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The New Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-G Family of Distributions with Applications

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Abstract In this paper, we propose a new family of generalized distributions called the Topp-Leone type II Exponentiated-Half Logistic-Marshall-Olkin-G (TL-TIIEHL-MO-G) distribution. The new distribution can be expressed as an infinite linear combination of exponentiated-G family of distributions. Some special models of the new family of distributions are explored. Statistical properties including the quantile function, ordinary and incomplete moments, stochastic orders, probability weighted moments, distribution of the order statistics and Rényi entropy are presented. The maximum likelihood method is used for estimating the model parameters and Monte Carlo simulation is conducted to examine the performance of the model. The flexibility and importance of the new family of distributions is demonstrated by means of applications to real data for censored and complete sets, respectively.

Keywords Topp-Leone-G, Type II Exponentiated-Half Logistic-G, Marshall-Olkin-G, Maximum Likelihood Estimation.

Mathematics Subject Classifications 62E99; 60E05

1. Introduction

The need for the construction and development of generalized models is of tremendous practical importance in probability and statistics as well as related area. In most cases, classical distributions do not provide adequate fits to real data hence the need for new generalized distributions. These new distributions have been generated by introducing one or more parameters to existing distributions. Examples of recently generalized distributions include the generalized odd Lindley-G by (Afify et al. [1]), Topp-Leone-Kumaraswamy-G by (Sule et al. [29]), type II half logistic-G by (Hassan et al. [12]), odd log logistic-Topp-Leone-G by (Alizadeh et al. [4]), new Topp-Leone-G by (Hassan et al. [13]), Marshall-Olkin generalized-G by (Yousof et al. [34]), Topp-Leone-Harris-G by (Chipepa et al. [21]), Marshall-Olkin Topp-Leone half-logistic-G by (Sengweni et al. [27]), Topp-Leone odd Burr X-G by (Oluyede et al. [23]), generalized Topp-Leone-G power series by (Warahena-Liyanage et al. [33]), type II Topp-Leone-G power series by (Makubate et al. [17]), exponentiated odd exponential half logistic- G power series by (Rannona et al. [25]), odd power generalized Weibull-G power series by (Oluyede et al. [22]), Lindley-Burr XII power series by (Makubate et al. [16]), half-logistic odd power generalized Weibull-G by (Peter et al. [24]), gamma odd Burr X-G by (Tlhaloganyang et al. [30]), exponentiated half logistic-Kumaraswamy-G by (Sengweni et al. [31]) and Topp-Leone-Marshall-Olkin-G by (Chipepa et al. [10]). Topp and Leone [32] presented a simple bounded J-shaped distribution as an alternative to the beta distribution. Al-Shomrani et al. [6] developed the Topp-Leone-G (TL-G) family of distributions with the cumulative distribution function (cdf) and probability distribution function (pdf) given by

$$F_{TL-G}(x;b,\xi) = \left[1-\bar{G}^2(x;\xi)\right]^b \qquad (1)$$

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and

$$f_{TL-G}(x;b,\xi) \quad = \quad 2bg(x;\xi)\bar{G}(x;\xi)\left[1-\bar{G}^2(x;\xi)\right]^{b-1}, \tag{2}$$

respectively, where $b>0$ is the shape parameter, $\bar{G}(x;\xi)=1-G(x;\xi)$ is the survival function, $G(x;\xi)$ is the baseline cdf and $g(x;\xi)$ is the baseline pdf which depends on the parameter vector ξ.

Another class of distributions, namely the type II exponentiated half logistic-G (TIIEHL-G) was proposed by Al-Mofleh et al. [5]. The TIIEHL-G cdf and pdf are given by

$$F_{TIIEHL-G}(x;\alpha,\xi) \quad = \quad 1-\left[\frac{1-G(x;\xi)}{1+G(x;\xi)}\right]^{\alpha} \tag{3}$$

and

$$f_{TIIEHL-G}(x;\alpha,\xi) \quad = \quad \frac{2\alpha g(x;\xi)\left(1-G(x;\xi)\right)^{\alpha-1}}{\left[1+G(x;\xi)\right]^{\alpha+1}}, \tag{4}$$

respectively, where $\alpha>0$ is the shape parameter, $G(x;\xi)$ is the baseline cdf and $g(x;\xi)$ is the baseline pdf which depends on the parameter vector ξ. Marshall and Olkin [18] introduced a family of distributions called Marshall-Olkin-G (MO-G) distribution with cdf and pdf given by

$$F_{MO-G}(x;\delta,\xi) \quad = \quad 1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \tag{5}$$

and

$$f_{MO-G}(x;\delta,\xi) \quad = \quad \frac{\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2}, \tag{6}$$

respectively, where $\delta>0$ is the shape parameter, $\bar{\delta}=1-\delta$ and $\bar{G}(x;\xi)=1-G(x;\xi)$.

In this paper, we propose a new family of distributions which exhibit different types of hazard rate functions including monotonic as well as non-monotonic shapes. Another motivation for developing the model in this paper is the construction of heavy-tailed distributions for modelling different real data sets, as well as distributions with symmetric, left-skewed, right-skewed and reverse-J shapes. We also seek to develop a new family of distributions which provide better fits than other generated distributions under the same transformation, and underlying baseline cdf.

This paper is organized as follows. In Section 2, we introduce the new family of distributions and its sub-families. In Section 3, mathematical properties of the new family are explored including expansion of the pdf, quantile function, moments, generating function and Rényi entropy. The estimation of the parameters are obtained using maximum likelihood estimation method for both complete and censored data in Section 4. Some special cases of the new family of distributions are given in Section 5. Also, in this section, we plot the density function, hazard rate function and present 3D plots of skewness and kurtosis. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates are presented in Section 6. Section 7 contains actuarial measures and numerical study of actuarial measures, while Section 8 gives applications of the new model to real data sets, and conclusions are given in Section 9.

2. The New Family

In this section, we present the new family, and its sub-families. The cdf, pdf and hazard rate function (hrf) of the Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-G (TLTIIEHL-MO-G) family of distributions are given by combining equations (1), (2), (3), (4), (5) and (6). The resulting cdf, pdf and hrf are given by

$$F(x;b,\alpha,\delta,\xi) = \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b}, \quad (7)$$

$$\begin{aligned} f(x;b,\alpha,\delta,\xi) &= \frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^2}\left[\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right]^{2\alpha-1} \\ &\times \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b-1} \quad (8) \end{aligned}$$

and

$$\begin{aligned} h(x;b,\alpha,\delta,\xi) &= \frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^2}\left[\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right]^{2\alpha-1} \\ &\times \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b-1} \\ &\times \left[1-\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b}\right]^{-1}, \quad (9) \end{aligned}$$

respectively, for $x > 0$, $b, \alpha, \delta > 0$, $\bar{\delta} = 1-\delta$ and parameter vector ξ.

2.1. Sub-Families

Sub-families of the new family of distributions are presented in this subsection.

- When $b = 1$, we obtain a family of distributions with cdf given by

$$F(x;\alpha,\delta,\xi) = 1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}$$

  $x > 0$, $\alpha, \delta > 0$, $\bar{\delta} = 1-\delta$ and parameter vector ξ. This is a new family of distributions.
- When $\alpha = 1$, we obtain a family of distributions with cdf given by

$$F(x;b,\delta,\xi) = \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2}\right]^{b}$$

  for $x > 0$, $b, \delta > 0$, $\bar{\delta} = 1-\delta$ and parameter vector ξ. This is a new family of distributions.
- When $\delta = 1$, we obtain a family of distributions with cdf given by

$$F(x;b,\alpha,\xi) = \left[1-\left(\frac{\bar{G}(x;\xi)}{1+G(x;\xi)}\right)^{2\alpha}\right]^{b}$$

  for $x > 0$, $b, \alpha > 0$ and parameter vector ξ. This is a new family of distributions.

- When $b=\alpha=1$, we obtain a new family of distributions with cdf given by

$$F(x;\delta,\xi) = 1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2}$$

for $x>0$, $\delta>0$, $\bar{\delta}=1-\delta$ and parameter vector ξ.
- When $b=\delta=1$, we obtain a new family of distributions with cdf given by

$$F(x;\alpha,\xi) = 1-\left(\frac{\bar{G}(x;\xi)}{1+G(x;\xi)}\right)^{2\alpha}$$

for $x>0$, $\alpha>0$, and parameter vector ξ.
- When $\alpha=\delta=1$, we obtain a new family of distributions with cdf given by

$$F(x;b,\xi) = \left[1-\left(\frac{\bar{G}(x;\xi)}{1+G(x;\xi)}\right)^{2}\right]^{b}$$

for $x>0$, $b>0$, and parameter vector ξ.
- When $b=\alpha=\delta=1$, we obtain a new family of distributions with cdf given by

$$F(x;\xi) = 1-\left(\frac{\bar{G}(x;\xi)}{1+G(x;\xi)}\right)^{2}$$

for $x>0$, and parameter vector ξ.

3. Some Statistical Properties

Statistical properties of the TL-TIIEHL-MO-G family of distributions are explored in this section. The statistical properties considered include expansion of the density function as well as the quantile function, probability weighted moments, generating function, distribution of order statistics, stochastic order, and Rényi entropy.

3.1. Linear Representation of the Density Function

In this subsection, we demonstrate that the TL-TIIEHL-MO-G density function can be expressed as an infinite linear combination of exponentiated-G (exp-G) densities. Consider the generalized binomial series expansion given by $(1-x)^{\alpha}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha}{k}x^{k}$, $|x|<1$ and using this expansion as follows:

$$\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha i},$$

then we have

$$\begin{aligned} f(x;b,\alpha,\delta,\xi) &= 4b\alpha\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\delta^{2\alpha i+2\alpha}\bar{G}^{2\alpha i+2\alpha-1}(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+1)} \\ &\times \left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha+1)} g(x;\xi). \end{aligned}$$

Consider the generalized binomial series expansions

$$\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha+1)} = \sum_{j=0}^{\infty}(-1)^j\binom{2\alpha i+2\alpha+j}{j}\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^j$$

and

$$\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^j = \sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\delta^k\bar{G}^k(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-k},$$

then the pdf of the TL-TIIEHL-MO-G family of distributions can be written as

$$\begin{aligned} f(x;b,\alpha,\delta,\xi) &= 4b\alpha\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k}\delta^{2\alpha i+2\alpha+k} \\ &\times \bar{G}^{2\alpha i+2\alpha+k-1}(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+k+1)}g(x;\xi). \end{aligned}$$

We note that

$$\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+k+1)} = \sum_{l=0}^{\infty}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\bar{\delta}^l\bar{G}^l(x;\xi),$$

so that the pdf of the TL-TIIEHL-MO-G family of distributions can be written as:

$$\begin{aligned} f(x;b,\alpha,\delta,\xi) &= 4b\alpha\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k} \\ &\times \frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\delta^{2\alpha i+2\alpha+k}\bar{\delta}^l\bar{G}^{l+2\alpha i+2\alpha+k-1}(x;\xi)g(x;\xi). \end{aligned}$$

Now consider

$$\left[1-G(x;\xi)\right]^{l+2\alpha i+2\alpha+k-1} = \sum_{m=0}^{\infty}(-1)^m\binom{l+2\alpha i+2\alpha+k-1}{m}G^m(x;\xi),$$

then the TL-TIIEHL-MO-G pdf becomes

$$\begin{aligned} f(x;b,\alpha,\delta,\xi) &= 4b\alpha\sum_{i,j,k,l,m=0}^{\infty}(-1)^{i+j+k+m}\binom{b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k} \\ &\times \binom{l+2\alpha i+2\alpha+k-1}{m}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\delta^{2\alpha i+2\alpha+k} \\ &\times \bar{\delta}^l g(x;\xi)G^m(x;\xi) \\ &= \sum_{m=0}^{\infty}t_{m+1}h_{m+1}(x;\xi), \end{aligned} \quad (10)$$

where

$$\begin{aligned} t_{m+1} &= 4b\alpha\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+m}}{m+1}\binom{b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k} \\ &\times \binom{l+2\alpha i+2\alpha+k-1}{m}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\delta^{2\alpha i+2\alpha+k}\bar{\delta}^l, \end{aligned} \quad (11)$$

and $h_{m+1}(x;\xi)=(m+1)g(x;\xi)G^m(x;\xi)$. The new density can be expressed as an infinite linear combination of exp-G densities. The mathematical and statistical properties of the new family of distributions follows directly from those of exp-G family of distributions.

3.2. Quantile Function

The quantile function of the TL-TIIEHL-MO-G family of distributions is obtained by inverting the non-linear equation

$$F(x;b,\alpha,\delta,\xi)=\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b}=u$$

for $0\leq u\leq 1$. Note that

$$\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}\left[2-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]=\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)},$$

which simplifies to

$$\bar{G}(x;\xi)=\frac{2\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}}{2\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}-\delta\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}+\delta}.$$

Consequently, the quantile function of the TL-TIIEHL-MO-G family of distributions is given by

$$Q_G(u;b,\alpha,\delta,\xi)=G^{-1}\left(1-\frac{2\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}}{2\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}-\delta\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}+\delta}\right). \quad (12)$$

Quantiles are obtained using equation (12) via a specified baseline cdf G using R software.

3.3. Probability Weighted Moments and Generating Functions

In this section, we present the probability weighted moments (PWMs) and generating functions of the TL-TIIEHL-MO-G family of distributions. Using equation (10), we can obtain the r^{th} moment of the TL-TIIEHL-MO-G family of distributions as follows:

$$E(X^{r}) \;=\; \int_{-\infty}^{\infty}x^{r}f(x;b,\alpha,\delta,\xi)dx=\sum_{m=0}^{\infty}t_{m+1}E(Y_{m+1}^{r}), \quad (13)$$

where $E(Y_{m+1}^{r})$ is the r^{th} moment of Y_{m+1} which follows exp-G distribution with power parameter $m+1$ and t_{m+1} is defined as equation (11). The moment generating function is given by

$$M_X(t)=E(e^{tX})=\sum_{m=0}^{\infty}t_{m+1}E(e^{tY_{m+1}}),$$

where $E(e^{tY_{m+1}})$ is the moment generating function of the exp-G distribution with power parameter $(m+1)$ and t_{m+1} is given by equation (11).

