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Extreme Value Stable Mixture Modelling with Applications to South African Atock Market Indices and Exchange Rate

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Abstract In recent times, there is a vested interest in the research and applications of extreme value mixture models in the stock market and insurance as well as medical industries. This study aims to evaluate the fit of two extreme value mixture models namely Stable-Normal-Stable (SNS) and Stable-KDE-Stable (SKS), where KDE represents the Kernel density estimator, for three FTSE/JSE indices namely All Share Index (ALSI), Banks Index, Mining Index and the USD/ZAR currency exchange rate. These novel models aim to capture the characteristics of South African financial data as compared to the existing commonly fitted extreme value mixture models. The results highlight the robustness of the SNS and SKS mixed model for each daily returns when conducting a graphical bulk model and comprehensive tail model analysis. Financial practitioners looking to curb losses and study alternatives for financial modeling in the South African financial industry using an extreme value mixed model approach may find the SNS and SKS model application beneficial.

Keywords Extreme Value Theory, Kernel Density Estimator Mixture models, Stable Distributions

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1. Introduction

In recent times, a plethora of extreme value mixture models were explored. Extreme value models are frequently used to describe the distribution of events that are rare in nature. Describing the likelihood of unusual behaviour has brought about the need for dependable and statistically justifiable models to be developed. The scarcity of extremal data should be noted therefore suggesting that making inferences is complicated. Applications of extreme value theory are seen extensively in number of industries such as hydrology, medicine as well as the insurance and finance industry.

It is well known that in the last decade South Africa has experienced bleak economic growth, high levels of crime, corruption and unemployment. These socio-economic issues have worsened with the unprecedented COVID-19 global pandemic and the Russia-Ukraine war. The requirement for reliable models that monitor the movement of volatile indices and exchange rates during globally disruptive events is of major importance in curtailing risk and implementing the relevant macroeconomic structural changes and governance essential for financial stability.

Literature references that financial data displays heavy tails and skewness and a possible solution of dealing with this problem is to recommend a volatility model that adequately describes the features of financial data. Noteworthy work by [17] proposed an extreme value mixture model that fits the GPD model at the upper and lower tail and the Normal distribution is fitted as the bulk model between the two tails. The mixture model created by [17] is an extension of literature by [10] where a two-stage model is developed where at the initial stage a Generalised Autoregressive Conditional Heteroskedasticity (GARCH) model is fitted to capture volatility clustering and the latter stage fits a Generalised Pareto Distribution (GPD) to the tails.

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This main contribution of this paper is to propose new extreme value mixture models namely, the Stable-Normal-Stable (SNS) and Stable-KDE-Stable (SKS), where KDE represents the Kernel density estimator, models to three FTSE/JSE stock market indices and the United States of American Dollar to the South African Rand. Distribution density plots of the fitted SNS and SKS models validate the adequacy of the models on each of the return series investigated in this study.

This study highlights the usefulness of extreme value mixture models in context and application of a South African financial perspective and recommends the SNS and SKS models as alternate distributions to consider when fitting large sets of economic data.

2. Literature Review

An extreme value mixture model where the bulk distribution is a parametric distribution and GPD is modeled at the tails was studied by [2]. The two distributions are spliced at some point known as the threshold that is treated as a parameter to be estimated. This paper used the truncated gamma distribution as the bulk model.

[15] adopted a semi-parametric Bayesian approach that includes for the bulk distribution, a mixture of weighted gamma densities and the GPD model for the tail. BIC or DIC statistics is used to determine the number of gamma components in the bulk model.

Like [15] and extending on the work of [2], [8] proposed a mixture model comprising a mixture of exponential distribution components below a threshold and the GPD is fitted for the threshold excess. The EM algorithm is utlised for parameter estimation where a mixture of two exponential distributions is used to model the Danish fire and a medical claim data sets.

[11] evaluated a data driven approach and proposed a two-tailed mixture model. The GPD model is suggested to fit the tails and a normal distribution is fitted as the bulk density. The estimation process involves the ordering and standardisation of data. [6] summarizes the 6-step estimation procedure.

A model similar to the work of [11] is investigated by [17], where Bayesian inference is used as the method of estimation. The Normal distribution is recommended as the bulk distribution and GPD for the lower and upper tails. A novel finding with this approach is the absence of influence of the bulk distribution on tail model fitting. A possible drawback pointed out by [6] deals with the misfit of the model, more especially the bulk model.

A flexible extreme value mixture modelling framework was defined by [9] where the standard kernel density estimator is the bulk model and the GPD is the tail models. The performance of the proposed two-tailed mixture extreme value model was evaluated by empirical analysis and simulation study for determining normal physiological measurements for pre-mature infants.

There is limited research on the topic of modelling Stable distributions to the tail component of extreme value mixture models to the best of our literature knowledge. The central theme of this study is to investigate the suitability of the SNS and SKS mixture model for the JSE financial data and USD/ZAR currency exchange rate that exhibit heavy tails and asymmetry.

3. Research and Methodology

The likelihood of extreme variation from the median probability distributions that are applied to understand uncommon events or anomalous behaviours is what is described by [6] as extreme value theory. [14] notes that the extrapolation of information from the extremal portions of distributions as opposed to the centers of distributions is the key idea behind extreme value theory.

Typically, parametric, semi-parametric and non-parametric methods are three approaches applied in extreme value theory. Extreme value mixture models usually have two components. The first component is a model that defines non-extremal data below a threshold known as the bulk model and the second component, known as the tail model, refers to the data above a certain threshold. In most cases, the bulk model is a parametric or non-parametric distribution, and the tail model is an adaptable threshold model like the GPD. This paper explores the possibility of fitting Nolan's S_0-paramterisation stable model.

3.1. Univariate stable distributions

Stable distributions are a four-parameter family of models based on stable laws. The literature of stable distributions was derived from the pioneering work of Paul Lévy in the 1920s.

[12] outlines stable laws as:

(i) The sum of two Normally distributed random variables yields a Normal random variable. If Y is Normal, then Y_1 and Y_2 are independent and identical to Y with any positive constants a and b.

$$aY_1 + bY_2 \stackrel{\mathrm{d}}{=} cY + d \tag{1}$$

for $c \geq 0$ and $d \in \mathbb{R}$ where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution.

(ii) Any random variable is *symmetrically stable* if it is stable and symmetrically distributed around 0, that is, $Y \stackrel{\mathrm{d}}{=} -Y$.

(iii) A random variable is *strictly stable* if $d = 0$.

There are three special cases where stable distributions that can be expressed as closed-form densities. The family of stable distributions is a rich class of models and includes the Normal, Cauchy, and Lévy distributions as subclasses described by their density functions. The Normal distribution is stable when $\alpha = 2$ and skewness $\beta = 0$, when $\alpha = 1$ and $\beta = 0$ then Cauchy Laws are described and Lévy distributions are stable with $\alpha = \dfrac{1}{2}$ and $\beta = 1$.

The next subsection discusses the characterization and parameterization of Stable distributions where stable parameters will be discussed in more detail.

