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A Weighted Exponentiated class of Distributions:Properties and Applications for Modelling Reliability Data

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Abstract lIn this study, we suggest the weighted Exp-G (WExp-G) continuous distributions as a novel class of continuous distributions with an additional shape parameter. Then we study the basic mathematical properties. We study Lindley and XGamma special cases. This model is flexible for modelling right skew data sets. The hazard rate of this model is decraesing, increasing and bathtub shape. By performing a simulation analysis, we compared different common methods of estimation. Finally we analyzed and used lifetime, failure time and stress real data sets to illustrate the purposes. This model is perfrom better than other two-parameter distribution.

Keywords Exp-G family; Lindley distribution; Moment; Quantile; Simulation.

AMS 2010 subject classifications 60Exx, 60E05

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1. Introduction

Modelling real lifetime data is important in various fields such as medicine, engineering, economics, and more. In many applications, we need extended types of distributions for tasks such as financial planning, life deposit, and lifetime analysis. As a result, several approaches for producing new distribution classes have been suggested in the literature. One such class is the exponentiated-G family (Exp-G) of type I, which was originally presented by Lehmann (1953). The survival function of Exp-G type I is given by:

$$S_{I,\alpha,\mathbf{\Phi}}(z) = 1 - F_{I,\alpha,\mathbf{\Phi}}(z) = 1 - G(z;\mathbf{\Phi})^{\alpha}\,|_{z\in\mathbf{R}}. \tag{1}$$

Here, $F_{I,\alpha,\mathbf{\Phi}}(z)$ is the cumulative distribution function (cdf), $\alpha > 0$ and $\mathbf{\Phi}$ is the vector of parameters for the parent cdf $G(.)$. Another family is the Lehmann type II family or the Exp-G of type II, which was originally presented by Gupta and Gupta (1998). The survival function of Exp-G type II is given by:

$$S_{II,\alpha,\mathbf{\Phi}}(z) = [1 - G(z;\mathbf{\Phi})]^{\alpha}\,|_{z\in\mathbf{R}}. \tag{2}$$

Several authors have extended the Exp-G of types I and II, such as the beta generated class of distributions (beta-G) by Jones (2004), the Kumaraswamy generated class of distributions (Kw-G) by Cordeiro and de Castro (2011), the Generalized beta generated (GB-G) by Alexander et al. (2012), and the extended Exp-G type I by Alizadeh et al. (2018).

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In this paper, we introduce a new family of distributions by mixing the Exp-G type I and type II families. The cumulative distribution function (cdf) of this new family, denoted as $WExp-G(\beta, \boldsymbol{\xi})$, is given by:

$$F(z) = G(z; \underline{\mathbf{\Phi}})^{\beta} \, [1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}]^{-1}. \tag{3}$$

Here, $\overline{G}(z; \underline{\mathbf{\Phi}}) = 1 - G(z; \underline{\mathbf{\Phi}})$ is the survival function and $\beta > 0$ is the shape parameter. The probability density function (pdf) and hazard rate function (hrf) of this new family are given by:

$$f(z) = \beta \, g(z; \underline{\mathbf{\Phi}}) \, G(z; \underline{\mathbf{\Phi}})^{\beta-1} \left[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta-1}\right] \left[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}\right]^{-2}, \tag{4}$$

$$h(z) = \frac{\beta \, g(z; \underline{\mathbf{\Phi}}) \, G(z; \underline{\mathbf{\Phi}})^{\beta-1} \left[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta-1}\right]}{[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}][1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta} - G(z; \underline{\mathbf{\Phi}})^{\beta}]}. \tag{5}$$

Patil et al. (1986) suggested a method for generating a weighted version of the Exp-G family. By using a non-negative function $w(z)$ with $E(w(Z)) < \infty$, the pdf of the weighted random variable is given by:

$$f_w(z) = \frac{f(z) w(z)}{E(w(Z))},$$

where $E(w(Z))$ represents the expected value of $w(Z)$. By choosing $g(z) = \frac{dG(z)}{dz}$ as a certain pdf with cdf $G(.)$ and

$$w(Z) = \beta \frac{G(z; \underline{\mathbf{\Phi}})^{\beta-1} \left[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta-1}\right]}{[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}]^2},$$

equation (4) presents a new weighted version of the Exp-G family. The main goal of this paper is to introduce one extra parameter to the G class of lifetime distributions, providing more flexibility to the Exp-G family. The WExp-G family has closed-form expressions for the cdf, pdf, and hrf, making it useful for practical applications.

The remainder of this work is organized as follows: Section 2 explores various properties of the WExp-G family. In section 3, we focus on the special cases of Lindley and XGamma distributions. Sections 4 and 5 present and compare various estimation methods through a simulation analysis. The outcomes of the new family are demonstrated in Section 6 through the analysis of lifetime data and failure time data. Finally, Section 7 provides some interesting remarks.

2. Main Properties

2.1. Quantile Function and Asymptotic

Let $U \sim U(0, 1)$. The root of the equation $G(z; \underline{\mathbf{\Phi}})^{\beta}[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}]^{-1} = u$ has a cumulative distribution function (CDF) given by (3). Let $\delta = \inf\{z | F(z; \underline{\mathbf{\Phi}}) > 0\}$. The asymptotic behavior of equations (3), (4), and (5) as $z \to \delta$ is as follows:

$$\begin{aligned} F(z; \underline{\mathbf{\Phi}}) &\sim \frac{1}{2} G(z; \underline{\mathbf{\Phi}})^{\beta}, \\ f(z; \underline{\mathbf{\Phi}}) &\sim \frac{\beta}{2} g(z; \underline{\mathbf{\Phi}}) G(z; \underline{\mathbf{\Phi}})^{\beta-1}, \\ h(z) &\sim \frac{\beta g(z; \underline{\mathbf{\Phi}}) G(z; \underline{\mathbf{\Phi}})^{\beta-1}}{2 - G(z; \underline{\mathbf{\Phi}})^{\beta}}. \end{aligned} \tag{6}$$

The asymptotic behavior of equations (3), (4), and (5) as $z \to \infty$ is given by:

$$\begin{aligned} 1 - F(z; \underline{\mathbf{\Phi}}) &\sim \beta \overline{G}(z; \underline{\mathbf{\Phi}}), \\ f(z; \underline{\mathbf{\Phi}}) &\sim \beta g(z; \underline{\mathbf{\Phi}}), \\ h(z) &\sim \frac{g(z; \underline{\mathbf{\Phi}})}{\overline{G}(z; \underline{\mathbf{\Phi}})}. \end{aligned} \tag{7}$$

These equations demonstrate the influence of the parameters on the tails of the proposed model.

2.2. Expansion for CDF and PDF

Here we obtain a linear combination for $F(z; \underline{\mathbf{\Phi}})$ using the generalized binomial expansion. For any $\beta > 0$, the generalized binomial expansion states that

$$\begin{aligned} F(z; \underline{\mathbf{\Phi}}) &= G(z; \underline{\mathbf{\Phi}})^{\beta} \left[1 + \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta}\right]^{-1} = G(z; \underline{\mathbf{\Phi}})^{\beta} \sum_{\tau=0}^{\infty} (-1)^{\tau}\, \overline{G}(z; \underline{\mathbf{\Phi}})^{\beta\,\tau} \\ &= \sum_{\tau, i_1=0}^{\infty} (-1)^{\tau+i_1} \binom{\beta\,\tau}{i_1} G(z; \underline{\mathbf{\Phi}})^{i_1+\beta} = \sum_{i_1=0}^{\infty} b_{i_1}\, G(z; \underline{\mathbf{\Phi}})^{i_1+\beta} = \sum_{i_1=0}^{\infty} b_{i_1}\, H_{i_1+\beta}(z), \end{aligned} \tag{8}$$

where

$$b_{i_1} = \sum_{\tau=0}^{\infty} (-1)^{\tau+i_1} \binom{\beta\,\tau}{i_1} \tag{9}$$

The PDF of Z follows by differentiating (8) as

$$f(z) = \sum_{i_1=0}^{\infty} b_{i_1}\, h_{i_1+\beta}(z), \tag{10}$$

where $h_{i_1+\beta}(z) = (i_1 + \beta) g(z)\, G(z)^{i_1+\beta-1}$ refers to the PDF of the Exp-G with power parameter $i_1 + \beta$. Using Equations (8) and (10), we can write the CDF (and PDF) of WExp-G as a linear combination of Exp-G type 1 distributions (and densities). Equations (10) and (8) are very useful for obtaining mathematical properties of the new model, such as moments, incomplete moments, moment generating function, and order statistics based on Exp-G properties.