The PWMs of a random variable X are defined by

$$\omega_{a,r}=E\Big(X^{a}[F(X)]^{r}\Big)=\int_{-\infty}^{\infty}x^{a}[F(x)]^{r}f(x)dx.$$

Let the cdf and pdf of the TL-TIIEHL-MO-G family of distributions be written as F(x) and f(x), respectively. From equations (7) and (8), we can write

$$\begin{aligned} f(x)[F(x)]^r &= \frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^2}\left[\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right]^{2\alpha-1} \\ &\times \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{br+b-1}. \end{aligned}$$

After some simplifications (see appendix), we have

$$\begin{aligned} f(x)[F(x)]^r &= 4b\alpha\sum_{i,j,k,l,m=0}^{\infty}(-1)^{i+j+k+m}\binom{br+b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k} \\ &\times \binom{l+2\alpha i+2\alpha+k+1}{m}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)} \\ &\times \delta^{2\alpha i+2\alpha+k}\bar{\delta}^l g(x;\xi)G^m(x;\xi) \\ &= \sum_{m=0}^{\infty}s_{m+1}h_{m+1}(x;\xi), \end{aligned}$$

where $h_{m+1}(x;\xi)=(m+1)g(x;\xi)G^m(x;\xi)$ and

$$\begin{aligned} s_{m+1} &= 4b\alpha\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+m}}{m+1}\binom{br+b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k} \\ &\times \binom{l+2\alpha i+2\alpha+k+1}{m}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\delta^{2\alpha i+2\alpha+k}\bar{\delta}^l. \end{aligned}$$

As a result, the PWMs of TL-TIIEHL-MO-G family of distributions is given by

$$\omega_{a,r}=\sum_{m=0}^{\infty}s_{m+1}\int_{-\infty}^{\infty}x^a h_{m+1}(x;\xi)dx.$$

3.4. Rényi Entropy

Rényi entropy of the TL-TIIEHL-MO-G family of distributions is given in this section. Rényi entropy [26] is a measure of uncertainty associated to a random variable X and is defined as

$$H_R(v)=\frac{1}{1-v}log\left(\int_0^{\infty}f^v(x)dx\right)$$

for $v>0$, $v\neq 1$. From equation (8), $f^v_{TL-TIIEHL-MO-G}(x;b,\alpha,\delta,\xi)=f^v(x)$ can be written as:

$$\begin{aligned} f^v(x) &= \left[\frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^2}\right]^v\left[\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right]^{v(2\alpha-1)} \\ &\times \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{v(b-1)}. \end{aligned}$$

After some simplifications (see the appendix), we can write

$$\begin{aligned} f^v(x) &= (4b\alpha)^v \sum_{i,j,k,l,m=0}^{\infty} (-1)^{i+j+k+m} \binom{v(b-1)}{i} \binom{2\alpha i+2\alpha v+v+j-1}{j} \binom{j}{k} \\ &\times \binom{l+2\alpha i+v(2\alpha-1)+k}{m} \frac{\Gamma(l+2\alpha i+2\alpha v+v+k)}{l!\Gamma(2\alpha i+2\alpha v+v+k)} \delta^{2\alpha i+v(2\alpha-1)+k} \\ &\times \bar{\delta}^l g^v(x;\xi) G^m(x;\xi). \end{aligned}$$

Consequently, Rényi entropy of the TL-TIIEHL-MO-G family of distributions can be written as

$$H_R(v) = \frac{1}{1-v} log\left[\sum_{m=0}^{\infty} a_m \left(\int_0^{\infty} g^v(x;\xi) G^m(x;\xi) dx\right)\right],$$

where

$$\begin{aligned} a_m &= (4b\alpha)^v \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+m} \binom{v(b-1)}{i} \binom{2\alpha i+2\alpha v+v+j-1}{j} \binom{j}{k} \\ &\times \binom{l+2\alpha i+v(2\alpha-1)+k}{m} \frac{\Gamma(l+2\alpha i+2\alpha v+v+k)}{l!\Gamma(2\alpha i+2\alpha v+v+k)} \delta^{2\alpha i+v(2\alpha-1)+k} \bar{\delta}^l. \end{aligned}$$

Note that, $\int_0^{\infty} g^v(x;\xi) G^m(x;\xi) dx$ can be obtained numerically. Also, Rényi entropy of the TL-TIIEHL-MO-G family of distributions can be obtained directly from that of the exponentiated-G distribution as follows:

$$H_R(v) = \frac{1}{1-v} log\left[\sum_{m=0}^{\infty} \phi_m e^{(1-v)I_{REG}}\right], \tag{14}$$

where

$$\begin{aligned} \phi_m &= (4b\alpha)^v \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+m}}{\left(\frac{m}{v}+1\right)^v} \binom{v(b-1)}{i} \binom{2\alpha i+2\alpha v+v+j-1}{j} \binom{j}{k} \\ &\times \binom{l+2\alpha i+v(2\alpha-1)+k}{m} \frac{\Gamma(l+2\alpha i+2\alpha v+v+k)}{l!\Gamma(2\alpha i+2\alpha v+v+k)} \delta^{2\alpha i+v(2\alpha-1)+k} \bar{\delta}^l \end{aligned}$$

and $I_{REG} = \frac{1}{1-v} log \int_0^{\infty} \left[\left(\frac{m}{v}+1\right) g(x;\xi) \left(G(x;\xi)\right)^{\frac{m}{v}}\right]^v dx$ is the Rényi entropy of the exp-G distribution with power parameter $\frac{m}{v}$.

3.5. *Distribution of Order Statistics*

Let $X_1, X_2, ..., X_n$ be independent and identically distributed TL-TIIEHL-MO-G random variables. The pdf of the i^{th} order statistic, $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{n! f(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}.$$

Using the results from the probability weighted moments in subsection 3.3, we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{p=0}^{\infty} (-1)^j \binom{n-i}{j} \omega_{p+1} h_{p+1}(x;\xi),$$

where $h_{p+1}(x;\xi) = (p+1)g(x;\xi)G^{p}(x;\xi)$ and

$$\begin{aligned}\omega_{p+1} &= 4b\alpha \sum_{k,l,m,n=0}^{\infty} \frac{(-1)^{k+l+m+p}}{p+1}\binom{b(i+j)-1}{k}\binom{2\alpha k+2\alpha+l}{l}\binom{l}{m} \\ &\times \binom{n+2\alpha k+2\alpha+m+1}{p}\frac{\Gamma(n+2\alpha k+2\alpha+m+1)}{n!\Gamma(2\alpha k+2\alpha+m+1)}\delta^{2\alpha k+2\alpha+m}\bar{\delta}^{n}.\end{aligned}$$

3.6. Stochastic Order

Let X and Y be two random variables with cdf $F_X(t)$ and $F_Y(t)$, respectively. Then X is stochastically smaller than Y if $\bar{F}_X(t) \le \bar{F}_Y(t)$ for all t, where $\bar{F}_X(t) = 1 - F_X(t)$ is the survival function. Stochastic ordering of X and Y is denoted by $X <_{st} Y$. A stronger ordering is the likelihood ratio order denoted by $X <_{lr} Y$, and given by $\frac{f_X(t)}{f_Y(t)}$ decreasing in t. Note that hazard rate order is stronger than stochastic order but weaker than likelihood ratio order. Hazard rate order is given by $h_X(t) \ge h_Y(t)$ for all t, and denoted by $X <_{hr} Y$. Thus, $X <_{lr} Y \implies X <_{hr} Y \implies X <_{st} Y$, (Shaked and Shanthikumar [28]).

Theorem: Suppose $X_1 \sim TL-TIIEHL-MO-G(b_1,\alpha,\delta,\xi)$ and $X_2 \sim TL-TIIEHL-MO-G(b_2,\alpha,\delta,\xi)$. If $b_1 < b_2$, then $\frac{f(x;b_1,\alpha,\delta,\xi)}{f(x;b_2,\alpha,\delta,\xi)}$ is decreasing in x.
Proof: After some simplifications, we have

$$\frac{f(x;b_1,\alpha,\delta,\xi)}{f(x;b_2,\alpha,\delta,\xi)} = \frac{b_1}{b_2}\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b_1-b_2}. \tag{15}$$

Let $W = \frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}$, then differentiating equation (15) with respect to x, we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{f(x;b_1,\alpha,\delta,\xi)}{f(x;b_2,\alpha,\delta,\xi)}\right) &= \frac{4b_1\alpha\delta g(x;\xi)(b_1-b_2)}{b_2\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2[1+(1-W)]^2} \\ &\times \left[\frac{W}{1+(1-W)}\right]^{2\alpha-1}\left[1-\left(\frac{W}{1+(1-W)}\right)^{2\alpha}\right]^{b_1-b_2-1}.\end{aligned}$$

If $b_1 < b_2$, then $\frac{\partial}{\partial x}\left(\frac{f(x;b_1,\alpha,\delta,\xi)}{f(x;b_2,\alpha,\delta,\xi)}\right) < 0$, which implies that likelihood ratio order exist among X_1 and X_2, that is, $X_2 <_{lr} X_1$. As a result, we conclude that X_1 and X_2 are stochastically ordered, that is, $X_1 <_{st} X_2$.

4. Maximum Likelihood Estimation

4.1. Estimation of Parameters for Uncensored Data

We present the maximum likelihood method of estimation for estimating the parameters of the TL-TIIEHL-MO-G family of distributions. Let $X \sim TL-TIIEHL-MO-G(b,\alpha,\delta,\xi)$ and $\Delta = (b,\alpha,\delta,\xi)^T$ be the vector of parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the TL-TIIEHL-MO-G family of distributions is given by

$$
\begin{aligned}
\ell_n(\Delta) &= n\log(4b\alpha\delta) + \sum_{i=1}^{n}\log\left[g(x_i;\xi)\right] + (2\alpha-1)\sum_{i=1}^{n}\log\left[\delta\bar{G}(x_i;\xi)\right] \\
&- (2\alpha+1)\sum_{i=1}^{n}\log\left[1-\bar{\delta}\bar{G}(x_i;\xi)\right] - (2\alpha+1)\sum_{i=1}^{n}\log\left[2-W\right] \\
&+ (b-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right],
\end{aligned}
$$

where $W = \frac{\delta\bar{G}(x_i;\xi)}{1-\bar{\delta}\bar{G}(x_i;\xi)}$. The elements of the score vector are given in the appendix. The maximum likelihood estimates of the parameters, denote by $\hat{\Delta}$ is obtained by solving the non-linear equation $\left(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \xi_k}\right)^T = \mathbf{0}$ using a numerical method such as Newton-Raphson procedure.

4.2. *Estimation of Parameters for Censored Data*

In lifetime experiment, there are limitations incurred by the researcher, be it cost or time, as such, the researcher cannot wait for all the subjects to die or all the items to fail. As a result, the researcher can wait until a fixed number of observations provide information or observe subjects for a fixed time period, these two are type II censoring and type I censoring respectively.

4.2.1. Type I Right Censoring In type I right censoring, the time is fixed before sample selection and number of survivors are random variables. In other words, each individual in the study has a fixed censoring time C_i which is the time between the date of entry and the end of the study. In this type of study, the data are indicated by pairs of random variables (T_i, ϑ_i), for $i = 1, 2, ..., n$. Let $T_1, T_2, ..., T_n$ be a sample of independent positive random variables so that T_i is associated with an indicator variable $\vartheta_i = 0$ if T_i is a censoring time. The likelihood function of type I right censored sample $(t_1, \vartheta_1), (t_2, \vartheta_2), ..., (t_n, \vartheta_n)$ from the TL-TIIEHL-MO-G family of distributions is given by

$$
\ell(\Delta) = \left[\prod_{i=1}^{n} f(x_i;\Delta)^{\vartheta_i}\right](1-F(x_i;\Delta))^{1-\vartheta_i}
$$

and the log likelihood function is given by

$$
\begin{aligned}
\ell_n(\Delta) &= \sum_{i=1}^{n}\vartheta_i \log\left(\left[\frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[2-W\right]^2}\right]\left[\frac{W}{2-W}\right]^{2\alpha-1}\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]^{b-1}\right]\right) \\
&+ \sum_{i=1}^{n}(1-\vartheta_i)\log\left(1-\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]^{b}\right),
\end{aligned}
$$

where $W = \frac{\delta\bar{G}(x_i;\xi)}{1-\bar{\delta}\bar{G}(x_i;\xi)}$.

4.2.2. Type II Censoring In type II censoring time is random variable and the number of survivors are fixed prior to sample selection. Assume that the lifetime of the first r failed items $x_1, x_2, ..., x_r$ are observed out of a random sample of n lifetimes $X_1, X_2, ..., X_n$ which are independent and identically distributed. Then for the vector of parameters $\Delta = (b, \alpha, \delta, \xi)^T$, the likelihood function of type II censoring from the from the TL-TIIEHL-MO-G family of distributions is given by

$$\ell(\mathit{\Delta}) = A\left[\prod_{i=1}^{r} f(x_i;\mathit{\Delta})\right](1-F(x_i;\mathit{\Delta}))^{n-r},$$

where A is the constant $\frac{n!}{(n-r)!}$, and the log-likelihood function is given by:

$$\begin{aligned}\ell_n(\mathit{\Delta}) &= r\log(4b\alpha\delta)+\sum_{i=1}^{r}\log\left[g(x_i;\xi)\right]+(2\alpha-1)\sum_{i=1}^{r}\log\left[\delta\bar{G}(x_i;\xi)\right]\\ &- (2\alpha+1)\sum_{i=1}^{r}\log\left[1-\bar{\delta}\bar{G}(x_i;\xi)\right]-(2\alpha+1)\sum_{i=1}^{r}\log\left[2-W\right]\\ &+ (b-1)\sum_{i=1}^{r}\log\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]+(n-r)\log\left(1-\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]^{b}\right),\end{aligned}$$

where $W=\frac{\delta\bar{G}(x_i;\xi)}{1-\bar{\delta}\bar{G}x_i;\xi)}$.

The Fisher information matrix given by

$$I\left(\mathit{\Delta}\right)=\left[\mathbf{I}_{\theta_i,\theta_j}\right]_{(p+3)\times(p+3)}=E\left(-\frac{\partial^2\ell_n}{\partial\theta_i\partial\theta_j}\right),$$

for $i,j=1,2,...,(p+3)$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}\left(\mathit{\Delta}\right)$ can be approximated by $\mathbf{J}_n\left(\hat{\mathit{\Delta}}\right)\approx\left[-\frac{\partial^2\ell_n}{\partial\theta_i\partial\theta_j}|_{\mathit{\Delta}=\hat{\mathit{\Delta}}}\right]_{(p+3)\times(p+3)}, i,j=1,2,...,(p+3)$.

Applying the large sample approximation, MLE of $\mathit{\Delta}$, that is, $\hat{\mathit{\Delta}}$ can be treated as being approximately normal, $N_{(p+3)}\left(\mathit{\Delta},J_n(\mathit{\Delta})^{-1}\right)$, where $J_n(\mathit{\Delta})^{-1}$ is the observed Fisher information matrix. The asymptotic multivariate normal $N_{(p+3)}\left(\mathit{\Delta},I_n(\hat{\mathit{\Delta}})^{-1}\right)$ distribution of $\hat{\mathit{\Delta}}$ can be used to construct confidence interval for the parameters. An $100(1-\alpha)\%$ asymptotic confidence interval for each parameter $\mathit{\Delta}_r$ is given by

$$ACI_r=\left(\hat{\mathit{\Delta}}_r\pm Z_{\frac{\alpha}{2}}\sqrt{\widehat{I}_{rr}}\right),$$

where $\widehat{I}_{rr}$ is the (r,r) diagonal element of $I_n(\hat{\mathit{\Delta}})^{-1}$ for r=1,2,...,(p+3) and $Z_{\frac{\alpha}{2}}$ is the $(\frac{\alpha}{2})^{th}$ percentile of the standard normal distribution.

5. Some Special Models

In this section, some special models of the TL-TIIEHL-MO-G family of distributions are presented. The baseline distributions considered include Weibull, Power, Burr III and Burr XII distributions, respectively.

