 are satisfied, there exist $\bar{\lambda} \in K^*$, $\bar{\mu} \in Q^*$, not both zero, such that

$$\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x})](x-\bar{x}) \geq 0, \forall x \in R^n$$

and

$$\bar{\mu}^T (goE)(\bar{x}) = 0$$

Now, suppose $\bar{\lambda} = 0$, then $\bar{\mu} \neq 0$ and (2) reduces to

$$\bar{\mu}^T \bigtriangledown (goE)(\bar{x})(x-\bar{x}) \geq 0 \tag{4}$$

Since, cone-E-constraint qualification holds at $\bar{x}$, there exists $\hat{x} \in R^n$ such that

$$(goE)(\bar{x}) + \bigtriangledown(goE)(\bar{x})(\hat{x}-\bar{x}) \in -intQ.$$

This implies,

$$\bar{\mu}^T [(goE)(\bar{x}) + \bigtriangledown(goE)(\bar{x})(\hat{x}-\bar{x})] < 0$$

which along with (3) contradicts (4). Hence, $\bar{\lambda} \in K^* \setminus \{0\}$. □

Now we prove that under certain cone-E-convexity assumptions on the functions involved, these necessary conditions become sufficient for a feasible solution to be a (weak) minimum of $(VOP)_E$ and hence obtain sufficient optimality conditions for (VOP).

Theorem 3 : Let $\bar{x} \in S_E$ be a feasible solution for $(VOP)_E$ and suppose there exist vectors $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$, such that (2) and (3) hold. Further, assume that f is K-E-convex and g is Q-E-convex at $\bar{x} \in S_E$ on S_0. Then $\bar{x}$ is a weak minimum of the problem $(VOP)_E$ and, thus, $E(\bar{x})$ is a weak minimum of the problem (VOP).

Proof : Assume on the contrary that $\bar{x}$ ia not a weak minimum of $(VOP)_E$, then there exists $\hat{x} \in S_o$ such that

$$f(E(\hat{x})) - f(E(\bar{x})) \in -intK$$

Since f is K-E-convex at $\bar{x}$ on S_0, by Proposition 1

$$f(E(\hat{x})) - f(E(\bar{x})) - \bigtriangledown f(E(\bar{x}))(E(\hat{x}) - E(\bar{x})) \in K$$

Adding the above two equations, we get

$$- \bigtriangledown f(E(\bar{x}))(E(\hat{x}) - E(\bar{x})) \in intK$$

Since $\bar{\lambda} \in K^* \setminus \{0\}$, we obtain

$$\bar{\lambda}^T \bigtriangledown f(E(\bar{x}))(E(\hat{x}) - (E(\bar{x})) < 0 \tag{5}$$

Also, since g is Q-E-convex at $\bar{x}$ on S_0, we have

$$g(E(\hat{x})) - g(E(\bar{x})) - \bigtriangledown g(E(\bar{x}))(E(\hat{x}) - E(\bar{x})) \in Q$$

As $\bar{\mu} \in Q^*$, we get

$$\bar{\mu}^T \{g(E(\hat{x})) - g(E(\bar{x})) - \bigtriangledown g(E(\bar{x}))(E(\hat{x}) - E(\bar{x}))\} \geq 0$$

However, $\hat{x} \in S_E$, $\bar{\mu} \in Q^*$ and (3) together give

$$\bar{\mu}^T \bigtriangledown g(E(\bar{x}))(E(\hat{x}) - E(\bar{x})) \leq 0 \tag{6}$$

Adding (5) and (6), we obtain a contradiction to (2). □

Theorem 4 : Let $\bar{x} \in S_E$ and suppose that there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that (2) and (3) hold. If f is K-E-pseudoconvex and g is Q-E-quasiconvex at $\bar{x}$ on S_E, then $\bar{x}$ is a weak minimum of $(VOP)_E$ and hence $E(\bar{x})$ is weak minimum of (VOP).