2.3. Order Statistics

Order statistics can be used in various mathematical theories and problems. Let $Z_1, Z_2, \ldots, Z_n$ represent a random sample from a WExp-G distribution. The τ-th order statistic is denoted by $Z_{\tau:n}$. The cumulative distribution function (cdf) of $Z_{\tau:n}$ is given by:

$$F_{\tau:n}(z) = \sum_{i_1=\tau}^{n} \binom{n}{i_1} F(z)^{i_1} \left[1 - F(z)\right]^{n-i_1} = \sum_{i_1=\tau}^{n} \sum_{i_2=0}^{n-i_1} (-1)^{i_2} \binom{n}{i_1} \binom{n-i_1}{i_2} F(z)^{i_1+i_2}. \tag{11}$$

This equation can be reformulated using the generalized binomial expansion as follows:

$$F(z)^{i_1+i_2} = \frac{G(z)^{\beta(i_1+i_2)}}{[1+\bar{G}(z)^{\beta}]^{i_1+i_2}} = G(z)^{\beta(i_1+i_2)} \sum_{l=0}^{\infty} (-1)^l \binom{i_1+i_2-1}{l} \bar{G}(z)^{\beta\, l}$$

$$= G(z)^{\beta(i_1+i_2)} \sum_{l,s=0}^{\infty} (-1)^{s+l} \binom{i_1+i_2-1}{l} \binom{\beta\, l}{s} G(z)^{s} = \sum_{l,s=0}^{\infty} (-1)^{s+l} \binom{i_1+i_2-1}{l} \binom{\beta\, l}{s} G(z)^{\beta(i_1+i_2)+s}. \tag{12}$$

Then, the cdf of $Z_{\tau:n}$ can be expressed as:

$$\begin{aligned} F_{\tau:n}(z) &= \sum_{i_1=\tau}^{n} \sum_{i_2=0}^{n-i_1} \sum_{l,s=0}^{\infty} (-1)^{j+s+l} \binom{n}{i_1} \binom{n-i_1}{i_2} \binom{i_1+i_2-1}{l} \binom{\beta\, l}{s} G(z)^{\beta(i_1+i_2)+s} \\ &= \sum_{i_1=\tau}^{n} \sum_{i_2=0}^{n-i_1} \sum_{s=0}^{\infty} w_{i_1,i_2,s} H_{\beta(i_1+i_2)+s}(z), \end{aligned} \tag{13}$$

where $w_{i_1,i_2,s}$ is defined as:

$$w_{i_1,i_2,s} = \sum_{l=0}^{\infty} (-1)^{i_2+s+l} \binom{n}{i_1} \binom{n-i_1}{i_2} \binom{i_1+i_2-1}{l} \binom{\beta\, l}{s}. \tag{14}$$

$H_{\beta(i_1+i_2)+s}(z)$ represents the exponential-Gamma type I density function with parameter $\beta(i_1+i_2)+s$. The probability density function (pdf) of the τ-th order statistic of any WExp-G distribution, using equation (13), can be expressed as:

$$f_{\tau:n}(z) = \sum_{i_1=\tau}^{n} \sum_{i_2=0}^{n-i_1} \sum_{s=0}^{\infty} w_{i_1,i_2,s} h_{\beta(i_1+i_2)+s}(z), \tag{15}$$

where $h_{\beta(i_1+i_2)+s}(z)$ is the exponential-Gamma type I density function with parameter $\beta(i_1+i_2)+s$. Equation (15) shows that the pdf of the τ-th order statistic can be expressed as a linear combination of exponential-Gamma type I densities. This allows us to derive statistical properties such as moments, incomplete moments, mean deviation, and Lorenz curves based on this equation.

2.4. Residual Entropy

Residual Entropy is an effective measure of information (Belzuce et al., 2004). The Residual entropy of Z is defined as follows:

$$\mathcal{E}(Z) = -\int_0^{\infty} F(z) \log(F(z)) dz. \tag{16}$$

After some algebraic manipulation by applying generalized binomial expansion and geometric expansion, we can obtain the following expression for $F(z)\log(F(z))$ for WExp-G($\beta, \boldsymbol{\xi}$):

$$F(z)\log(F(z)) = \sum_{\tau,i_1,i_2=0}^{\infty} v_{\tau,i_1,i_2} G(z)^{\beta+i_2}, \tag{17}$$

where

$$v_{\tau,i_1,k} = \frac{\beta\,(-1)^{i_1+1+i_2}\binom{\beta\,i_1+\tau+1}{i_2} - (-1)^{\tau+i_1+2+i_2}\binom{\beta(\tau+i_1)}{i_2}}{\tau+1}. \tag{18}$$

Finally, we can obtain the expression for $\mathcal{E}(Z)$ as follows:

$$\mathcal{E}(Z) = \sum_{\tau,j,i_2=0}^{\infty} v_{\tau,i_1,i_2} I(\beta+i_2), \tag{19}$$

where $I(\beta+i_2) = \int_{-\infty}^{\infty} G(z)^{\beta+i_2} dz$.

3. Special cases

In this section, we study Lindley and XGamma as two special members of the WExp-G family.

3.1. Lindley case

In this section, we study WExp-Lindley by taking the survival function (sf) of the Lindley model, where

$$S(z;\underline{\mathbf{\Phi}}) = \frac{1}{\alpha+1}\,(\alpha\,z+\alpha+1)\,e^{-\alpha\,z},$$

and $g(z;\underline{\mathbf{\Phi}}) = \frac{\alpha^2}{\alpha+1}(1+\mathrm{z})\mathrm{e}^{-\alpha\,z}$ for $z>0$ and $\alpha>0$ as the Lindley sf and pdf in equations (3,4 and 5). We denote it by WExp-Li(α,β). The cumulative distribution function (cdf) and probability density function (pdf) of WExp-Li(α,β) are provided by

$$F(z;\alpha,\beta) = \frac{\left[1-\mathrm{e}^{-\alpha\,z}\left(\frac{\alpha\,z+\alpha+1}{\alpha+1}\right)\right]^{\beta}}{1+\left[\mathrm{e}^{-\alpha\,z}\left(\frac{\alpha\,z+\alpha+1}{\alpha+1}\right)\right]^{\beta}}.$$

The pdf and hazard rate function (hrf) of WExp-Li(α,β) can be easily derived. Figures 1 present some plots for the pdf and hrf of WExp-Li for chosen parameter values. Figure 1 shows that the pdf of WExp-Li(α,β) is a unimodal right-skewed distribution. The hrf of WExp-Li(α,β) can be monotonically decreasing, monotonically increasing, upside-down bathtub, or bathtub-upside down shape.