5.1. *Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-Weibull (TL-TIIEHL-MO-W) Distribution*

Suppose we take the baseline distribution to be the one parameter Weibull distribution with the cdf and pdf given by $G(x;\lambda)=1-e^{-x^{\lambda}}$ and $g(x;\lambda)=\lambda x^{\lambda-1}e^{-x^{\lambda}}$, respectively, then from equations (7) and (8), we obtain the cdf and pdf of the TL-TIIEHL-MO-W distribution as

$$F(x;b,\alpha,\delta,\lambda) = \left[1-\left(\frac{\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}}{1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)}\right)^{2\alpha}\right]^{b}$$

and

$$f(x;b,\alpha,\delta,\lambda) = \frac{4b\alpha\delta\lambda x^{\lambda-1}e^{-x^{\lambda}}}{\left[1-\bar{\delta}e^{-x^{\lambda}}\right]^{2}\left[1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)\right]^{2}}\left[\frac{\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}}{1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)}\right]^{2\alpha-1}$$
$$\times\left[1-\left(\frac{\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}}{1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)}\right)^{2\alpha}\right]^{b-1},$$

for $b,\alpha,\delta,\lambda > 0$ and $\bar{\delta} = 1-\delta$.

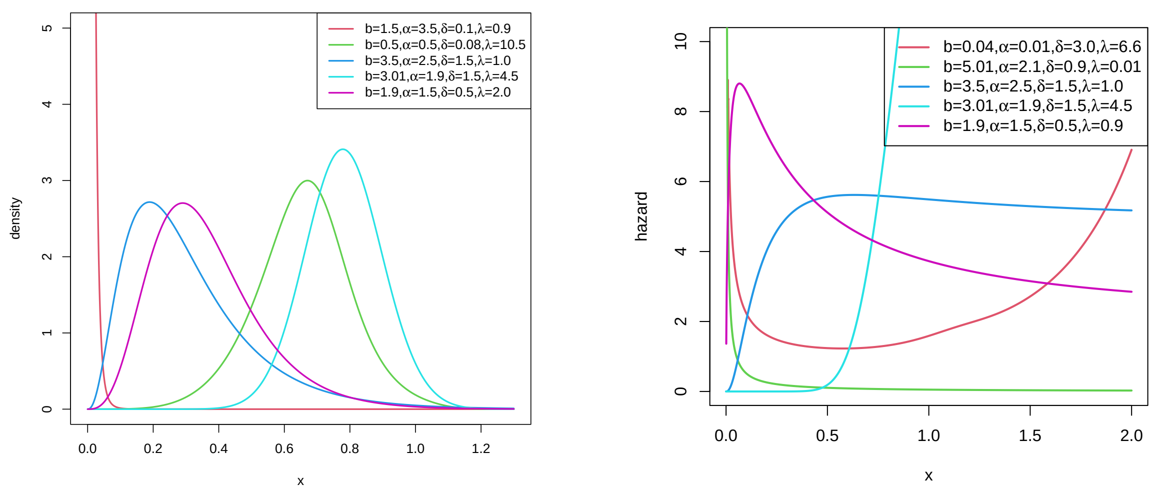

Figure 1. Plots of the Density and Hazard Rate Functions for the TL-TIIEHL-MO-W Distribution

Figure 13 shows the plots of the pdf and hrf of the TL-TIIEHL-MO-W distribution. The pdf of the TL-TIIEHL-MO-W distribution can take several shapes including left-skewed, right-skewed, almost symmetric and reverse-J shapes, while the hrf displays increasing, decreasing, bathtub and upside down bathtub shapes. For the same parameter values, hazard rate function plots were done for different nested models including the Weibull, TIIEHL-W, TIIEHL-MO-W and MO-W distributions. The plots were placed in the appendix. From these plots, we noticed that the Weibull distribution and MO-W distribution display monotone shapes only, that is decreasing and increasing shapes, the TIIEHL-W and the TIIEHL-MO-W show increasing, decreasing and bathtub shapes. We can then see that the newly developed model, that is, the TL-TIIEHL-MO-W has an additional shape (upside-down bathtub) of the hazard rate function plot, hence improved flexibility. Tables 1 and 2 give the quantiles and moments for TL-TIIEHL-MO-W distribution for selected parameter values.

Table 1. **Quantiles for TL-TIIEHL-MO-W Distribution**

	$(b, \alpha, \delta, \lambda)$				
u	$(1, 1.5, 1.3, 3.5)$	$(1.7, 1.1, 1.8, 2.5)$	$(1.0, 0.5, 2.5, 0.5)$	$(2.0, 0.3, 2.9, 0.4)$	$(9.9, 1, 1.5, 0.5)$
0.1	0.3380	0.4306	0.0091	0.0077	0.1503
0.2	0.4217	0.5287	0.0116	0.0248	0.2249
0.3	0.4837	0.6009	0.0324	0.0532	0.2989
0.4	0.5383	0.6678	0.0692	0.0908	0.3814
0.5	0.5880	0.7311	0.1227	0.1480	0.4728
0.6	0.6395	0.7972	0.2109	0.2319	0.5906
0.7	0.6944	0.8684	0.3562	0.3643	0.7506
0.8	0.7563	0.9545	0.6175	0.5984	0.9950
0.9	0.8455	1.0763	1.2160	1.1304	1.4690

Table 2. **Moments for TL-TIIEHL-MO-W Distribution**

	$(b, \alpha, \delta, \lambda)$				
u	$(2, 1.5, 7.3, 1.9)$	$(1.3, 1.2, 0.6, 2.3)$	$(2.1, 2.9, 2.7, 1.8)$	$(1.1, 1.5, 2.5, 1.6)$	$(2.1, 1.5, 1.7, 1.9)$
E(X)	1.0251	0.4340	0.5167	0.5263	0.6187
E(X^2)	1.1488	0.2303	0.3084	0.3740	0.4438
E(X^3)	1.3809	0.1438	0.2065	0.3239	0.3595
E(X^4)	1.7590	0.1030	0.1522	0.3249	0.3229
E(X^5)	2.3550	0.0832	0.1219	0.3664	0.3173
E(X^6)	3.2944	0.0745	0.1048	0.4556	0.3375
SD	0.3131	0.2047	0.2035	0.3115	0.2469
CV	0.3054	0.4717	0.3938	0.5917	0.3991
CS	0.0750	0.8662	0.5239	0.8234	0.6246
CK	2.8482	4.1229	3.2400	3.6645	3.4805

Figures 2 and 3 show the 3D-plots of skewness and kurtosis for TL-TIIEHL-MO-W distribution. When we fix δ and λ, the skewness and kurtosis of the TL-TIIEHL-MO-W distribution decrease as b and α increase. The plots show that when we fix the parameter b and α, the skewness and kurtosis of the TL-TIIEHL-MO-W distribution increases as δ and λ increase.

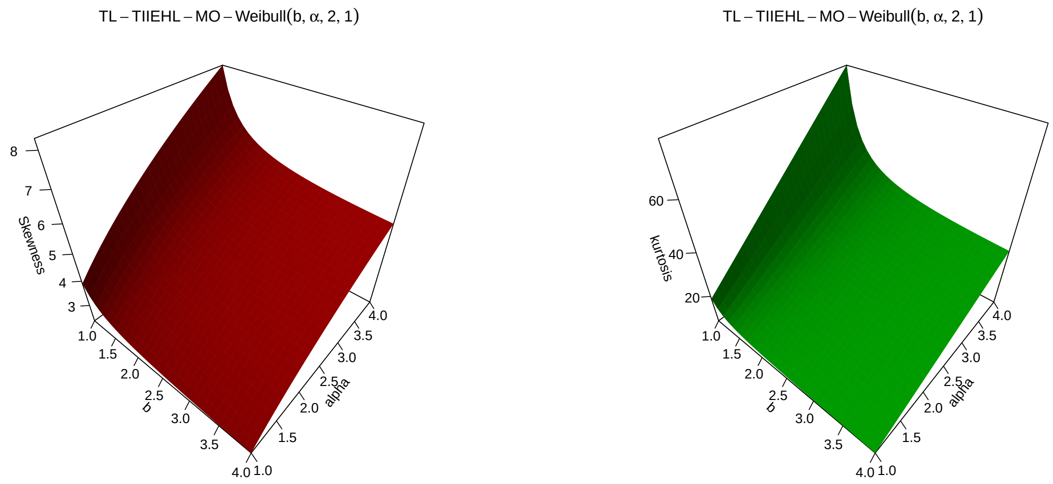

Figure 2. Plots of Skewness and Kurtosis for TL-TIIEHL-MO-Weibull Distribution.

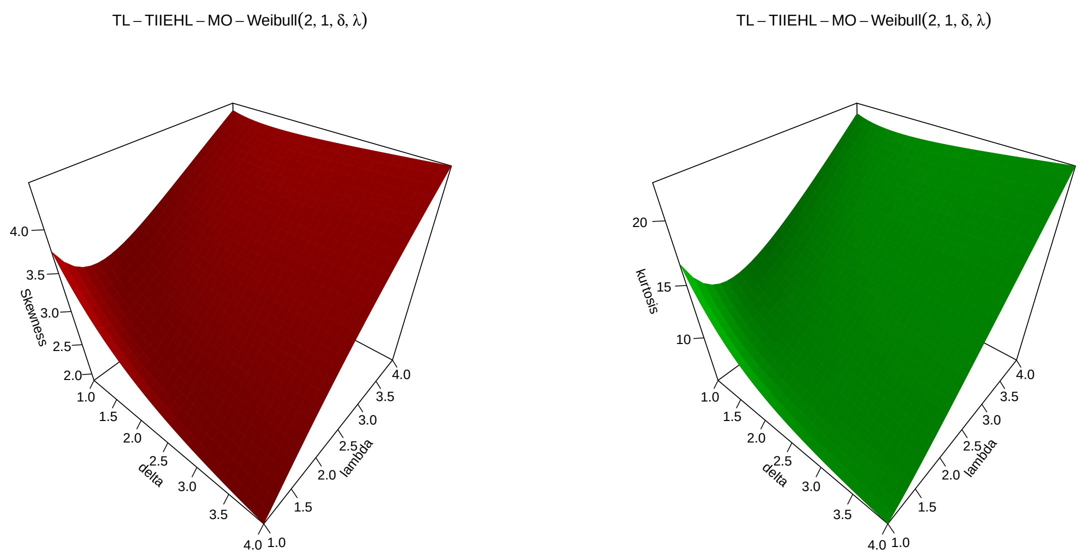

Figure 3. Plots of Skewness and Kurtosis for TL-TIIEHL-MO-W Distribution

5.2. *Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-Power (TL-TIIEHL-MO-P) Distribution*

Suppose we take the baseline distribution as power distribution with the cdf and pdf given by $G(x;\theta,k)=(\theta x)^k$ and $g(x;\theta,k)=k\theta^k x^{k-1}$, respectively. From equations (7) and (8), we obtain the cdf and pdf of the TL-TIIEHL-MO-P distribution as

$$F(x;b,\alpha,\delta,\theta,k) \quad = \quad \left[1-\left(\frac{\frac{\delta\left[1-(\theta x)^k\right]}{1-\bar{\delta}[1-(\theta x)^k]}}{1+\left(1-\frac{\delta[1-(\theta x)^k]}{1-\bar{\delta}[1-(\theta x)^k]}\right)}\right)^{2\alpha}\right]^b,$$

and

$$\begin{aligned} f(x;b,\alpha,\delta,\theta,k) \quad &= \quad \frac{4b\alpha\delta k\theta^k x^{k-1}}{\left(1-\bar{\delta}\left(1-(\theta x)^k\right)\right)^2\left[1+\left(1-\frac{\delta[1-(\theta x)^k]}{1-\bar{\delta}[1-(\theta x)^k]}\right)\right]^2} \\ &\times \left[\frac{\frac{\delta\left[1-(\theta x)^k\right]}{1-\bar{\delta}[1-(\theta x)^k]}}{1+\left(1-\frac{\delta[1-(\theta x)^k]}{1-\bar{\delta}[1-(\theta x)^k]}\right)}\right]^{2\alpha-1}\left[1-\left(\frac{\frac{\delta\left[1-(\theta x)^k\right]}{1-\bar{\delta}[1-(\theta x)^k]}}{1+\left(1-\frac{\delta[1-(\theta x)^k]}{1-\bar{\delta}[1-(\theta x)^k]}\right)}\right)^{2\alpha}\right]^{b-1}, \end{aligned}$$

for $b,\alpha,\delta,\theta,k>0$, and $\bar{\delta}=1-\delta$.

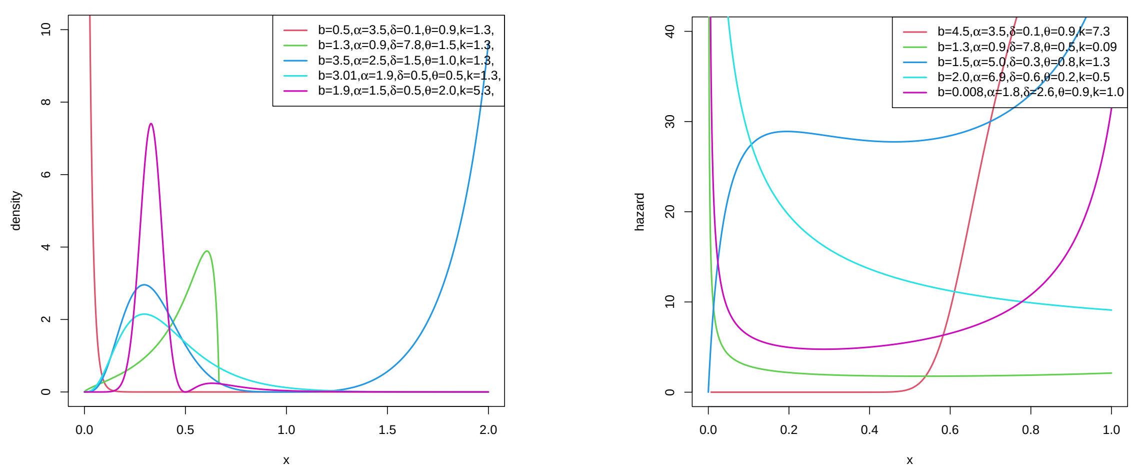

Figure 4. Plots of the Density and Hazard Rate Functions for the TL-TIIEHL-MO-P Distribution

Figure 4 shows the plots of the pdf and hrf of the TL-TIIEHL-MO-P distribution. The pdf of the TL-TIIEHL-MO-P distribution can take several shapes including bimodal, left-skewed, right-skewed, almost symmetric and reverse-J shapes, while the hrf displays increasing, decreasing, bathtub, and upside down bathtub followed by bathtub shapes. Table 3 gives the quantiles for TL-TIIEHL-MO-P distribution for selected parameter values.