3.1.1 Characterization and parameterization of Stable distributions

Stable distributions are defined by four parameters, namely α, β, γ and δ. The index of stability is explained by the parameter α where $0 < \alpha < 2$. Skewness is denoted by the parameter β where $-1 < \beta < 1$. The distribution is symmetric if $\beta = 0$. If $\beta > 0$, the distribution is skewed to the right and if $\beta < 0$, then the distribution is skewed to the left. The scale parameter is denoted by $\gamma > 0$. The parameter δ denotes the rightward or leftward shift of the distribution. It is called the location parameter. The distribution has a leftward shift if $\delta < 0$. On the other hand, the distribution has a rightward shift if $\delta > 0$. Multiple parameterizations are used to describe stable laws. This is due to historical evolution and the many challenges observed when investigating stable distributions. The notation $S(\alpha, \beta, \gamma, \delta, k)$ is used to label the class of stable laws. The four parameters α, β, γ and δ need to be estimated and the integer k distinguishes between the different parameterizations [13].

[13] describes the S_0-parameterization as: A random variable Y is $S(\alpha, \beta, \gamma, \delta; 0)$ if

$$Y \stackrel{\mathrm{d}}{=} \begin{cases} \gamma\left(Z - \beta\tan\frac{\pi\alpha}{2}\right) + \delta, & \alpha \neq 1; \\ \gamma Z + \delta, & \alpha = 1. \end{cases} \tag{2}$$

where $Z \equiv Z(\alpha, \beta)$. In this case Y has characteristic function:

$$E\left(e^{itY}\right) = \begin{cases} \exp(-\gamma^{\alpha}|t|^{\alpha}\left[1 + i\beta\left(\tan\frac{\pi\alpha}{2}\right)(\mathrm{sign}(t)) \times (|\gamma t|^{1-\alpha} - 1)\right] + i\delta t), & \alpha \neq 1; \\ \exp(-\gamma|t|\left[1 + i\beta\frac{2}{\pi}(\mathrm{sign}(t)) \times \log(\gamma|t|)\right] + i\delta t), & \alpha = 1. \end{cases} \tag{3}$$

[13] mentions using the S_0-parameterization for statistical inferences, and numerical functions, as it has the simplest form for the characteristic function that is continuous in all four parameters. The S_0-parameterization acknowledges a location-scale family. If $Z \sim S(\alpha, \beta, \gamma, \delta; 0)$, then for $\alpha \neq 0$, $b \in \mathbb{R}$, $aZ + b \sim S(\alpha, \mathrm{sign}(\alpha)\beta, |a|\gamma, a\delta + b; 0)$.

Stable parameter estimation

[13] positions that many standard parameter estimation procedures fail to work for data due to the scarcity of closed-form densities for stable distributions. The are several methods of stable parameter estimation and one cannot consider any specific method as superior or most effective, however, the maximum likelihood method is the most often used for stable parameter estimation.

In this study, stable parameters are estimated using maximum likelihood method for the lower and upper tails and thereafter the Anderson-Darling goodness-of-fit test is applied as part of a tail model analysis to test the suitability of the fitted univariate stable models at the tail component.

4. Stable-Normal-Stable mixture model

This study investigates the fit of the Stable-Normal-Stable (SNS) model using Nolan's S_0 - parmeterisation on the South African currency and stocks. The SNS mixture model is a two-tailed model where the Normal distribution is fitted as the bulk model and the stable model is fitted to the tails. The SNS model is described as:

$$F\left(x|\theta\right)=\begin{cases}\phi\left(u_l|\mu,\sigma\right)\left[1-G\left(-x|\alpha_l,\beta_l,\gamma_l,\delta l\right)\right] & x\leq u_l;\\ \phi\left(u_l|\mu,\sigma\right), & u_l<x<u_r;\\ \phi\left(u_r|\mu,\sigma\right)+\left[1-\phi\left(u_r|\mu,\sigma\right)G\left(-x|\alpha_r,\beta_r,\gamma_r,\delta r\right)\right] & x\geq u_r.\end{cases} \tag{4}$$

where, $\theta=\left(u_l,\alpha_l,\beta_l,\gamma_l,\delta_l,\mu,\sigma,u_r,\alpha_r,\beta_r,\gamma_r\right), G\left(\textbf{.}\mid\alpha,\beta,\gamma,\delta,u\right)$ are the fitted Stable distribution function for the upper (shown by subscript r) and lower tail (shown by subscript l) with ϵ, the four stable parameters and threshold $u.\phi(\textbf{.}\mid\mu,\sigma)$ is the Normal distribution with mean μ and σ as standard deviation.

4.1. *SNS parameter estimation*

In this study, a piecewise method of estimation is used to create the SNS model. The basic principle behind the piecewise method is the assumption that data follows different distributions over its range therefore the mixture model should be modelled in "pieces". Also, to the best of our knowledge, analytic software that makes use of stable models at the tails of extreme value mixture models have not been explored before. In this context, the thresholds u_l and u_r need to be pre-determined before proceeding to fit the respective stable tail and Normal bulk model.

5. Kernel Density Estimation (KDE)

5.1. *Univariate Kernel Density Estimator*

[6] describes the traditional kernel density estimator as:

$$\hat{f}\left(y;y,\lambda\right)=\frac{1}{n\lambda}\sum_{i=1}^{n}K((y-y_i)/\lambda) \tag{5}$$

where $K(\textbf{.})$ is the kernel density function which is usually symmetric and λ is bandwidth parameter. $K(y)\geq 0$ and $\int K(y)dy=1$ are two conditions that are satisfied by the kernel function.

Scale notation $K_\lambda(y)=\lambda^{-1}K\left(\frac{y}{\lambda}\right)$ is used to denote another formula for the kernel function:

$$\hat{f}\left(y;y,\lambda\right)=\frac{1}{n}\sum_{i=1}^{n}K_\lambda\left(y-y_i\right) \tag{6}$$

An interpretation of the kernel estimate at a point y by [16] is the average of n kernel ordinates at that specific point. Uniform, Normal and bi-weight among other functions are suggested by literature for kernel functions. The Gaussian probability density function is a popular choice for the Kernel function, where the bandwidth λ takes the role of standard deviation and controls the spread of the kernel. [16] also note that the choice of the kernel function is not as critical as the choice of the bandwidth.

5.2. *Selecting the Bandwidth*

Various methods for choosing the bandwidth λ is discussed in literature. The simplest way to select the bandwidth parameter λ is to plot various values and choose the parameter value that best suits the underlying density. The other possible method to select the bandwidth parameter is to minimize an error criterion, for example, the Mean Square Error (MSE).

5.3. Stable-KDE-Stable (SKS) mixture model

A two-tailed Kernel Stable model, namely, the Stable-KDE-Stable model is evaluated in this study. [9] proposed a two-tailed model by joining a standard kernel density estimator between two extreme value GPD tail models. [6] describes the two-tailed model introduced by [9] as:

$$F\left(y|\theta\right)=\begin{cases}\phi_{u_l}\left[1-G\left(-y|\epsilon_l,\sigma_{u_l},-u_{u_l}\right)\right], & y\leq u_l;\\ \phi_{u_l}+\left(1-\phi_{u_l}-\phi_{u_r}\right)\dfrac{H\left(y|X,\lambda\right)-H\left(u_l\|X,\lambda\right)}{H\left(u_r|X,\lambda\right)-H\left(u_l|X,\lambda\right)} & u_l<y<u_r;\\ \left(1-\phi_{u_r}\right)+\phi_{u_r}G\left(y|\epsilon_r,\sigma_{u_r},u_r\right) & y\geq u_r.\end{cases} \tag{7}$$

where $\theta=\left(X,\lambda,u_l,\sigma_{u_l},\epsilon_l,\phi_{u_l},u_2,\sigma_{u_r},\epsilon_r,\sigma_{u_r}\right)$ is the parameter vector. ϕ_{u_l} and σ_{u_r} are estimated as sample proportions less than the lower threshold u_l and above the upper threshold u_r. $G\left(-y|\epsilon_l,\sigma_{u_l},-u_{u_l}\right)$ and $G\left(y|\epsilon_r,\sigma_{u_r},u_r\right)$ represents the unconditional GPD function for $y<u_l$ and $y>u_r$ respectively. $H(\centerdot|y,\lambda)$ is the distribution function of the kernel density estimator where λ is the bandwidth.