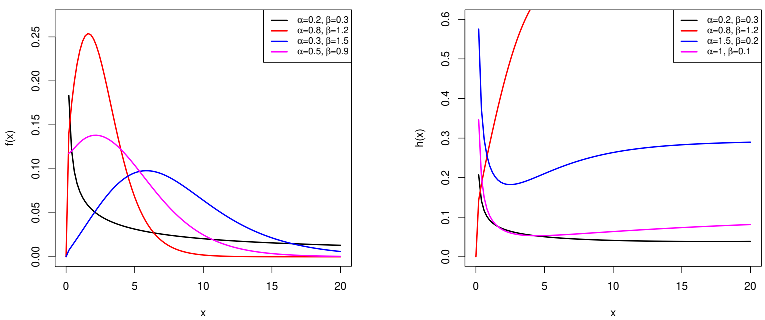

Figure 1. Density and hazard rate of WExp-Li(α,β) for some subset of parameters

3.2. Asymptotics

As $z \to 0$, the asymptotic values of the pdf, cdf, and hrf of the WExp-Li(α, β) distribution are given by

$$F(z) \sim \alpha^{\beta}\, z^{\beta},$$
$$f(z) \sim \beta\, \alpha^{\beta}\, z^{\beta-1},$$
$$h(z) \sim \frac{\beta\, \alpha^{\beta}\, z^{\beta-1}}{1-\alpha^{\beta}\, z^{\beta}}.$$

The asymptotic of the cdf, pdf, and hrf of the WExp-Li(α, β) distribution as $z \to+\infty$ are, respectively, as follows

$$1-F(z) \sim \frac{\alpha\, \beta\, z}{\alpha+1} \mathrm{e}^{-\alpha\, \mathrm{z}},$$
$$f(z) \sim \frac{\alpha^2\, \beta\, z}{\alpha+1} \mathrm{e}^{-\alpha\, \mathrm{z}},$$
$$h(z) \sim \alpha.$$

These equations show how the parameters affect the tails of the WExp-Li distribution.

3.3. Extreme Value and the Norming Constants

If $\bar{Z} = \frac{Z_1+\ldots+Z_\tau}{\tau}$ represents the sample mean, then according to the "usual central limit theorem", $\sqrt{\tau}(\bar{Z} - E(Z))/\sqrt{\mathrm{Var}(Z)}$ approaches the standard normal distribution as $\tau \to+\infty$. The asymptotic behavior of the maximum value $M_\tau = \max(Z_1, ..., Z_\tau)$ and $m_\tau = \min(X_1, ..., X_\tau)$ might be of interest. We obtain the following equations for the cumulative distribution function (cdf) of $Q(z) = \frac{1}{\alpha}$ as

$$\lim_{t\to 0} \frac{F(t\, z)}{F(t)} = \lim_{t\to 0} \frac{\left[1-(1+\frac{\alpha\, t\, z}{\alpha+1})\mathrm{e}^{-\alpha\, \mathrm{t\, z}}\right]^{\beta}}{\left[1-(1+\frac{\alpha\, t}{\alpha+1})\mathrm{e}^{-\alpha\, \mathrm{t}}\right]^{\beta}} = z^{\beta},$$

and

$$\lim_{t\to+\infty} \frac{1-F(t+x\, Q(t))}{1-F(t)} = \mathrm{e}^{-z}.$$

Thus, according to Leadbetter et al. (2012), there exist norming constants $a_\tau > 0, \quad b_\tau, \quad c_\tau > 0$ and d_τ such that

$$Pr\left[a_\tau(M_\tau - b_\tau) \leq x\right] \to \mathrm{e}^{-\mathrm{e}^{-\mathrm{x}}} | \tau \to +\infty,$$

and

$$Pr\left[c_\tau(m_\tau - d_\tau) \leq x\right] \to 1-\mathrm{e}^{-\mathrm{x}^{\beta}} | \tau \to +\infty,$$

It is possible to determine the form of the "norming constants". According to Leadbetter et al. (2012)'s Corollary 1.6.3, one can see that $b_\tau = F^{-1}(1-\frac{1}{\tau})$ and $a_\tau =$, where $F^{-1}(\cdot)$ represents the inverse function of $F(\cdot)$.

3.4. Moments

Let $\mu'_n = E(Z^n) = \int_0^\infty z^n\, f(z)dz$ denote the nth moment of the WExp-Li distribution. The mean and variance of Z can be easily obtained using the first two moments. The central moments $\mu_\ell = E(Z-\mu)^\ell$ of the WExp-Li(α, β) distribution can be obtained as

$$\mu_\ell = E(Z-\mu)^\ell = \sum_{m=0}^{\ell} \binom{\ell}{m} \mu'_m (-\mu)^{\ell-m}. \tag{20}$$

where $\mu'_\ell = E(Z^\ell)$, $\mu = \mu'_1 = E(Z)$ and ℓ is an integer value.

The skewness and kurtosis measures are defined by

$$Skewness = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3}{(\mu'_2 - \mu_1'^2)^{3/2}},$$

,

$$Kurtosis = \frac{\mu_4}{\mu_2^2} = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu_1'^2\mu'_2 - 3\mu_1'^4}{(\mu'_2 - \mu_1'^2)^2}.$$

In order to obtain the nth moment, we first define

$$\mathcal{A}(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \alpha) = \int_0^{+\infty} z^{\alpha_1}\,(1+z)^{\alpha_2}\,\mathrm{e}^{-\alpha_3\,z}\left[1 - (\frac{\alpha + 1 + \alpha z}{\alpha + 1})\mathrm{e}^{-\alpha z}\right]^{\alpha_4} \mathrm{d}z.$$

By using the generalized binomial expansion, it can be shown that

$$\mathcal{A}(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \alpha) = \sum_{\ell, r=0}^{+\infty}\sum_{k=0}^{\ell}(-1)^\ell \binom{\ell}{k}\binom{\alpha_4}{\ell}\binom{\alpha_2}{r}\alpha_\ell \times \frac{\Gamma(\varsigma^*)}{(\alpha\,\ell + \alpha_3)^{\varsigma^*}},$$

where $\varsigma^* = k + a_1 + r + 1$ and $\alpha_\ell = \left(\frac{\alpha}{\alpha+1}\right)^\ell$. So, the nth moment of the WExp-Li distribution is given by

$$E\left[Z^n\right] = \frac{\alpha^2}{\alpha+1}\sum_{i_1=0}^{+\infty}(i_1 + \beta)\,a_{i_1}\;\mathcal{A}(n, 1, \alpha, i_1 + \beta - 1; \alpha). \tag{21}$$

Figure 2 displays the skewness and kurtosis measures of WExp-Li(α, β) distribution. The following results are concluded: (i) when the parameters α increase, the skewness incrase; (ii) when the parameters β increase, the skewness and kurtosis decrease.

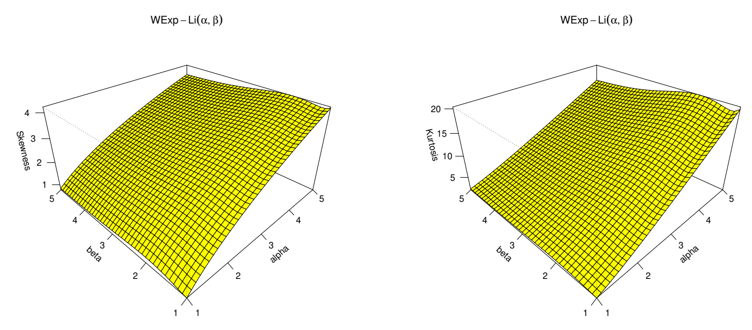

Figure 2. Skewness and Kurtosis of WExp-Li(α, β) for some subset of parameters

Moreover, it is simple to check that the moment generating function for the WExp-Li(α, β) distribution is given by

$$M_Z(t) = E\left[\mathrm{e}^{\mathrm{t}\,\mathrm{Z}}\right] = \frac{\alpha^2}{\alpha+1}\sum_{i_1=0}^{+\infty}(i_1 + \beta)\,b_{i_1}\,\mathcal{A}(0, 1, \alpha - t, \beta + i_1 - 1; \alpha).$$