Table 3. **Quantiles for TL-TIIEHL-MO-P Distribution**

	$(b,\alpha,\delta,\theta,k)$				
u	$(3.5, 1.5, 1.0, 1.5, 1.5)$	$(0.4, 1.0, 1.1, 1.2, 4.2)$	$(1.0, 1.5, 1.5, 1.1, 1.8)$	$(7.1, 1.5, 0.9, 1.9, 8.3)$	$(9.9, 1, 1.5, 0.5, 2.9)$
0.1	0.1482	0.1558	0.1191	0.4303	1.5442
0.2	0.1849	0.2352	0.1808	0.4410	1.6175
0.3	0.2181	0.3007	0.2327	0.4505	1.6704
0.4	0.2440	0.3583	0.2835	0.4555	1.7139
0.5	0.2730	0.4161	0.3343	0.4636	1.7500
0.6	0.3026	0.4683	0.3845	0.4704	1.7873
0.7	0.3341	0.5262	0.4388	0.4791	1.8232
0.8	0.3748	0.5845	0.5044	0.4827	1.8579
0.9	0.4246	0.6584	0.5945	0.4908	1.9078

5.3. *Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-Burr III (TL-TIIEHL-MO-BIII) Distribution*

Suppose the baseline distribution is the Burr III distribution with the cdf and pdf given by $G(x;\lambda,\beta)=(1+x^{-\lambda})^{-\beta}$ and $g(x;\lambda,\beta)=\lambda\beta x^{-(\lambda+1)}(1+x^{-\lambda})^{-(\beta+1)}$, respectively, for $\lambda,\beta,x>0$. From equations (7) and (8), we obtain the cdf and pdf of the TL-TIIEHL-MO-BIII distribution as

$$F(x;b,\alpha,\delta,\lambda,\beta) = \left[1-\left(\frac{\frac{\delta\left[1-(1+x^{-\lambda})^{-\beta}\right]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}}{1+\left(1-\frac{\delta[1-(1+x^{-\lambda})^{-\beta}]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}\right)}\right)^{2\alpha}\right]^{b}$$

and

$$\begin{aligned} f(x;b,\alpha,\delta,\lambda,\beta) &= \frac{4b\alpha\delta\lambda\beta x^{-(\lambda+1)}(1+x^{-\lambda})^{-(\beta+1)}}{\left(1-\bar{\delta}\left[1-(1+x^{-\lambda})^{-\beta}\right]\right)^2\left[1+\left(1-\frac{\delta[1-(1+x^{-\lambda})^{-\beta}]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}\right)\right]^2} \\ &\times \left[\frac{\frac{\delta\left[1-(1+x^{-\lambda})^{-\beta}\right]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}}{1+\left(1-\frac{\delta[1-(1+x^{-\lambda})^{-\beta}]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}\right)}\right]^{2\alpha-1} \\ &\times \left[1-\left(\frac{\frac{\delta\left[1-(1+x^{-\lambda})^{-\beta}\right]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}}{1+\left(1-\frac{\delta[1-(1+x^{-\lambda})^{-\beta}]}{1-\bar{\delta}[1-(1+x^{-\lambda})^{-\beta}]}\right)}\right)^{2\alpha}\right]^{b-1}, \end{aligned}$$

for $b,\alpha,\delta,\lambda,\beta>0$, and $\bar{\delta}=1-\delta$.

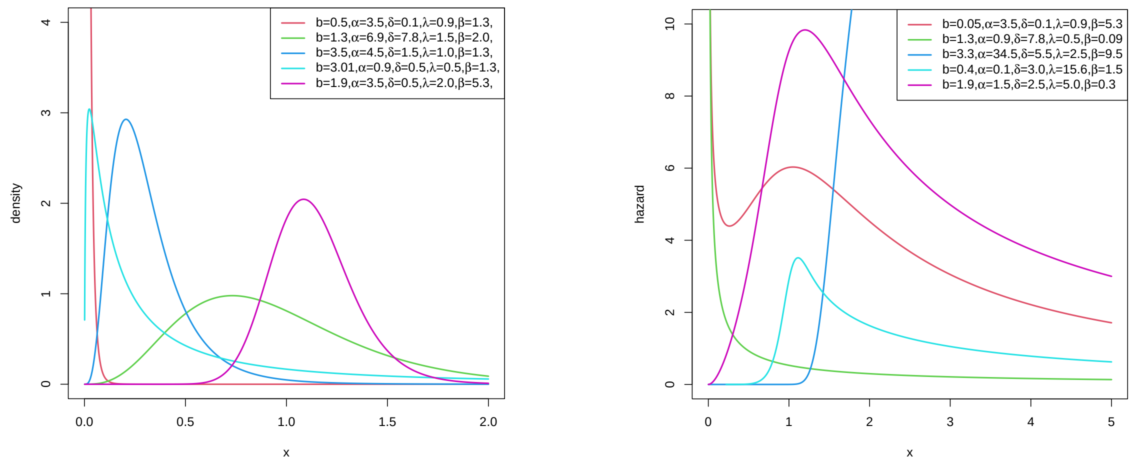

Figure 5. Plots of the Density and Hazard Rate Functions for the TL-TIIEHL-MO-BIII Distribution

Figure 5 shows the plots of the pdf and hrf of the TL-TIIEHL-MO-BIII distribution. The pdf of the TL-TIIEHL-MO-BIII distribution can take several shapes including left-skewed, right-skewed, almost symmetric and reverse-J shapes, while the hrf displays increasing, decreasing, upside down bathtub and bathtub followed by upside down bathtub shapes. Tables 4 and 5 give the quantiles and moments of TL-TIIEHL-MO-BIII distribution for selected parameter values.

Table 4. **Quantiles for TL-TIIEHL-MO-BIII Distribution**

			$(b, \alpha, \delta, \lambda, \beta)$		
u	$(2.1, 1.5, 1.0, 1.5, 1.5)$	$(1.5, 1.1, 1.5, 3.5, 1.0)$	$(1.0, 0.5, 1.7, 0.5, 2.8)$	$(1.0, 0.3, 2.9, 5.4, 15.2)$	$(0.8, 4.9, 2.3, 1.6, 0.9)$
0.1	0.3391	0.4998	0.0646	1.5168	0.0289
0.2	0.4313	0.5918	0.1280	1.6685	0.0583
0.3	0.5079	0.6589	0.2068	1.8027	0.0869
0.4	0.5849	0.7238	0.2994	1.9442	0.1158
0.5	0.6650	0.7852	0.4217	2.0963	0.1491
0.6	0.7543	0.8522	0.5804	2.2827	0.1873
0.7	0.8720	0.9257	0.8165	2.5253	0.2297
0.8	0.0305	1.0250	1.2040	2.8881	0.2932
0.9	0.3082	1.1834	2.0350	3.5989	0.3893

Table 5. **Moments for TL-TIIEHL-MO-BIII Distribution**

	$(b, \alpha, \delta, \lambda, \beta)$				
u	$(2, 1.5, 7.3, 1.9)$	$(1.3, 1.2, 0.6, 2.3)$	$(2.1, 2.9, 2.7, 1.8)$	$(1.1, 1.5, 2.5, 1.6)$	$(2.1, 1.5, 1.7, 1.9)$
E(X)	0.3806	0.8217	0.3112	1.0413	1.0399
E(X^2)	0.1707	0.7582	0.1178	1.0931	1.1635
E(X^3)	0.0889	0.7876	0.0519	1.1568	1.3917
E(X^4)	0.0537	0.9329	0.0259	1.2336	1.7716
E(X^5)	0.0375	1.2971	0.0144	1.3254	2.3911
E(X^6)	0.0304	2.2386	0.0088	1.4345	3.4122
SD	0.1604	0.2881	0.1450	0.0945	0.2864
CV	0.4213	0.3507	0.4659	0.0907	0.2754
CS	1.0782	1.1755	0.7026	-0.0762	0.4787
CK	5.3908	6.9988	3.6878	3.0952	3.4418

5.4. Topp-Leone-Type II Exponentiated Half Logistic-Marshall-Olkin-Burr XII (TL-TIIEHL-MO-BXII) Distribution

Suppose the baseline distribution is the Burr XII distribution with the cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$ and $g(x; c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}$, respectively, for $c, k, x > 0$. From equations (7) and (8), we obtain the cdf and pdf of the TL-TIIEHL-MO-BXII distribution as

$$F(x; b, \alpha, \delta, c, k) = \left[1 - \left(\frac{\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}}{1 + \left(1 - \frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)}\right)^{2\alpha}\right]^{b}$$

and

$$\begin{aligned} f(x; b, \alpha, \delta, c, k) &= \frac{4b\alpha\delta ckx^{c-1}(1 + x^c)^{-(k+1)}}{\left[1 - \bar{\delta}(1 + x^c)^{-k}\right]^2 \left[1 + \left(1 - \frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)\right]^2} \\ &\times \left[\frac{\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}}{1 + \left(1 - \frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)}\right]^{2\alpha-1} \\ &\times \left[1 - \left(\frac{\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}}{1 + \left(1 - \frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)}\right)^{2\alpha}\right]^{b-1}, \end{aligned}$$

for $b, \alpha, \delta, c, k > 0$ and $\bar{\delta} = 1 - \delta$.

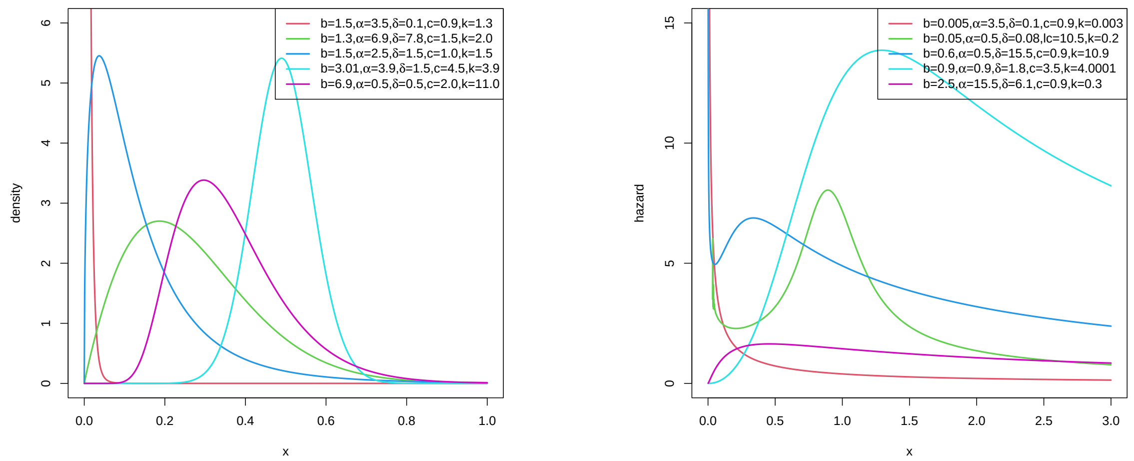

Figure 6. Plots of the Density and Hazard Rate Functions for the TL-TIIEHL-MO-BXII Distribution

Figure 6 shows the plots of the pdf and hrf of the TL-TIIEHL-MO-BXII distribution. The pdf of the TL-TIIEHL-MO-BXII distribution can take several shapes including left-skewed, right-skewed, almost symmetric and reverse-J shapes, while the hrf displays decreasing, upside down bathtub and bathtub followed by upside down bathtub shapes. Table 6 and 7 gives the quantiles and moments of TL-TIIEHL-MO-BXII distribution for selected parameter values.

Table 6. **Quantiles for TL-TIIEHL-MO-BXII Distribution**

	$(b, \alpha, \delta, c, k)$				
u	$(3.1, 1.5, 1.0, 1.5, 1.5)$	$(1.5, 1.1, 1.5, 3.5, 1.0)$	$(1.0, 0.2, 1.5, 1.2, 2.8)$	$(1.9, 29.9, 27.2, 4.1, 1.1)$	$(10.9, 1.0, 1.5, 2.8, 1.7)$
0.1	0.1843	0.4998	0.1145	0.5272	0.7725
0.2	0.2333	0.5918	0.2305	0.5899	0.8456
0.3	0.2793	0.6589	0.3632	0.6350	0.8968
0.4	0.3211	0.7238	0.5325	0.6734	0.9483
0.5	0.3671	0.7852	0.7515	0.7094	1.0050
0.6	0.4211	0.8522	1.0585	0.7454	1.0612
0.7	0.4875	0.9257	1.5254	0.7815	1.1271
0.8	0.5734	1.0250	2.3621	0.8253	1.2181
0.9	0.7218	1.1834	4.4833	0.8848	1.3613

Table 7. **Moments for TL-TIIEHL-MO-BXII Distribution**

	(b,α,δ,c,k)				
	$(3.1,1.5,1,1.5,1.5)$	$(1.0,43.1,0.1,3.5,0.1)$	$(2,6.5,2.5,3.5,1.8)$	$(1.0,0.3,2.9,9.4,10.2)$	$(1.9,1.0,1.5,0.5,2.9)$
E(X)	0.3077	0.8498	0.2600	0.2076	0.4533
E(X^2)	0.3077	0.8498	0.2600	0.2076	0.4533
E(X^3)	0.3077	0.8498	0.2600	0.2076	0.4533
E(X^4)	0.3077	0.8498	0.2600	0.2076	0.4533
E(X^5)	0.3077	0.8498	0.2600	0.2076	0.4533
E(X^6)	0.3077	0.8498	0.2600	0.2076	0.4533
SD	0.4615	0.3572	0.4386	0.4056	0.4978
CV	1.5001	0.4204	1.6871	1.9539	1.0982
CS	0.8335	-1.9585	1.0943	1.4421	0.1876
CK	1.6948	4.8358	2.1976	3.0797	1.0352

6. Simulation Study

The performance of the TL-TIIEHL-MO-W distribution is examined by conducting various simulations for different sample sizes (n=100, 200, 400, 800 and 1600) via the R package. We simulate $N = 3000$ samples for the true parameter values given in Table 8. The tables list the mean MLEs of the model parameters along with the respective average bias (ABias) and root mean square errors (RMSEs). The ABias and RMSE for the estimated parameter, say $\hat{\theta}$ are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

The results in Tables 8 and 9 show that the mean estimates of the parameters are approaching the true parameter value as the sample size increases. Moreover, the RMSEs and the average bias decrease with increasing sample size thereby suggesting consistency of the MLE of the model parameters.

7. Actuarial Measures

Actuaries are mostly concerned with evaluating the exposure of market risk in a portfolio of instruments. Risk measures are calculated in this section, and these includes value at risk (VaRq), tail value at risk ($TVaR_q$), tail variance (TV_q) and tail variance premium (TVP_q) for the TL-TIIEHL-MO-G family of distributions. These risk measures play a very important role in portfolio optimization under uncertainty.

7.1. Risk Measures

- **Value at Risk Measure**

  Let X follow the TL-TIIEHL-MO-G family of distributions with pdf (8), then the VaR_q, where q is a specified level of significance, is given by

$$VaR_q = x_q = G^{-1}\left(1 - \frac{2\left(1-q^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}}}{2\left(1-q^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}} + \delta\left(1-q^{\frac{1}{b}}\right)^{\frac{1}{2\alpha}} + \delta}\right),$$

  for $0 \leq q \leq 1$.