The rationale behind choosing the SNS and SKS models is primarily because of the flexible characteristics of the 0-parameterization stable model. Stable distributions are known to have a robust fit to the underlying data and therefore is an apt choice when considering the tail component of EVT mixture models. With this, the prediction of extreme events is likely to be more accurate as compared to traditional EVT modelling approaches.

5.4. Anderson-Darling (AD) test

The Anderson-Darling goodness of fit test was the result of extensive research by T.W. Anderson and D.A. Darling [1]
The Anderson-Darling test statistic A^2 is defined as:

$$A_n^2=-\sum_{i=1}^{n}\frac{2i-1}{n}\left(\ln(\hat{F}(x_{(i)}))+\ln(1-\hat{F}(x_{(n+1-i)}))\right)-n \tag{8}$$

where $x_{(1)}<\cdots<x_{(m)}$ is the ordered sample size m from smallest to largest and $F(x)$ is the underlying theoretical cumulative distribution to which the sample is compared.

The null hypothesis $x_{(1)}<\cdots<x_{(m)}$ comes from the underlying distribution of $F(x)$. The null hypothesis is rejected at a specified level of significance α , if the test statistic A^2 is greater than the critical value for a table of critical values at different sample sizes. Generally, critical values of the Anderson-Darling test statistic depend on the distribution being tested. The goodness-of-fit of several distributions may evaluated using the Anderson-Darling test.

6. Empirical data and analysis

The data sets used in this study are the daily FTSE/JSE All-Share Index, FTSE/JSE Banks Index, FTSE/JSE Mining Index and USD/ZAR prices obtained from McGregor BFA and were recorded over the period from 13 August 2010 to 14 August 2020. The return series for each index is calculated as the first backward differences of the index values' natural logarithm. For day t, the daily log return r_t is defined as:

$$r_t=ln\left(P_t\right)-\left(P_{t-1}\right) \tag{9}$$

where, P_t is the price at day t.

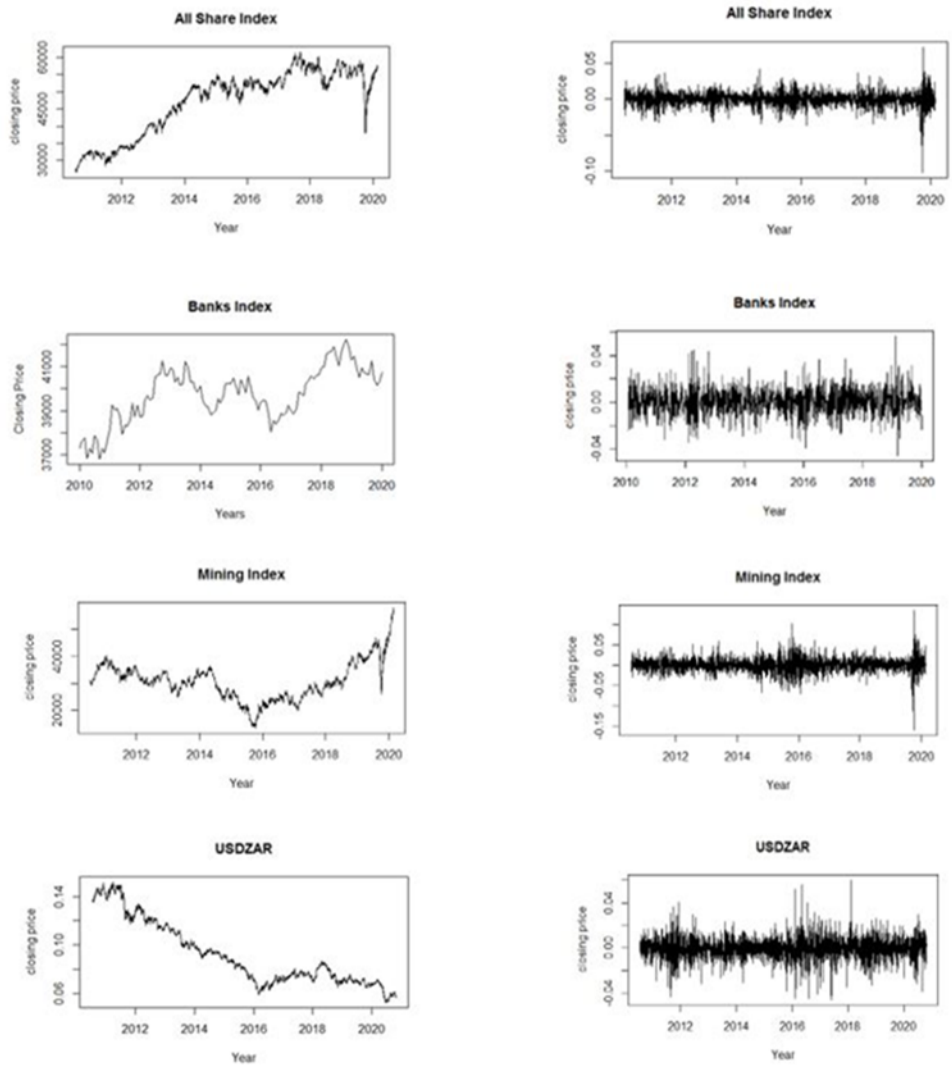


Figure 1. Time series plot of JSE Indices and USDZAR exchange rate (left) and one day returns (right).

In Figure 1, the plots specify several trends in mean and variance over time indicating non-stationarity. The variance diverges over time demonstrating heteroscedasticity and volatility clustering which is probable when dealing with financial data. Isolated extreme returns caused by shocks to financial markets are apparent, such as the 2015 stock market crash and the 2019-2020 global COVID-19 pandemic.