In order to obtain the nth incomplete moment of the WExp-Li(α, β) distribution let us define

$$B(\alpha_1, \alpha_2, \alpha_3, \alpha_4; y, \alpha) = \int_0^y z^{\alpha_1}\,(1+z)^{\alpha_2}\,\mathrm{e}^{-\alpha_3\,z}\left[1-\left(\frac{1+\alpha z+\alpha}{\alpha+1}\right)\mathrm{e}^{-\alpha z}\right]^{\alpha_4}\mathrm{d}z$$
$$= \sum_{\ell,r=0}^{+\infty}\sum_{k=0}^{\ell}(-1)^{\ell}\binom{\ell}{k}\binom{\alpha_4}{\ell}\binom{\alpha_2}{r}\alpha_\ell \times \frac{\gamma(\varsigma^*, \frac{y}{\alpha\,\ell+\alpha_3})}{(\alpha\,\ell+\alpha_3)^{\varsigma^*}}, \tag{22}$$

where $\gamma(\alpha, z) = \int_0^z t^{\alpha-1}\,\mathrm{e}^{-t}\,dt$ stands for the incomplete gamma function. Note that the second equality of (22) is obtained by generalized binomial expansion. Hence, using (22) the nth incomplete moment of the WExp-Li(α, β) distribution is derived by

$$m_n(y) = E\left[Z^n\,|Z<y\right] = \frac{\int_0^y z^n\,f(y)dy}{F(y)} = \frac{\alpha^2}{(\alpha+1)F(y)}\sum_{i_1=0}^{+\infty}(i_1+\beta)\,b_{i_1}\;B(n, 1, \alpha, i_1+\beta-1; y, \alpha).$$

In what follows, we provide two measures of deviation, i.e. Mean deviations abot the mean (δ_1) and mean δ_2 respectively. By definition of these measures, it is easy to show that

$$\delta_1\,(Z) = 2\mu\,F(\mu) - 2\int_0^{\mu} z\,f(z)\,\mathrm{d}z,$$

and

$$\delta_2\,(Z) = \mu - 2\int_0^{M} z\,f(z)\,\mathrm{d}z,$$

since M is the median of Z. Therefore, it can be verified that measures $\delta_1\,(Z)$ and $\delta_2\,(Z)$ are given by

$$\delta_1\,(Z) = 2\mu\,F(\mu) - \frac{\alpha^2}{\alpha+1}\sum_{j=0}^{+\infty}(j+\beta)b_j\,\mathcal{A}(1, 1, \alpha, j+\beta-1; \alpha),$$

and

$$\delta_2\,(Z) = \mu - \frac{2\,\alpha^2}{\alpha+1}\sum_{j=0}^{+\infty}(j+\beta)b_j\,B(1, 1, \alpha, \beta+j-1; M, \alpha).$$

3.5. XGamma case

Sen et al. (2016), introduced XGamma distribution. The cdf and pdf of XGamma are defined by

$$G(z;\alpha) = 1 - (1 + \frac{\alpha\,z}{1+z} + \frac{0.5\,\alpha^2\,z^2}{1+\alpha})\mathrm{e}^{-\alpha\,z}, \tag{23}$$

and

$$g(z;\alpha) = \frac{\alpha^2}{1+\alpha}(1+0.5\alpha\,z^2)\mathrm{e}^{-\alpha\,z}, \tag{24}$$

where $z>0$ and $\alpha>0$ are the shape parameter. In this subsection we study WExp-XGamma by taking the cdf and pdf of XGamma in equations (3,4 and 5). We show it by WExp-XGamma(α, β). The cdf and pdf of WExp-XGamma(α, β) are provided by

$$F(z;\alpha,\beta) = \frac{\left[1-\mathrm{e}^{-\alpha\,z}(\frac{0.5\alpha^2\,z^2+\alpha\,z+\alpha+1}{\alpha+1})\right]^{\beta}}{1+\left[\mathrm{e}^{-\alpha\,z}(\frac{0.5\alpha^2\,z^2+\alpha\,z+\alpha+1}{\alpha+1})\right]^{\beta}}.$$

The pdf and hrf of WExp-XGamma(α, β) can be easily derived. Figure 3 presents some plots for WExp-Xgamma pdf and WExp-Xgamma hrf for chosen parameter values. Figure 3 shows that the pdf of WExp-XGamma(α, β) is unimodal right skew distribution. The hrf of WExp-XGamma(α, β) can be monotonically decreasing, monotonically increasing and bathtube shape.

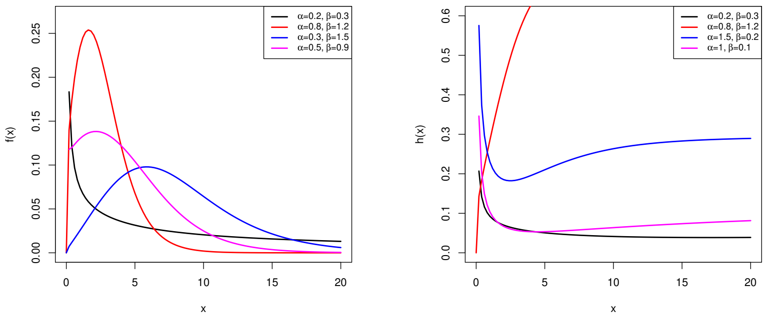

Figure 3. Density and hazard rate of WExp-XGamma(α, β) for some subset of parameters

3.6. Asymptotics

As $z \to 0$, Asymptotic values of pdf, cdf, and hrf of the WExp-XGamma(α, β) distribution are given by

$$\begin{aligned} F(z) &\sim \alpha^\beta\, z^\beta, \\ f(z) &\sim \beta\, \alpha^\beta\, z^{\beta-1}, \\ h(z) &\sim \frac{\beta\, \alpha^\beta\, z^{\beta-1}}{1-\alpha^\beta\, z^\beta}. \end{aligned}$$

The asymptotic of cdf, pdf and hrf of the WExp-XGamma(α, β) distribution as $x \to +\infty$ are, respectively, as follows

$$\begin{aligned} 1-F(z) &\sim \frac{0.5\,\alpha^2\,\beta\, z^2}{\alpha+1} \mathrm{e}^{-\alpha\, z}, \\ f(z) &\sim \frac{0.5\,\beta\alpha^3\, z^2}{\alpha+1} \mathrm{e}^{-\alpha\, z}, \\ h(z) &\sim \alpha. \end{aligned}$$

These equations show how parameters affect the tails of the WExp-XGamma distribution.

3.7. extreme value and the norming constants

If $\bar{Z} = \frac{Z_1+\ldots+Z_\tau}{\tau}$ represents the sample mean, then by the "usual central limit theorem", $\sqrt{\tau}(\bar{Z} - E(Z))/\sqrt{\mathrm{Var}(Z)}$ approaches the standard normal distribution as $\tau \to +\infty$. The asymptotic of the maximum value $M_\tau = \max(Z_1, ..., Z_\tau)$ and $m_\tau = \min(X_1, ..., X_\tau)$ might be of interest. .We obtain the following equations for the cdf in of $Q(z) = \frac{1}{\alpha}$ as

$$\lim_{t\to 0} \frac{F(t\,z)}{F(t)} = \lim_{t\to 0} \frac{\left[1-(1+\frac{\alpha\, t\, z}{\alpha+1}+\frac{0.5\,\alpha^2\, z^2\, t^2}{1+\alpha})\mathrm{e}^{-\alpha\, t\, z}\right]^\beta}{\left[1-(1+\frac{\alpha\, t}{\alpha+1}+\frac{0.5\,\alpha^2\, z^2}{1+\alpha})\mathrm{e}^{-\alpha\, t}\right]^\beta} = z^\beta,$$

and

$$\lim_{t\to+\infty}\frac{1-F(t+x\,Q(t))}{1-F(t)}=\mathrm{e}^{-z}.$$

Thus, from Leadbetter et al. (2012)', there exist norming constants $a_\tau>0,\quad b_\tau,\quad c_\tau>0$ and d_τ such that

$$Pr\left[a_\tau(M_\tau-b_\tau)\le x\right]\to\mathrm{e}^{-\mathrm{e}^{-\mathrm{x}}}|\tau\to+\infty,$$

and

$$Pr\left[c_\tau(m_\tau-d_\tau)\le x\right]\to 1-\mathrm{e}^{-\mathrm{x}^{\beta}}|\tau\to+\infty,$$

It is possible to determine the form of the "norming constants". Due to Leadbetter et al. (2012)'s Corollary 1.6.3, one can see that $b_\tau=F^{-1}(1-\frac{1}{\tau})$ and $a_\tau=,\ F^{-1}(\cdot)$ represents the inverse function of $F(\cdot)$.