Table 8. **Simulation Results**

		$(4.1, 0.1, 0.1, 2.1)$			$(1.0, 0.1, 1.0, 1.3)$		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias
b	100	5.3787	3.5259	0.4788	1.8450	2.4900	0.8450
	200	5.1118	2.0734	0.2118	1.4298	1.4006	0.4298
	400	5.0500	1.5686	0.1500	1.1771	0.7765	0.1771
	800	5.0422	1.3369	0.1422	1.0949	0.4237	0.0949
	1600	4.9765	1.0881	0.0765	1.0538	0.2231	0.0538
α	100	0.1203	0.0768	0.0203	0.1614	0.1609	0.0614
	200	0.1069	0.0303	0.0069	0.1350	0.1209	0.0350
	400	0.1023	0.0190	0.0023	0.1156	0.0833	0.0156
	800	0.1011	0.0138	0.0011	0.1057	0.0292	0.0057
	1600	0.1002	0.0109	0.0002	0.1032	0.0170	0.0032
δ	100	0.4402	2.1932	0.3402	1.6577	2.2090	0.6577
	200	0.2323	0.6150	0.1323	1.4261	1.7708	0.4261
	400	0.1631	0.4009	0.0631	1.1608	1.0344	0.1608
	800	0.1257	0.1456	0.0257	1.0186	0.5122	0.0186
	1600	0.1172	0.1088	0.0172	0.9721	0.3100	-0.0279
λ	100	2.0482	0.2704	-0.0518	1.2565	0.3381	-0.0435
	200	2.0834	0.2154	-0.0166	1.2643	0.2341	-0.0357
	400	2.0997	0.1524	-0.0003	1.2868	0.1577	-0.0132
	800	2.1005	0.1010	0.0005	1.2941	0.1036	-0.0059
	1600	2.1033	0.0826	0.0033	1.2957	0.0724	-0.0043

- **Tail Value at Risk Measure**
  This measure is used to determine the expected value of loss given that an event outside a given probability level has occurred. Let X have TLTIIEHLMO-G pdf, then using equations (10) and (11), $TVaR_q$ of X is computed as

$$\begin{aligned} TVaR_q(X) &= \frac{1}{1-q}\int_{VaR_q}^{\infty} xf(x)dx \\ &= \frac{1}{1-q}\sum_{m=0}^{\infty}\int_{VaR_q}^{\infty} xt_{m+1}h_{m+1}(x;\xi)dx, \end{aligned}$$

  where t_{m+1} is defined as equation (11) and $h_{m+1}(x;\xi) = (m+1)\,g(x;\xi)G^m(x;\xi)$ is the exp-G density with power parameter $(m+1)$ and parameter vector ξ.
- **Tail Variance Measure**
  Tail variance (TV_q) is an actuarial measure that pays attention to the tail variance beyond VaR_q. TV_q is given

Table 9. **Simulation Results**

		$(17.5, 0.1, 17.5, 1.3)$			$(1.0, 0.1, 0.1, 1.3)$		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias
b	100	19.3075	11.3343	1.8075	1.5666	1.3439	0.5666
	200	18.4400	4.7341	0.9400	1.3793	0.8679	0.3793
	400	17.9752	3.2051	0.4752	1.2957	0.6633	0.2957
	800	17.7162	1.9504	0.2162	1.2038	0.3934	0.2038
	1600	17.600	1.1395	0.0995	1.1969	0.2651	0.1969
α	100	0.1461	0.1617	0.0461	0.1229	0.0569	0.0229
	200	0.1282	0.1112	0.0282	0.1127	0.0347	0.0127
	400	0.1138	0.0790	0.0138	0.1095	0.0241	0.0095
	800	0.1049	0.0289	0.0049	0.1074	0.0165	0.0074
	1600	0.1022	0.0156	0.0022	0.1077	0.0125	0.0077
δ	100	17.8596	1.6761	0.3596	0.1567	0.3110	0.0567
	200	17.7310	1.1851	0.2310	0.1115	0.1286	0.0115
	400	17.6105	0.7925	0.1105	0.0914	0.0617	-0.0086
	800	17.5509	0.4825	0.0509	0.0842	0.0406	-0.0158
	1600	17.5218	0.2741	0.0218	0.0773	0.0350	-0.0227
λ	100	1.2747	0.2247	-0.0253	1.2638	0.2131	-0.0362
	200	1.2762	0.1726	-0.0238	1.2758	0.1531	-0.0244
	400	1.2878	0.1206	-0.0122	1.2785	0.1078	-0.0215
	800	1.2943	0.0801	-0.0057	1.2813	0.0732	-0.0187
	1600	1.2965	0.0511	-0.0035	1.2738	0.0546	-0.0262

as

$$\begin{aligned} TV_q &= E\left(X^2|X > x_q\right) - (TVaR_q)^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f(x)dx - (TVaR_q)^2 \\ &= \frac{1}{1-q}\sum_{m=0}^{\infty}\int_{VaR_q}^{\infty} x^2 t_{m+1} h_{m+1}(x;\xi)dx - (TVaR_q)^2, \end{aligned}$$

where t_{m+1} is defined as equation (11) and $h_{m+1}(x;\xi) = (m+1)\, g(x;\xi)G^m(x;\xi)$ is the exp-G density with power parameter $(m+1)$ and parameter vector ξ.

- **Tail Variance Premium Measure**
  Tail variance premium (TVP_q) is one of the important actuarial measures and is given by $TVP_q = TVaR_q + \delta\,(TV_q)$ for $0 < \delta < 1$.

7.2. Numerical Study of Actuarial Measures

In this section, numerical study of actuarial measures is done for the TL-TIIEHL-MO-W, TL-TIIEHL-MO-W nested models when $\alpha = 1$ and when $b = 1$, Weibull and Topp-Leone Marshall-Olkin-Weibull (TL-MO-W) distributions for different sets of parameters. A random sample of size $n = 100$ is generated from these distributions, and the maximum likelihood method of estimation is used to estimate the model parameters. Secondly, a repetition of 1000 iterations are made in order to find the values of the risk measures for the distributions. The model with the highest values of risk measures has the heavier tail. From the table, we note that TL-TIIEHL-MO-W distribution has heavier tails as compared to other distributions as it has the highest values of risk measures.

Table 10. **Simulation Results of VaR, TVaR, TV and TVP**

Distribution	Level of Significance	VaR	TVaR	TV	TVP
TL-TIIEHL-MO-W	0.7	2.8848	7.7835	94.6708	74.0530
	0.75	3.5640	8.6982	101.1911	84.5915
α=0.8	0.8	4.4599	9.8754	109.3414	97.3485
δ=1.3	0.85	5.7158	11.4843	120.1144	113.5816
b=0.59	0.9	7.6707	13.9216	135.7812	136.1247
λ=0.3	0.95	11.4902	18.5304	163.7639	174.1061
TL-TIIEHL-MO-W	0.7	0.0922	0.5672	15.5559	11.4563
α=1	0.75	0.1141	0.6602	18.6099	14.6175
δ=1.3	0.8	0.1497	0.7926	23.1641	19.3239
b=0.59	0.85	0.2166	0.9971	30.6948	27.0877
λ=0.3	0.9	0.3742	1.3543	45.5987	42.3931
	0.95	0.9488	2.1208	89.8090	87.4393
TL-TIIEHL-MO-W	0.7	0.0099	0.4341	20.3403	14.6724
α=0.8	0.75	0.0180	0.5182	24.3615	18.7894
δ=0.5	0.8	0.0348	0.6415	30.3682	24.9360
b=1	0.85	0.0741	0.8381	40.3208	35.1108
λ=0.3	0.9	0.1886	1.1971	60.0598	55.2509
	0.95	0.7180	2.0184	118.6663	114.7513
	0.7	0.0298	0.6120	13.8932	10.3373
	0.75	0.0503	0.7266	16.5931	13.1714
Weibull	0.8	0.0885	0.8914	20.6055	17.3758
λ=0.3	0.85	0.1666	1.1477	27.2111	24.2771
	0.9	0.3559	1.5986	40.2050	37.7832
	0.95	1.0219	2.5902	78.4084	77.0782
TL-MO-W	0.7	0.3272	1.8236	4.3453	4.8654
b=0.8	0.75	0.5246	2.1049	4.7299	5.6523
δ=1.3	0.8	0.8634	2.4616	5.2604	6.6699
λ=0.9	0.85	1.4918	2.9011	6.1887	8.1615
γ=0.3	0.9	2.8359	3.3172	8.3351	10.8187
	0.95	6.7990	2.2429	5.1144	7.1015
	0.7	1.0342	1.9792	1.0071	2.6842
APW	0.75	1.1985	2.1524	1.0240	2.9204
$\alpha = 1.5$	0.8	1.4021	2.3664	1.0443	3.2018
$\beta = 0.9$	0.85	1.6686	2.6457	1.0697	3.5549
$\lambda = 1.3$	0.9	2.0515	3.0452	1.1044	4.0392
	0.95	2.7237	3.7431	1.1613	4.8464
APTLW	0.7	0.6018	1.5085	-0.4191	1.2151
$\alpha = 1.9$	0.75	0.6429	1.5485	-0.3342	1.2978
$\beta = 0.7$	0.8	0.6891	1.5958	-0.1967	1.4384
$c = 1.1$	0.85	0.7433	1.6447	0.0769	1.7100
$\lambda = 0.9$	0.9	0.8118	1.6600	0.7850	2.3665
	0.95	0.9137	1.3451	3.6001	4.7652

8. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the TL-TIIEHL-MO-W distribution for data modelling. Goodness-of-fit statistics are used to compare TL-TIIEHL-MO-W distribution to the nested and non-nested models. These include: -2log-likelihood statistic (-2ln(L)), Akaike Information Criterion (AIC = 2p - 2ln(L)), Consistent Akaike Information Criterion $\left(CAIC = AIC + \frac{2p(p+1)}{n-p-1}\right)$, Bayesian Information Criterion (BIC = pln(n) -2ln(L)), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. We also used the Cramér-von Mises (W^*), Anderson-Darling (A^*), the Kolmogorov-Smirnov (K-S) statistic as well as its associated p-value and Sum of Squares from the probability plots to assess goodness-of-fit. The Sum of Squares (SS) described by Chen and Balakrishnan [8] is given by $SS = \sum_{j=1}^{n}\left[F(x_{(j)}) - \left(\dfrac{j-0.375}{n+0.25}\right)\right]^2$, where j=1,2,...,n and $x_{(j)}$ are ordered values of the observed data. In general, the smaller the values of goodness-of-fit and the highest p-value for the K-S statistic, the better the fit.

The TL-TIIEHL-MO-W distribution is fitted to data sets and these fits are compared to several non-nested models including; beta odd-Lindley-Uniform (BOLU) distribution by (Chipepa et al. [10]), odd exponentiated half logistic-Burr XII (OEHLBXII) distribution by (Aldahlan and Afify [3]), Weibull-Lomax (WLx) distribution by (Afify et al. [2]), exponentiated Lindley odd log-logistic Weibull (ELOLLW) distribution by (Korkmaz et al. [15]), Weibull-Burr XII (WBXII) distribution by (Afify et al. [2]), Topp-Leone-Marshall-Olkin-Weibull (TL-MO-W) distribution by (Chipepa et al. [9]), type II generalized inverse exponential Burr III distribution by (Jamal et al. [14]) and Topp-Leone-odd Burr III-Log-Logistic (TL-OBIII-LLoG) by (Moakofi et al. [19]). The pdfs of these distributions are given as follows:

$$\begin{aligned} f_{BOLU}(x;a,b,\lambda,\theta) &= \frac{1}{B(a,b)}\left[1-\frac{\lambda+\left(1-\frac{x}{\theta}\right)}{(1+\lambda)\left(1-\frac{x}{\theta}\right)}e^{-\lambda\frac{x}{\theta-x}}\right]^{a-1} \\ &\times \left[\frac{\lambda+\left(1-\frac{x}{\theta}\right)}{(1+\lambda)\left(1-\frac{x}{\theta}\right)}e^{-\lambda\frac{x}{\theta-x}}\right]^{b-1}\frac{\lambda^2\theta^2}{(1+\lambda)(\theta-x)^3}e^{-\lambda\frac{x}{\theta-x}}, \end{aligned}$$

for $a, b, \lambda, \theta > 0$,

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b) = \frac{2\alpha\lambda abx^{a-1}e^{\lambda[1-(1+x^a)^b]}\left(1-e^{\lambda[1-(1+x^a)^b]}\right)^{\alpha-1}}{(1+x^a)^{-b+1}\left(1+e^{\lambda[1-(1+x^a)^b]}\right)^{\alpha+1}},$$

for $\alpha, \lambda, a, b > 0$,

$$\begin{aligned} f_{WLx}(x;a,b,\lambda,\beta) &= \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\alpha-1}\left(1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right)^{b-1} \\ &\times e^{(-a([1+(\frac{x}{\beta}t)]^{\alpha}-1)^b)}, \end{aligned}$$

for $a, b, \lambda, \beta > 0$,

$$\begin{aligned} f_{ELOLLW}(x;\beta,\lambda,\theta,\gamma) &= \frac{\theta^2\gamma\lambda^\gamma x^{\gamma-1}e^{-(\lambda x)^\gamma}\left(e^{-(\lambda x)}\right)^{\theta-1}}{(\theta+\beta)\left[\left(1-e^{-(\lambda x)^\gamma}\right)+e^{-(\lambda x)^\gamma}\right]^{\theta-1}} \\ &\times \left(1-\beta log\left[\frac{e^{-(\lambda x)^\gamma}}{\left(1-e^{-(\lambda x)^\gamma}\right)+e^{-(\lambda x)^\gamma}}\right]\right), \end{aligned}$$

for $\beta, \lambda, \theta, \gamma > 0$,

$$f_{WBXII}(x;a,b,\alpha,\beta) = ab\alpha\beta x^{\alpha-1}\frac{\left[1-(1+x^{\alpha})^{-\beta}\right]^{b-1}}{(1+x^{\alpha})^{-\beta b+1}}e^{(-a[(1+x^{\alpha})^{\beta-1}]^{b})},$$

for $a, b, \alpha, \beta > 0$,

$$f_{TL-MO-W}(x;b,\delta,\lambda,\gamma) = \frac{2b\delta^{2}\gamma\lambda^{\gamma}x^{\gamma-1}e^{-2(\lambda x)^{\gamma}}}{\left(1-\bar{\delta}e^{-(\lambda x)^{\gamma}}\right)^{3}}\left[1-\left(\frac{\delta e^{-(\lambda x)^{\gamma}}}{1-\bar{\delta}e^{-(\lambda x)^{\gamma}}}\right)^{2}\right)^{b-1}$$

for $b, \delta, \lambda, \gamma > 0$ and $x > 0$,

$$\begin{aligned} f_{TIIGIEBIII}(x;\lambda,\theta,c,k) &= \frac{\lambda\theta ckx^{-c-1}\left(1-[1+x^{-c}]^{-k}\right)^{\theta-1}}{\left(1-\left(1-[1+x^{-c}]^{-k}\right)^{\theta}\right)^{2}} \\ &\times \exp\left(\frac{\left(1-[1+x^{-c}]^{-k}\right)^{\theta}}{1-\left(1-[1+x^{-c}]^{-k}\right)^{\theta}}\right), \end{aligned} \tag{16}$$

for $\lambda, \theta, c, k > 0$ and $x > 0$, and

$$\begin{aligned} f_{TL-OBIII-LLoG}(x;\alpha,\beta,b,\lambda) &= 2\alpha\beta b\left[1-\left(1-\left[1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right]^{-\beta}\right)^{2}\right]^{b-1} \\ &\times \left(1-\left[1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right]^{-\beta}\right)\frac{g(x;\lambda)}{\left((1+x^{\lambda})^{-1}\right)^{2}} \\ &\times \left[1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right]^{-\beta-1}\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha-1}, \end{aligned}$$

for $\alpha, \beta, b, \lambda > 0$ and $x > 0$.