Proof : Let if possible, $\bar{x}$ be not a weak minimum of $(VOP)_E$. Then there exists $\hat{x} \in S_E$ such that

$$f(E(\hat{x})) - f(E(\bar{x})) \in -intK$$

Since f is K-E-pseudoconvex at $\bar{x}$ on S_E, we get

$$\bigtriangledown(foE)(\bar{x})(E(x) - E(\bar{x})) \in -intK$$

Now, since $\bar{\lambda} \in K^* \setminus \{0\}$, we obtain (5).

Also, since $\hat{x} \in S_E$ and $\bar{\mu} \in Q^*$, we have $\bar{\mu}^T(goE)(\hat{x}) \leq 0$.

On using (3), we get

$$\bar{\mu}^T\{(goE)(\hat{x}) - (goE)(\bar{x})\} \leq 0. \tag{7}$$

If $\bar{\mu} \neq 0$, then (7) implies $\{(goE)(\hat{x}) - (goE)(\bar{x})\} \notin intQ$.

Since g is Q-E-quasiconvex at $\bar{x}$, therefore $- \bigtriangledown (goE)(\bar{x})(E(\hat{x}) - E(\bar{x})) \in Q$,

so that

$$\bar{\mu}^T \bigtriangledown (goE)(\bar{x})(E(\hat{x}) - E(\bar{x})) \leq 0 \tag{8}$$

If $\bar{\mu} = 0$, then also (8) holds. Now proceeding as in Theorem 3, we get a contradiction. □

Theorem 5 : Let $\bar{x} \in S_E$ and suppose that there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that (2) and (3) hold. If f is strictly K-E-pseudoconvex and g is Q-E-quasiconvex at $\bar{x}$ on S_E then $\bar{x}$ is a minimum of $(VOP)_E$ and hence $E(\bar{x})$ is a minimum of (VOP).

Proof : Assume on the contrary that $\bar{x}$ is not a minimum of $(VOP)_E$. Then there exists $\hat{x} \in S_E$ such that

$$f(E(\hat{x})) - f(E(\bar{x})) \in -K \setminus \{0\}$$

Since f is strictly K-E-pseudoconvex at $\bar{x}$ on S_E, we get

$$\bigtriangledown(foE)(\bar{x})(E(\hat{x}) - E(\bar{x})) \in -\text{int}K.$$

Now proceeding as in Theorem 3, we obtain a contradiction. □

4. Unified E-duality

This section is devoted to the formulation and study of a unified E-dual for the vector optimization problem $(VOP)_E$ which enables us to study both the Mond-Weir type and the Wolfe type duals in a unified framework. We associate the following unified E-dual $(VUD)_E$ with the primal problem $(VOP)_E$.

$$(VUD)_E \qquad K\text{-maximize}(foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)$$

$$\text{subject to} \qquad [\lambda^T \bigtriangledown (foE)(y) + \mu^T \bigtriangledown (goE)(y)](x-y) \geq 0$$

$$\delta\mu^T(goE)(y) \geq 0$$

where $y \in S$, $\lambda \in K^* \setminus \{0\}$, $\mu \in Q^*$, $\delta \in \{0,1\}$, $l \in intK$.

Note that if $\delta = 0$, then the above dual reduces to Mond-Weir type dual and if $\delta = 1$, then it takes the form of Wolfe type dual.

Let S_E^U denote the feasible set of $(VUD)_E$.

To proceed, we first introduce the notion of a E-weak maximizer of $(VUD)_E$.

Definition 11 : A feasible solution $(\bar{y}, \bar{\lambda}, \bar{\mu})$ of $(VUD)_E$ is said to be a E-weak maximizer of $(VUD)_E$ if for every feasible solution (y, λ, μ) of $(VUD)_E$,

$$\{foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)\} - \{(foE)(\bar{y}) + \frac{l}{\bar{\lambda}^T l}(1-\delta)\bar{\mu}^T(goE)(\bar{y})\} \notin \text{int}K$$

Let $T_E = \{y \in R^n : (y, \lambda, \mu) \in S_E^U\}$

Theorem 6 : Unified Weak Duality between $(VOP)_E$ and $(VUD)_E$

Let x be feasible for $(VOP)_E$ and (y, λ, μ) be feasible for $(VUD)_E$. Assume that, f and g are K-E-convex and Q-E-convex respectively at y on $S_E \cup T_E$. Then,

$$\{(foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)\} - (foE)(x) \notin intK$$