Table 1. Descriptive summary statistics of daily return of financial stock market indices and exchange rate

	ALSI		Banks Index		Mining Index		USDZAR	
Panel A:Descriptive Statistics								
Number of observations	2499.00		2499.00		2499.00		2675.00	
Minimum	0.1023		2.3021		0.1589		-0.0460	
Maximum	0.0726		0.0991		0.1346		0.0603	
Mean	0.0003		-0.0008		0.0002		-0.0003	
Skewness	-0.7310		-41.0875		-0.1605		-0.1671	
Excess Kurtosis	8.8822		1919.8771		6.0443		2.7766	
Panel B: Testing for normality, autocorrelation and heteroscedasticity								
Test	Statistic	*p*-value	Statistic	*p*-value	Statistic	*p*-value	Statistic	*p*-value
Jarque-Bera	8455.2667	$< 0.0001^{***}$	385117.132	$< 0.0001^{***}$	3823.93	$< 0.0001^{***}$	874.3992	$< 0.0001^{***}$
Ljung Box	67.2900	$< 0.0001^{***}$	7.0024	0.9967	47.62	0.0005	11.7751	0.9236
ARCH LM Test	936.0966	$< 0.0001^{***}$	0.0000	0.9900	50.0300	0.0000	15.4100	0.0000
Testing for unit root and stationarity								
Unit root test	Statistic	*p*-value	Statistic	*p*-value	Statistic	*p*-value	Statistic	*p*-value
ADF Test	–13.6259	0.01	–12.9576	0.01	–13.5509	0.01	–14.7402	$< 0.0001^{***}$
PP Test	-2586.908	0.01	-2548.281	0.01	-2430.247	0.01	-2688.028	0.01
KPSS Test	0.1303	0.10	0.0914	0.10	0.2394	0.10	0.0431	0.10
*** symbolizes a very small *p*-value								

Descriptive Statistics for the daily closing prices of the FTSE/JSE financial stock indices returns and USD/ZAR are shown in Table 1 in Panel A. The excess kurtosis value indicates the leptokurtic behaviour of these return series. This means that the empirical distribution of the daily returns is much heavier than the frequently fitted normal distribution. Excessively large values for skewness and excess kurtosis are seen for the Banks Index returns and perhaps describe the inadequate performance of the South African economy [4]. The Jarque-Bera test for normality gives a p-value less than 0.0001 for all four returns, thus rejecting the normality assumption at all levels of significance. Panel B displays tests for normality, autocorrelation and heteroscedasticity are shown. The null hypothesis of normality for the Jarque-Bera test is rejected at 5% level of significance for all stock and currency returns. This deduces the use of heavy tailed models when analyzing the returns series should be considered.

The Ljung box test provides mixed results. The significant p-values of the Ljung box test for ALSI and the Mining Index suggest rejecting the null hypothesis of no autocorrelation. The null hypothesis for FTSE/JSE Banks Index and USD/ZAR exchange rate is rejected inferring that the return series exhibit serial correlation.

Viewing the results from Panel C, it can be deduced at a 5% level of significance, the null hypothesis of a unit root is rejected, and it can be decided that all return series are stationary. The KPSS test showed that all returns are stationary since all p-values are 0.1 which is greater than 0.05 thus the null hypothesis of stationarity is rejected.

6.1. Results and discussion

SNS parameter estimates are shown in Table 2 where $\hat{\mu}$ denotes the estimated
Normal mean, $\hat{\sigma}$ is the estimated Normal standard deviation of the Normal bulk model. $\hat{\alpha}_l, \hat{\beta}_l, \hat{\gamma}_l, \hat{\delta}_l$ represents the lower stable tail estimated parameters. Similarly, $\hat{\alpha}_r, \hat{\beta}_r, \hat{\gamma}_r, \hat{\delta}_r$ describes the upper stable tail estimated parameters. Table 3 shows the bulk parameter estimates for the SKS model. Since a piecewise estimation approach was adopted, the fitted stable table tail model is the same as Table 2 for each quantile threshold point.

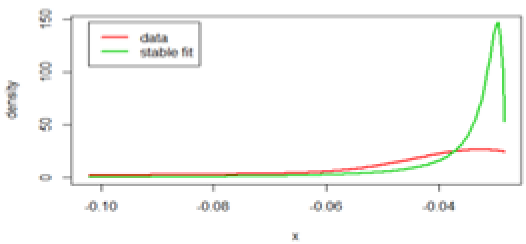
Table 2. ML parameter estimates of the SNS mixture model.

SNS parameter	Financial Stock Indices and exchange rate returns															
	FTSE/JSE ALSI				FTSE/JSE Banks Index				FTSE/JSE Mining Index				USD/ZAR			
	Quantiles															
	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$
$\hat{\alpha}_l$	0.7046	0.8555	0.9106	0.9757	0.6116	0.7299	0.7153	0.9576	0.0091	0.9772	0.9267	0.8950	1.9999	1.9999	1.0880	0.9889
$\hat{\beta}_l$	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.000	-1.0000	-1.0000	-1.0000	-1.0000	-0.4954	0.0683	-1.0000	-1.0000
$\hat{\gamma}_l$	0.0022	0.0024	0.0022	0.0022	0.0050	0.0040	0.0026	0.0032	0.0045	0.0042	0.0037	0.0032	0.0039	0.0045	0.0025	0.0021
$\hat{\delta}_l$	-0.0309	-0.0241	-0.0193	-0.0158	-0.0505	-0.0363	-0.0288	-0.0232	-0.0532	-0.0414	-0.0324	-0.0244	-0.0350	-0.0285	-0.0199	-0.0149
	Bulk Model															
$\hat{\mu}$	0.0004	0.0004	0.0004	0.0003	0.0002	0.0002	0.0002	0.0003	0.0003	0.0003	0.0002	0.0003	-0.0003	-0.0002	-0.0002	-0.0001
$\hat{\sigma}$	0.0089	0.0080	0.0070	0.0059	0.0141	0.0127	0.0113	0.0091	0.0153	0.0137	0.0120	0.0098	0.0086	0.0078	0.0069	0.0056
	Quantiles															
	$\hat{q}_{0.99}$	$\hat{q}_{0.975}$	$\hat{q}_{0.95}$	$\hat{q}_{0.90}$	$\hat{q}_{0.99}$	$\hat{q}_{0.975}$	$\hat{q}_{0.95}$	$\hat{q}_{0.90}$	$\hat{q}_{0.99}$	$\hat{q}_{0.975}$	$\hat{q}_{0.95}$	$\hat{q}_{0.90}$	$\hat{q}_{0.99}$	$\hat{q}_{0.975}$	$\hat{q}_{0.95}$	$\hat{q}_{0.90}$
$\hat{\alpha}_r$	0.9738	0.8633	0.8532	0.9300	1.1938	0.7726	0.8786	0.9580	0.9670	1.0251	1.0707	0.9118	0.7644	0.9562	0.9539	0.9769
$\hat{\beta}_r$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{\gamma}_r$	0.0023	0.0022	0.0020	0.0018	0.0071	0.0037	0.0032	0.0033	0.0050	0.0040	0.0044	0.0037	0.0017	0.0020	0.0018	0.0016
$\hat{\delta}_l$	0.0311	0.0230	0.0180	0.0140	0.0559	0.0363	0.0294	0.0226	0.0535	0.0426	0.0340	0.0241	0.0255	0.0206	0.0166	0.0132

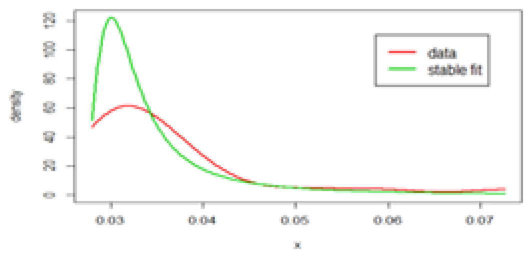
Table 3. ML parameter estimates of the SKS mixture model.