Figure 4 displays the skewness and kurtosis measures of WExp-XGamma(α,β) distribution. The following results are concluded: (i) when the parameters α increase, the skewness incrase; (ii) when the parameters β increase, the skewness and kurtosis decrease.

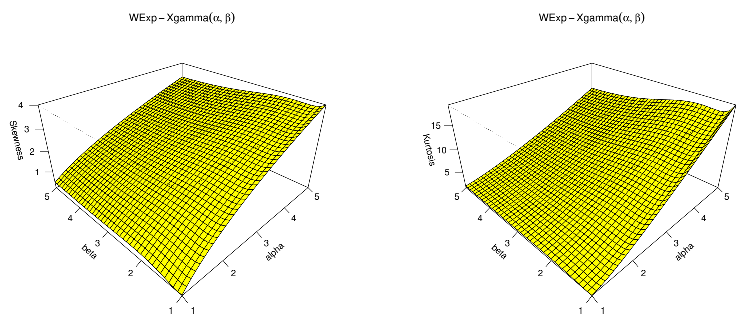

Figure 4. Skewness and Kurtosis of WExp-XGamma(α,β) for some subset of parameters

4. Estimation

In this section we study various methods for estimating the parameters of the WExp-Li($\underline{\mathbf{\Phi}}$). In first subsection we study the maximum-likelihood estimation (MLE). In the second subsection we study the least estimation, weighted least squares estimation and Cramer-von-Mises estimation . In the third subsection, we compare all of these methods by graphical simulation study.

4.1. MLE method

Let $z_1,z_2,\ldots,z_{\mathfrak{m}}$ be a any RS of volume $\mathfrak{n}$ from the WExp-Li($\underline{\mathbf{\Phi}}$) model. The function the log-likelihood ($l_{\mathfrak{m}}\left(\underline{\mathbf{\Phi}}\right)$) for a vector of parameters can be written as

The log-likelihood function , say $l_{\mathfrak{m}}\left(\underline{\mathbf{\Phi}}\right)$, for the parameters vector of the WExp-Li($\underline{\mathbf{\Phi}}$) model can be written as

$$\begin{aligned} l_{\mathfrak{m}}\left(\underline{\mathbf{\Phi}}\right) &= \mathfrak{m}\log\left(\frac{\beta\,\alpha^2}{1+\alpha}\right)-\alpha\sum_{\zeta=1}^{\mathfrak{m}}z_\zeta+(\beta-1)\sum_{\zeta=1}^{\mathfrak{m}}\log(\varkappa_\zeta)+\sum_{\zeta=1}^{\mathfrak{m}}\log(1+z_\zeta) \\ &+ \sum_{\zeta=1}^{\mathfrak{m}}\log\left[1+(1-\varkappa_\zeta)^{\beta-1}\right]-2\sum_{\zeta=1}^{\mathfrak{m}}\log\left[1+(1-\varkappa_\zeta)^{\beta}\right], \end{aligned} \tag{25}$$

where $\varkappa_\zeta = 1 - \mathrm{e}^{-\alpha\, z_\zeta}(1 + \frac{\alpha\, z_\zeta}{1+\alpha})$ is a transformed observation. The common score vector components $U\left(\underline{\mathbf{\Phi}}\right)$ are

$$U_\alpha\left(\underline{\mathbf{\Phi}}\right) = \frac{2\mathfrak{m}}{\alpha} - \frac{\mathfrak{m}}{1+\alpha} - \sum_{\zeta=1}^{\mathfrak{m}} z_\zeta + (\beta-1)\sum_{\zeta=1}^{\mathfrak{m}} \frac{\varkappa_{\mathfrak{m}}^{(\alpha)}}{\varkappa_\zeta} - (\beta-1)\sum_{\zeta=1}^{\mathfrak{m}} \frac{\varkappa_\zeta^{(\alpha)}\,(1-\varkappa_\zeta)^{\beta-2}}{1+(1-\varkappa_\zeta)^{\beta-1}} + 2\beta\sum_{\zeta=1}^{\mathfrak{m}} \frac{\varkappa_\zeta^{(\alpha)}\,(1-\varkappa_\zeta)^{\beta-1}}{1+(1-\varkappa_\zeta)^{\beta}},$$

and

$$U_\beta\left(\underline{\mathbf{\Phi}}\right) = \frac{\mathfrak{m}}{\beta} + \sum_{\zeta=1}^{\mathfrak{m}} \log(\varkappa_\zeta) + \sum_{\zeta=1}^{\mathfrak{m}} \frac{(1-\varkappa_\zeta)^{\beta-1}\log(1-\varkappa_\zeta)}{1+(1-\varkappa_\zeta)^{\beta-1}} - 2\sum_{\zeta=1}^{\mathfrak{m}} \frac{(1-\varkappa_\zeta)^{\beta}\log(1-\varkappa_\zeta)}{1+(1-\varkappa_\zeta)^{\beta}},$$

where

$$t_\zeta^{(\alpha)} = \frac{dt_\zeta}{d\alpha} = \frac{\alpha\, z_\zeta}{(1+\alpha)^2}(\alpha + 2 + \mathrm{z}_\zeta)\mathrm{e}^{-\alpha\, z_\zeta}.$$

4.2. Other Estimators

4.2.1. Least square and Weighted least square estimators Least Square estimators (LSEs) and Weighted Least Sqaure estimators (WLSEs) have been suggested by Swain et al. (1988). These estimators are obtained by minimizing the following equations

$$LS\left(t_{\zeta:\mathfrak{m}};\underline{\mathbf{\Phi}}\right) = \sum_{\zeta=1}^{\mathfrak{m}} \left(F_{WExp-Li}\left(t_{\zeta:\mathfrak{m}};\left(\underline{\mathbf{\Phi}}\right)\right) - \left(1+\mathfrak{m}\right)^{-1}\zeta\right)^2 |_{\zeta=1,2,\ldots,\mathfrak{m}},$$

and

$$WLS\left(t_{\zeta:\mathfrak{m}};\underline{\mathbf{\Phi}}\right) = \sum_{\zeta=1}^{\mathfrak{m}} \xi_\zeta\left(\mathfrak{m}\right)\left(F_{WExp-Li}\left(t_{\zeta:\mathfrak{m}};\underline{\mathbf{\Phi}}\right) - \left(1+\mathfrak{m}\right)^{-1}\zeta\right)^2,$$

where

$$\xi_\zeta\left(m\right) = \frac{(\mathfrak{m}+2)}{\zeta(\mathfrak{m}-\zeta+1)}(\mathfrak{m}+1)^2.$$

4.2.2. CM estimators Cramér-von-Mises estimators (CM) has been suggested by Choi and Bulgren (1968). This estimator is obtained by minimizing the following equation

$$S_{\mathrm{CM}}\left(t_{\zeta:\mathfrak{m}};\underline{\mathbf{\Phi}}\right) = \frac{1}{12\mathfrak{m}} + \sum_{\zeta=1}^{\mathfrak{m}} \left(F_{WExp-Li}\left(t_{\zeta:M};\underline{\mathbf{\Phi}}\right) - \frac{1}{2\mathfrak{m}}\left(2\,\zeta-1\right)\right)^2.$$

4.3. Simulation study

For $\left(\underline{\mathbf{\Phi}}\right) = (0.3, 0.5), (1, 3), (0.5, 1.5), (0.8, 1.2), (2, 2)$, we performed simulation analysis according to the following algorithm

- Generate N=10^4 samples of the size ζ from (3) for Lindley case by solving equation $F_{WExp-Li}(x) = u$ where $u \sim U(0,1)$.
- Compute the estimates for the N=10^4 samples, say $\left(\hat{\alpha}_\zeta, \hat{\beta}_\zeta\right)$ for $\zeta = 1, 2, ..., 10^4$.
- Compute the "biases" and "mean squared errors".