8.1. Monthly Actual Taxes Revenue Data

The dataset by Owoloko et al. [20] represents monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010. The data are: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

Table 11. **Parameter estimates and standard errors for monthly actual taxes revenue data**

	Estimates			
Distribution	α	δ	b	λ
TL-TIIEHL-MO-W	3.1360×10^{-2}	2.2368×10^{4}	6.3028×10^{-1}	1.1523
	(1.1184×10^{-2})	(5.4359×10^{-7})	(1.6828×10^{-1})	(9.5334×10^{-2})
	a	b	λ	θ
BOLU	4.1986	2.0023	6.4991×10^{5}	6.7906×10^{6}
	(7.7075×10^{-1})	(3.4466×10^{-1})	(1.0966×10^{-6})	(1.0495×10^{-7})
	α	λ	a	b
HLBXII	0.4238	0.004	0.220	0.4146
	(9.9031×10^{-6})	(5.3626×10^{-6})	(7.9375×10^{-1})	(2.7485×10^{-2})
	a	b	α	β
WLx	1.0085×10^{3}	3.9950	8.2644×10^{-2}	2.3766
	(5.8146×10^{-4})	(1.1958)	(1.0840×10^{-2})	(2.6689)
	b	λ	θ	γ
ELLOW	4.6910×10^{2}	1.9444	2.6754×10^{-2}	1.1041×10^{-1}
	(1.1247×10^{-4})	(4.7698)	(8.5543×10^{21})	(1.1846×10^{-1})
	a	b	α	β
WBXII	3.1271	3.8340	7.2981	0.0283
	(3.6981)	(0.4076)	(3.5795)	(0.0139)
	b	δ	λ	γ
TLMOW	1.8119×10^{1}	1.4327×10^{1}	3.1140	2.2770×10^{-1}
	(4.6107×10^{-4})	(9.6691×10^{-3})	(1.5534×10^{-1})	(2.1943×10^{-2})
	λ	θ	c	k
TIIGIEBIII	2.9882×10^{2}	4.8520×10^{1}	2.0899×10^{-1}	4.5252
	(2.5384×10^{-4})	(4.2122×10^{-3})	(1.8687×10^{-2})	(1.5073×10^{-1})
	α	β	b	λ
TLOBIIILLoG	0.2128	1.8442	10.6021	3.4638
	(0.0433)	(0.2938)	(2.3694)	(0.7041)

Table 12. **Goodness-of-fit statistics for various models for monthly actual taxes revenue data**

Distribution	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
TL-TIIEHL-MO-W	375.7873	383.7872	384.5280	392.0970	0.0267	0.2001	0.0527	0.9967	0.0258
BOLU	384.6959	392.7924	393.5331	401.1025	0.1752	1.0752	0.1458	0.1627	0.2524
HLBXII	428.8007	436.8007	437.5415	445.1109	0.5026	3.2171	0.1907	0.0274	0.7101
WLx	386.9186	394.9186	395.6593	403.2287	0.1999	1.2443	0.1258	0.3077	0.1805
ELLOW	389.1109	397.1109	397.8516	405.4211	0.2288	1.4391	0.1349	0.2334	0.2127
WBXII	385.4740	393.4739	394.2147	401.7841	0.1826	1.1382	0.1254	0.3115	0.1763
TLMOW	376.5911	384.5911	385.3319	392.9013	0.0442	0.2651	0.0648	0.9655	0.0444
TIIGIEBIII	376.5622	384.5622	385.3029	392.8723	20.0432	118.3365	0.9942	2.2×10^{-16}	19.8957
TLOBIIILLoG	410.3740	495.3846	496.3846	503.9540	0.0538	0.3121	0.6096	2.2×10^{-16}	0.6414

Table 13. **Parameter estimates and goodness-of-fit statistics for nested models for the monthly actual taxes revenue data**

	Estimates											
Distribution	α	δ	b	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value
TLTIIEHLMOW $(\alpha, 1, b, \lambda)$	0.0313	1	2.9802	1.2695	386.5603	392.5603	392.9967	398.7929	0.1988	1.2311	0.1418	0.1863
	(0.0238)	-	(1.4035)	(0.2037)								
TLTIIEHLMOW $(1, \delta, b, 1)$	1	7.377×10^{12}	5.0993×10^{-18}	1	556.9123	4819.3490	4819.5640	4823.5040	2.8671	15.1110	1.0000	2.2×10^{-16}
	-	(6.5280×10^{-2})	(8.4987×10^{3})	-								
TLTIIEHLMOW $(1, \delta, 1, \lambda)$	1	263.4774	1	0.5560	399.0322	403.2465	403.2465	407.1872	0.3406	2.1013	0.1297	0.2739
	-	(87.7731)	-	(0.0244)								
TLTIIEHLMOW $(\alpha, 1, b, 1)$	0.0906	1	6.5497	1	382.9535	386.9538	387.1678	391.1086	0.1502	0.9070	0.1162	0.4035
	0.0116	-	(1.7752)	-								

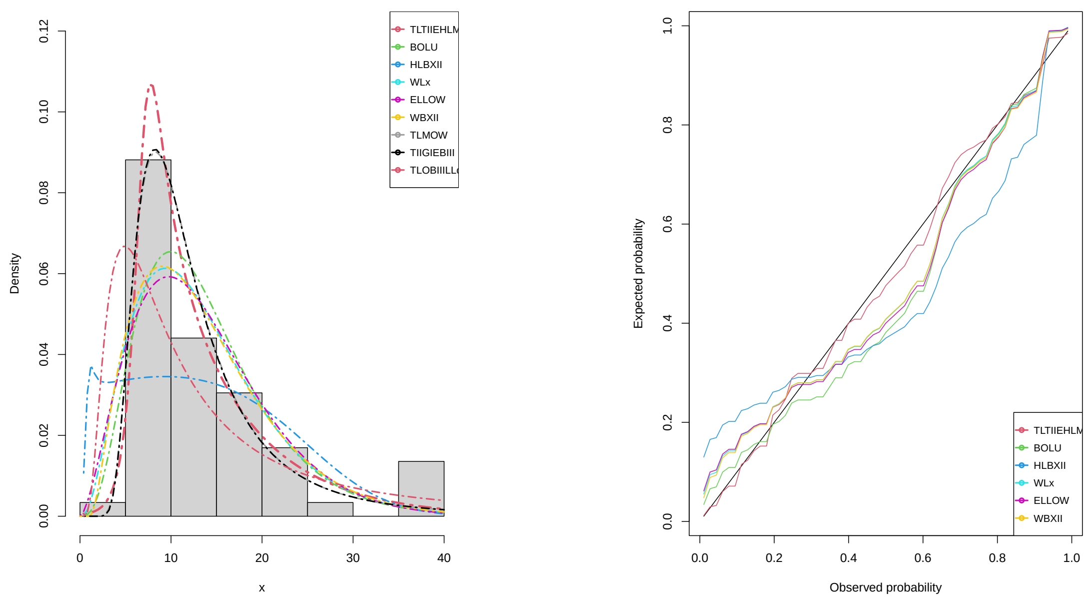

Figure 7. Fitted densities and probability plots for monthly actual taxes revenue data

The MLEs of the parameters with standard errors in parenthesis and the values of the goodness-of-fit statistics $-2\log(L)$, AIC, AICC, BIC, W^*, A^*, K-S, p-value of the K-S statistic and the SS are given in Tables 11 and 13.

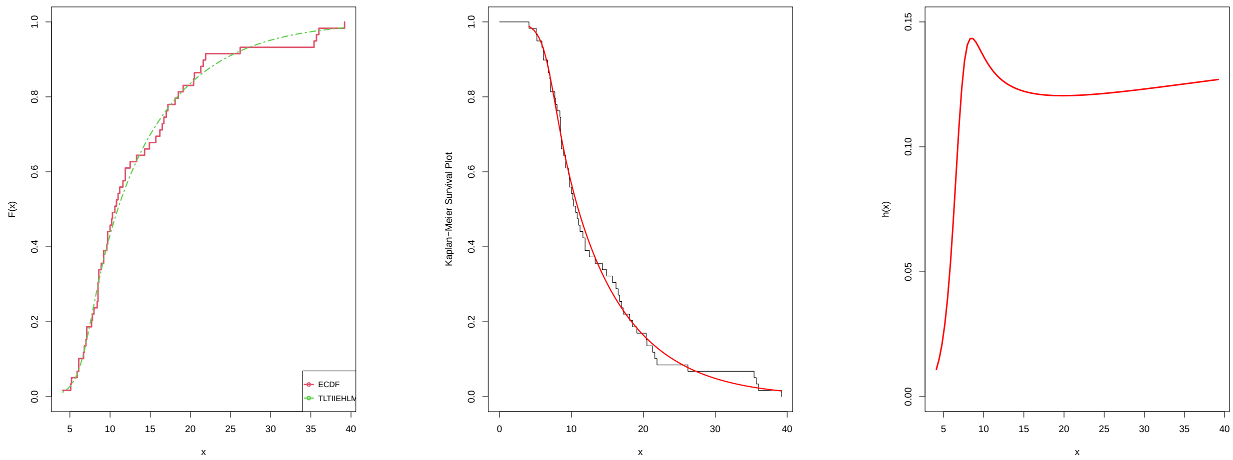

Figure 8. Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate function plot of the TL-TIIEHL-MO-W distribution for monthly actual taxes revenue data

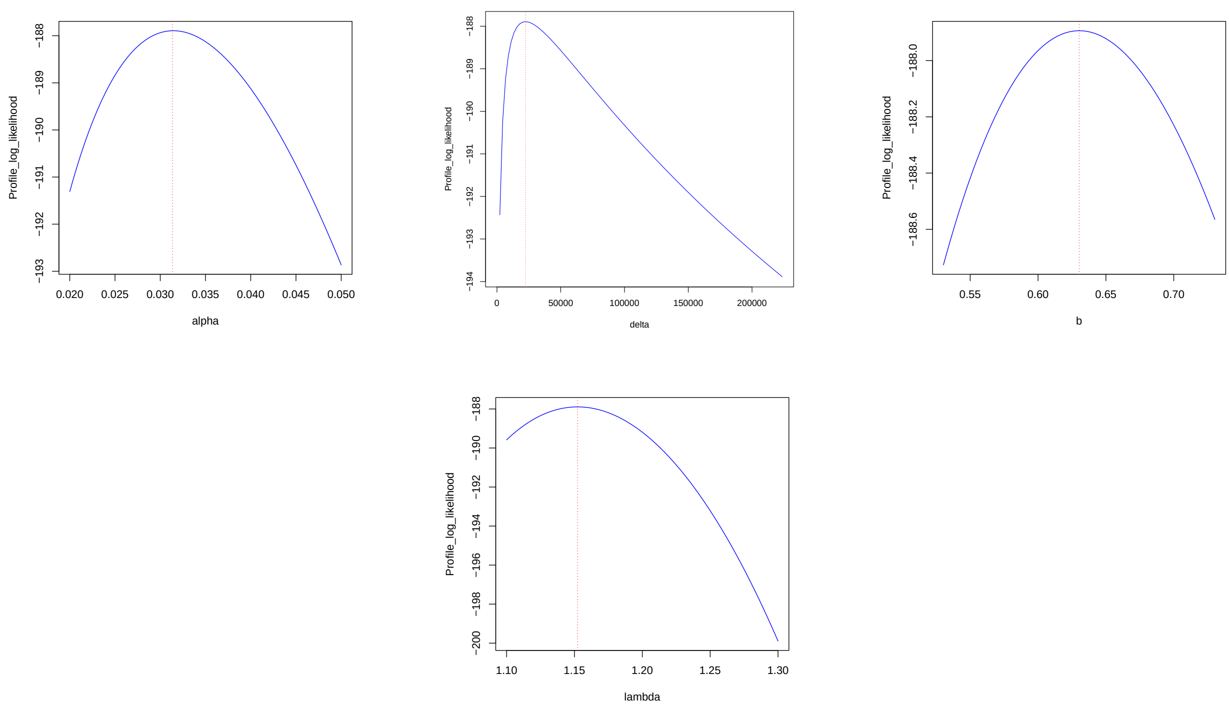

Figure 9. Profile plots of the TL-TIIEHL-MO-W distribution for monthly actual taxes revenue data

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 7 and 8, we can conclude that the TL-TIIEHL-MO-W distribution provide a better fit compared to the other models. The estimated variance-covariance matrix for monthly actual taxes revenue data is given by,

$$\begin{bmatrix} 1.2509\times10^{-4} & -2.2874\times10^{-9} & -9.2039\times10^{-4} & -8.9266\times10^{-4} \\ -2.2874\times10^{-9} & 2.9549\times10^{-8} & 9.0535\times10^{-8} & 3.6893\times10^{-8} \\ -9.2039\times10^{-4} & 9.0535\times10^{-8} & 2.8318\times10^{-2} & 1.2914\times10^{-2} \\ -8.9266\times10^{-4} & 3.6893\times10^{-8} & 1.2914\times10^{-2} & 9.0886\times10^{-3} \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters α, δ, b and λ are: $3.1360\times10^{-2} \pm 2.1921\times10^{-2}$, $2.2368\times10^{4} \pm 1.0654\times10^{-6}$, $6.3028\times10^{-1} \pm 3.2983\times10^{-1}$ and $1.1523 \pm 1.8686\times10^{-1}$, respectively.

8.2. Strength of Glass Fibre Data

The dataset by Bourguignon et al. [7] represents strengths of 1.5 cm glass fibres. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 14. **Parameter estimates and standard errors for various models for strength of glass fibre data**

	Estimates			
Distribution	α	δ	b	λ
TL-TIIEHL-MO-W	1.9142×10^{-1}	5.0193×10^{2}	4.3334×10^{-1}	3.6921
	(8.8239×10^{-2})	(1.9065×10^{-4})	(6.4697×10^{-2})	(2.6555×10^{-1})
	a	b	λ	θ
BOLU	3.7867	6.7321×10^{1}	2.0304×10^{-1}	2.9970
	(1.1993)	(3.4167×10^{-4})	(6.6882×10^{-2})	(2.9072×10^{-1})
	α	λ	a	b
HLBXII	0.3225	0.0030	11.8172	0.8356
	(0.0670)	(0.0036)	(0.0075)	(0.1347)
	a	b	α	β
WLx	7.8577×10^{-2}	3.5565	8.8711×10^{2}	1.3052×10^{3}
	(2.4487×10^{-2})	(3.4240×10^{-1})	(5.9992×10^{-4})	(4.0690×10^{-4})
	b	λ	θ	γ
ELLOW	0.8916	0.7976	0.4768	4.9441
	(1.1036)	(0.6241)	(0.1496)	(1.1094)
	a	b	α	β
WBXII	0.0404	0.4550	6.7113	2.1037
	(2.0150×10^{-6})	(8.1007×10^{-4})	(9.2488×10^{-3})	(9.7161×10^{-5})
	b	δ	λ	γ
TLMOW	0.9632	80.0679	1.2887	2.2492
	(0.3603)	(246.1160)	(1.6534)	(1.2011)
	λ	θ	c	k
TIIGIEBIII	7.1957	8.6050×10^{3}	2.4668×10^{-1}	1.2591×10^{1}
	(6.0534)	(3.2315×10^{-4})	(4.8895×10^{-2})	(6.2896×10^{-1})
	α	β	b	λ
TLOBIIILLoG	6.6524	9.0034	0.4501	0.5263
	(58.6495)	(5.4007)	(0.31064)	(4.6403)

Table 16. **Parameter estimates and goodness-of-fit statistics for nested models for strength of glass fibre data**

	Estimates											
Distribution	α	δ	b	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value
TLTIIEHLMOW $(\alpha, 1, b, \lambda)$	3.8012	1	1.7111×10^{5}	3.1669×10^{-1}	84.3866	90.3865	90.7933	96.8159	1.1088	5.9196	0.2565	0.0005
	(4.3197×10^{-2})	-	(8.6511×10^{-8})	(2.6054×10^{-2})								
TLTIIEHLMOW $(1, \delta, b, 1)$	1	1.2269	31.0220	1	68.4735	72.4735	72.6735	76.7598	0.7845	4.2750	0.2219	0.0040
	-	(0.5786)	(23.2634)	-								
TLTIIEHLMOW $(1, \delta, 1, \lambda)$	1	92.9772	1	2.4886	24.6102	28.6102	28.8102	32.8965	0.0954	0.5511	0.1076	0.4597
	-	(27.2280)	-	(0.1335)								
TLTIIEHLMOW $(\alpha, 1, b, 1)$	1.1045	1	60.9815	1	67.4617	71.4617	71.6617	75.7479	0.8549	4.6416	0.2318	0.0023
	(0.0972)	-	(20.9903)	-								