SNS parameter	Financial Stock Indices and exchange rate returns															
	FTSE/JSE ALSI				FTSE/JSE Banks Index				FTSE/JSE Mining Index				USD/ZAR			
	Quantiles															
	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$	$\hat{q}_{0.01}$	$\hat{q}_{0.025}$	$\hat{q}_{0.05}$	$\hat{q}_{0.1}$
$\hat{\lambda}_l$	0.0016	0.0011	0.0010	0.0004	0.0025	0.0014	0.0010	0.0010	0.0023	0.0016	0.0008	0.0005	0.0004	0.0007	0.0009	0.0009

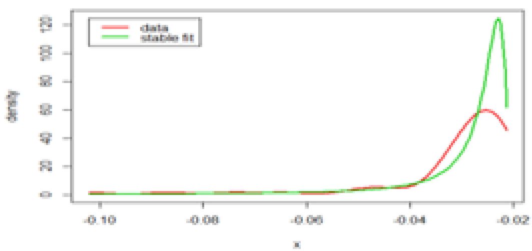
6.1.1. Fitted stable tail densities



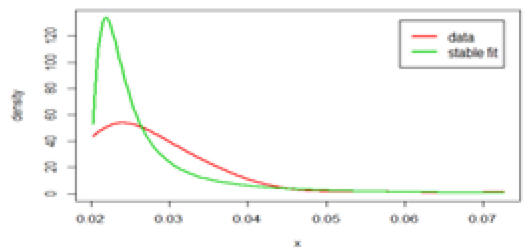
a) 1% lower tail



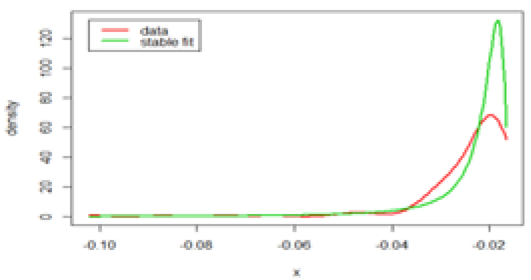
b) 99% upper tail



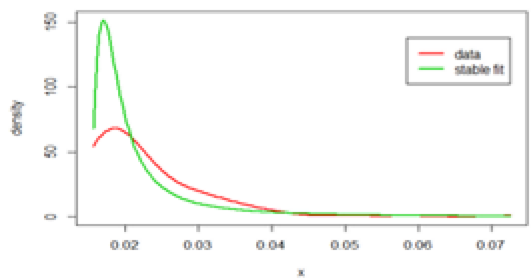
c) 2.5% lower tail



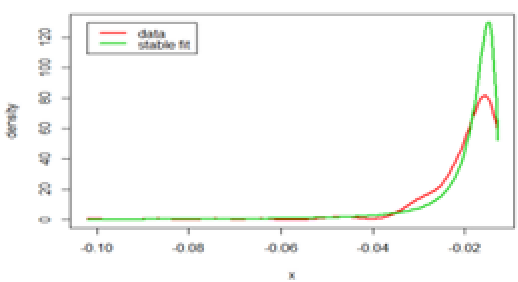
d) 97.5% upper tail



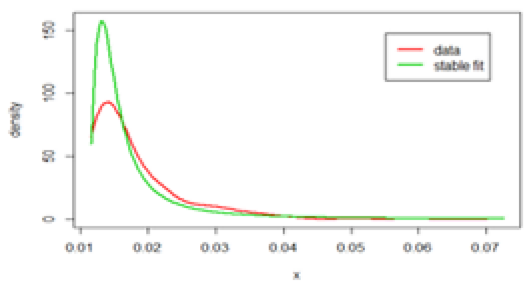
e) 5% lower tail



f) 95% upper tail



g) 10% lower tail



h) 90% upper tail

Figure 2. FTSE/JSE ALSI stable tail model density plots.

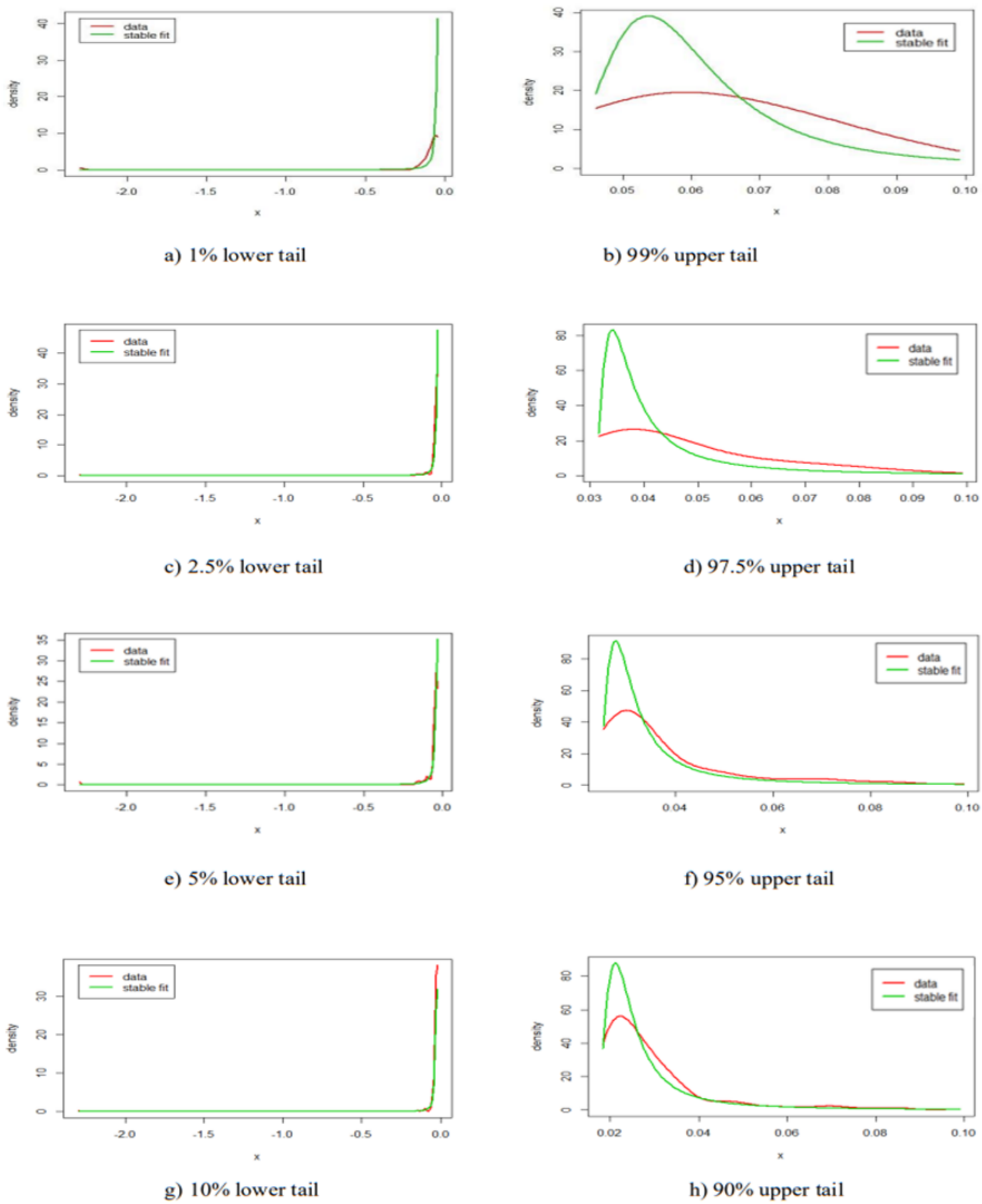


Figure 3. FTSE/JSE Banks Index stable tail model density plots.