The $bias_\varepsilon(\zeta)$ as well as the $MSE_\varepsilon(\zeta)$ for $\zeta = 30, 80, \cdots 480$ are computed using the "R-optim function" and the Nelder-Mead method. Tables 1-10 display the results. From tables 1-10, the biases for estimating α are mostly negative and the biases for estimating β are mostly positive. The bias approach zero and the MSEs decreases as n increases. It shows that the consistency of these methods. Also MLE is better than other methods.

Table 1. Estimated Biases for $(\alpha, \beta) = (0.3, 0.5)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	-0.03825	-0.03694	-0.03802	-0.03234	0.03052	0.03574	0.06630	0.06342
80	-0.02881	-0.02821	-0.02873	-0.02572	0.02055	0.02600	0.03286	0.03624
130	-0.02216	-0.02141	-0.02212	-0.01899	0.01504	0.01923	0.02240	0.02546
180	-0.01922	-0.01870	-0.01919	-0.01666	0.01410	0.01679	0.01940	0.02101
230	-0.01701	-0.01665	-0.01698	-0.01517	0.00752	0.01057	0.01160	0.01622
280	-0.01516	-0.01457	-0.01514	-0.01268	0.00338	0.00657	0.00668	0.01138
330	-0.01491	-0.01440	-0.01490	-0.01270	0.00478	0.00744	0.00759	0.01197
380	-0.01301	-0.01267	-0.01300	-0.01125	0.00215	0.00491	0.00457	0.00921
430	-0.01218	-0.01176	-0.01216	-0.01027	0.00099	0.00332	0.00313	0.00759
480	-0.01202	-0.01162	-0.01201	-0.01024	0.00282	0.00443	0.00475	0.00833

Table 2. Estimated MSE for $(\alpha, \beta) = (0.3, 0.5)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0.00236	0.00219	0.00234	0.00194	0.01745	0.01599	0.02466	0.01935
80	0.00124	0.00117	0.00124	0.00105	0.00606	0.00547	0.00708	0.00546
130	0.00075	0.00070	0.00075	0.00060	0.00350	0.00319	0.00390	0.00302
180	0.00058	0.00053	0.00058	0.00046	0.00245	0.00222	0.00269	0.00217
230	0.00045	0.00042	0.00045	0.00037	0.00138	0.00124	0.00148	0.00135
280	0.00040	0.00036	0.00040	0.00029	0.00132	0.00111	0.00138	0.00106
330	0.00034	0.00031	0.00034	0.00026	0.00097	0.00082	0.00102	0.00081
380	0.00027	0.00025	0.00027	0.00021	0.00084	0.00066	0.00087	0.00065
430	0.00024	0.00022	0.00024	0.00018	0.00072	0.00059	0.00074	0.00056
480	0.00023	0.00021	0.00023	0.00018	0.00057	0.00046	0.00059	0.00048

Table 3. Estimated Biases for $(\alpha, \beta) = (1, 3)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	-0.03787	-0.02961	0.01796	0.02413	0.02413	0.47184	1.07728	0.83389
80	-0.03048	-0.02122	-0.00735	-0.00264	-0.00264	0.14310	0.28723	0.27239
130	-0.03348	-0.02783	-0.01901	-0.01650	-0.01650	0.05317	0.13820	0.13549
180	-0.02584	-0.01996	-0.01513	-0.01395	-0.01395	0.06909	0.11696	0.10886
230	-0.02835	-0.02280	-0.01985	-0.01520	-0.01520	0.02249	0.05418	0.07851
280	-0.03054	-0.02393	-0.02352	-0.01823	-0.01823	0.00276	0.01599	0.03563
330	-0.02525	-0.01975	-0.01920	-0.01439	-0.01439	0.01138	0.02134	0.04894
380	-0.02694	-0.02320	-0.02165	-0.01933	-0.01933	-0.01096	0.00735	0.01346
430	-0.01840	-0.01477	-0.01369	-0.01149	-0.01149	0.04153	0.05670	0.06010
480	-0.02100	-0.01688	-0.01673	-0.01208	-0.01208	0.00527	0.01145	0.03803

Table 4. Estimated MSE for $(\alpha, \beta) = (1, 3)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0.04030	0.03974	0.04452	0.02954	8.10141	6.75801	13.22829	4.41817
80	0.01393	0.01136	0.01357	0.00845	1.0353	5.86673	1.21895	0.71547
130	0.01091	0.00887	0.01036	0.00642	0.69058	0.56198	0.74943	0.44217
180	0.00716	0.00531	0.00681	0.00433	0.43844	0.33435	0.46817	0.28974
230	0.00571	0.00447	0.00535	0.00329	0.35318	0.28111	0.36580	0.22007
280	0.00592	0.00418	0.00557	0.00287	0.34061	0.23861	0.34684	0.15619
330	0.00416	0.00305	0.00390	0.00248	0.22792	0.17033	0.23151	0.14608
380	0.00407	0.00304	0.00382	0.00240	0.21801	0.15972	0.21986	0.12563
430	0.00320	0.00231	0.00305	0.00188	0.18603	0.13831	0.19100	0.11470
480	0.00281	0.00191	0.00265	0.00144	0.15495	0.10665	0.15604	0.08450

Table 5. Estimated Biases for $(\alpha, \beta) = (0.5, 1.5)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	-0.01354	-0.01223	0.00572	0.00073	0.33100	0.32871	0.64487	0.46906
80	-0.01503	-0.01473	-0.00843	-0.01181	0.12046	0.12116	0.21267	0.14059
130	-0.01501	-0.01465	-0.01148	-0.01312	0.05420	0.05865	0.10293	0.07061
180	-0.01269	-0.01263	-0.01014	-0.01149	0.04234	0.03973	0.07715	0.04591
230	-0.01122	-0.01129	-0.00927	-0.01095	0.04200	0.03852	0.06860	0.03771
280	-0.01183	-0.01176	-0.01035	-0.01111	0.01948	0.01902	0.03971	0.02431
330	-0.00996	-0.00966	-0.00869	-0.00886	0.02800	0.03062	0.04532	0.03649
380	-0.01199	-0.01122	-0.01095	-0.01040	0.00478	0.01107	0.01911	0.01514
430	-0.01089	-0.01036	-0.00997	-0.00974	0.00803	0.01252	0.02069	0.01452
480	-0.00955	-0.00918	-0.00872	-0.00858	0.01302	0.01629	0.02443	0.01868

Table 6. Estimated MSE for $(\alpha, \beta) = (0.5, 1.5)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0.00550	0.00540	0.00725	0.00514	1.67216	1.46195	2.86449	1.55369
80	0.00167	0.00148	0.00183	0.00118	0.28863	0.24197	0.38889	0.18328
130	0.00091	0.00079	0.00091	0.00065	0.13068	0.10404	0.15808	0.07714
180	0.00065	0.00052	0.00065	0.00045	0.08593	0.06030	0.10169	0.04664
230	0.00046	0.00039	0.00046	0.00034	0.06357	0.04617	0.07356	0.03377
280	0.00040	0.00034	0.00038	0.00032	0.04076	0.02989	0.04472	0.02613
330	0.00029	0.00025	0.00027	0.00022	0.03365	0.02665	0.03688	0.02350
380	0.00032	0.00027	0.00030	0.00025	0.02681	0.02135	0.02848	0.01887
430	0.00026	0.00023	0.00025	0.00021	0.02371	0.01826	0.02505	0.01487
480	0.00022	0.00019	0.00021	0.00018	0.02060	0.01580	0.02178	0.01393