Proof: On the contrary, assume that, for some feasible solution x of $(VOP)_E$ and (y, λ, μ) of $(VUD)_E$,

$$\{(foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)\} - (foE)(x) \in intK \tag{9}$$

Since f is K-E-convex at y on $S_E \cup T_E$, therefore

$$f(E(x)) - f(E(y)) - \bigtriangledown f(E(y))(E(x) - E(y)) \in K \tag{10}$$

Adding (9) and (10), we get

$$\frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y) - \bigtriangledown f(E(y))(E(x) - E(y)) \in intK$$

As $\lambda \in K^* \setminus \{0\}$, we obtain

$$(1-\delta)\mu^T(goE)(y) - \lambda^T \bigtriangledown f(E(y))(E(x) - E(y)) > 0 \tag{11}$$

Also, since g is Q-E-convex at y on $S_E \cup T_E$ and $\mu \in Q^*$,

$$\mu^T[(goE)(x) - (goE)(y) - \bigtriangledown(goE)(y)(E(x) - E(y))] \geq 0 \tag{12}$$

Adding (11) and (12), we get

$$\mu^T(g(E(x)) - \delta\mu^T(g(E(y))) > (\lambda^T \bigtriangledown f(E(y)) + \mu^T \bigtriangledown g(E(y)))(E(x) - E(y)) \geq 0$$

as (y, λ, μ) is feasible for $(VUD)_E$.

This implies, $\delta\mu^T g(E(y)) < \mu^T(g(E(x)) \leq 0$, since x is feasible for $(VOP)_E$.

But this contradicts the feasibility of (y, λ, μ). Hence the result. □

Theorem 7 : Unified Weak Duality between (VOP) and $(VUD)_E$

Let $E(x)$ be feasible for (VOP) and (y, λ, μ) be feasible for $(VUD)_E$. Further, assume that f is *K*-*E*-convex and g is *Q*-*E*-convex at y on $S_E \cup T_E$. Then, unified weak duality holds between (VOP) and $(VUD)_E$, that is

$$\{(foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)\} - (foE)(x) \notin intK.$$

Proof: Since $E(x)$ is feasible for (VOP), by Lemma 1, x is a feasible solution of $(VOP)_E$. Thus all the conditions of Theorem 6 are satisfied and hence unified weak duality holds between (VOP) and $(VUD)_E$. □

Theroem 8 : Unified Strong Duality between $(VOP)_E$ **and** $(VUD)_E$ **(and hence between** (VOP) **and** $(VUD)_E$**)**

Let $\bar{x} \in S_E$ be a weak minimum of $(VOP)_E$ at which the cone-*E*-constraint qualification holds. Then there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for the problem $(VUD)_E$. Moreover, if Weak Duality holds between $(VOP)_E$ and $(VUD)_E$, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak maximum for $(VUD)_E$. In other words, if $E(\bar{x}) \in S_o$ is a weak minimum of the problem (VOP), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak maximum of the problem $(VUD)_E$ and hence Strongly Duality holds between (VOP) and $(VUD)_E$.

Proof : Let $\bar{x} \in S_E$ be a weak minimum of $(VOP)_E$ at which the cone-*E*-constraint qualification holds. Then, by Theorem 2, there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that

$$[\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x})](x - \bar{x}) \geq 0, \forall x \in R^n$$

$$\bar{\mu}^T(goE)(\bar{x}) = 0.$$

Since the above inequality holds for every $x \in R^n$, we conclude that

$$\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x}) = 0,$$

$$\bar{\mu}^T(goE)(\bar{x}) = 0.$$

Thus, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for the problem $(VUD)_E$.