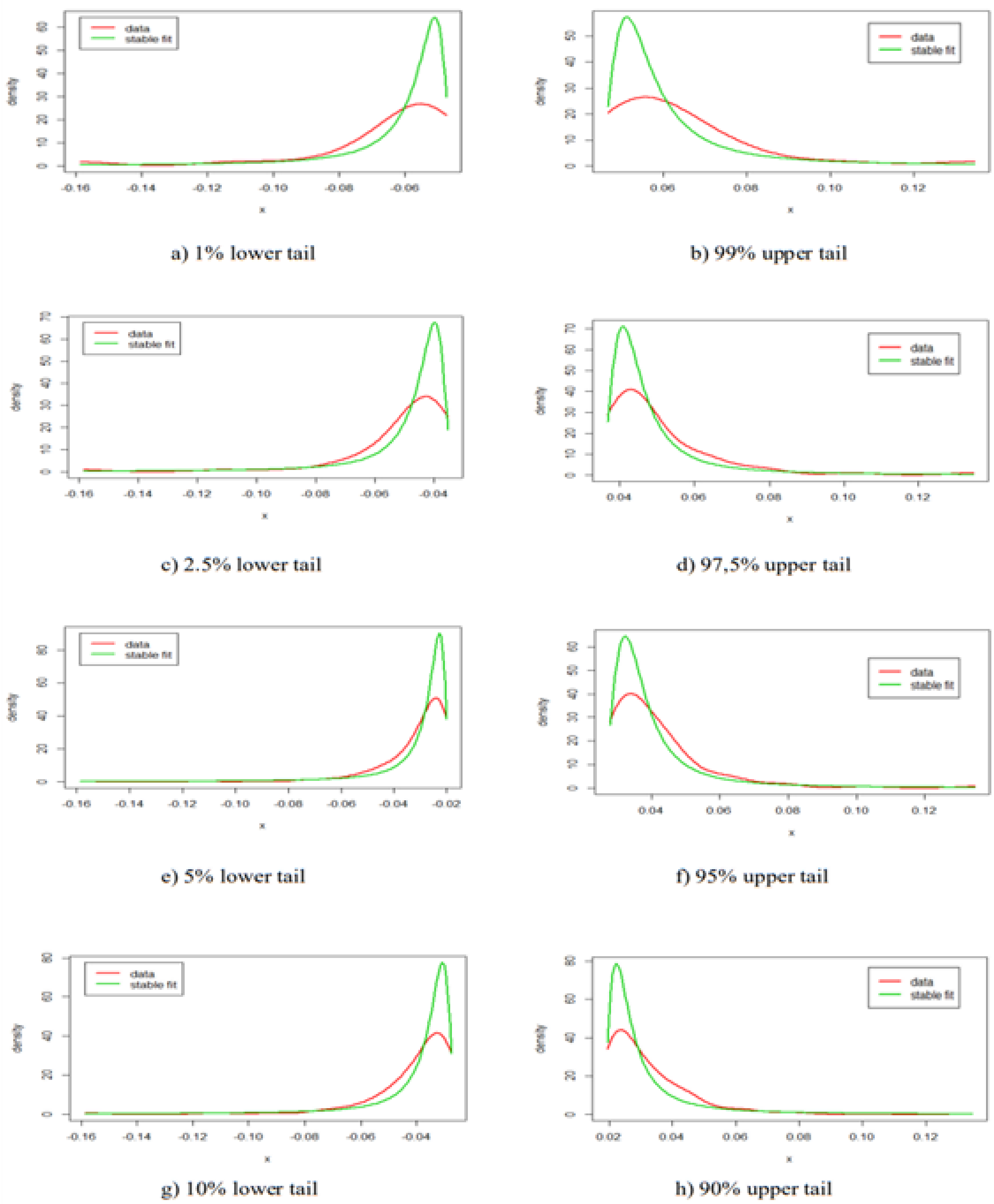


Figure 4. FTSE/JSE Mining Index stable tail model density plots.

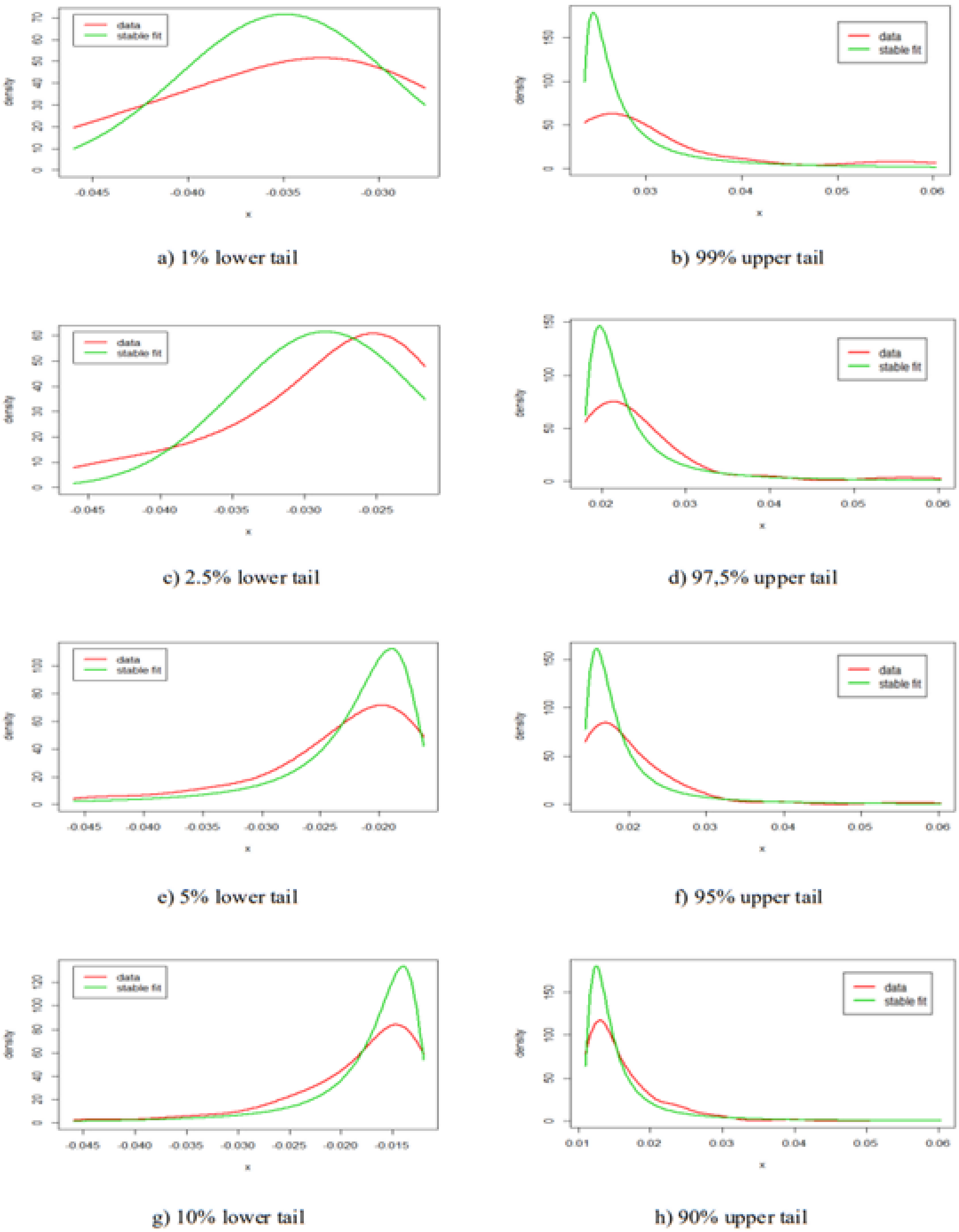


Figure 5. USD/ZAR stable tail model density plots.