Table 7. Estimated Biases for $(\alpha, \beta) = (0.8, 1.2)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	-0.02443	-0.02075	-0.00463	-0.00347	0 .17153	0.18543	0.35671	0.29351
80	-0.02971	-0.02756	-0.02382	-0.02295	0.07347	0.07939	0.12614	0.08877
130	-0.02696	-0.02610	-0.02379	-0.02286	0.03569	0.04006	0.06419	0.05484
180	-0.02388	-0.02257	-0.02168	-0.02024	0.03192	0.03807	0.05187	0.04084
230	-0.02216	-0.02145	-0.02046	-0.01985	0.03178	0.03244	0.04727	0.03576
280	-0.02005	-0.01887	-0.01870	-0.01735	0.02178	0.02712	0.03412	0.03107
330	-0.01841	-0.01791	-0.01727	-0.01679	0.01314	0.01476	0.02352	0.01758
380	-0.01717	-0.01656	-0.01618	-0.01537	0.01455	0.01706	0.02355	0.01747
430	-0.01749	-0.01693	-0.01662	-0.01626	0.01452	0.01630	0.02244	0.01603
480	-0.01771	-0.01714	-0.01693	-0.01589	0.01660	0.01924	0.02370	0.02058

Table 8. Estimated MSE for $(\alpha, \beta) = (0.8, 1.2)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0.00949	0.00870	0.01205	0.00941	0.66367	0.58224	1.10613	.70575
80	0.00292	0.00256	0.00289	0.00221	0.11336	0.09493	0.14831	0.07132
130	0.00171	0.00155	0.00160	0.00139	0.04116	0.03314	0.04824	0.03233
180	0.00116	0.00103	0.00108	0.00093	0.02565	0.02077	0.02912	0.01741
230	0.00094	0.00086	0.00088	0.00081	0.02089	0.01802	0.02320	0.01752
280	0.00079	0.00071	0.00074	0.00066	0.01593	0.01334	0.01726	0.01177
330	0.00066	0.00059	0.00062	0.00055	0.01287	0.01052	0.01371	0.00970
380	0.00055	0.00049	0.00052	0.00045	0.01196	0.00942	0.01267	0.00859
430	0.00057	0.00052	0.00055	0.00049	0.01020	0.00832	0.01077	0.00778
480	0.00055	0.00051	0.00053	0.00047	0.00781	0.00649	0.00828	0.00609

5. Applications

In this Section, we introduce and analyse three real data applications by comparing the fits of the proposed WTL-Li model with some well-known models. For all examples, the criteria of the Cramér–von Mises (W^*), the criteria of

Table 9. Estimated Biases for $(\alpha, \beta) = (2, 2)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0.09551	0.09937	0.20618	0.16255	0.40113	0.37844	0.83434	0.50914
80	0.00564	0.00519	0.04796	0.02564	0.11656	0.11120	0.25249	0.16489
130	-0.00964	-0.00482	0.01627	0.01277	0.05117	0.06735	0.13216	0.12067
180	-0.00643	-0.00715	0.01230	-0.00548	0.08021	0.07464	0.13880	0.07086
230	-0.00980	-0.00981	0.00528	-0.01154	0.07904	0.07622	0.12588	0.06516
280	-0.01254	-0.01345	-0.00018	-0.01225	0.07214	0.07066	0.11036	0.07262
330	-0.01469	-0.01243	-0.00425	-0.00933	0.05028	0.05586	0.08240	0.06104
380	-0.01915	-0.01850	-0.01004	-0.01613	0.05057	0.05233	0.07855	0.05661
430	-0.01389	-0.01283	-0.00578	-0.01116	0.05283	0.05531	0.07763	0.05651
480	-0.02141	-0.02210	-0.01428	-0.02115	0.03538	0.03228	0.05723	0.03162

Table 10. Estimated MSE for $(\alpha, \beta) = (2, 2)$

n	$LSE(\hat{\alpha})$	$WLSE(\hat{\alpha})$	$CME(\hat{\alpha})$	$MLE(\hat{\alpha})$	$LSE(\hat{\beta})$	$WLSE((\hat{\beta})$	$CME(\hat{\beta})$	$MLE(\hat{\beta})$
30	0..20436	0.18217	0.28501	0.16880	6.24131	3.96150	11.09270	1.62092
80	0.05517	0.04874	0.06529	0.04145	0.51683	0.42480	0.65623	0.33335
130	0.02982	0.02365	0.03313	0.02053	0.28755	0.22016	0.33737	0.19233
180	0.03046	0.02406	0.03286	0.01744	0.27798	0.20488	0.31416	0.13417
230	0.02097	0.01648	0.02234	0.01279	0.20122	0.14986	0.22549	0.09920
280	0.01889	0.01435	0.01980	0.01073	0.17214	0.12394	0.18973	0.08689
330	0.01235	0.00995	0.01285	0.00834	0.12609	0.09601	0.13774	0.07111
380	0.01077	0.00828	0.01103	0.00703	0.09878	0.07160	0.10759	0.05534
430	0.01025	0.00789	0.01056	0.00662	0.09278	0.06897	0.10036	0.05487
480	0.00999	0.00755	0.01012	0.00603	0.08356	0.05828	0.08927	0.04208

the Anderson-Darling (A^*) and p-value for Kolmogorov-Smirnow-test are considered. The MLE method was used to estimate the model parameters.

The standard Weibull distribution (W), the standard Lindley (Li) distribution (Ghitany et al., 2008), Generalized Lindley (GL) (Nadarajah et al., 2011), the standard Power Lindley (PL) distribution (Ghitany et al., 2013), the Topp-Leone Lindley distribution (TL-Li) (Al-Shomarni et al., 2016),Odd log-logistic Lindley (OLL-Li) model (Ozel et al, 2017) and Generalized Exponential model (GE) (Gupta and Kundu, 1999), the new odd log-logistic (NOLLHL) model (Alizadeh et al., 2019), the modified half-logistic (MHL) and the extended odd weibull Lindley (ExOW-Li)(Alizadeh et al., 2018) model (Mohammad, 2021) are choosed for comparison in two real data sets. The survival function of the mentioned models are given in Appendix.

The data provided by Murthy (2004, p100), which show the lifetimes of twenty electronic components, will be approved as the first data. The data are: 0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09 .

The data provided by Murthy (2004, p195), which show the failure time of fifty items, will be approved as the second data.

The third data represent the accelerated life testing of (n = 40) items with change in stress from 100 to 150 at t = 15. These data has been studied by Cordeiro et al. (2016).

An overview of the estimated MLE's and fitted information criteria for both data sets using various models can be seen in Tables 11 and 12. The WExp-Li distribution provide better fit than other competitive models with additional criteria. The histogram of lifetime data set, as well as the fitted pdf plots, are shown in Figures 6 and 7 . All compuattion in this section has been done by the R software.