Now assume on the contrary that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a weak maximum of $(VUD)_E$, then there exists a feasible solution (y, λ, μ) of $(VUD)_E$ such that

$$\{(foE)(y) + \frac{l}{\lambda^T l}(1-\delta)\mu^T(goE)(y)\} - \{(foE)(\bar{x}) + \frac{l}{\bar{\lambda}^T l}(1-\delta)\bar{\mu}^T(goE)(\bar{x})\} \in intK$$

which on using $\bar{\mu}^T(goE)(\bar{x}) = 0$ contradicts Weak Duality Theorem 6. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak maximum for $(VUD)_E$.

Strong Duality between (VOP) and $(VUD)_E$ follows directly from Lemma 1. □

5. Applications

We now present an example of a nonconvex nonsmooth vector optimization problem (VOP) over cones and we use one of the aforementioned sufficiency conditions to characterize its optimality.

Example 3 : Consider $f : R \to R^2$ and $g : R \to R^2$ where

$$f(x) = \left(-\sqrt[3]{x^2}, 1 - e^{\sqrt[3]{x}}\right)$$

$K = \{(x, y) : y \leq 0, y \leq x\}$

$$g(x) = \left(\frac{3}{32}\sqrt[3]{x}, -\frac{1}{3}x - \frac{1}{4}\sqrt[3]{x}\right)$$

$Q = \{(x, y) : x \leq 0, y \geq -x\}$.

The problem (VOP) is given by

$$(VOP) \qquad K\text{-minimize} \quad f(x) = (-\sqrt[3]{x^2}, 1 - e^{\sqrt[3]{x}})$$

$$\text{subject to} \qquad -g(x) = -(\frac{3}{32}\sqrt[3]{x}, -\frac{1}{3}x - \frac{1}{4}\sqrt[3]{x}) \in Q$$

Let $E : R \to R$ be defined as $E(x) = x^3$

The transformed differentiable vector optimization problem is:

$$(VOP)_E \qquad K\text{-minimize} \quad (foE)(x) = (-x^2, 1 - e^x)$$

$$\text{subject to} \qquad -(goE)(x) = -(\frac{3}{32}x, -\frac{1}{3}x^3 - \frac{1}{4}x) \in Q.$$

Note that $\bar{x} = 0$ is a feasible solution in $(VOP)_E$

$K^* = \{(x, y) : x \geq 0, y \leq -x\}$, $Q^* = \{(x, y) : x \leq y, y \geq 0\}$

There exist $\bar{\lambda} = (\frac{19}{30}, \frac{-31}{32}) \in K^* \setminus \{0\}$ and $\bar{\mu} = (\frac{1}{3}, 4) \in Q^*$ such that

$$\bar{\lambda}^T \bigtriangledown (foE)(\bar{x}) + \bar{\mu}^T \bigtriangledown (goE)(\bar{x})](x - \bar{x}) \geq 0, \forall x \in R^n$$

and

$$\bar{\mu}^T (goE)(\bar{x}) = 0$$

.

Now as shown in Example 2, f is K-E-pseudoconvex at $\bar{x}$.

Also, g is Q-E-quasiconvex at $\bar{x}$, because for every $x \in R$, we have

$$\{(goE)(x) - (goE)(\bar{x})\} = (\frac{3}{32}x, -\frac{1}{3}x^3 - \frac{1}{4}x) \notin intQ \Rightarrow x \geq 0 \Rightarrow -\bigtriangledown (goE)(\bar{x})\{E(x) - E(\bar{x})\} = (-\frac{3}{32}x^3, \frac{1}{4}x^3) \in Q.$$

Thus, by Theorem 4 , $\bar{x}$ is a weak minimum of $(VOP)_E$ and hence $E(\bar{x}) = 0$ is a weak minimum of (VOP).

6. Conclusion

This paper investigates a new method to solve a nonconvex nondifferentiable vector optimization problem over arbitrary cones making use of an operator $E : R^n \to R^n$ which transforms the considered problem to a differentiable problem. We introduce new classes of cone-E-convex functions which extend several generalized convexity notions introduced in the literature. We then employ the defined functions to study optimality and duality results for a vector optimization problem over cones.

It will be interesting to establish similar results under weaker assumptions. Other classes of optimization problems may also be explored in an analogous framework.

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