Table 15. **Goodness-of-fit statistics for various models for strength of glass fibre data**

Distribution	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
TL-TIIEHL-MO-W	21.9785	29.9785	30.6882	38.5510	0.0370	0.2572	0.0650	0.9530	0.0353
BOLU	29.9557	37.9557	38.6454	46.5282	0.2026	1.1312	0.1427	0.1536	0.1915
HLBXII	50.3244	58.3244	59.0141	66.8969	0.2417	1.3747	0.1423	0.1558	0.3093
WLx	28.9130	36.9130	37.6027	45.4855	0.1874	1.0450	0.1403	0.1672	0.1758
ELLOW	28.4214	36.4214	37.1111	44.9940	0.1953	1.0753	0.1367	0.1896	0.1627
WBXII	27.3639	35.3639	36.0536	43.9365	0.1449	0.8150	0.1283	0.2507	0.1386
TLMOW	24.5685	32.5685	33.2581	41.1410	0.0918	0.5322	0.1063	0.4746	0.0818
TIIGIEBIII	64.6483	72.6490	73.3387	81.2216	17.4783	119.6339	0.99996	2.2×10^{-16}	17.0399
TLOBIIILLoG	62.6225	70.6225	71.3122	71.1951	0.7957	4.3547	0.2448	0.0011	0.8324

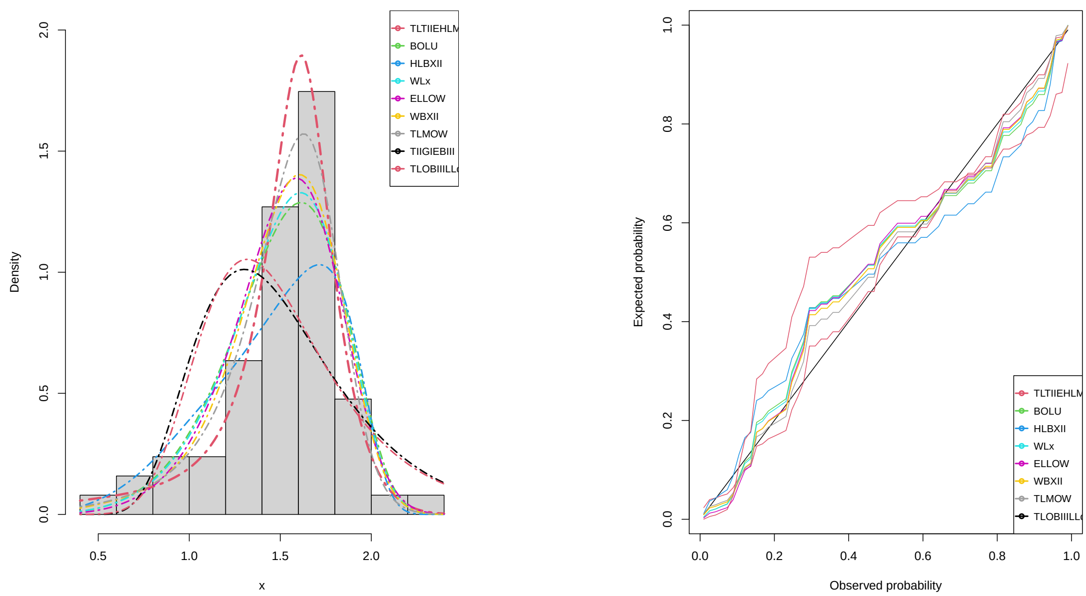

Figure 10. Fitted densities and probability plots for strength of glass fibre data

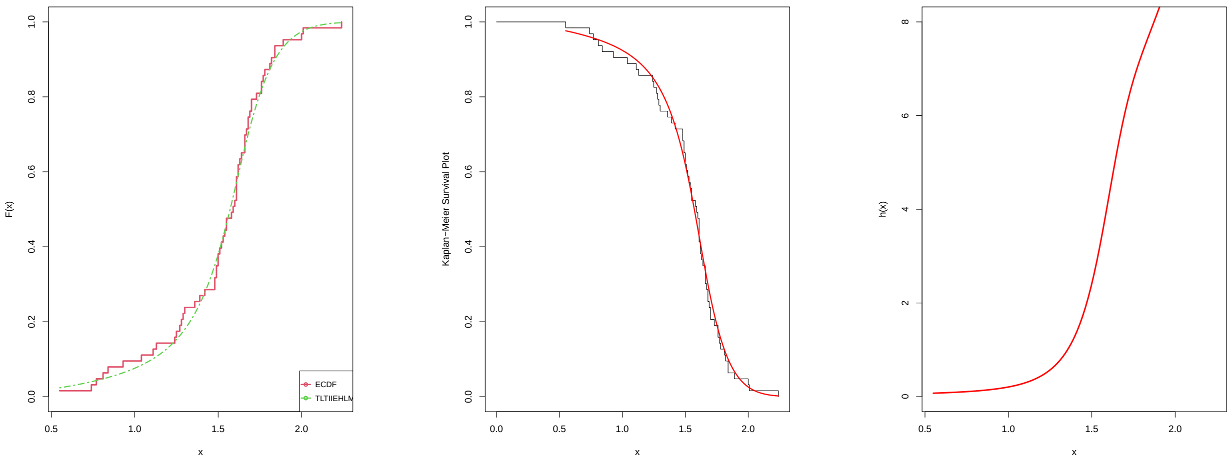

Figure 11. Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate function plot of the TL-TIIEHL-MO-W distribution for strength of glass fibre data

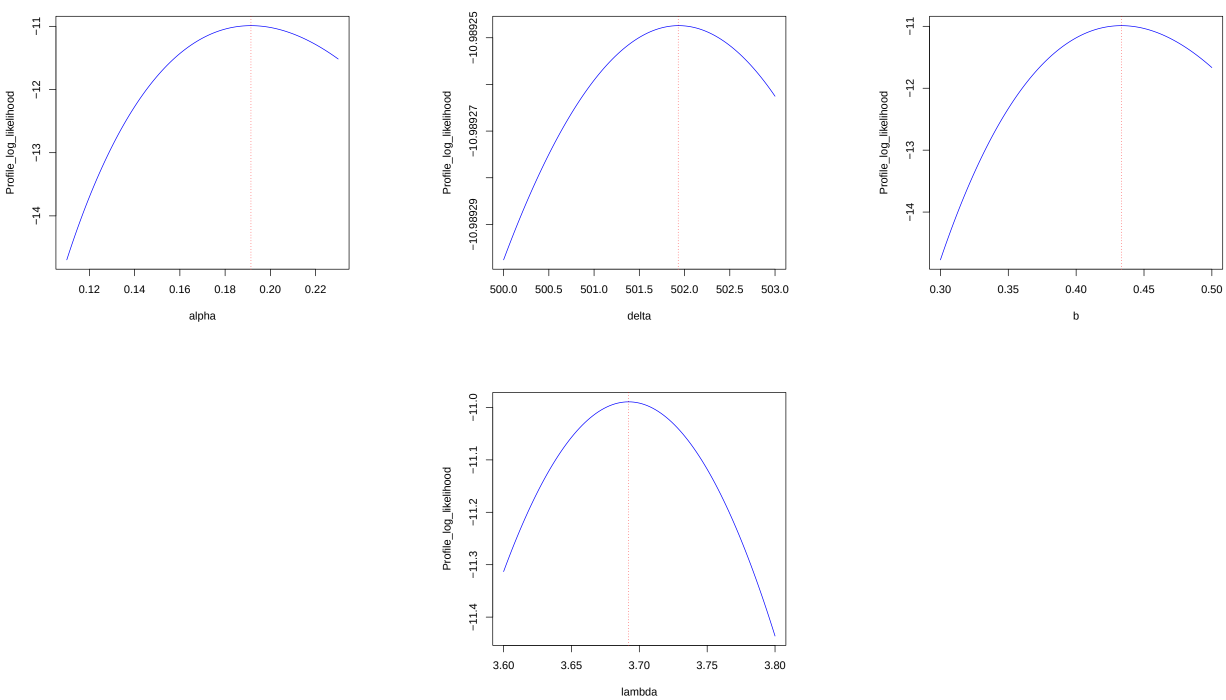

Figure 12. Profile plots of the TL-TIIEHL-MO-W distribution for strength of glass fibre data

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 10 and 11, we can conclude that the TL-TIIEHL-MO-W distribution provide a better fit compared to the other models. The estimated variance-covariance matrix for strength of glass fibre data is given by,

$$\begin{bmatrix} 7.7862 \times 10^{-3} & 1.5341 \times 10^{-5} & 2.3160 \times 10^{-3} & -2.0731 \times 10^{-2} \\ 1.5341 \times 10^{-5} & 3.6348 \times 10^{-8} & 3.4459 \times 10^{-6} & -5.0420 \times 10^{-5} \\ 2.3160 \times 10^{-3} & 3.4459 \times 10^{-6} & 4.1857 \times 10^{-3} & -3.3549 \times 10^{-3} \\ -2.0731 \times 10^{-2} & -5.0420 \times 10^{-5} & -3.3549 \times 10^{-3} & 0.0705 \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters α, δ, b and λ are: $1.9142\times10^{-1} \pm 0.1729$, $5.0193 \times10^{2} \pm 0.0004$, $4.3334 \times10^{-1} \pm 0.1268$ and 3.6921 ± 0.5205, respectively.

Table 17. Likelihood Ratio Test Results

		Monthly Actual Taxes Revenue Data	Strength of Glass Fibre Data
Model	df	$\chi^2(p-value)$	$\chi^2(p-value)$
TL-TIIEHLMO-W(α, 1, b, λ)	1	10.7730(0.00103)	62.408($<$0.00001)
TL-TIIEHLMO-W(1, δ, b, 1)	2	181.125($<$0.00001)	46.495($<$0.00001)
TL-TIIEHLMO-W(α, 1, b, 1)	2	7.1662(0.027792)	45.4832($<$0.00001)

The likelihood ratio test results in Table 17 show that the TL-TIIEHL-MO-W distribution is significantly different and performs better than its nested models at 5 percent level of significance, since all p-values are small and less than 0.05 for all the data sets considered.

8.3. *Applications for Censored Data*

8.3.1. Type I Right Censoring The maintenance dataset containing both the complete dataset and censored dataset was analysed by Fagbamigbe et al [11]. The complete data are: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Type I right censored maintenance data are: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4.

Table 18. **Parameter estimates and goodness-of-fit statistics for various models for maintenance data**

	Estimates							
Distribution	α	δ	b	λ	-2log(L)	AIC	CAIC	BIC
TL-TIIEHL-MO-W	3.6339×10^{2}	2.0108×10^{2}	2.8212×10^{5}	4.4166×10^{-2}	165.6286	173.6287	174.6043	180.9433
	(1.0421)	(1.8656)	(1.0935×10^{-4})	(5.3741×10^{-3})				
	a	b	λ	θ				
BOLU	7.8827×10^{-1}	1.3657	2.1053×10^{5}	1.5480×10^{6}	186.4674	194.4675	195.4431	201.7820
	(1.5594×10^{-1})	(3.3468×10^{-1})	(1.9916×10^{-6})	(2.7085×10^{-7})				
	α	λ	a	b				
HLBXII	0.2534	0.5700	7.1640	0.0606	173.5862	181.5862	182.5618	188.9008
	(0.0968)	(1.2635)	(3.1805)	(0.0605)				
	a	b	α	β				
WLx	4.3100×10^{3}	1.9468	5.6840×10^{-3}	3.4444×10^{-1}	170.1022	178.1021	179.0777	185.4167
	(2.3135×10^{-5})	(6.5230×10^{-1})	(5.9779×10^{-3})	(3.8001×10^{-1})				
	b	λ	θ	γ				
ELLOW	3.4400×10^{3}	6.9556×10^{1}	1.0359×10^{-1}	5.4301×10^{-1}	179.0476	187.0476	188.0232	194.3622
	(3.6407×10^{-10})	(3.1994×10^{-5})	(3.9733×10^{-2})	(6.5847×10^{-2})				
	a	b	α	β				
WBXII	0.4088	22.7554	0.0343	0.9939	183.3187	191.3187	192.2943	198.6333
	(2.0599)	(0.0464)	(0.0046)	(0.1592)				
	λ	θ	c	k				
TIIGIEBIII	6.1693×10^{-5}	7.8035×10^{-2}	2.6610×10^{-2}	1.0147×10^{-2}	267.4038	275.4009	276.3765	282.7155
	(1.0029×10^{-5})	(4.0142×10^{-7})	(4.2707×10^{-3})	(3.8564×10^{-8})				
	α	β	b	λ				
TLOBIIILLoG	9.9576×10^{3}	1.7594×10^{-1}	9.5948×10^{1}	6.1994×10^{-5}	166.2650	174.2650	175.2406	181.5796
	(3.9381×10^{-11})	(1.3858×10^{-2})	(1.3439×10^{-5})	(7.7697×10^{-6})				

The results on Table 18 show that the TL-TIIEHL-MO-W distribution performs better than other equi-parameter models, as it has the lowest values of the goodness-of-fit statistics: $-2log(L), AIC, CAIC$ and BIC.

8.3.2. Type II Right Censoring Type II right censored maintenance data are: 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4.

Table 19. **Parameter estimates and goodness-of-fit statistics for various models for maintenance data**

	Estimates							
Distribution	α	δ	b	λ	-2log(L)	AIC	CAIC	BIC
TL-TIIEHL-MO-W	1.2737e $\times 10^{3}$	7.3720×10^{2}	2.5443×10^{5}	5.2458×10^{-2}	147.8421	155.8421	156.8177	163.1567
	(4.2835)	(7.3840)	(1.8093×10^{-3})	(6.9523×10^{-3})				
	a	b	λ	θ				
BOLU	9.6997×10^{-1}	2.0323	2.5185×10^{5}	2.7347×10^{6}	169.3003	177.3003	178.2759	184.6149
	(2.0301×10^{-1})	(5.5514×10^{-1})	(4.3407×10^{-6})	(3.9975×10^{-7})				
	α	λ	a	b				
HLBXII	0.3021	0.4968	13.3801	0.0356	151.8897	159.8897	160.8653	167.2042
	(0.1130)	(1.0320)	(6.9801)	(0.0316)				
	a	b	α	β				
WLx	1.4219×10^{4}	2.1330	4.9147×10^{-3}	4.8994×10^{-1}	155.2298	163.2298	164.2054	170.5444
	(2.1490×10^{-6})	(7.2450×10^{-1})	(5.4446×10^{-3})	(5.6285×10^{-1})				
	b	λ	θ	γ				
ELLOW	2.6150×10^{3}	3.1441×10^{1}	9.9670×10^{-2}	6.1241×10^{-1}	163.8388	171.8388	172.8145	179.1534
	(7.7660×10^{-10})	(7.6716×10^{-5})	(3.9890×10^{-2})	(7.7721×10^{-2})				
	a	b	α	β				
WBXII	0.2934	28.0449	0.0313	0.9949	168.2056	176.2056	177.1812	183.5202
	(1.4247)	(0.0240)	(0.0043)	(0.1246)				
	λ	θ	c	k				
TIIGIEBIII	9.7616	33.9838	0.1983	3.8026	149.4696	157.4696	158.4452	164.7842
	(23.8112)	(45.3576)	(0.1647)	(0.6371)				
	α	β	b	λ				
TLOBIIILLoG	3.9328×10^{2}	2.4424	1.6663	2.1621×10^{-3}	151.2862	159.2862	160.2618	166.6007
	(4.1594×10^{-7})	(1.3130×10^{-1})	(1.3947×10^{-1})	(2.7346×10^{-4})				

The TL-TIIEHL-MO-W distribution performs better than competing models when applied to censored data. Results in Table 19 show that TL-TIIEHL-MO-W distribution has lower values of $-2log(L), AIC, CAIC$ and BIC as compared to other models, hence showing better fit.