Figure 5. **Fitted densities for lifetime data set.**

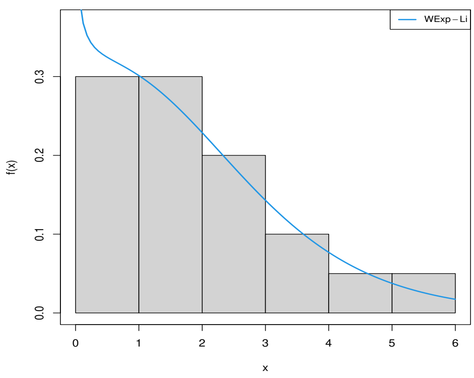

Figure 6. **Fitted densities for failure time data set.**

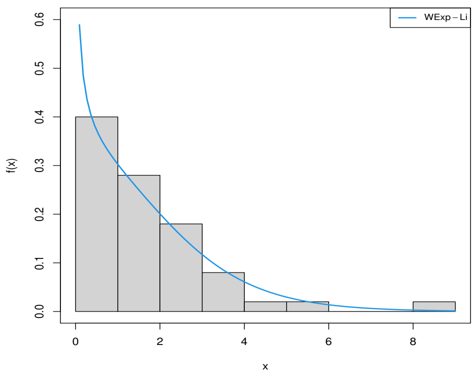

Figure 7. **Fitted densities for stress data set.**

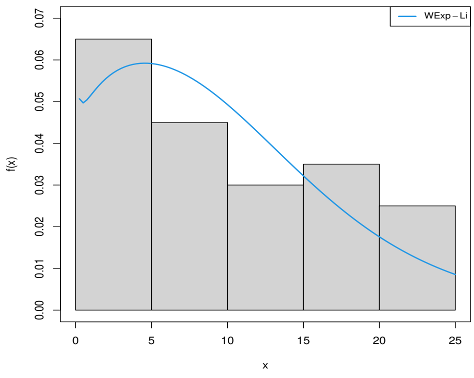

Table 11. Estimated parameters with standard errors in parenthesis

model	lifetime data			failure time data			stress data		
WExp-Li (α, β)	1.056	0.844		1.250	0.683		0.230	0.795	
	(0.173)	(0.215)		(0.157)	(0.108)		(0.026)	(0.139)	
TLLi(α, β)	0.469	0.910		0.471	0.730		0.099	0.808	
	(0.100)	(0.271)		(0.068)	(0.134)		(0.015)	(0.170)	
PL(α, β)	0.761	1.063		0.994	0.882		0.183	0.984	
	(0.174)	(0.187)		(0.127)	(0.096)		(0.053)	(0.110)	
Li(α)	0.803			0.910			0.177		
	(0.133)			0.096			(0.019)		
GL(α, β)	0.785	0.955		0.790	0.767		0.164	0.869	
	(0.179)	(0.287)		(0.122)	(0.142)		(0.026)	(0.187)	
OLL-Li(α, β)	0.810	0.978		0.979	0.835		0.183	0.908	
	(0.155)	(0.207)		(0.132)	(0.112)		(0.024)	(0.133)	
W(α, β)	0.426	1.196		0.610	0.976		0.038	1.349	
	(0.135)	(0.224)		(0.105)	(0.111)		(0.019)	(0.179)	
GE(α, β)	0.559	1.319		0.560	0.903		0.115	1.360	
	(0.154)	(0.332)		(0.105)	(0.162)		(0.021)	(0.288)	
MHL(α, β)	1.031	0.633		0.786	0.655		1.947	0.116	
	(0.360)	(0.177)		(0.170)	(0.122)		(0.748)	(0.016)	
NOLLHL(α, β)	0.939	0.697		0.796	0.743		1.627	0.122	
	(0.296)	(0.712)		(0.147)	(0.126)		(0.541)	(0.017)	
ExOW-Li(α, β, c)	7.472	2.445	36.598	4.468	1.387	11.277	1.237	2.430	40.534
	(0.002)	(1.119)	(18.560)	(0.002)	(0.346)	(2.227)	(0.002)	(0.863)	(15.486)

Table 12. Goodness-of-fit statistics for lifetime data and failure time data

	lifetime data			failure time data			stress data		
model	W^*	A^*	$p-value$	W^*	A^*	$p-value$	W^*	A^*	$p-value$
WExp-Li	**0.043**	**0.267**	**0.947**	**0.027**	**0.183**	**0.973**	**0.065**	**0.496**	**0.837**
TLLi	0.057	0.342	0.853	0.038	0.245	0.930	0.081	0.591	0.773
PL	0.059	0.354	0.939	0.063	0.375	0.818	0.093	0.666	0.739
Li	0.064	0.381	0.848	0.049	0.300	0.865	0.092	0.659	0.731
GL	0.063	0.378	0.809	0.046	0.285	0.887	0.090	0.647	0.693
OLL-Li	0.064	0.381	0.811	0.049	0.306	0.855	0.091	0.654	0.619
W	0.071	0.418	0.863	0.079	0.468	0.723	0.087	0.630	0.746
GE	0.084	0.493	0.648	0.076	0.448	0.656	0.118	0.821	0.455
MHL	0.069	0.411	0.441	0.045	0.275	0.716	0.231	1.498	$4.55e-06$
NOLLHL	0.067	0.396	0.338	0.041	0.256	0.715	0.184	1.248	$4.83e-06$
ExOW-Li	0.070	0.424	0.415	0.0706	0.468	0.424	0.094	0.680	0.151

6. Conclusions

In above work, we presented a new weighted family of Exponentiated distributions named the weighted Exp-G (WExp-G) class of distributions. Some essential properties of this class, such as the quantile function, the asymptotic, residual entropy index and the order statistics are obtained. Then, we focus on Lindley and XGamma special cases. We used many estimation methods for estimate the unknown parameters. The estimated Bias and MSE of parameters for all methods of estimation, converge to zero by increasing the sample size, it show that all methods are useful for large samples. Then the flexibility of the proposed model are shown by comparing the fitted densites by other competitive models.

Appendix

Survival function of competitive models in application section.

$$
\begin{aligned}
&\mathcal{H}_\alpha(t) = 1-\exp(-\alpha\, t)(1+\alpha+\alpha\, t)(1+\alpha)^{-1}\\
&S_{Li(\alpha)}(t) = 1-\mathcal{H}_\alpha(t)|_{t>0,\alpha>0},\\
&S_{TL-Li(\alpha,\beta)}(t) = 1-\left[1-(1-\mathcal{H}_\alpha(t))^2\right]^\beta \;|_{t>0,\alpha,\beta>0},\\
&S_{OLL-Li(\alpha,\beta)}(t) = (1-\mathcal{H}_\alpha(t))^\beta \left[\mathcal{H}_\alpha(t)^\beta+(1-\mathcal{H}_\alpha(t))^\beta\right]^{-1} \;|_{t>0,\alpha,\beta>0},\\
&S_{PL(\alpha,\beta)}(t) = \exp(-\alpha\, t^\beta)\left(\frac{1+\alpha+\alpha\, t^\beta}{1+\alpha}\right) |_{t>0,\alpha,\beta>0},\\
&S_{GL(\alpha,\beta)}(t) = 1-\mathcal{H}_\alpha(t)^\beta \;|_{t>0,\alpha,\beta>0},\\
&S_{GE(\alpha,\beta)}(t) = 1-(1-\exp(-\alpha\, t))^\beta \;|_{t>0,\alpha,\beta>0},\\
&S_{W(\alpha,\beta)}(t) = \exp(-\alpha\, t^\beta) \;|_{t>0,\alpha,\beta>0}.\\
&S_{MHL(\alpha,\beta)}(t) = \frac{1+\exp(-\beta\, t)-(1-\exp(-t))^\alpha}{1+\exp(-\beta\, t)} \;|_{t>0,\alpha,\beta>0},\\
&S_{NOLLHL(\alpha,\beta)}(t) = \frac{(2\exp(-t))^\beta}{(2\exp(-t))^\beta+(1-\exp(-t))^\alpha} \;|_{t>0,\alpha,\beta>0}.\\
&S_{ExOW-Li(\alpha,\beta,c)}(t) = \left\{1+c\left[\frac{\mathcal{H}_\alpha(t)}{1-\mathcal{H}_\alpha(t)}\right]^\beta\right\}^{\frac{-1}{c}} \;|_{t>0,\alpha,\beta>0,c>0}.
\end{aligned}
\tag{26}
$$

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