Figures 2, 3, 4, and 5 shows that the fitted stable tail segment of the SNS and SKS model fits best for all returns at all tails quantile specifically the extreme data values. Overfitting is noticed as the tail section reaches the bulk component which is expected behaviour of stable models where variability is noticed. Mixed results are seen for the 99% upper tail of the Banks index where descriptive statistics have previously shown extreme kurtosis and heavy tails. Generally, this suggests the stable model for the tails as a worthy alternative to the tail model for risk-averse practitioners.Table 4 and Table 5 shows the results of the Anderson Darling Goodness-of-fit test on the tail components of the fitted SNS and SKS models.

Table 4. Anderson Darling Goodness of fit test of the stable lower tail model for the SNS and SKS mixture models.

Lower Tail Quantiles				
	1% lower tail	**2.5% lower tail**	**5% lower tail**	**10% lower tail**
FTSE/JSE ALSI				
AD Test Statistic	0.2363	0.5988	0.8688	1.0311
p-value	0.9771	0.6485	0.4337	0.3413
FTSE/JSE Banks Index				
AD Test Statistic	0.2298	0.4069	0.7775	1.3615
p-value	0.9800	0.8414	0.4973	0.2133
FTSE/JSE Mining Index				
AD Test Statistic	0.2286	0.4062	0.9290	2.0496
p-value	0.9805	0.8421	0.3966	0.0863
USD/ZAR				
AD Test Statistic	0.6671	3.2959	0.7694	1.7949
p-value	0.5855	0.0195	0.5034	0.1195

The results from the Anderson-Darling goodness-of fit tests in Table 4 confirm the results seen in the stable density lower tail plots where the suggested stable model serves as a good fit at all quantile thresholds for the lower tail. For the each of the four return index series, the fitted stable tail model favours the null hypothesis of model adequacy at a 5% level of significance for all quantiles thus confirming formally that the stable models ought to be considered for the tail portion of extreme value mixture models.

Table 5. Anderson Darling Goodness of fit test of the stable upper tail model for the SNS and SKS mixture models.

Upper Tail Quantiles				
	99% lower tail	**97.5% lower tail**	**95% lower tail**	**90% lower tail**
FTSE/JSE ALSI				
AD Test Statistic	0.1315	0.8604	1.2868	1.3250
p-value	0.9996	0.4390	0.2366	0.2244
FTSE/JSE Banks Index				
AD Test Statistic	0.5916	0.8516	0.5783	1.5101
p-value	0.6543	0.85164	0.6685	0.1741
FTSE/JSE Mining Index				
AD Test Statistic	0.2703	0.3339	0.7417	2.729
p-value	0.9584	0.9104	0.5247	0.03771
USD/ZAR				
AD Test Statistic	0.2674	0.6262	0.9962	1.1420
p-value	0.9602	0.6230	0.3591	0.2906

Similar to Table 4, results from the Anderson-Darling goodness-of fit in Table 5 confirm the suggested stable upper tail model serves as a good fit at all quantile thresholds except for the 90% upper tail quantile threshold for the FTSE/JSE Mining Index return series. The fitted stable tail model fails to reject the null hypothesis of model adequacy at a 5% level of significance. At a 1% level of significance the upper tail fitted stable model is deemed to be a good fit at all upper quantile threshold points, once again confirming the flexibility and robustness of fitted stable distributions for the tail portion of extreme value mixture models.

6.1.2. Fitted bulk densities

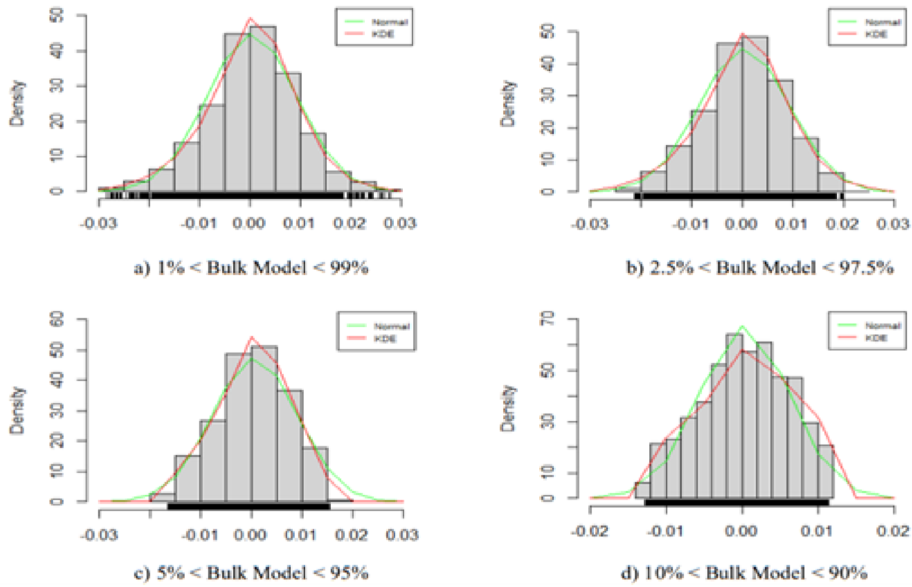


Figure 6. FTSE/JSE ALSI returns Bulk density plots for fitted models.

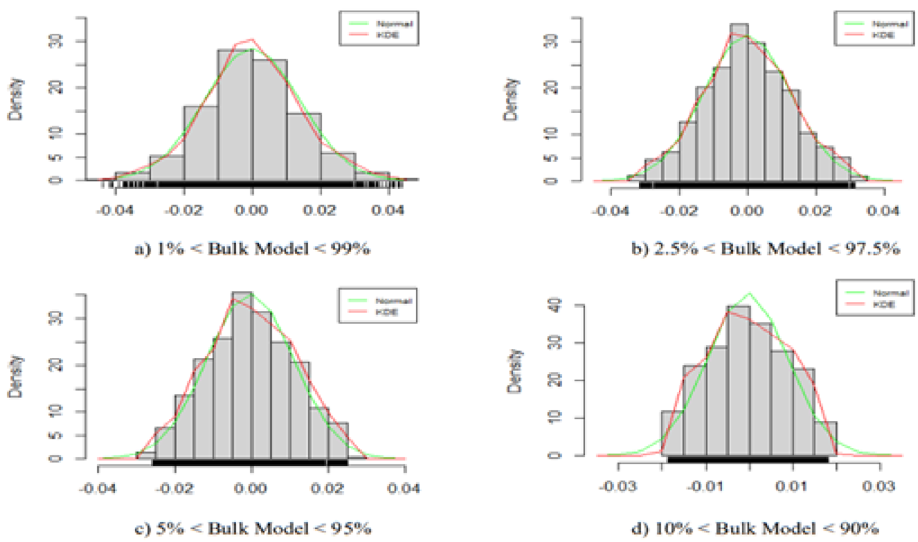


Figure 7. FTSE/JSE Banks Index returns Bulk density plots for fitted models.