9. Conclusions

A new generalized family of distributions called the Topp-Leone type II Exponentiated Half Logistic-Marshall-Olkin-G (TL-TIIEHL-MO-G) distribution was developed and presented. The density of the new family of distributions can be expressed as an infinite linear combination of exponentiated-G densities. The hazard rate function of the new family of distributions can be quiet flexible for specified baseline distributions. We also obtain closed form expressions for the moments, distribution of order statistics, probability weighted moments and Rényi entropy. The method of maximum likelihood estimation (MLE) was used to estimate the model parameters. Performance of the special case of the TL-TIIEHL-MO-W distribution was examined by conducting various simulations for different sample sizes. Finally, the special case of TL-TIIEHL-MO-W distribution was fitted to real data (both censored and complete) sets to illustrate the applicability and usefulness of the proposed family of distributions.

Appendix

Probability Weighted Moments
Using the generalized binomial series expansion, we obtain

$$\begin{aligned}\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{br+b-1} &= \sum_{i=0}^{\infty}(-1)^i\binom{br+b-1}{i}\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha i}\\ &= \sum_{i=0}^{\infty}(-1)^i\binom{br+b-1}{i}\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{2\alpha i}\\ &\times \left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-2\alpha i},\end{aligned}$$

and

$$\begin{aligned}f(x)[F(x)]^r &= 4b\alpha\sum_{i=0}^{\infty}(-1)^i\binom{br+b-1}{i}\delta^{2\alpha i+2\alpha}\bar{G}^{2\alpha i+2\alpha-1}(x;\xi)\\ &\times \left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+1)}\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha+1)} g(x;\xi).\end{aligned}$$

Considering

$$\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha+1)} = \sum_{j=0}^{\infty}(-1)^j\binom{2\alpha i+2\alpha+j}{j}\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^j$$

and

$$\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^j = \sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\delta^k\bar{G}^k(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-k},$$

we can write

$$\begin{aligned}f(x)[F(x)]^r &= 4b\alpha\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{br+b-1}{i}\binom{2\alpha i+2\alpha+j}{j}\binom{j}{k}\delta^{2\alpha i+2\alpha+k}\\ &\times \bar{G}^{2\alpha i+2\alpha+k-1}\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+k+1)} g(x;\xi).\end{aligned}$$

Note that

$$\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha+k+1)} = \sum_{l=0}^{\infty}\frac{\Gamma(l+2\alpha i+2\alpha+k+1)}{l!\Gamma(2\alpha i+2\alpha+k+1)}\bar{\delta}^l\bar{G}^l(x;\xi)$$

and

$$[1-G(x;\xi)]^{l+2\alpha i+2\alpha+k+1} = \sum_{m=0}^{\infty}(-1)^m\binom{l+2\alpha i+2\alpha+k+1}{m}G^m(x;\xi).$$

Order Statistics

Distribution of order statistics is derived as follows; From equations (7) and (8), we have

$$f(x)[F(x)]^{i+j-1} = \frac{4b\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^2}\left[\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right]^{2\alpha-1} \times \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b(i+j)-1}.$$

Using the generalized binomial series expansion, we obtain

$$\begin{aligned}\left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{b(i+j)-1} &= \sum_{k=0}^{\infty}(-1)^k\binom{b(i+j)-1}{k} \times \left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha k} \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{b(i+j)-1}{k}\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{2\alpha k} \times \left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-2\alpha k},\end{aligned}$$

so that

$$\begin{aligned}f(x)[F(x)]^{i+j-1} &= 4b\alpha\sum_{k=0}^{\infty}(-1)^k\binom{b(i+j)-1}{k}\delta^{2\alpha k+2\alpha}\bar{G}^{2\alpha k+2\alpha-1}(x;\xi) \\ &\times \left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha k+2\alpha+1)} \\ &\times \left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha k+2\alpha+1)} g(x;\xi).\end{aligned}$$

Consider

$$\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha k+2\alpha+1)} = \sum_{l=0}^{\infty}(-1)^l \times \binom{2\alpha k+2\alpha+l}{l}\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^l,$$

$$\left[1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^l = \sum_{m=0}^{\infty}(-1)^m\binom{l}{m}\delta^m\bar{G}^m(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-m},$$

$$\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha k+2\alpha+m+1)} = \sum_{n=0}^{\infty}\frac{\Gamma(n+2\alpha k+2\alpha+m+1)}{n!\Gamma(2\alpha k+2\alpha+m+1)}\bar{\delta}^n\bar{G}^n(x;\xi),$$

and

$$[1-G(x;\xi)]^{n+2\alpha k+2\alpha+m+1} = \sum_{p=0}^{\infty}(-1)^p\binom{n+2\alpha k+2\alpha+m+1}{p}G^p(x;\xi).$$

We can write

$$\begin{aligned} f(x)[F(x)]^{i+j-1} &= 4b\alpha \sum_{k,l,m,n,p=0}^{\infty} (-1)^{k+l+m+p} \binom{b(i+j)-1}{k} \binom{2\alpha k+2\alpha+l}{l} \binom{l}{m} \\ &\times \binom{n+2\alpha k+2\alpha+m+1}{p} \frac{\Gamma(n+2\alpha k+2\alpha+m+1)}{n!\Gamma(2\alpha k+2\alpha+m+1)} \\ &\times \delta^{2\alpha k+2\alpha+m} \bar{\delta}^{n} g(x;\xi) G^{p}(x;\xi) \\ &= \sum_{p=0}^{\infty} s_{p+1} g_{p+1}(x;\xi). \end{aligned}$$

Rényi Entropy
Considering

$$\begin{aligned} \left[1-\left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha}\right]^{v(b-1)} &= \sum_{i=0}^{\infty}(-1)^{i}\binom{v(b-1)}{i} \\ &\times \left(\frac{\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}}{1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)}\right)^{2\alpha i}, \end{aligned}$$

we have

$$\begin{aligned} f^{v}(x) &= (4b\alpha)^{v}\sum_{i=0}^{\infty}(-1)^{i}\binom{v(b-1)}{i}\delta^{2\alpha i+v(2\alpha-1)}\bar{G}^{2\alpha i+v(2\alpha-1)}(x;\xi) \\ &\times \left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha v+v)}\left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha v+v)} g^{v}(x;\xi). \end{aligned}$$

Consider

$$\begin{aligned} \left[1+\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)\right]^{-(2\alpha i+2\alpha v+v)} &= \sum_{j=0}^{\infty}(-1)^{j}\binom{2\alpha i+2\alpha v+v+j-1}{j} \\ &\times \left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{j} \end{aligned}$$

and

$$\left(1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{j} = \sum_{k=0}^{\infty}(-1)^{k}\binom{j}{k}\delta^{k}\bar{G}^{k}(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-k},$$

we have

$$\begin{aligned} f^{v}(x) &= (4b\alpha)^{v}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{v(b-1)}{i}\binom{2\alpha i+2\alpha v+v+j-1}{j}\binom{j}{k}\delta^{2\alpha i+v(2\alpha-1)+k} \\ &\times \bar{G}^{2\alpha i+v(2\alpha-1)+k}(x;\xi)\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha v+v+k)} g^{v}(x;\xi). \end{aligned}$$

Now, with

$$\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^{-(2\alpha i+2\alpha v+v+k)} = \sum_{l=0}^{\infty}\frac{\Gamma(l+2\alpha i+2\alpha v+v+k)}{l!\Gamma(2\alpha i+2\alpha v+v+k)}\bar{\delta}^{l}\bar{G}^{l}(x;\xi)$$

and

$$\left[1-G(x;\xi)\right]^{l+2\alpha i+v(2\alpha-1)+k} = \sum_{m=0}^{\infty}(-1)^m\binom{l+2\alpha i+v(2\alpha-1)+k}{m}G^m(x;\xi),$$

Elements of the Score Vector

The first derivatives of the log-likelihood function with respect to each of the parameters in $\Delta=(b,\alpha,\delta,\xi)^T$ and are given by

$$\frac{\partial\ell_n}{\partial b} = \frac{n}{b}+\sum_{i=1}^{n}log\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right],$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\alpha} &= \frac{n}{\alpha}+2\sum_{i=1}^{n}log\left[\delta\bar{G}(x;\xi)\right]-2\sum_{i=1}^{n}log\left[1-\bar{\delta}\bar{G}(x;\xi)\right]-2\sum_{i=1}^{n}log\left[2-W\right]\\ &- (b-1)\sum_{i=1}^{n}\frac{2\left(\frac{W}{2-W}\right)^{2\alpha}\ell_n\left(\frac{W}{2-W}\right)}{\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\delta} &= \frac{n}{\delta}+(2\alpha-1)\sum_{i=1}^{n}\frac{1}{\delta}-(2\alpha+1)\sum_{i=1}^{n}\frac{\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}+(2\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial W}{\partial\delta}}{2-W}\\ &- (b-1)\sum_{i=1}^{n}\frac{\frac{\partial}{\partial\delta}\left(\frac{W}{2-W}\right)^{2\alpha}}{\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell_n}{\partial\xi_k} &= \sum_{i=1}^{n}\frac{[g(x;\xi)]'}{g(x;\xi)}+(2\alpha-1)\sum_{i=1}^{n}\frac{\left[\bar{G}(x;\xi)\right]'}{\bar{G}(x;\xi)}+(2\alpha+1)\sum_{i=1}^{n}\frac{\bar{\delta}\left[\bar{G}(x;\xi)\right]'}{1-\bar{\delta}\bar{G}(x;\xi)}\\ &+ (2\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial W}{\partial\xi_k}}{2-W}-(b-1)\sum_{i=1}^{n}\frac{\frac{\partial}{\partial\xi_k}\left(\frac{W}{2-W}\right)^{2\alpha}}{\left[1-\left(\frac{W}{2-W}\right)^{2\alpha}\right]},\end{aligned}$$

where $\frac{\partial W}{\partial\delta}=\frac{\bar{G}(x;\xi)G(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2}$, $[g(x;\xi)]'=\frac{\partial g(x;\xi)}{\partial\xi_k}$, $\left[\bar{G}(x;\xi)\right]'=\frac{\partial\bar{G}(x;\xi)}{\partial\xi_k}$ and

$$\frac{\partial}{\partial\delta}\left(\frac{W}{2-W}\right)^{2\alpha} = 2\alpha\left(\frac{W}{2-W}\right)^{2\alpha-1}\left(\frac{\frac{2\bar{G}(x;\xi)G(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2}}{\left[2-W\right]^2}\right).$$

Hazard Rate Function Plots for Nested Models

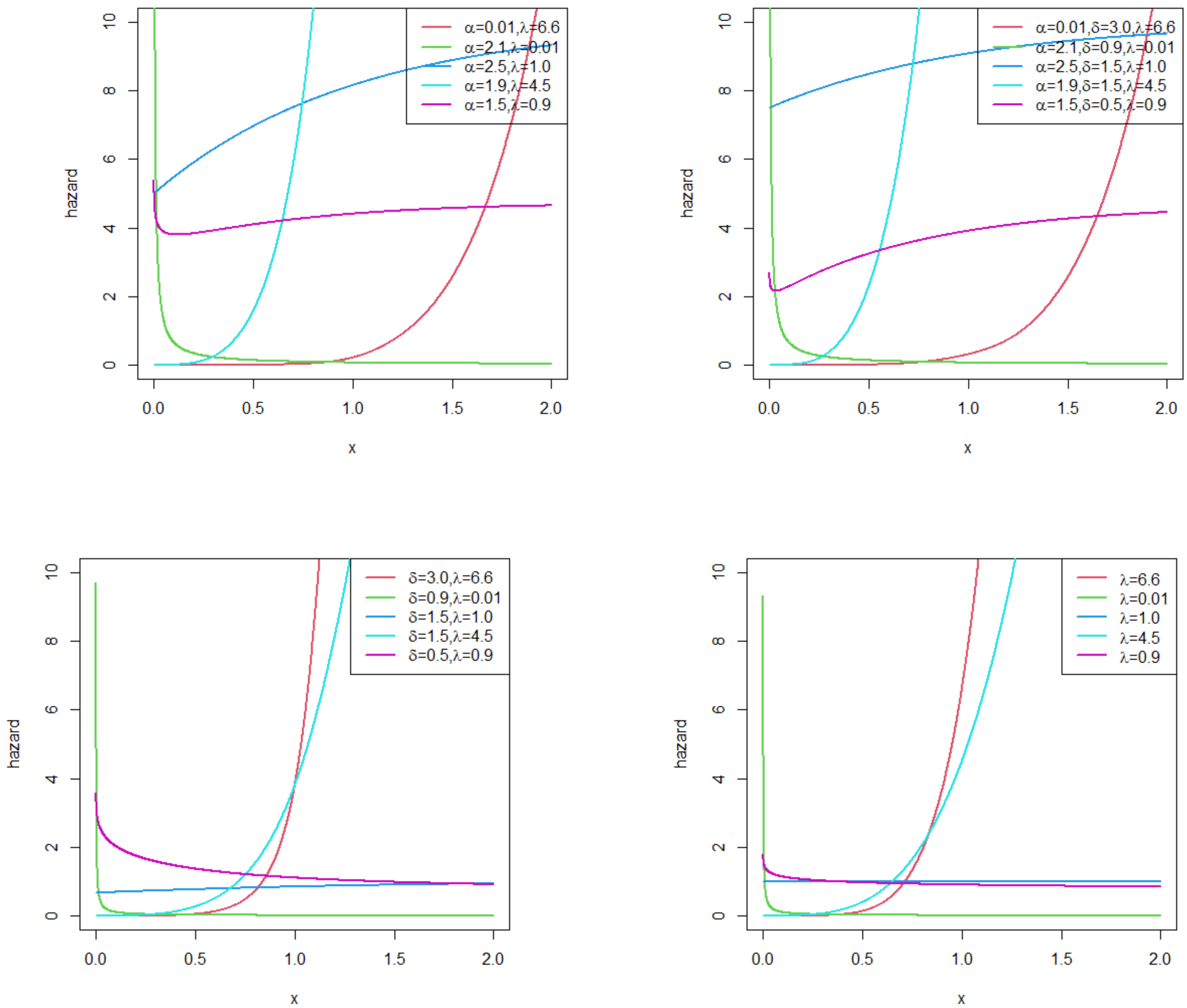

Figure 13. Plots of the Hazard Rate Function for the TIIEHLW, TIIEHLMOW, MOW and Weibull Distributions

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