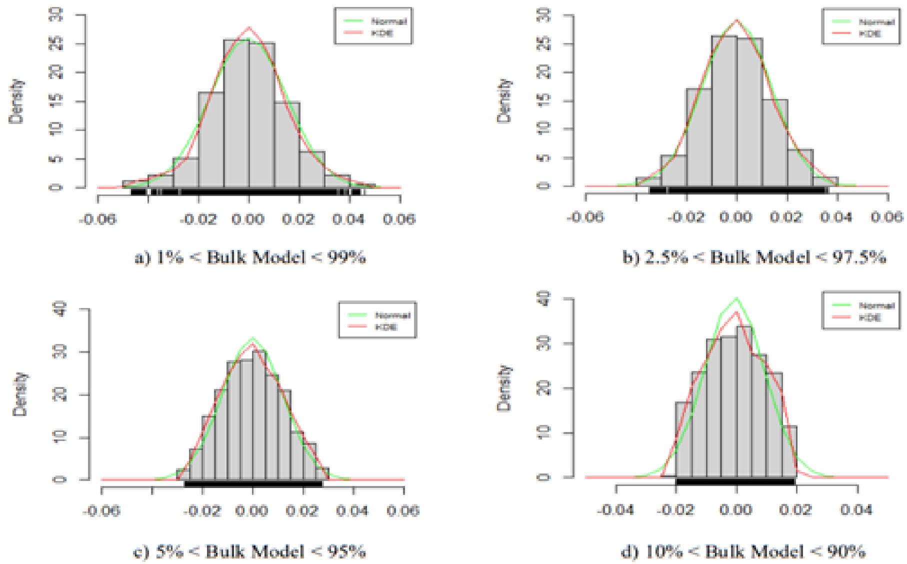


Figure 8. FTSE/JSE Mining Index returns Bulk density plots for fitted models.

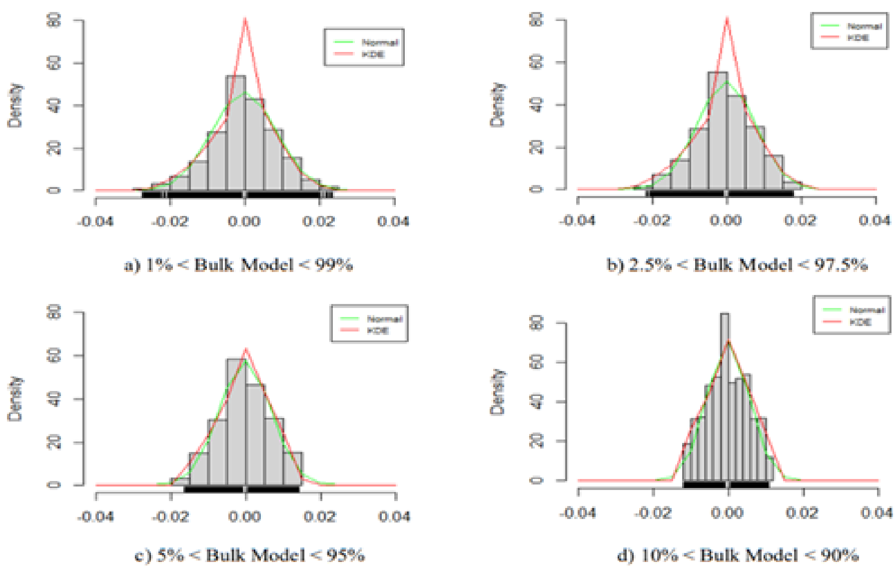


Figure 9. USD/ZAR returns Bulk density plots for fitted models.

The fitted Normal and KDE model density plots for the SNS and SKS models are shown in Figures 6, 7, 8 and 9 above for each stock indices and South African exchange rate. Evidently, the KDE model is a good fit across the entire range of the bulk data as compared to the commonly used Normal distribution, however, the Normal model is a close second alternative to model bulk data over the bulk return series range. This seems to be the opposite for the FTSE/JSE Banks Index where the fitted Normal model out-performs the KDE model. This study places more focus on the stable tail component where model robustness is of great interest especially from an extreme value mixture modelling perspective.

7. Discussion

Extreme Value theory (EVT) is prevalently used to study the cases of unusual occurrences where mathematically and statistically sound models are developed and extrapolated. Statistical inference has become more challenging due to the well documented fact of heavy tails and skewness prominent in financial data. The traditional Gaussian framework commonly used in financial applications for many years has shown to be inadequate by several empirical studies in capturing the characteristics of financial data. This study highlights that extreme events in finance reveal the likelihood of not only extreme losses (seen in the lower tail) but also extreme gains (seen in the upper tail). Results from this study show that the monthly data series of South African Industrial Index (J520) data rejects the normality assumption indicating that the return series is fat-tailed and that the fitted stable tail model provides an adequate fit to extreme returns. Research by [9] highlights a drawback with proposed mixture models in industry is the misspecification of the bulk model that has large impact of the fit of the tails. The non-parametric KDE model is applied to the bulk model to deal with computational and inference challenges. In this regard, new EVT mixture models, namely the SNS and SKS models are proposed as possible alternative models to be explored, since Nolan's S_0 stable distribution is a flexible class of distributions that account for asymmetry and heavy tails, provides a reasonable model for tail behaviour. This study delves into novel models that prove to be sound alternatives to exploring Extreme Value theory in a South African financial context, however, with novelty comes limitations. The SNS and SKS models in the current state can only be considered in a piece-wise manner on not in its entirety due to computational challenges as Stable distributions describe a family of models, this includes the Gaussian model among other parametric distributions such as the Cauchy and Levy model. Model validation also becomes a challenge due to computational complexities. The data sets used in this study has minimal influence on the methodologies of thes models as the ideology behind the SNS and SKS models were derived from the extensive research and empirical analysis of univariate stable distributions that lead to a natural progression to explore mixture model applications. In essence, both the SNS and SKS models can be explored further with other datasets in various industries that exhibit heavy tails and skewness.

8. Conclusion

Statistical analyses from this study propose the new SNS and SKS mixed model as an alternate for modeling extreme occurrences often observed in financial industries as these models prove to capture the complexities of financial returns. Stable distributions are a wealthy family of models that are ideal for handling the heavy tail and skewness properties seen in financial return series. For each return series, SNS and SKS parameters were estimated and subsequently the distribution density plots, and Anderson Darling test analyzed the fit which mostly inferred a good fit throughout the data overall. This article provides valuable findings to researchers or financial practitioners who are interested with extreme value mixture modeling within the context of the South African financial setting as an alternate to traditional models which, fails to capture the empirical characteristics of financial data. As further research, as a comprehensive simulation study on the proposed mixture models should be considered with applications to data outside of finance to understand and measure the versatility and robustness of the newly proposed SNS and SKS models. Also, although the proposed models have a computational challenge, the robust fit is worth exploring when analysts consider EVT mixed models. Essentially, this study recommends the practice of extreme value mixture models in achieving a better understanding of the South African financial industry where financial modeling strategies are applied and validated. This study urges the consideration of applying the SNS and SKS models across the African continent with similar economies or in other industries where the novel mixed models that depict real world occurrences are highly vital.

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Conceptualization, KN and KC;
methodology, KN ;
software, KN and KC;
validation, KC and RC;
formal analysis, KN;
investigation, KN and RC;
data curation, KN;
writing— original draft preparation, KN
writing—review and editing, KN, KC and RC;
visualisation, KN;
supervision, KC and RC;
project administration, KC and RC.

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